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Resolvability vs. almost resolvability

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ABSTRACT

A space X is *κ-resolvable* (resp. *almost κ-resolvable*) if it contains κ dense sets that are pairwise disjoint (resp. almost disjoint over the ideal of nowhere dense subsets of X). Answering a problem raised by Juhász, Soukup, and Szentmiklóssy, and improving a consistency result of Comfort and Hu, we prove, in ZFC, that for every infinite cardinal κ there is an almost 2^κ-resolvable but not ω_1-resolvable space of dispersion character κ.

A space X is said to be *κ-resolvable* if it contains κ dense sets that are pairwise disjoint. X is called *maximally resolvable* iff it is $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|G|: G \neq \emptyset \text{ open}\}$ is the *dispersion character* of X.

V. Malykhin, in [4], was the first to suggest studying families of dense sets of a space X that, rather than disjoint, are merely *almost disjoint* with respect to the ideal $\mathcal{N}(X)$, where $\mathcal{N}(X)$ denotes the family of all nowhere dense subsets of the space X. He called a space X *extraresolvable* if it has $\Delta(X)^+$ many dense sets such that any two of them have nowhere dense intersection. This idea was generalized in [3], where the natural notion of *almost κ-resolvability* was introduced: A space X is called *almost κ-resolvable* if it contains κ dense sets that are pairwise almost disjoint over the ideal $\mathcal{N}(X)$ of nowhere dense subsets of X. (Actually, this concept was given a different name in [3], namely: "κ-extraresolvable", but we think the terminology given here is much better.)

Note that this makes good sense for $\kappa \leqslant \Delta(X)$ as well. But while "almost ω-resolvable" is clearly equivalent to "ω-resolvable", the analogous question for higher cardinals remained open. In particular, the following natural problem was formulated in [3]:

Problem 1. Let X be an extraresolvable (T_2, T_3, or Tychonov) space with $\Delta(X) \geqslant \omega_1$. Is X then ω_1-resolvable?

(The assumption $\Delta(X) \geqslant \omega_1$ is clearly necessary to make this problem non-trivial.)

Comfort and Hu, see [2, Corollary 3.6], gave a negative answer to this problem, assuming the failure of the continuum hypothesis, CH. More precisely they got the following result:

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Theorem. *If κ is an infinite cardinal such that GCH first fails at κ then there is a 0-dimensional T_2 space X with $|X| = \Delta(X) = \kappa^+$ such that X is κ-resolvable, extraresolvable but not κ^+-resolvable, hence not maximally resolvable and if $\kappa = \omega$ then not ω_1-resolvable.*

Our aim in this note is to give the following "final" answer to the above problem, in ZFC.

Theorem 2. *For every cardinal κ there is a 0-dimensional T_2 space of dispersion character κ that is extraresolvable but not ω_1-resolvable.*

We shall actually prove a bit more. Note that no space X can be almost $(2^{\Delta(X)})^+$-resolvable, moreover "almost $2^{\Delta(X)}$-resolvable" can be strictly stronger than "extraresolvable $\equiv$ almost $\Delta(X)^+$-resolvable".

Theorem 3. *For every cardinal κ there is an almost 2^κ-resolvable (and so extraresolvable) but not ω_1-resolvable 0-dimensional T_2 space of cardinality and dispersion character κ. In fact, our example is a κ-dense subspace of the Cantor cube of weight 2^κ.*

To prove this theorem we shall make use of the method of constructing $\mathcal{D}$-forced spaces that was introduced in [3]. Therefore, we first recall some definitions and results from [3].

Let $\mathcal{D}$ be a family of dense subsets of a space X. A subset $M \subset X$ is called a *$\mathcal{D}$-mosaic* iff there is a maximal disjoint family $\mathcal{V}$ of open subsets of X and for each $V \in \mathcal{V}$ there is $D_V \in \mathcal{D}$ such that

$$M = \bigcup\{V \cap D_V\colon\ V \in \mathcal{V}\}.$$

Clearly, every $\mathcal{D}$-mosaic is dense. We say that the space X (or its topology) is *$\mathcal{D}$-forced* iff every dense subset of X includes a $\mathcal{D}$-mosaic.

Let S be any set and $\mathbb{B} = \{\langle B^0_\zeta, B^1_\zeta\rangle\colon\ \zeta < \mu\}$ be a family of 2-partitions of S. We denote by $\tau_{\mathbb{B}}$ the (obviously zero-dimensional) topology on S generated by the subbase $\{B^i_\zeta\colon\ \zeta < \mu,\ i < 2\}$, moreover we set $X_{\mathbb{B}} = \langle S, \tau_{\mathbb{B}}\rangle$.

Given a cardinal κ, we have $\Delta(X_{\mathbb{B}}) \geqslant \kappa$ iff $\mathbb{B}$ is *κ-independent*, i.e.,

$$\mathbb{B}[\varepsilon] \stackrel{\text{def}}{=} \bigcap\bigl\{B^{\varepsilon(\zeta)}_\zeta\colon\ \zeta \in \operatorname{dom}\varepsilon\bigr\}$$

has cardinality at least κ whenever $\varepsilon \in \mathrm{Fn}(\mu, 2)$.

Note that $X_{\mathbb{B}}$ is Hausdorff iff $\mathbb{B}$ is *separating*, i.e. for each pair $\{\alpha, \beta\} \in [S]^2$ there are $\zeta < \mu$ and $i < 2$ such that $\alpha \in B^i_\zeta$ and $\beta \in B^{1-i}_\zeta$.

A set $D \subset X$ is said to be *κ-dense* in the space X iff $|D \cap U| \geqslant \kappa$ for each non-empty open set $U \subset X$. Thus D is dense iff it is 1-dense. Also, it is obvious that the existence of a κ-dense set in X implies $\Delta(X) \geqslant \kappa$.

Theorem. *([3, Main Theorem 3.3]) Assume that κ is an infinite cardinal and we are given $\mathbb{B} = \{\langle B^0_\xi, B^1_\xi\rangle\colon\ \xi < 2^\kappa\}$, a κ-independent family of 2-partitions of κ, moreover a non-empty family $\mathcal{D}$ of κ-dense subsets of the space $X_{\mathbb{B}}$. Then there is a separating κ-independent family $\mathbb{C} = \{\langle C^0_\xi, C^1_\xi\rangle\colon\ \xi < 2^\kappa\}$ of 2-partitions of κ such that*

(1) *every $D \in \mathcal{D}$ is also κ-dense in $X_{\mathbb{C}}$ (and so $\Delta(X_{\mathbb{C}}) = \kappa$),*
(2) *$X_{\mathbb{C}}$ is $\mathcal{D}$-forced.*

Actually, the space $X_{\mathbb{C}}$ has other interesting properties as well but we shall not make use of those here. We are now ready to prove our promised result.

Proof of Theorem 3. Let κ be an arbitrary infinite cardinal. It is well known, see e.g. [3, Fact 3.2], that we can find two disjoint families $\mathbb{B} = \{\langle B^0_i, B^1_i\rangle\colon\ i < 2^\kappa\}$ and $\mathbb{D} = \{\langle D^0_i, D^1_i\rangle\colon\ i < 2^\kappa\}$ of 2-partitions of κ such that their union $\mathbb{B} \cup \mathbb{D}$ is κ-independent, that is, for any $\eta, \varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ we have

$$\bigl|\mathbb{D}[\eta] \cap \mathbb{B}[\varepsilon]\bigr| = \kappa.$$

In other words, this means that

$$\mathcal{D} = \bigl\{\mathbb{D}[\eta]\colon\ \eta \in \mathrm{Fn}\bigl(2^\kappa, 2\bigr)\bigr\}$$

is a family of κ-dense subsets of $X_{\mathbb{B}}$, hence we may apply Theorem 4 to this $\mathbb{B}$ and $\mathcal{D}$ to get a family $\mathbb{C}$ of 2^κ many 2-partitions of κ that satisfies conditions (1) and (2) above.

The space that we need will be a further refinement of $X_{\mathbb{C}}$. To obtain that, we next fix a 2-partition $\langle I, J\rangle$ of the index set 2^κ such that $|I| = |J| = 2^\kappa$. For every unordered pair $a \in [I]^2$ we shall write $a^+ = \max a$ and $a^- = \min a$, so that $a = \{a^-, a^+\}$.

Let $\{j(a,m)\colon a \in [I]^2,\ m < \omega\}$ be pairwise distinct elements of J. For any $a \in [I]^2$ and $m < \omega$ we then define the sets

$$E^0_{a,m} = D^0_{j(a,m)} \setminus \left(D^0_{a^-} \cap D^0_{a^+}\right) \quad \text{and} \quad E^1_{a,m} = \kappa \setminus E^0_{a,m}.$$

Clearly, then we have

$$E^1_{a,m} = D^1_{j(a,m)} \cup \left(D^0_{a^-} \cap D^0_{a^+}\right).$$

In this way we obtained a new family

$$\mathbb{E} = \left\{\langle E^0_{a,m}, E^1_{a,m}\rangle\colon a \in [I]^2,\ m < \omega\right\}$$

of 2-partitions of κ. We shall show that the space $X_{\mathbb{C}\cup\mathbb{E}}$ satisfies all the requirements of Theorem 3.

Claim 3.1. *For any finite function $\eta \in \mathrm{Fn}([I]^2 \times \omega, 2)$ and any ordinal $\alpha \in I$ there is a finite function $\varphi \in \mathrm{Fn}(2^\kappa, 2)$ such that $\alpha \notin \operatorname{dom}\varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$.*

Proof. For each $a \in [I]^2$ let us pick $a^* \in a$ with $a^* \neq \alpha$. Then we have

$$\begin{aligned}
\mathbb{E}[\eta] &= \bigcap_{\eta(a,m)=0} E^0_{a,m} \cap \bigcap_{\eta(a,m)=1} E^1_{a,m} \\
&\supset \bigcap_{\eta(a,m)=0} \left(D^0_{j(a,m)} \setminus \left(D^0_{a^-} \cap D^0_{a^+}\right)\right) \cap \bigcap_{\eta(a,m)=1} D^1_{j(a,m)} \\
&\supset \bigcap_{\eta(a,m)=0} \left(D^0_{j(a,m)} \cap D^1_{a^*}\right) \cap \bigcap_{\eta(a,m)=1} D^1_{j(a,m)} \\
&= \bigcap_{\eta(a,m)=0} D^1_{a^*} \cap \bigcap_{\langle a,m\rangle \in \operatorname{dom}\eta} D^{\eta(a,m)}_{j(a,m)}.
\end{aligned}$$

The expression in the last line above is, however, equal to $\mathbb{D}[\varphi]$ for a suitable $\varphi \in \mathrm{Fn}(2^\kappa, 2)$ because j is an injective map of $[I] \times \omega$ into J and $a^* \neq \alpha$ belongs to $I = \kappa \setminus J$ for all $a \in [I]^2$. □

Claim 3.2. *$\mathbb{C} \cup \mathbb{E}$ is κ-independent, hence $\Delta(X_{\mathbb{C}\cup\mathbb{E}}) = \kappa$.*

Proof. Let $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ and $\eta \in \mathrm{Fn}([I]^2 \times \omega, 2)$ be picked arbitrarily. By Claim 3.1 there is $\varphi \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\mathbb{D}[\varphi] \in \mathcal{D}$ we have $|\mathbb{C}[\varepsilon] \cap \mathbb{D}[\varphi]| = \kappa$ because $\mathbb{C}$ satisfies condition (1). Consequently, we have $|\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]| = \kappa$ as well. □

Claim 3.3. *The family $\{D^0_\alpha\colon \alpha \in I\}$ witnesses that $X_{\mathbb{C}\cup\mathbb{E}}$ is almost 2^κ-resolvable.*

Proof. First we show that D^0_α is dense in $X_{\mathbb{C}\cup\mathbb{E}}$ whenever $\alpha \in I$. So fix $\alpha \in I$, moreover let $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ and $\eta \in \mathrm{Fn}([I]^2 \times \omega, 2)$. By Claim 3.1 there is $\varphi \in \mathrm{Fn}(2^\kappa, 2)$ such that $\alpha \notin \operatorname{dom}\varphi$ and $\mathbb{E}[\eta] \supset \mathbb{D}[\varphi]$. Since $\alpha \notin \operatorname{dom}\varphi$ we have $D^0_\alpha \cap \mathbb{D}[\varphi] \in \mathcal{D}$. Hence, as $\mathbb{C}$ has property (1),

$$\emptyset \neq \left(D^0_\alpha \cap \mathbb{D}[\varphi]\right) \cap \mathbb{C}[\varepsilon] \subset D^0_\alpha \cap \left(\mathbb{E}[\eta] \cap \mathbb{C}[\varepsilon]\right)$$

as well. So D^0_α intersects every basic open subset of $X_{\mathbb{C}\cup\mathbb{E}}$, i.e. D^0_α is dense in $X_{\mathbb{C}\cup\mathbb{E}}$.

Next we show that $D_\alpha \cap D_\beta$ is nowhere dense in the space $X_{\mathbb{C}\cup\mathbb{E}}$ whenever $a = \{\alpha, \beta\} \in [I]^2$. Indeed, let $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ be again a basic open set with $\varepsilon \in \mathrm{Fn}(2^\kappa, 2)$ and $\eta \in \mathrm{Fn}([I]^2 \times \omega, 2)$ and let us pick $m < \omega$ such that $\langle a, m\rangle \notin \operatorname{dom}\eta$. Then

$$\eta' = \eta \cup \left\{\left\langle \langle a, m\rangle, 0\right\rangle\right\} \in \mathrm{Fn}\left([I]^2 \times \omega, 2\right),$$

hence $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$ is a (non-empty) basic open set in the space $X_{\mathbb{C}\cup\mathbb{E}}$. Moreover, $E^0_{a,m} = D^0_{j(a,m)} \setminus (D^0_\alpha \cap D^0_\beta)$ implies

$$(D_\alpha \cap D_\beta) \cap \mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta'] \subset (D_\alpha \cap D_\beta) \cap \left(D^0_{j(a,m)} \setminus (D_\alpha \cap D_\beta)\right) = \emptyset,$$

consequently, $D_\alpha \cap D_\beta$ is not dense in $\mathbb{C}[\varepsilon] \cap \mathbb{E}[\eta]$. □

Finally, the following simple claim will complete the proof of our theorem.

Claim 3.4. *The space $X_{\mathbb{C}}$ is ω_1-irresolvable, that is, not ω_1-resolvable.*

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Proof. Assume that $\{F_\zeta\colon \zeta < \omega_1\}$ is a family of dense subsets of $X_{\mathbb{C}}$. By condition (2) the topology of $X_{\mathbb{C}}$ is $\mathcal{D}$-forced, so every F_ζ includes a $\mathcal{D}$-mosaic in $X_{\mathbb{C}}$, consequently for all $\zeta < \omega_1$ there are $\varepsilon_\zeta \in \mathrm{Fn}(2^\kappa, 2)$ and $\phi_\zeta \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \subset F_\zeta$. By the well-known Δ-system lemma we may then find $\zeta < \xi < \omega_1$ such that $\varepsilon = \varepsilon_\zeta \cup \varepsilon_\xi \in \mathrm{Fn}(2^\kappa, 2)$ and $\phi = \phi_\zeta \cup \phi_\xi \in \mathrm{Fn}(2^\kappa, 2)$. (Actually, much more is true: there is an uncountable set $S \in [\omega_1]^{\omega_1}$ such that the members of both $\{\varepsilon_\zeta\colon \zeta \in S\}$ and $\{\phi_\zeta\colon \zeta \in S\}$ are pairwise compatible.) But then we have

$$F_\zeta \cap F_\xi \supset \mathbb{D}[\phi_\zeta] \cap \mathbb{C}[\varepsilon_\zeta] \cap \mathbb{D}[\phi_\xi] \cap \mathbb{C}[\varepsilon_\xi] = \mathbb{D}[\phi] \cap \mathbb{C}[\phi] \neq \emptyset. \qquad \square$$

To conclude our proof, it suffices to recall the obvious fact that if a topology on a set is λ-resolvable then so is any coarser topology. Hence the ω_1-irresolvability of $X_{\mathbb{C}}$ implies that of $X_{\mathbb{C}\cup\mathbb{E}}$. $\square$

Let us point out that as extraresolvability implies almost ω-resolvability that is equivalent to ω-resolvability, any counterexample to Problem 1 is automatically an example of an ω-resolvable but not maximally resolvable space, hence it is a solution to the celebrated problem of Ceder and Pearson from [1]. The first Tychonov ZFC examples of such spaces were given in [3] and the spaces constructed in Theorem 3 extend the supply of such examples.

References

[1] J.I. Ceder, T. Pearson, On products of maximally resolvable spaces, Pacific J. Math. 22 (1967) 31–45.
[2] W.W. Comfort, W. Hu, Resolvability properties via independent families, Topology Appl. 154 (2007) 205–214.
[3] István Juhász, Lajos Soukup, Zoltán Szentmiklóssy, $\mathcal{D}$-forced spaces: A new approach to resolvability, Topology Appl. 153 (11) (2006) 1800–1824.
[4] V.I. Malykhin, Irresolvability is not descriptively good, manuscript.