The Singular Cardinals Problem
Independence Results

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Abstract: Assuming the consistency of a supercompact cardinal, we prove the consistency of

1) $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\alpha+1}$, $\alpha < \omega_1$ arbitrary;

2) $\aleph_{\omega_1}$ strong limit, $2^{\aleph_{\omega_1}} = \aleph_{\alpha+1}$, $\alpha < \omega_2$ arbitrary;

3) $\aleph_\delta$ strong limit, cf $\delta = \aleph_0$, $2^{\aleph_\delta}$ arbitrarily large before the first inaccessible cardinal; for $\aleph_\delta$ "large" enough.

Our work continues that of Magidor [Mg 1][Mg 2].

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Notation:

Let $i,j,\alpha,\beta,\gamma,\xi,\zeta$ be ordinals, δ a limit ordinal, λ,μ,κ,χ cardinals (usually infinite) ℓ,k,n,m, natural numbers.

Let $P_{<\kappa}(A) = \{t\colon t \text{ a subset of cardinality } < \kappa\}$. We let t,s denote members of $P_{<\kappa}(\lambda)$.

Notation on forcing:

$\underline{P},Q,R$ denote forcing notions, i.e. partial orders, $R \subseteq Q$ means every element of R is an element of Q and on R the partial orders are equal. Let $R < Q$ mean $R \subseteq Q$, any two elements of R are compatible in R iff they are compatible in Q and every maximal antichain of R is a maximal antichain of Q.

Let $Col(\lambda,<\kappa) = \{f\colon f \text{ a partial function from } (\kappa-\lambda) \times \lambda$ to κ, $f(\alpha,i) < \alpha$, f has power $< \lambda\}$.

We let π,σ denote members of $\underline{P}$, q,r members of Q or R. We say π,σ are compatible if they have a common upper bound and equivalent if they are compatible with the same members of $\underline{P}$. Note that π,σ are equivalent iff for any generic G, $\pi \in G \iff \sigma \in G$.

For any forcing notion, let $\emptyset$ be its minimal element.

§1.

1.1. Framework: In our universe V, κ is λ_n-supercompact for $n < \omega$, $\lambda_n \le \lambda_{n+1}$, moreover the λ_n-supercompactness is preserved by any κ-directed complete forcing notion (see Laver [L]). R_n is a κ-complete forcing notion. $R < R_{n+1}$, $\Vdash_{R_n}$ "$\lambda_n^{<\kappa} = \lambda_n$". So if we force by R_n, κ is still λ_n-supercompact (more exactly $|\lambda_n|$-supercompact, as maybe λ_n was collapsed), so there is an R_n-name $\underset{\sim}{E}_n$ of a normal fine ultrafilter on $P_{<\kappa}(\lambda_n) = \{t\colon t$ a subset of λ_n of power $<\kappa\}$. Note that $P_{<\kappa}(\lambda_n)^{V^{R_n}}$ belongs to V, is included in it and is $P_{<\kappa}(\lambda_n)^V$; as forcing by R_n does not add sequences of ordinals of length $<\kappa$. But the members of $\underset{\sim}{E}_n$ (which are subsets of $P_{<\kappa}(\lambda_n)$) are not necessarily from V.

Let for $t \in P_{<\kappa}(\lambda_n)$, $\alpha < \lambda_n$, $\alpha(t)$ be the order type of $\alpha \cap t$.

Let $I_n \subseteq P_{<\kappa}(\lambda_n)$, $\Vdash_{R_n}$ "$I_n \in \underset{\sim}{E}_n$".$t \in I_n \Rightarrow [\lambda_n(t)^{<\kappa(t)} = \lambda_n(t)$ and $t \cap \kappa$ strongly inaccessible] possible as we have assumed $\Vdash_{R_n}$ "$\lambda_n^{<\kappa} = \lambda_n$".

Let $t \underset{\sim}{\subseteq} s$ mean $t \subseteq s$, $|t| < \kappa(s)$.

We let $\underset{\sim}{C}$ be an R_n-name for every n, i.e. $\underset{\sim}{C}_n$ is an $\underset{\sim}{R}_n$-name for every n, and $\Vdash_{R_n}$ "$\underset{\sim}{C}_n = \underset{\sim}{C}_{n+1}$".

<u>1.2 The forcing notion</u>: The forcing notion $\underline{P}$ we shall use is defined as follows:

An element π of $\underline{P}$ has the form:

$$<r, \bar{t}, \bar{f}, \bar{A}, \bar{G}>$$

where for some $n < \omega$:

A) $r \in R_0$

B) $\bar{t} = <t_1,\ldots,t_n>$, $t_\ell \in I_\ell$, $t_\ell \underset{\sim}{\subseteq} t_{\ell+1}$

C) $\bar{f} = <\underset{\sim}{f}_0,\ldots,\underset{\sim}{f}_n>$, $\underset{\sim}{f}_\ell$ an $R_{\ell+1}$-name

D) Let $\kappa_\ell = \kappa(t_\ell)$ - (which is the order-type of $\kappa \cap t_\ell$), then $\underset{\sim}{f}_0 \in Col(\aleph_1, < \kappa_1)$, $\underset{\sim}{f}_\ell \in Col(\lambda_n(t_n)^+, \kappa_{\ell+1})$ for $1 \leq \ell < n$ and $\underset{\sim}{f}_n \in Col(\lambda_n(t_n)^+, < \kappa)$ (i.e. those things are forced, but $\underset{\sim}{f}_\ell$ is a name of an element of V, so we omit the ~ if we know the value and write $\bar{f} \in V (f_\ell \in V)$.)

E) $\bar{A} = <\underset{\sim}{A}_\ell : n < \ell < \omega>$, $\underset{\sim}{A}_\ell$ is an R_ℓ-name of a member of $\underset{\sim}{E}_\ell$

F) $\bar{G} = <G_\ell : n < \ell < \omega>$, $\underset{\sim}{G}_\ell$ is an $R_{\ell+1}$-name of a function with domain I_ℓ, and $\underset{\sim}{G}_\ell(t) \in Col(\lambda_\ell(t)^+, <\kappa)$.

We write $n = n[\pi]$, $t_i = t_i[\pi]$ etc., or $n = n^\pi$, etc.

1.2(A) The order on $\underset{\sim}{P}$:

The order is natural: $\pi \le \sigma$ iff

A) $r^\pi \le r^\sigma$ (in R_0)

B) $n^\pi \le n^\sigma$ and $t_\ell^\pi = t_\ell^\sigma$ for $\ell = 1,\dots,n^\pi$

C) $\underset{\sim}{f}_\ell^\pi \subseteq \underset{\sim}{f}_\ell^\sigma$ for $\ell = 0,\dots,n^\pi$ (i.e. $r^\sigma \Vdash_{R_0}$ "$\underset{\sim}{f}_\ell^\pi \subseteq \underset{\sim}{f}_\ell^\sigma$)"

D) $r^\sigma \Vdash_{R_0}$ "$t_\ell^\sigma \in \underset{\sim}{A}_\ell^\pi$ and $G_\ell^\pi(t_\ell^\sigma) \subseteq \underset{\sim}{f}_\ell^\sigma$ "

for $\ell = n^\pi + 1,\dots,n^\sigma$

E) $\underset{\sim}{A}_\ell^\pi \supseteq A_\ell^\sigma$, $(\forall t \in I_\ell)$ $[\underset{\sim}{G}_\ell^\pi \subseteq \underset{\sim}{G}_\ell^\sigma]$ (i.e., this is forced by r^σ)

1.2 (B) Claim: The set of $\pi \in \underset{\sim}{P}$, such that $f^\pi \in V$ is a dense subset of $\underset{\sim}{P}$. So usually we deal with such π only.

1.3 Technical Definitions on the forcing conditions:

1.3 A Definition: For $\pi,\sigma \in \underset{\sim}{P}$ we call σ a j-direct extension of π if

a) $\pi \le \sigma$

b) $\underset{\sim}{f}_\ell[\pi] = \underset{\sim}{f}_\ell[\sigma]$ for $j \le \ell \le n^\pi$

c) $q[\sigma] \Vdash_{R_0}$ "$\underset{\sim}{G}(t_\ell) = \underset{\sim}{f}_\ell[\sigma]$" for $n^\pi < \ell \le n^\sigma$

d) $\underset{\sim}{A}_\ell[\pi] = \underset{\sim}{A}_\ell[\sigma]$ for $\ell > n^\sigma$

e) $\underset{\sim}{G}_\ell[\pi] = \underset{\sim}{G}_\ell[\sigma]$ for $\ell > n^\sigma$

Convention:
We omit j when $j = n^\pi + 1$.

1.3 B Definition: For $\pi,\sigma \in \underset{\sim}{P}$ we call σ a j-length preserving extension of π if

(a) $\pi \le \sigma$

(b) $n^\pi = n^\sigma$

(c) $\underset{\sim}{f}_\ell^\pi = \underset{\sim}{f}_\ell^\sigma$ for $\ell < j$

Convention

We omit j when $j = n^\pi + 1$

1.3.C Definition: 1) For $\pi, \sigma \in \underline{P}$, we call σ an R-extension of π if $\pi \le \sigma$, $\bar{t}^\pi = \bar{t}^\sigma, \bar{f}^\pi = \bar{f}^\sigma$, $\underset{\sim}{A}^\pi = \underset{\sim}{A}^\sigma$, $\underset{\sim}{G}^\pi = \underset{\sim}{G}^\sigma$, or at least if r^σ forces those inequalities.

2) For $\pi, \sigma \in \underline{P}$ we call σ an R-constant extension of π if $\pi \le \sigma$, $r^\pi = r^\sigma$.

1.3.D Claim and Definition:

If $j < \omega$, $\pi_1 \le \pi_2$ then there is a unique π such that π is a j-direct extension of π_1, π_2 is a j-length preserving extension of π, and $r[\pi_1] = r[\pi_2]$ [and $r[\pi] = r[\pi_1]$].

This unique π is called the upper [lower] j-interpolant of π_1, π_2. If $j = n[\pi_2] + 1$ we omit j.

1.4 The Inner Model:

The forcing $\underline{P}$ gives too much, e.g. it collapses all cardinals which are both $\le \sum_n \lambda_n$ and $> \kappa$, and maybe also κ. But we shall use an inner model. Define some $\underline{P}$-names: $\underset{\sim}{t}_n, \underset{\sim}{\kappa}_n$ are $t_n^\pi, \kappa(t_n^\pi)$ (for every large enough π in the generic set), $\underset{\sim}{F}_\ell = \cup \{f_\ell^\pi : \pi$ in the generic set$\}$, and $\underset{\sim}{C}$ as an R_0-name is a $\underline{P}$-name.

For a generic $G \subseteq \underline{P}$, we shall be interssted in the inner model $V[\langle\kappa_\ell[G], \underset{\sim}{F}_\ell[G] : \ell < \omega\rangle, \underset{\sim}{C}[G]]$; let $\underset{\sim}{V}_f$ be a P-name of this class.

1.5 Automorphism of P:

The proofs of the following are well known.

1.5 A Claim: Suppose H is an automorphism of $\underset{\sim}{P}$, then it induces naturally a permutation of the set of $\underset{\sim}{P}$-names $a \to \underset{\sim}{a}^H$ and if $\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ are $\underset{\sim}{P}$-names, $\phi(x_1, \ldots, x_n)$ a first order formula, $\pi \in \underset{\sim}{P}$, then $\pi \Vdash$ "$\phi(\underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n)$" iff $H(\pi) \Vdash_{\underset{\sim}{P}}$ "$\phi(\underset{\sim}{a}_1^H, \ldots, \underset{\sim}{a}_n^H)$". We say that H preserves a if $\Vdash_{\underset{\sim}{P}}$ "$\underset{\sim}{a} = \underset{\sim}{a}^H$".

1.5 B Claim: If $\pi \Vdash_P$ "$\underset{\sim}{a} \in \underset{\sim}{V}_f$" then there is a P-name b, such that $\pi \Vdash_P$ "$\underset{\sim}{a} = \underset{\sim}{b}$", and every automorphism of $\underset{\sim}{P}$ which preserves $\underset{\sim}{V}_f$ (i.e. preserve $\underset{\sim}{\kappa}_\ell$, $\underset{\sim}{f}_\ell$, $\underset{\sim}{C}$) preserves also $\underset{\sim}{b}$.

1.5 C Claim: Let H be a permutation of $\bigcup_{n<\omega} \lambda_n$, which maps $\lambda_n (n < \omega)$ onto themselves, $H \restriction \kappa =$ the identity:

1) H induces an automorphism of $\underset{\sim}{P}$, which we denote by H too, as follows:

$$\text{for } t \in I_n (n < \omega) \text{ let } H(t) = \{H(i) : i \in t\}$$

$$r^{H(\pi)} = r^\pi$$

$$\bar{t}^{H(\pi)} = \langle H(t_\ell^\pi) : 1 \leq \ell \leq n^\pi \rangle$$

$$\bar{f}^{H(\pi)} = \bar{f}$$

$$\underset{\sim}{A}_\ell^{H(\pi)} = \{H^{-1}(t) : t \in \underset{\sim}{A}_\ell\}$$

$$\underset{\sim}{G}_\ell^{H(\pi)}(t) = \underset{\sim}{G}_\ell^\pi(H^{-1}(t))$$

2) Note also that $\Vdash_{R_n}$ "$\{t \in I_n : H(t) = t\} \in \underset{\sim}{E}_n$" and that H preserves $\underset{\sim}{V}_f$.

1.5 D Definition: We call $\underset{\sim}{a}$ a $\underset{\sim}{V}_f$-name if it is a $\underset{\sim}{P}$-name and is preserved by any automorphism of $\underset{\sim}{P}$ preserving $\underset{\sim}{V}_f$, and $\Vdash_{\underset{\sim}{P}}$ "$\underset{\sim}{a} \in \underset{\sim}{V}_f$".

1.5 E Claim: Suppose H is a permutation of $\bigcup \lambda_n$, mapping each λ_n onto itself, $H \restriction \kappa =$ the identity.

Then for any $\pi \in \underset{\sim}{P}$, $H(\pi)$ and $H*(\pi)=\langle r^{\pi},\bar{t}^{H(\pi)},\bar{f}^{H(\pi)},\bar{A}^{\pi},G^{\pi}\rangle$ are compatible. Moreover suitable increasing of the $\bar{\underset{\sim}{A}}$ makes them equivalent.

<u>Proof</u>: By 1.5 C(2), remembering that the $\underset{\sim}{E}_{\ell}$'s are ultrafilters.

<u>1.6 Claim</u>:

Suppose a) $\pi,\sigma \in \underset{\sim}{P}$, $n(\pi) = n(\sigma) = n$, $\bar{t}^{\pi} = \bar{t}^{\sigma}, \bar{f}^{\pi} = \bar{f}^{\sigma}$ and is in V, $\underset{\sim}{A}^{\pi} = \underset{\sim}{A}^{\sigma}$ and $\underset{\sim}{G}^{\pi} = \underset{\sim}{G}^{\sigma}$.

b) every $r \in R_n$ compatible with $r[\sigma]$ is compatible with $r[\pi]$

c) $\underset{\sim}{a}$ is a $\underset{\sim}{V}_f$-name.

<u>Then</u> if $\pi \Vdash_P$ "$\underset{\sim}{a} = a$" (for some $a \in V$) then $\sigma \Vdash_{\underset{\sim}{P}}$ "$\underset{\sim}{a} = a$".

<u>Proof</u>: Easy.

<u>1.7 Definition.</u> <u>Good Cardinals for P</u>

A cardinal μ is good for R_n or for n in short, <u>if</u> $\mu = \mu^{<\kappa}$ and there are forcing notions Q_m^0, Q_m^1, $R_m = Q_m^0 * \underset{\sim}{Q}_m^1$, Q_m^0 is μ^+-complete and $\underset{\sim}{Q}_m^1$ satisfies the μ^+-chain condition (i.e. $\Vdash_{Q_m^0}$ "$\underset{\sim}{Q}_m^1$ satisfies the μ^+-chain condition").

<u>Remark</u>: 1) Really $Q_m^0 * \underset{\sim}{Q}_m^1 > R_m$ is sufficient.

2) Note that $\underset{\sim}{Q}_m^1$ is not required to be κ-complete, so $R_m = Q_m^0 \times Q_m^1$, Q_m^0 μ^+-complete, Q_m^1 satisfying the μ^+-chain condition is sufficient for this definition.

<u>1.8 The Main Lemma</u>:

A) If κ is good for every $n \geq n_0$, then in V_f κ is strong limit. Moreover every subset of $\lambda_n(\underset{\sim}{t}_n)$ belong to $L[\langle F_{\ell}:\ell \leq n\rangle]$, hence $2^{\lambda_n(t_n)} = \lambda_n(t_n)^+$ (the + is in V_f and in V too).

B) If $\mu \geq \kappa$ is good for every $n \geq n_0$, $\Vdash_{R_n}$ "μ is regular", then in V_f μ^+ is still a regular cardinal. Moreover for any function g from μ to ordinals, for some $A \in V$, $V \models$ "$|A| = \mu$" and Range(g) $\subseteq A$.

The proof is broken to a series of claims.

1.9. Notation:

For $m < \omega$, let $\underset{\sim}{P}_m^{nt} = \{<\bar{t}^\pi, \bar{f}^\pi>: \pi \in \underset{\sim}{P}, \bar{f}^\pi \in V \text{ and } n^\pi = m\}$. For any μ, $\kappa \leq \mu \leq \lambda$ we define an equivalence relation $\approx_\mu^m$ on $\underset{\sim}{P}_m^{nt}$: $<\bar{t}^1, \bar{f}^1> \approx_\mu^m <\bar{t}^2, \bar{f}^2>$ iff $t_\ell^1 \cap \mu = t_\ell^2 \cap \mu$, $f_\ell^1 = f_\ell^2$, and there is a permutation of λ_m which is the identity on μ, preserves $\kappa, \lambda_0, \ldots, \lambda_m$ and maps t_ℓ^1 onto t_ℓ^2 (all for $1 \leq \ell \leq m$).

1.9 A Claim: $\approx_\mu^m$ has $\mu^{<\kappa}$ equivalence classes, and we can find $<\bar{t}^{i,j}, \bar{f}^{i,j}>$ ($i < \mu^{<\kappa}, j < \mu^+$) such that: $<t^{i,j}, \bar{f}^{i,j}> / \approx_\mu^m$ depend on i only, every $\approx_\mu^m$-equivalence class is represented by some $<\bar{t}^{i,j}, \bar{f}^{i,j}>$, and if $\lambda_m > \mu$, there are $\alpha_{i,j} < \lambda_m$ defined when $t_m^{i,j} \not\subseteq \mu$ which belong to $t_m^{i',j'}$ iff $i = i'$, $j = j'$. We can assume moreover $t_m^{i,j} \cap t_m^{i(1),j(1)} \subseteq \mu$.

1.10. Claim:

Suppose $\pi \in \underset{\sim}{P}$, $m < \omega$, g a $\underset{\sim}{V}_f$-name of a function from μ to ordinals, $\pi \Vdash_{\underset{\sim}{P}}$ "$\underset{\sim}{t}_m \not\subseteq \mu$". Suppose further that μ is good for m and $\mu = \mu^{<\kappa}$.

Then there is a length preserving extension σ of π, such that if $\sigma \leq \sigma' \in \underset{\sim}{P}$, $i < \mu$, $\sigma' \Vdash_{\underset{\sim}{P}}$ "$g(i) = \sigma$", $m[\sigma'] = m$ then also the upper interpolant σ^{up} of σ and σ' force this. Moreover, for some set A of ordinals, $|A| \leq \mu$ ($A \in V$ of course), for every

R-extension σ^* of the lower interpolant of σ and σ', if $\sigma^* \Vdash_{\underline{P}}$ "$g(i) = \alpha$", then $\alpha \in A$.

Proof of Claim 1.10:

So $R_m = Q_m^0 * \underset{\sim}{Q}_m^1$, and $R_0 < R_m$, hence there is a pair $\langle q^0, \underset{\sim}{q}^1 \rangle \in Q_m^0 * \underset{\sim}{Q}_m^1$, such that every extension of it in $Q_m^0 * \underset{\sim}{Q}_m^1$ is compatible with r^π (which belong to R_0).

We now define for $i < \mu$, ordinals $\alpha_i < \mu^+$, and for $j < \alpha_i$ a condition $\pi_{i,j}$, such that

A) $r[\pi_{i,j}] = \langle q^0_{i,j}, \underset{\sim}{q}^1_{i,j} \rangle$, $q^0_{i,j} \le q^0_{\xi,\zeta}$ when $\langle i,j \rangle < \langle \xi,\zeta \rangle$ (i.e $i < \xi$ or $i = \xi$, $j < \zeta$).

B) for each i, $\{q^1_{i,j+1} : j+1 < \alpha_i\}$ is a maximal antichain (of $\underset{\sim}{Q}^1_m$) (i.e. $q^0_{i+1,0}$ forced this).

C) $\bar{t}[\pi_{i,j+1}] = \bar{t}^{i,j}$, $\bar{f}[\pi_{i,j+1}] = \bar{f}^{i,j}$, $n[\pi_{i,j}] = m$ (see Claim 1.9A).

D) $\pi_{0,0}$ is a direct extension of π.

E) For each $\alpha \in t^{i,j}_m$ either $\pi_{i,j+1}$ determines (i.e. forces) a value for $\underset{\sim}{g}(\alpha)$ or there is no length preserving $\pi' \ge \pi_{i,j}$ which does so.

F) For $\ell > m$, $\Vdash_{R_\ell} (\forall t \in \underset{\sim}{A}_t[\pi_{i,j+1}])(t^{i,j}_m \underset{\sim}{\subseteq} t)$.

G) For $\ell > m$, $\underset{\sim}{G}_\ell[\pi_{i,j}]$ increases, i.e. if $\langle i,j \rangle \le \langle \xi,\zeta \rangle$, $t \in I_\ell$ then $\Vdash_{R_\ell}$ " if (1) $t \in \underset{\sim}{A}_\ell[\pi_{i,j}]$ and $t \in \underset{\sim}{A}_\ell[\pi_{\xi,\zeta}]$ or (2) $i = j = 0$ then $\underset{\sim}{G}_\ell[\pi_{i,j}](t) \subseteq \underset{\sim}{G}_\ell[\pi_{\xi,\zeta}](t)$".

H) $\underset{\sim}{G}_\ell[\pi_{i,j}]$ does not increase unnecessarily, i.e. $\Vdash_{R_\ell}$ "if $t \notin \underset{\sim}{A}_\ell[\pi_{i,j}]$ or j is not a successor or $\underset{\sim}{q}^1_{i,j}$ is not in the generic set then $G_\ell[\pi_{i,j}](t)$ is the union of $G_\ell[\pi_{\xi,\zeta}]$ $\langle \xi\ \zeta \rangle < \langle i,j \rangle$, $t \in \underset{\sim}{A}[\pi_{\xi,\zeta}]$ or $i = j = 0$".

Note that $\underset{\sim}{G}_\ell[\pi_{i,j}](t)$ is increased only when $t_m^{i,j} \underset{\sim}{\subsetneq} t$ so this occurs $\leq |t|^{<\kappa(t)} = |t| = \lambda_\ell(t)$ times, but $\mathrm{col}(\lambda_\ell(t)^+, < \kappa)$ is $\lambda_\ell(t)^+$-complete, so we can continue to define. So there is no problem to carry the construction by induction on $\langle i,j\rangle$ (the α_i's are defined as $\underset{\sim}{Q}_m^1$ satisfies the μ^+-chain condition). In the end we have to define σ. For $r[\sigma]$, note first that $\{q_{i,j}^0 : i < \mu, j < \alpha_i\}$ has an upper bound $q_\mu^0 \in Q_\mu^0$ as Q_μ^0 is μ^+-complete (and by A)), and $r[\pi]$, $(q_\mu^0, \underset{\sim}{q}^1)$ are compatible by the choice of $(q^0, \underset{\sim}{q}^1)$, and let $r[\sigma]$ be any upper bound of $r, (q_\mu^0, \underset{\sim}{q}^1)$. Obviously, $\bar{t}^\sigma = \bar{t}^\pi$, $\bar{f}^\sigma = \bar{f}^\pi$. Now $\underset{\sim}{A}_\ell^\sigma = \{t \in I_\ell : t \in \underset{\approx}{A}_\ell(\pi)$ and for any $i < \mu$, $j < \alpha_i$, $t^{i,j} \underset{\sim}{\subsetneq} t$ implies $t \in \underset{\sim}{A}_\ell[\pi_{i,j+1}]$. Moreover if $s \underset{\sim}{\subsetneq} t$, $s \approx_\mu^m t^{i,j}$, $f_m^{i,j} \subseteq \kappa(t) \times \kappa(t)$, H a permutation of $\bigcup_i \lambda_i$ which is the identity except interchanging $t^{i,j}$ and s then $t = H(t) \in H[\underset{\sim}{A}_\ell[\pi_{i,j}]].\}$ $\underset{\sim}{A}_\ell$ is an R_ℓ-name as each $\underset{\sim}{A}_\ell[\pi_{i,j}]$ is, and is forced to be in $\underset{\sim}{E}_\ell$ by the normality of $\underset{\sim}{E}_\ell$ (a conclusion of it, more exactly).

Now $\underset{\sim}{G}_\ell[\sigma](t)$ is the union of $G_\ell[\pi_{i,j}](t)$, $t \in \underset{\approx}{A}[\pi_{i,j}]$ or $i = j = 0$. It is easy to check everything, because $\underset{\sim}{g} \in \underset{\sim}{V}_f$, (and use 5D, 6).

<u>1.11. Claim:</u>

Suppose $\pi \in \underline{P}$, $\underset{\sim}{g}$ a $\underset{\sim}{V}_f$-name of a function from π to ordinals, $\pi \Vdash_{\underline{P}}$ "for every $m > n^\pi$, $\underset{\sim}{t}_m \not\subseteq \mu$" and $\mu = \mu^{<\kappa}$.

Then there is a length preserving extension σ of π, such that if $\sigma \leq \sigma' \in \underline{P}$, μ good for $n[\sigma']$, $i < \mu$, $i \in t_{n[\sigma']}[\sigma']$, and $\sigma' \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(i) = \alpha$" then also the upper interpolant σ'' of σ and σ'

forces this. Also $\underline{\text{if}}$ for some $<A_i : i < \mu> \in V$, $|A_i| \le \mu$, and $\sigma \le \sigma' \in \underline{P}$, μ is good for $R_{n[\sigma']}$, $i \in \mu \cap t_{n[\sigma']}[\sigma']$, $\sigma' \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(i) = \alpha$" $\underline{\text{then}}$ $\alpha \underset{\sim}{\in} A_i$. Hence if μ is good for arbitrarily large m, if $\sigma \le \sigma' \in \underline{P}$, $\sigma' \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(i) = \alpha$" then for some direct extension σ'' of σ', the upper interpolant σ''' of σ, σ'' forces this.

$\underline{\text{Proof}}$: Repeat claim 1.10 ω times.

$\underline{\text{1.12 Proof of the Main Lemma 1.8B}}$.

Quite easy from Claim 11, because $\Vdash_{\underline{P}}$ "$\mu \subseteq \bigcup_{n<\omega} \underset{\sim}{t}_n$" and $\{\pi : \Vdash_{\underline{P}}$ "for every $m > n^\pi$, $\underset{\sim}{t}_m \not\subseteq \mu$"$\}$ is a dense subset of $\underline{P}$.

$\underline{\text{1.13 Claim}}$: Suppose $\pi \in \underline{P}$, $m > n^\pi$, $\Vdash_{\underline{P}}$ "$\underset{\sim}{g} \in \underset{\sim}{V}_f$, $\underset{\sim}{g}$ a function from μ to ordinals".

Then there is $\sigma \in \underline{P}$, $\pi \le \sigma$ such that

A) π and σ are identical except that possibly $\underset{\sim}{G}_m[\sigma]$ is not equal to $\underset{\sim}{G}_m[\pi]$,

B) Suppose $\sigma_1 \in P$, $n[\sigma_1] = m$, $r[\sigma_1] \ge r[\sigma]$, $t_\ell \in I_\ell$ for $1 \le \ell \le m$, $r[\sigma_1] \Vdash_{R_0}$ "$\underset{\sim}{G}_m(t_m) = f_m[\sigma_1]$" and $\underset{\sim}{G}_\ell[\sigma_1] = \underset{\sim}{G}_\ell[\sigma]$, $\underset{\sim}{A}_\ell[\sigma_1] = \underset{\sim}{A}_\ell[\sigma]$ for $m < \ell < \omega$.

Suppose further σ_2 is an (m-1)-length preserving extension of σ_1, $n[\sigma_2] = n[\sigma_1] = m$, $\bar{t}[\sigma_2] = \bar{t}[\sigma_1]$, $f_\ell[\sigma_2] = f_\ell[\sigma_1]$ for $\ell < m$, $\underset{\sim}{\bar{A}}[\sigma_2] = \underset{\sim}{\bar{A}}[\sigma_1]$, $\underset{\sim}{\bar{G}}[\sigma_2] = \underset{\sim}{\bar{G}}[\sigma_1]$, $i \in t_m \cap \mu$, $\sigma_2 \Vdash_{\underline{P}}$ "$g(i) = \alpha$". Then $<r[\sigma_2], \bar{t}[\sigma_1], \bar{f}[\sigma_1], \underset{\sim}{\bar{A}}[\sigma_1], \underset{\sim}{\bar{G}}[\sigma_1]>$ forces this too.

$\underline{\text{Proof}}$: Just note that for each t_m the number of $<t_\ell : 1 \le \ell < m>$,

$\langle f_\ell : \ell < m\rangle$ we have to consider is $\le |t_m|^{<\kappa(t_m)} = |t_m| \equiv \lambda_m(t_m)$ whereas $\mathrm{Col}(\lambda_m(t_m)^+, < \kappa)$ is $\lambda_m(t_m)^+$-complete.

Note also that we in fact, are interested in R_{m+1} only (not R_0).

<u>1.13A Claim</u>: The parallel claim to 13 holds for $m = n^\pi$.

<u>1.14. Corollary</u>. Suppose $\underset{\sim}{g}$ is a $\underset{\sim}{V}_f$-name, μ good for every $m \ge n[\pi]$, $\underset{\sim}{g}$ a function from μ to ordinals. Then there is π_1, a length-preserving extension of π such that:

<u>if</u> $\pi_1 \le \sigma \in P$, $i \in t_m[\sigma] \cap \mu$, $\sigma \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(i) = \alpha$" <u>then</u> also $\sigma^* \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(i) = \alpha$" where σ^* is defined by

$$r[\sigma^*] = r[\sigma]$$

$$n[\sigma^*] = n[\sigma],\ t_\ell[\sigma^*] = t_\ell[\sigma] \text{ for } \ell = 1,\dots,n[\sigma]$$

$$f_\ell[\sigma^*] = f_\ell[\sigma] \text{ for } \ell = 0,\dots,m-1$$

$$f_\ell[\sigma^*] = \underset{\sim}{G}_\ell[\pi_1](t_\ell) \text{ for } \ell = m,\dots,n[\sigma]$$

$$\underset{\sim}{A}_\ell[\sigma^*] = \underset{\sim}{A}_\ell[\pi_1] \text{ for } n[\sigma] < \ell < \omega$$

$$\underset{\sim}{G}_\ell[\sigma^*] = \underset{\sim}{G}_\ell[\pi_1] \text{ for } n[\sigma] < \ell < \omega.$$

<u>Proof</u>: Use Claim 1.11 and then 1.13 for all m.

<u>1.15 Claim</u>: Suppose π_1, g, $n = n[\pi_1]$ are as in corollary 1.14, $\mu \le \lambda(t_n^\pi)$ and $m < n[\pi_1]$, and $\underset{\sim}{g}$ is a function from μ to $\{0,1\}$ (i.e. $\underset{\sim}{g} \in V[<\kappa_\ell[G], f_\ell[G]: n \le \ell < \omega>, C]$, and note that we can get $\underset{\sim}{V}_f[G]$ from this universe by forcing by the product of n Lévy collapsing).

Then there is $\pi_2 \in \underline{P}$ such that:

A) $\pi_1 \le \pi_2$, $\bar{t}[\pi_1] = t[\pi_2]$, $\bar{\underset{\sim}{G}}[\pi_1] = \bar{\underset{\sim}{G}}[\pi_2]$

B) If σ is an m-direct extension of π_2, $\alpha < \mu$, $i < 2$

$\sigma \Vdash_{\underline{P}}$ "$g(\alpha) = i$" then also π_2 forces this.

Proof: Fix α.

Let $W_k = \{\langle\bar{t},\bar{f}\rangle$: for some m-direct extension σ of π_2, $n[\sigma] = k$,
$\bar{t}^{\pi} = \bar{t}$, $f^{\pi}\restriction k = \bar{f} \in V\}$.

For every $w = (\bar{t},\bar{f}) \in W_k$ we define an R_{k+1}-name $\underset{\sim}{V}_k(\bar{t},\bar{f})$ of an ordinal < 3: for $r \in R_k$

$r \Vdash$ "$\underset{\sim}{V}_k(\bar{t},\bar{f}) = \ell$" iff

A) $\ell < 2$, $\langle r, \bar{t},\bar{f}^{\wedge}\langle\underset{\sim}{G}_k(t_k)\rangle, \underset{\sim}{\bar{A}}\restriction[k+1,\omega), \underset{\sim}{\bar{G}}\restriction[k+1],\omega)\rangle \Vdash_{\underline{P}}$ "$\underset{\sim}{g}(\alpha) = \ell$"

or

B) $\ell = 2$, and for no r', $r \le r' \in R_{k+1}$, $\ell' < 2$
$\langle r',\bar{t},\bar{f}^{\wedge} < \underset{\sim}{G}_k(t_k)\rangle, \underset{\sim}{\bar{A}}\restriction[k+1,\omega), \underset{\sim}{\bar{G}}\restriction[k+1,\omega)\rangle \Vdash_{\underline{P}}$ "$g(\alpha) = \ell$".

Now for every $k < \omega$ we define by downward induction on $\ell \le k$, for every $(\bar{t},\bar{f}) \in W_\ell$ an $R_{\ell+1}$-name $\underset{\sim}{V}_k(\bar{t},\bar{f})$ of an ordinal < 3 and $\underset{\sim}{B}_\ell(\bar{t},\bar{f})$ of a member of $\underset{\sim}{E}_{\ell+1}$.

For $\ell = k$ we have defined $\underset{\sim}{V}_k(\bar{t},\bar{f})$, and let $\underset{\sim}{B}_k(\bar{t},\bar{f}) = I_{k+1}$.

Now let $(\bar{t},\bar{f}) \in W_\ell$, $\ell < k$, $\underset{\sim}{V}_k(\bar{t}',\bar{f}')$ is defined for $(\bar{t}',\bar{f}') \in W_{\ell+1}$. Then $\underset{\sim}{V}_k(\bar{t},\bar{f})$ is the unique $i < 3$ such that:
$(\bar{t} = \langle t_1,\ldots,t_\ell\rangle)$

$$\{t_{\ell+1} \in I_{\ell+1} : i = \underset{\sim}{V}_k(\bar{t}^{\wedge}\langle t_{\ell+1}\rangle, \bar{f}^{\wedge}\langle\underset{\sim}{G}_\ell(t_\ell)\rangle)\} \in \underset{\sim}{E}_{\ell+1}$$

(note all the names in the expression above are $\underset{\sim}{R}_{\ell+1}$-names)

and

$$\underset{\sim}{B}_k(\bar{t},\bar{f}) = \{t_{\ell+1} \in I_{\ell+1} : \underset{\sim}{V}_k(\bar{t},\bar{f}) = \underset{\sim}{V}_k(\bar{t}^{\wedge}\langle t_{\ell+1}\rangle, \bar{f}^{\wedge}\langle\underset{\sim}{G}_\ell(t_\ell)\rangle)\}$$

Now we can define $\underset{\sim}{A}_\ell[\pi_2]$:

$\underset{\sim}{A}_\ell[\pi_2] = \{t \in I_k$: for every $(\bar{t},\bar{f}) \in W_{\ell-1}$, such that $t_{\ell-1} \underset{\sim}{\subseteq} t$, and $k \ge \ell$, $t \in B_k(\bar{t},\bar{f})$, and of course $t \in \underset{\sim}{A}_\ell[\pi_1]\}$

This is fine for α, and as there are "few" (i.e. μ) α's there is no problem to prove the claim.

§2. Applications

For this section we make the following hypothesis.

2.1 Hypothesis: There is a universe V satisfying ZFC + G.C.H., in which κ is supercompact, moreover the supercompactness is preserved by κ-directed complete forcing.

Remark: We can weaken "κ is supercompact" by "κ is λ-supercompact" for λ suitable for each theorem, bur as long as we cannot get inner models with supercompact this is not so interesting, and anyhow clearly we get by our proof the expected results (or almost, replacing λ by λ^+).

Similarly for assuming G.C.H. - it is expected that violating C.G.H. is "harder" so we do not lose generality, and so though it seemed that we can get rid of it, there is no point in doing this.

Notation: $(\aleph_\alpha)^{+i} = \aleph_{\alpha+i}$ so $\lambda^{+1} = \lambda^+$.

2.2. Theorem: 1) For any $\alpha < \aleph_1$, there is an extension V_f of V in which κ is $\aleph_\omega$, is strong and $2^{\aleph_\omega} = \aleph_{\alpha+1}$.

2) Moreover in V_f there are $f_i \in \prod_{n<\omega} \aleph_n$ for $i < \aleph_{\alpha+1}$, such that for $i < j$, $f_i <^* f_j$, i.e., $\{n: f_i(n) < f_j(n)\}$ is co-finite.

Proof: 1) For $\alpha < \omega$ this is done in Magidor [Mg 1], so let $\alpha = \delta + k$, δ a limit ordinal, let $\delta = \bigcup_{n<\omega} D_n$, D_n finite, increasing. Let Q be any κ-complete forcing adding $\kappa^{+\alpha+2}$ subsets to κ and satisfying the κ^+-chain condition e.g. $Q = \{f: f \text{ a partial function from } \kappa^{+\alpha+2} \text{ to } \{0,1\} \text{ of power } < \kappa\}$.

Let $T_n = \{(\beta,\gamma): 0 \le \beta < \gamma < \delta$ and $(\forall \xi \in D_n) \neg (\beta \le \xi \le \gamma)\}$

$$R_n = (\prod_{(\beta,\gamma)\in T_n} \mathrm{Col}(\kappa^{+\beta}, < \kappa^{+\gamma})) \times Q.$$

$$\lambda_n = \kappa^{+\alpha+2}.$$

Now clearly:

Fact A: R_n is κ-directed complete, $\Vdash_{R_n} 2^\kappa < \kappa^{+\omega}$, hence for every $t \in I_n$, $\lambda_n(t) < \kappa(t)^{+\omega}$.

Fact B: $\kappa^{+\beta}$ is good for R_n if for some γ, $\beta = \gamma + 2$, $\gamma + 1 \in D_n$ or $\beta > \delta + 1$, hence every $\gamma + 2 \le \alpha + 1$ is good for R_n for every n large enough.

Fact C: In V_f (from §1) κ is $\aleph_\omega$, $2^{\kappa_n} \le \chi_n$, $2^{\chi_n} = \lambda_n^+$ and $\kappa_n < \kappa$, where $\chi_n = \lambda_n(\underset{\sim}{t}_n)$ so κ is $\aleph_\omega$ and strong limit.

Now the theorem is immediate.

2) Just change Q to add $f_i:\kappa \to \kappa$, $(i < \kappa^{+\alpha+2})$, such that $f_i <^* f_j$ for $i < j$.

2.3. Theorem: 1) For any $\alpha < \aleph_2$ there is an extension V_f of V in which κ is $\aleph_{\omega_1}$, and is strong limit, and $2^{\aleph_{\omega_1}} = \aleph_{\alpha+1}$.

2) For any fixed $\xi < \omega_1$ we can assume that $H(\aleph_\xi)^V = H(\aleph_\xi)^{V_f}$ ($H(\lambda)$ is the family of set of hereditary cardinality $< \lambda$).

Proof: 1) Just amalgamate the proof of 2.2 and of Magidor [Mg 1] §5 (see [Mg 3] too).

2) Just let in §1 $\underset{\sim}{f}_0 \in \mathrm{Col}(\aleph_\xi, < \kappa_1(t_1))$.

Remark: By Magidor [Mg 4] (improving Galvin and Hajnal) if Chang's conjecture holds, $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$. So using a method which gives 2.3(2), we cannot improve 2.3(1).

Remark: By [Sh 2], we cannot improve 2.2(1) result for $\alpha > (2^{\aleph_0})^+$ if the method gives 2.2(2) too.

2.4. Definition: 1) For a monotonic function f from ordinals to ordinals we define a function $f^{[i]}$ from ordinals to ordinals by induction on i:

$$f^{[0]}(\alpha) = \alpha$$

$$f^{[\alpha+1]}(\alpha) = f(f^{[i]}(\alpha) + 1)$$

$$f^{[\delta]}(\alpha) = \bigcup_{i<\delta} f^{[i]}(\alpha)$$

2) For a function f from ordinals to ordinals we define a function f^* from ordinals to ordinals

$$f^*(\alpha) = f^{[\alpha]}(0)$$

3) For a class C of ordinals we define by induction on α a function Suc_C^α from ordinals

$$Suc_C^0(i) = \min\{\xi: \xi \in C, \xi > i\}$$

$$Suc_C^{\alpha+1}(i) = (Suc_C^\alpha)^*(i)$$

$$Suc_C^\delta(i) = \bigcup_{\alpha<\delta} Suc_C^\alpha(i)$$

4) In 3) if C is the class of infinite cardinals, then we omit it.

2.5. Lemma: Suppose λ has cofinality $\aleph_0$, and for $\chi < \lambda$, $n < \omega$, $Suc^n(\chi) < \lambda$. Suppose further $\mu \geq \lambda$ but there is no weakly inaccessible cardinal κ, $\lambda \le \kappa \le \mu$. Then there are D_n, $D_n \subseteq D_{n+1}$, $\cup D_n \supseteq \{\chi: \lambda \le \chi \le \mu, \chi \text{ a cardinal}\}$, and $\mathrm{Suc}^n_{D_n}(\kappa) \ge \mu$.

<u>Proof</u>: We prove, by induction on μ, the existence of $\langle D_n(\mu): n < \omega\rangle$ as required.

<u>Case I</u>: $\mu = \lambda$.

We let $D_n(\mu) = \{\lambda\}$, and there is no problem.

<u>Case II</u>: $\mu = \chi^+$ for some χ, or even $\chi < \mu \le \beth_\chi$.

Let $D^* = \{\kappa: \chi < \kappa\}$, $D_0(\mu) = \{\kappa: k \ge \mu\}$, $D_1(\mu) = D^*$

$D_{n+2}(\mu) = D_n(\chi) \cup D^*$.

So $D_n(\mu)$ $(n < \omega)$ is increasing, with union $\supseteq \{\kappa: \lambda \le \kappa \le \mu\}$; $\mathrm{Suc}^n_{D_n}(\chi)^{(\lambda)} \ge \chi$, hence $\mathrm{Suc}^{n+2}_{D_n(\chi)}(\lambda) \ge \beth_\chi$ hence is $\ge \mu$, so $\mathrm{Suc}^n_{D_n(\mu)}(\lambda) \ge \mu$ for $n > 1$; for $n = 0,1$ this is true too by the definition.

<u>Case III</u>: μ is a limit cardinal, but for $\chi < \mu$, $\beth_\chi < \mu$.

Let $\mu = \sum_{i<\chi} \mu_i$, $\chi = \mathrm{cf}\mu$, $\mu_i < \mu$, μ_i increasing continuous, $\mu_0 = \beth_0$.

Let $D_0(\mu) = D_1(\mu) = \{\kappa: \kappa \ge \mu\}$, $D_{n+2}(\mu) = \{\chi:$ for some i, $\mu_i \le \chi < \mu_{i+1}$, $\chi \in D_n(\mu_{i+1})$ and $\beth_i \in D_n(\beth_\chi)\}$. The checking is easy.

<u>2.6. Theorem</u>: 1) For any $\mu > \kappa$, $\mu^\kappa = \mu$, smaller than the first inaccessible cardinal $> \kappa$, there is a forcing extension V_f of V, in which κ is $\mathrm{Suc}^\omega(\beth_0)$, is strong limit, $2^\kappa = \mu$, and no cardinal in $[\kappa,\mu]$ is collapsed, except possibly successors of singular cardinals.

2) Moreover we can assume that in V_f there are functions $f_i \in \prod_{n<\omega} \kappa_n$ ($i < \mu$ where $\kappa = \sum_{n<\omega} \kappa_n$, $\kappa_n < \kappa_{n+1}$) such that for $i < j$, $f_i <^* f_j$.

<u>Proof</u>: Similar to the proof of 2.2., using lemma 2.5 (so for $t \in I_n$, $\mathrm{Suc}^n(\kappa(t)) \geq \lambda(t)$).

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