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Kulikov's problem on universal torsion-free abelian groups revisited

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Abstract

Let T be a torsion abelian group. We consider the class of all torsion-free abelian groups G satisfying $\mathrm{Ext}(G, T) = 0$ and search for λ-universal objects in this class. We show that, for certain T, there is no ω-universal group. However, for uncountable cardinals λ there is always a λ-universal group if we assume $(V = L)$. Together with results by the second author this solves completely a problem by Kulikov.

Introduction

We consider Kulikov's problem on the existence of λ-universal groups. As defined in the abstract, a torsion-free abelian group G is λ-universal for a torsion abelian group T and a cardinal λ if G is of rank less than or equal to λ, $\mathrm{Ext}(G, T) = 0$ and every torsion-free abelian group H of rank less than or equal to λ satisfying $\mathrm{Ext}(H, T) = 0$ embeds into G. Kulikov [**6**] asked if, for any (uncountable) λ and arbitrary T, there is always a λ-universal group for T. The first attempt for a solution was made by the second author in [**5**], where it was proved that the answer to Kulikov's question is yes whenever the torsion group T has only finitely many non-trivial bounded primary components and either λ is countable or we are working in Gödel's universe L. It was then later shown in [**4**] that indeed the answer can consistently be no in other models of Zermelo–Fraenkel axioms of set theory and the axiom of choice (ZFC).

Here we treat the remaining cases and prove that for torsion groups T with infinitely many non-trivial bounded primary components there is no ω-universal group for T but always a λ-universal group if λ is uncountable and $V = L$ holds.

Our notations are standard and for unexplained notions we refer to [**3**] or [**2**]. All groups under consideration are abelian, the set of primes is denoted by Π and all rational groups $R \subseteq \mathbb{Q}$ are assumed to contain the 1. By Ext we denote the first derived functor $\mathrm{Ext}^1_{\mathbb{Z}}$ of the Hom functor and by $\mathrm{rk}(G)$ we mean the rank $\dim_{\mathbb{Q}}(G \otimes \mathbb{Q})$ of a torsion-free group G.

1. *λ-universal groups*

In this section, we recall the notion of λ-universal group for a given torsion group T and state some results from previous papers.

Definition 1.1. If T is a torsion group and λ a cardinal, then a group G is called *λ-universal for T* if the following conditions are satisfied:

(1) $\mathrm{Ext}(G, T) = 0$;

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(2) $\mathrm{rk}(G) \leqslant \lambda$;
(3) if H is torsion-free with $\mathrm{rk}(H) \leqslant \lambda$ and $\mathrm{Ext}(H, T) = 0$, then H embeds into G.

Kulikov [6, Question 1.66] raised the following problem.

QUESTION 1.2. Do λ-universal groups exist for all uncountable cardinals λ and for all torsion groups?

Obviously, if T is a cotorsion group, that is, $\mathrm{Ext}(\mathbb{Q}, T) = 0$, then there is a λ-universal group $G_\lambda = \mathbb{Q}^{(\lambda)}$ for T for any cardinal λ. However, in general the answer was not known until the following was proved by the second author in [5].

PROPOSITION 1.3. *Let T be a torsion abelian group. Then the following properties are satisfied.*

(1) *For $n \in \omega$ there is an n-universal group for T if and only if T has only finitely many non-trivial bounded p-components.*

(2) *If T has only finitely many non-trivial bounded p-components, then there is an ω-universal group for T.*

(3) *Assuming $(V = L)$ and that T has only finitely many non-trivial bounded p-components, there is a λ-universal group for T for any uncountable cardinal λ.*

It thus seems reasonable that the structure of T and also some additional set theory plays a role in giving the answer to Kulikov's question. In fact, it was shown in [4] that the non-existence of λ-universal groups (for any λ and any T not cotorsion) is consistent with Zermelo–Fraenkel axioms of set theory and the axiom of choice.

2. *ω-universal groups*

For our investigations we need the following lemma, which is well known. However, we include the proof for the convenience of the reader.

First recall that the basic subgroup B of a torsion group T is the direct sum $B = \bigoplus_{p \in \Pi} B_p$ of the basic subgroups B_p of the p-components T_p; for each prime p, B_p is a direct sum of cyclic p-groups, B_p is a pure subgroup of T_p and the quotient T_p/B_p is divisible (see [3]).

LEMMA 2.1. *Let T be a torsion group and $B \subseteq T$ be a basic subgroup of T. Then, for any group G, we have $\mathrm{Ext}(G, T) = 0$ if and only if $\mathrm{Ext}(G, B) = 0$.*

Proof. The short exact sequence $0 \to B \to T \to T/B \to 0$ induces the exact sequence $\mathrm{Ext}(G, B) \to \mathrm{Ext}(G, T) \to \mathrm{Ext}(G, T/B) = 0$ where the last term is zero since T/B is divisible. Thus, $\mathrm{Ext}(G, B) = 0$ implies $\mathrm{Ext}(G, T) = 0$.

Conversely, assume $\mathrm{Ext}(G, T) = 0$. By Fuchs [3, Theorem 36.1], B is an epimorphic image of T and thus $\mathrm{Ext}(G, B) = 0$ by the closure properties of the Ext-functor. □

The main ingredient of the proofs in [5] is the following useful result due to Baer [1].

Lemma 2.2 (Baer). *Let T be a torsion and G be a torsion-free group such that* $\mathrm{Ext}(G,T)=0$. *Then the following hold.*

(B-i) *If $Q=\{p_1,\ldots,p_i,\ldots\}$ is an infinite set of different primes for which $p_iT<T$, then G contains no pure subgroup S of finite rank such that G/S has elements not equal to* 0 *divisible by all $p\in Q$.*

(B-ii) *If, for some prime p, the reduced part of the p-component of T is unbounded, then G contains no pure subgroup S of finite rank such that G/S has elements not equal to* 0 *divisible by all powers of p.*

Moreover, if G is countable, then (B-i) *and* (B-ii) *suffice for* $\mathrm{Ext}(G,T)$ *to be zero.*

We first show that the above conditions are equivalent to the following two conditions, which will become more handy in our situation.

Lemma 2.3. *Let T be a torsion and G be a torsion-free group such that* $\mathrm{Ext}(G,T)=0$. *Then the following hold.*

(S-i) *If $Q=\{p_1,\ldots,p_i,\ldots\}$ is an infinite set of different primes for which $p_iT<T$, then G contains no finite rank free subgroup L such that $t_p(L_*/L)\neq 0$ for all $p\in Q$.*

(S-ii) *If, for some prime p, the reduced part of the p-component of T is unbounded, then G contains no pure subgroup S of finite rank such that G/S has elements not equal to* 0 *divisible by all powers of p.*

Moreover, if G is countable, then (S-i) *and* (S-ii) *suffice for* $\mathrm{Ext}(G,T)$ *to be zero.*

Proof. In order to prove the statement, it suffices to show that conditions (i) and (ii) from Lemma 2.2 and conditions (i) and (ii) from Lemma 2.3 are equivalent. Obviously conditions (B-ii) and (S-ii) are the same and without loss of generality we may assume that T is reduced.

Assume first that condition (B-i) holds and let L be a finite rank free subgroup of G such that $t_p(L_*/L)\neq 0$ for all $p\in Q$ for some infinite set of primes Q with $pT<T$ for $p\in Q$. We have to obtain a contradiction. Suppose that L has rank $n>0$ and let $L=\bigoplus_{l<n}\mathbb{Z}x_l$. For $k<n$ put $L^k=\bigoplus_{l<k}\mathbb{Z}x_l$, hence $L^0=\{0\}$ and $L^n=L$. By assumption, for each $p\in Q$, there is $y_p\in L_*\backslash L$ such that $py_p\in L$. Let k_p be minimal such that $y_p\in L_*^{k_p+1}$. Without loss of generality, we may assume that there is no $y'\in L_*\backslash L$ such that $py'\in L$ and $y'\in L_*^{k'_p+1}$ for some $k'_p<k_p$ (possibly by changing the choice of y_p). Now $L_*^{k_p+1}/L_*^{k_p}$ is a torsion-free group of rank 1 and $x_{k_p}+L_*^{k_p}$ and $y_p+L_*^{k_p}$ are non-zero elements of $L_*^{k_p+1}/L_*^{k_p}$. By the choice of k_p we conclude that $py_p\in\mathbb{Z}x_{k_p}+L_*^{k_p}$ and let

$$(+)\quad py_p=a_px_{k_p}+z_p,$$

for some $z_p\in L_*^{k_p}$. We distinguish two cases.

Case 1: If p does not divide a_p for infinitely many $p\in Q$ (and without loss of generality for all $p\in Q$), then by a pigeonhole argument we may assume that $k_p=k$ for a fixed k and all $p\in Q$. Hence, the above equation (+) reduces to $py_p=a_px_k+z_p$ and shows that $x_k+L_*^k\in L_*^{k+1}/L_*^k$ is divisible by all $p\in Q$, contradicting condition (B-ii).

Case 2: If p divides a_p for (almost) all primes $p\in Q$, then the above equation (+) shows that z_p is divisible by p and we let $z_p=pw_p$ for some $w_p\in L_*^{k_p}$. Now, on the one hand, $py_p\in L^{k_p+1}$ and $a_px_{k_p}\in L^{k_p+1}$, so $z_p=py_p-a_px_{k_p}\in L^{k_p+1}$. But $z_p\in L_*^{k_p}$, hence $z_p\in L^{k_p}$.

On the other hand, dividing equation (+) by p shows that $y_p=(a_p/p)x_{k_p}+w_p$. Hence, if $w_p\in L^{k_p}$, then $y_p=(a_p/p)x_{k_p}+w_p\in L^{k_p+1}\subseteq L$, contradicting the fact that $y_p\notin L$. Thus, $w_p\notin L^{k_p}$ but $w_p\in L_*^{k_p}$ and $pw_p\in L^{k_p}$, which is a contradiction to the choice of y_p and k_p.

Assume now that condition (S-ii) holds and let S be a finite rank pure subgroup of G such that G/S contains an element $g + S \neq 0$ that is divisible for all $p \in Q$ for some infinite set of primes Q with $pT < T$ for $p \in Q$. We must show a contradiction to condition (S-ii). Choose a maximal free subgroup $L' = \bigoplus_{l<n}$ of S and put $L = L' \oplus \mathbb{Z}g$. Moreover, let $g + S = px_p + S$ for some $x_p \in G$ and all $p \in Q$. Using the fact that $x_p \in L_*$ for all $p \in Q$, it is now easy to check that $t_p(L_*/L) \neq \{0\}$ for all $p \in Q$, contradicting condition (S-ii). This completes the proof. □

It will be helpful to see what conditions (S-i) and (S-ii) mean for subgroups $G \subseteq \mathbb{Q}$ of the rational numbers. It is easy to see that in this case (S-i) and (S-ii) reduce to the following conditions.

(1) If $1/p \in G$ for some infinite set of primes Q, then $T_p = 0$ for almost all primes $p \in Q$.
(2) If $1/p^n \in G$ for some prime p and all $n \in \omega$, then T must be bounded.

Now let T be any torsion group and put $R_T = \langle 1/p^n : n \in \omega,\ T_p \textit{ bounded}\rangle$. Then the above implies that any rank 1 group $S \subseteq \mathbb{Q}$ such that $\mathrm{Ext}(S, T) = 0$ embeds into R_T. However, $\mathrm{Ext}(R_T, T) = 0$ if and only if T has only finitely many non-trivial bounded p-components. This motivates in some way the results from Proposition 1.3.

We now use the above characterization to show that, in case the torsion group T has infinitely many non-trivial bounded p-components, there is no ω-universal group for T. This complements the results in [**5**] and solves Kulikov's problem completely in the countable case.

THEOREM 2.4. *Let T be a torsion abelian group that has infinitely many bounded non-trivial p-components. Then there is no λ-universal group for T for any cardinal $\aleph_0 \leqslant \lambda < 2^{\aleph_0}$.*

Proof. Let T be a (reduced) torsion group and assume that $P = \{p : T_p \neq 0 \textit{ but } T_p \textit{ is bounded}\}$ is infinite. Suppose $\aleph_0 \leqslant \lambda < 2^{\aleph_0}$ is a cardinal and there is a λ-universal group G for T. We list the elements from P by $P = \{p_n : n \in \omega\}$ and let

$$D = \bigoplus_{m\in\omega} \mathbb{Q}x_m \oplus \mathbb{Q}x_*$$

be a vector space over the field of rational numbers of countably infinite dimension. For each sequence $\eta \in {}^{\omega}2$ we now define a countable torsion-free group G_η such that $\mathrm{Ext}(G_\eta, T) = 0$. Let G_η be the subgroup of D generated by

$$G_\eta := \langle \{x_*, x_n : n \in \omega\} \cup \{y^\eta_{n,k} : n, k \in \omega\}\rangle,$$

where

(1) $y^\eta_{n,0} = x_n$;
(2) $y^\eta_{n,k+1} = (1/p_n)(y^\eta_{n,k} - \eta(k))x_*$;

for $n \in \omega$. It is now a straightforward calculation that $\mathrm{Ext}(G_\eta, T) = 0$ by Lemma 2.3 since the involved primes come from P. It now suffices to prove that at most λ of the groups G_η can be embedded into G, which contradicts the universality of G since there are $2^{\aleph_0}$ many groups G_η. If not, then there is a subset $\Delta \subseteq {}^{\omega}2$ of cardinality bigger than λ and embeddings

$$h_\eta : G_\eta \longrightarrow G$$

from G_η into G for every $\eta \in \Delta$. Clearly, there are only λ possibilities for the image $h_\eta(x_*) \in G$, hence without loss of generality we may assume that $h_\eta(x_*) = y \in G$ for some fixed $y \in G$ and all $\eta \in \Delta$. Let

$$Q = \{p \in P : t_p((\mathbb{Z}y)_*/\mathbb{Z}y) \neq 0\}.$$

By Lemma 2.3 condition (S-i) and $\mathrm{Ext}(G, T) = 0$, we conclude that Q must be finite. Choose $q = p_{n_*}$ such that $q \in P\backslash Q$. Again there are at most λ choices for the image $h_\eta(x_{n_*}) \in G$, so without loss of generality we have

$$h_\eta(x_{n_*}) = w \in G,$$

for all $\eta \in \Delta$ and some fixed $w \in G$. Let $\eta \neq \nu \in \Delta$ and let $k = \min\{l \in \omega : \eta(l) \neq \nu(l)\}$. We claim that

$$h_\eta(y^\eta_{n_*,l}) = h_\nu(y^\nu_{n_*,l}),$$

for all $l \leqslant k$. For $l = 0$ this is trivial since

$$h_\eta(y^\eta_{n_*,0}) = h_\eta(x_{n_*}) = w = h_\nu(x_{n_*}) = h_\nu(y^\nu_{n_*,o}).$$

If $l < k$ and $h_\eta(y^\eta_{n_*,l}) = h_\nu(x^\nu_{n_*,l})$ by induction, then

$$\begin{aligned} h_\eta(y^\eta_{n_*,l+1}) &= h_\eta\left(\frac{1}{p_{n_*}}(y^\eta_{n_*,l} - \eta(l)x_*)\right) = \frac{1}{p_{n_*}}(h_\eta(y^\eta_{n_*,l}) - \eta(l)y) \\ &= \frac{1}{p_{n_*}}(h_\nu(y^\nu_{n_*,l}) - \eta(l)y) = h_\nu(y^\nu_{n_*,l+1}). \end{aligned}$$

We now compute in G the element

$$\begin{aligned} h_\eta(y^\eta_{n_*,k+1}) - h_\nu(y^\nu_{n_*,k+1}) &= \frac{1}{p_{n_*}}(h_\eta(y^\eta_{n_*,k} - \eta(k)y)) - \frac{1}{p_{n_*}}(h_\nu(y^\nu_{n_*,k} - \nu(k)y)) \\ &= \pm\frac{1}{p_{n_*}}y. \end{aligned}$$

Hence, y is divisible by p_{n_*} in G, but this contradicts the choice of y. □

3. *λ-universal groups in L*

In contrast to Theorem 2.4 and completing the results from [5], we now show that, for uncountable λ, there are always λ-universal groups for any torsion group T if we assume $(V = L)$.

Theorem 3.1. *Assume $(V = L)$, and let T be an abelian torsion group and λ be an uncountable cardinal. Then there is a λ-universal group for T.*

Proof. Assume that T is a torsion group and that λ is uncountable. The following proposition is well known (see [2]).

Proposition 3.2 $(V = L)$. *Let G be a torsion-free group of infinite rank and T be a torsion group. Then $\mathrm{Ext}(G, T) = 0$ if and only if G is the union of a continuous well-ordered ascending chain $\{G_\alpha : \alpha < \lambda\}$ of subgroups $(G_0 = 0)$ such that $G_{\alpha+1}/G_\alpha$ is countable, $|G_\alpha| < |G|$ and $\mathrm{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ for all $\alpha < \lambda$.*

We now proceed by constructing a tree of torsion-free groups $\{G_\eta : \eta \in {}^{\lambda>}\lambda\}$ such that the following hold for any $\eta \in {}^{\lambda>}\lambda$:

(i) $G_{\langle\rangle} = 0$ where $\langle\rangle$ denotes the empty function;
(ii) $|G_\eta| \leqslant |lg(\eta)| + \aleph_0$ where $lg(\eta)$ denotes the length of η;

(iii) $\mathrm{Ext}(G_\eta, T) = 0$;
(iv) $\langle G_{\eta\restriction\alpha} : \alpha \leqslant lg(\eta)\rangle$ is purely continuously increasing;
(v) if $\eta \in {}^{\alpha+1}\lambda$ for some $\alpha < \lambda$, then $G_\eta/G_{\eta\restriction\alpha}$ is countable and satisfies $\mathrm{Ext}(G_\eta/G_{\eta\restriction\alpha}, T) = 0$;
(vi) if $\eta \in {}^{\lambda>}\lambda$ and G_η is a pure subgroup of some torsion-free group H with $\mathrm{Ext}(H/G_\eta, T) = 0$ and H/G_η countable, then there is some $\epsilon \in \lambda$ such that there is an isomorphism from H onto $G_{\eta^\wedge\epsilon}$ extending the identity on G_η.

It is easy to see that this construction can be carried out since the number of countable torsion-free groups is $2^{\aleph_0} = \aleph_1 \leqslant \lambda$. We now want to construct the universal group. Therefore, let, for any $\eta \in {}^{\lambda>}\lambda$ and $\epsilon \in \lambda$,

$$G_{\eta^\wedge\epsilon}/G_\eta = \{x_{\eta^\wedge\epsilon,n} : n \in \omega\}.$$

Without loss of generality, we may define a $\mathbb{Q}$-vector space

$$\bar{G} = \bigoplus_{\eta\in{}^{\lambda>}\lambda,\epsilon\in\lambda} \bigoplus_{n\in\omega} \mathbb{Q}x_{\eta^\wedge\epsilon,n},$$

and can assume that $G_\eta \subseteq \bar{G}$ for all $\eta \in {}^{\lambda>}\lambda$. Let

$$G_{\text{univ}} = \left\langle \bigcup_{\eta\in{}^{\lambda>}\lambda} G_\eta \right\rangle_{\bar{G}} \subseteq \bar{G}.$$

We claim that G_{univ} is λ-universal for T. Certainly, we have $\mathrm{rk}(G_{\text{univ}}) \leqslant \lambda$ since $2^{<\lambda} = \lambda$ in L and thus it remains to prove that $\mathrm{Ext}(G_{\text{univ}}, T) = 0$ and that any other torsion-free group H with $\mathrm{rk}(H) \leqslant \lambda$ and $\mathrm{Ext}(H, T) = 0$ embeds into G_{univ}.

We first prove that G_{univ} satisfies $\mathrm{Ext}(G_{\text{univ}}, T) = 0$. Therefore, let

$${}^{\lambda>}\lambda = \langle \eta_\alpha : \alpha \leqslant \alpha_*\rangle$$

be an enumeration of ${}^{\lambda>}\lambda$ such that $\eta_\alpha \lhd \eta_\beta$ implies $\alpha < \beta$. For $\alpha \leqslant \alpha_*$ we then choose

$$H_\alpha = \langle x_{\eta^\wedge\epsilon,n} : n \in \omega,\ \eta \in {}^{\lambda>}\lambda,\ \epsilon \in \lambda,\ \eta^\wedge\epsilon \in \{\eta_\gamma : \gamma < \alpha\}\rangle_{G_{\text{univ}}}.$$

Then $H_0 = 0$ and $G_{\text{univ}} = H_{\alpha_*} = \bigcup_{\alpha<\alpha_*} H_\alpha$ is the union of the purely continuously increasing chain of subgroups H_α. Since $\mathrm{Ext}(H_{\alpha+1}/H_\alpha, T) = 0$ for all $\alpha < \alpha_*$, we conclude that $\mathrm{Ext}(G_{\text{univ}}, T) = 0$. Note that $H_{\alpha+1}/H_\alpha \cong G_{\eta^\wedge\alpha}/G_\eta$.

Finally, we have to prove that any torsion-free group H with $\mathrm{Ext}(H, T) = 0$ and $\mathrm{rk}(H) \leqslant \lambda$ embeds into G_{univ}. However, if H is such a group, then $H = \bigcup_{\alpha<\mu} H_\alpha$ with $H_{\alpha+1}/H_\alpha$ countable and $\mathrm{Ext}(H_{\alpha+1}/H_\alpha, T) = 0$ for all $\alpha < \mu$. Now, by construction there is a branch $\eta \in {}^{\lambda>}\lambda$ such that η resembles this filtration and hence $G_\eta \cong H$. So H embeds into G_{univ} as claimed and G_{univ} is indeed universal. □

References

1. R. Baer, 'The subgroup of the elements of finite order of an abelian group', *Ann. of Math.* 37 (1936) 766–781.
2. P. C. Eklof and A. Mekler, *Almost free modules, set-theoretic methods (revised edition)* (North-Holland Mathematical Library, Amsterdam, 2002).
3. L. Fuchs, *Infinite abelian groups* I, II (Academic Press, New York, 1970/73).
4. S. Shelah and L. Strüngmann, 'On Kulikov's problem and universal torsion-free abelian groups', *J. London Math. Soc.* 67 (2003) 626–642.

5. L. Strüngmann, 'On problems by Baer and Kulikov using $V = L$', *Illinois J. Math.* 46 (2002) 477–490.
6. E.I. Khukhro and V.D. Mazurov, *The Kourovka Notebook: unsolved questions in group theory*, 14th edn (Russian Academy of Sciences, Novosibirsk, Russia, 1999).

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