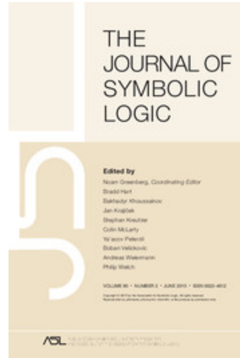

Uniformization principles

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UNIFORMIZATION PRINCIPLES

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Abstract. It is consistent that for many cardinals λ there is a family of at least λ^+ unbounded subsets of λ which have uniformization properties. In particular if it is consistent that a supercompact cardinal exists, then it is consistent that $\aleph_\omega$ has such a family. We have applications to point set topology, Whitehead groups and reconstructing separable abelian p-groups from their socles.

§0. Introduction. In [DS] a combinatorial principal called "uniformization" was defined. This principle contains some of the consequences of MA + $\neg$CH, but may hold in a much wider context than MA + $\neg$CH. This principle has applications to abelian group theory and topology. Aside from the interest of the applications, we view uniformization principles as a natural set-theoretic study.

From a purely set-theoretic point of view uniformization principles can be viewed as a strong denial of $\diamond$ or diamond-like principles. Unlike MA + $\neg$CH, uniformization principles can be true under GCH and on sets which MA + $\neg$CH forbids. The uniformization result obtained in [S0] was the second example of an MA + $\neg$CH phenomenon which was shown to be consistent with CH. Unlike the first (which was Jensen's proof [DJ] of the consistency of Suslin's hypothesis with CH), the results in [S0] show that there may be some difference among the stationary subsets of ω_1. Namely there is a stationary subset of ω_1 for which $\diamond$ is true and another with the property that any colouring in ω colours of any ladder system can be uniformized (see below for definitions). Also the instances of uniformization at cardinals above the continuum (as presented here) is somewhat more satisfactory than the generalizations of Martin's axiom to those cardinals (e.g. there are fewer restrictions on the cardinal arithmetic).

DEFINITION. Suppose $\langle A_i : i \in I\rangle$ is a family of ordered sets. Further suppose h with domain $\bigcup A_i$ is a function with ordinal values. A *colouring* of $\langle A_i : i \in I\rangle$ is a sequence $\langle c_i : i \in I\rangle$ of ordinal valued functions where the domain of c_i is A_i. We say $\langle c_i : i \in I\rangle$ is *dominated* by h if for all i and $a \in A_i$, $c_i(a) < h(a)$. (When h is a partial function and $\bigcup A_i = \mu$, then $h(\alpha)$ is considered to be μ if $h(\alpha)$ is undefined.) A

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colouring $\langle c_i : i \in I\rangle$ is *uniformized* by g with domain $\bigcup A_i$ if, for all i, $\{a \in A_i : g(a) = c_i(a)\}$ is coinitial in A_i. We write $\mathrm{Unif}_h\langle A_i : i \in I\rangle$ to denote that any colouring of $\langle A_i : i \in I\rangle$ dominated by h can be uniformized. If each A_i is an unbounded subset of μ, we write $\mathrm{Unif}\langle A_i : i \in I\rangle$, where h is the function which is constantly μ.

This is not the most general uniformization property which can be considered. However it will suffice for our considerations. The roots of uniformization principles can be traced back to Martin and Solovay's [MaSo] proof that assuming MA $+ \neg$CH, $2^{\aleph_0} = 2^{\aleph_1}$. They choose $\aleph_1$ almost disjoint subsets $\{A_\alpha : \alpha < \omega_1\}$ and wish to code an arbitrary subset $X \subseteq \omega_1$ by a subset of ω. In terms of uniformization, the natural approach is to let c_α be constantly 0 on A_α if $\alpha \in X$ and let c_α be constantly 1 if $\alpha \notin X$. If $h: \omega \to 2$ uniformizes $\langle c_\alpha : \alpha < \omega_1\rangle$ then h would code X. Unfortunately this approach does not quite work, since the appropriate uniformization principle fails assuming MA $+ \neg$CH. However, Martin and Solovay show that MA $+ \neg$CH implies there is a function which uniformizes all the c_α which are constantly 0 and agrees with any other c_α on an infinite set. (The proof we give of Proposition 6.1 is the Martin-Solovay argument.)

Independently the general topologists, in trying to understand normal Moore spaces or spaces which were ω_1-collectionwise Hausdorff (see §6), were led to consider uniformization problems. It should be noted that the colourings they wished to uniformize as well as the ones associated in [MaSo] are *monochromatic* in the sense that each c_α is a constant function.

Shelah was led to formulate the notion of uniformization principles by his work on the Whitehead problem. Whitehead asked which Abelian groups A have the property that any exact sequence

$$0 \to \mathbf{Z} \to B \xrightarrow{\pi} A \to 0$$

of abelian groups splits (i.e. there is a homomorphism $\sigma: A \to B$ so that $\pi \circ \sigma$ is the identity on A). Such a group A is called a Whitehead group. It turns out that uniformization principles are the set-theoretic analogue to the existence of nonfree Whitehead groups (see [S3] or [E, Chapter 10] for more details). Here the colourings are not monochromatic.

There are uniformization principles which are implied by MA $+ \neg$CH. In [DS] it was shown that, assuming MA $+ \neg$CH, if for each $\lambda \in \lim(\omega_1)$, A_λ is an increasing cofinal sequence of order type ω, then $\mathrm{Unif}_{\boldsymbol{\omega}}\langle A_\lambda : \lambda \in \lim \omega_1\rangle$ (where $\boldsymbol{\omega}$ is the function which is constantly ω). (The sequence of sets $\langle A_\lambda : \lambda \in \lim \omega_1\rangle$ is called a *ladder system*. So the result is sometimes stated as asserting that any colouring in ω colours of any ladder system on $\lim \omega_1$ can be uniformized.) In [S0] the same statement was shown consistent with CH, if $\lim(\omega_1)$ is replaced by a fixed stationary subset of ω_1.

More typical of our considerations in this paper is the following theorem of Shelah [S3].

0.1. THEOREM. *Let $g: \omega \to \omega$. Define $h: {}^{<\omega}\omega$ by $h(\eta) = g(l(\eta))$ (where $l(\eta)$ is the length of η). Then it is consistent that there is a sequence $\langle \eta_i : i < \omega_1\rangle$ of distinct elements of ${}^{\omega}\omega$ such that $\mathrm{Unif}_h\langle A_i : i < \omega_1\rangle$ holds, where $A_i = \{\eta_i \upharpoonright n : n < \omega\}$.*

By [S3], Theorem 0.1 fails if MA $+ \neg$CH holds. Of course the A_i's may be regarded as subsets of ω. We are interested in generalizing the above theorem to uncountable cardinals. In general the results are easier to prove for regular cardinals than for singular cardinals. In fact for singular cardinals we will have to assume the consistency of the existence of supercompact cardinals. Our proofs all follow the same strategy. First we force the existence of the desired sequence $\langle A_i : i \in I \rangle$. Then we iteratively force functions which uniformize the relevant colourings. For results at singular cardinals we change the cofinality or collapse a supercompact cardinal.

In §1 we present the basic construction, which shows, for a regular uncountable λ, that it is consistent that there is $\langle A_i : i < \lambda^+ \rangle$ such that each A_i is an unbounded subset of λ and $\mathrm{Unif}\langle A_i : i < \lambda^+ \rangle$ holds. Further, if λ is supercompact then it remains supercompact in the forcing extension. §2 is concerned with variations of this result. In §§3 and 4 we deal with singular cardinals. In §3 we consider the sequence $\langle A_i : i < \lambda^+ \rangle$ provided in §1, where λ is supercompact. We then use Prikry forcing to change the cofinality of λ to ω. Then we show $\mathrm{Unif}_h\langle A_i : i < \lambda^+ \rangle$ holds, where $h(\alpha)$ is the least element of Prikry sequence greater than α. In §4 we use Magidor's forcing to collapse λ to $\aleph_\omega$. Again we show $\langle A_i : i < \lambda^+ \rangle$ retains uniformization properties.

§5 represents a variation of our theme. Uniformization properties can be thought of as strong negations to diamond-like principles (cf. [DS]). In [S1] a diamond-like principle $D(X, \lambda)$ which always holds for ${}^\omega\mu$ and $\lambda \leq \mu$ is defined. (The notation $D(X, \lambda)$ is introduced for our convenience in this paper.) We show that it is consistent for uncountable $\lambda \leq \mu$ that the ideal of sets $X \subseteq {}^\omega\mu$ such that $D(X, \lambda)$ does not hold is not λ^+-complete. We do this by showing it is consistent that ${}^\omega\mu$ can be partitioned into λ sets each of which satisfies a uniformization property. Finally, in §6, we give some applications of our consistency results to abelian group theory and point set topology. The most interesting of these results are for singular cardinals of cofinality ω, particularly $\aleph_\omega$. We show it is consistent that examples which were known to consistently exist in power ω_1 can also consistently exist in power $\aleph_{\omega+1}$.

§§2–5 depend either on the results or methods of §1, and §4 depends on the methods of §3. Otherwise the sections of the paper can be read independently. Of course the reader who only reads §6 will have to accept the quoted consistency results on faith.

Our set-theoretic notation is standard and roughly follows that of [S4]. The reader should note that by $p \leq q$ we mean that q is a stronger condition than p. The mathematical content of this paper is due to Shelah. While he was visiting Simon Fraser University during the summer of 1985, he explained these results to Mekler. Mekler agreed to write the paper and flesh out the proofs. It should be mentioned that Shelah also could prove results analogous to those of §3 when the cofinality of λ is changed to some uncountable $\kappa < \lambda$ using the methods of [M2]. However Mekler's industry did not extend to writing this section. We hope the interested reader who understands [M2] and §§3 and 4 of this paper will be able to reconstruct these results.

§1. Uniformization at inaccessible cardinals. Throughout this section λ will be a strongly inaccessible cardinal and $\mu \geq \lambda$. We will define a poset $\mathbf{P}$ so that forcing with $\mathbf{P}$ adds $\langle A_\alpha : \alpha < \mu \rangle$, where each A_α is an unbounded subset of λ and $\mathrm{Unif}\langle A_\alpha : \alpha < \mu \rangle$ holds. We will define a sequence $\mathbf{P}_i$ of posets ($i \leq \lambda^\mu$). To avoid complicating the notation we will make promises about how the $\mathbf{P}_i$'s are to be defined.

Promises. (1) The poset $\mathbf{Q}_0$ will add μ functions g_ν from λ to λ such that, for all $\alpha < \lambda$ and $\nu < \mu$, $g_\nu(\alpha) \in \omega_{\alpha+1}$. More formally $\mathbf{Q}_0$ is the poset whose elements are functions $q: \mu \to {}^{<\lambda}\lambda$ where for all but $\langle \lambda$ ν's, $q(\nu) = 0$ and for all $\alpha \in \operatorname{dom} q(\nu)$, $q(\nu)(\alpha) \in \omega_{\alpha+1}$. For $p, q \in \mathbf{Q}_0$ $p \leq q$, if for all ν $p(\nu) \subseteq q(\nu)$. We identify a $\mathbf{Q}_0$-generic set $\langle g_\nu: \nu < \mu \rangle$ with a sequence $\langle A_\alpha: \alpha < \mu \rangle$ where each $A_\nu = \{g_\nu \restriction \alpha + 1: \alpha < \lambda\}$. Alternately for each α we can enumerate $\prod_{\beta \leq \alpha} \omega_\beta$ by the ordinals in $(\beth_\alpha, \beth_{\alpha+1})$. Fixing such enumerations we can identify each A_ν with a set of ordinals such that for all α, $|A_\nu \cap (\beth_\alpha, \beth_{\alpha+1})| = 1$.

Note. The A_ν's so constructed will be *tree-like* (i.e. for all $\nu, \tau < \mu$ if $\gamma \in A_\nu \cap A_\tau$ then, for all $\rho < \gamma$, $\rho \in A_\nu$ iff $\rho \in A_\tau$). We let $\tilde{A}_\nu$ denote the canonical name for A_ν.

(2) Suppose $i > 0$; then we will choose $\langle \tilde{f}^i_\xi: \xi < \mu \rangle$ such that for each ξ, $\Vdash_{\mathbf{P}_\alpha} \tilde{f}^i_\xi: \tilde{A}_\xi \to \lambda$. We let Q_i be the set whose elements are the partial functions from μ to λ with domain of cardinality $<\lambda$. (Our notation varies from that of [S4]. There Q_i is the name of the ith iterand.)

For $i \leq (\lambda^\mu)^+$, $\mathbf{P}_i$ will be the set of functions p with domain i satisfying the conditions below:

(i) for all but $<\lambda$ β's, $p(\beta) = 0$;

(ii) $p(0) \in \mathbf{Q}_0$;

(iii) for all $\beta > 0$, $p(\beta) \in Q_\beta$ and $p \restriction \beta \Vdash_{\mathbf{P}_\beta}$ "for all ν, $\gamma \in \operatorname{dom} p(\beta)$ if $\eta \in (\tilde{A}_\nu \setminus {}^{<p(\beta)(\nu)}\lambda) \cap (\tilde{A}_\gamma \setminus {}^{<p(\beta)(\gamma)}\lambda)$ then $\tilde{f}^\beta_\nu(\eta) = \tilde{f}^\beta_\gamma(\eta)$."

$\mathbf{P}_i$ is ordered in the obvious way.

It is useful to understand clause (iii). Suppose G is $\mathbf{P}_{\beta+1}$-generic. Then if we let

$$f = \bigcup_{\substack{p \in G \\ \nu < \mu}} (\tilde{f}^\beta_\nu)^G \restriction ((\tilde{A}_\nu)^G \setminus {}^{<p(\beta)(\nu)}\lambda),$$

f uniformizes the sequence $\langle (\tilde{f}^\beta_\nu)^G: \nu < \mu \rangle$.

We are using a $\langle\lambda$-support iteration, so it is easy to see that $\mathbf{P}_i$ is λ-directed complete for all $i \leq \lambda^\mu$. The next lemma follows easily from λ-completeness.

1.1. LEMMA. *Suppose $i \leq \lambda^\mu$ and $\mathbf{P}_i$ is as described above. Then for all $p \in \mathbf{P}_i$ there is $q \geq p$ satisfying the following properties*:

($*$) *For all $0 < \beta < i$, if* $\operatorname{dom} q(\beta) \neq 0$ *then, for all $\nu < \mu$, $q \restriction \beta$ determines $\tilde{f}^\beta_\nu$ on $q(0)(\nu)$. (I.e. for all $\alpha \in \operatorname{dom} q(0)(\nu)$ there is τ such that $q \restriction \beta \Vdash_{\mathbf{P}_\beta} \tilde{f}^\beta_\nu(\alpha) = \tau$.)*

($**$) *There is an ordinal $\tau < \lambda$ so that*: (a) $\omega_{\tau+1} > |\{\nu: q(0)(\nu) \neq 0\}|$; (b) *for all ν if $q(0)(\nu) \neq 0$ then* $\operatorname{dom} q(0)(\nu) = \tau$; *and* (c) *if $q(0)(\nu) \neq 0$ then* $\operatorname{dom} q(\beta) = \{\nu: q(0)(\nu) \neq 0\}$.

(For future use note that given any stationary set S we can choose τ above so that $\tau \in S$.)

1.2. LEMMA. *For all $i \leq \lambda$ and $\mathbf{P}_i$ as above, $\mathbf{P}_i$ has the λ^+-c.c.*

PROOF. Suppose $\{q_\nu: \nu < \lambda^+\} \subseteq \mathbf{P}_i$. We can assume the q_ν's satisfy ($*$) and ($**$) above. Also we can assume there is an ordinal τ which witnesses that ($**$) holds for all the q_ν. Applying the Δ-lemma, we can assume there is a set $X \subseteq i$ such that, for $\nu \neq \nu'$, $\{\beta: q_\nu(\beta) \neq 0 \neq q_{\nu'}(\beta)\} = X$. Further there is a set $X_0 \subseteq \mu$ so that, for all $\nu \neq \nu'$, $\{\gamma: q_\nu(0)(\gamma) \neq 0 \neq q_{\nu'}(0)(\gamma)\} = X_0$. For $\nu < \lambda^+$, let $T_\nu = \{q_\nu(0)(\gamma) \restriction \rho + 1: \rho < \tau$ and $q_\nu(0)(\gamma) \neq 0\}$. We can assume the T_ν's form a Δ-system with root T. Next for each $\beta \in X$ and $\nu < \lambda^+$ choose a function $f_{\nu\beta}: T_\nu \to \lambda$ such that

$$q_\nu \restriction \beta \Vdash \text{“for all } q_\nu(\beta)(\gamma) \leq \rho < \tau,\ \tilde{f}^\beta_\gamma(q_\nu(0)(\gamma) \restriction \rho + 1) = f_{\nu\beta}(q_\nu(0)(\gamma) \restriction \rho + 1)\text{”},$$

whenever $q_\nu(0)(\gamma) \neq 0$. Finally as $\lambda^{<\lambda} = \lambda$, we can assume for all $\beta \in X, \gamma \in X_0, \eta \in T$ and $\nu, \nu' < \lambda^+$ that $q_\nu(\beta)(\gamma) = q_{\nu'}(\beta)(\gamma)$, $q_\nu(0)(\gamma) = q_{\nu'}(0)(\gamma)$ and $f_{\nu\beta}(\eta) = f_{\nu'\beta}(\eta)$.

Consider now q_0 and q_1. We can now define a condition p stronger than q_0 and q_1. Let $\{\gamma_i: i < \omega_\tau\}$ enumerate $\{\gamma: q_0(0)(\gamma) \neq 0$ or $q_1(0)(\gamma) \neq 0\}$. Define p by letting $p(0)(\gamma) = 0$ if $\gamma \notin \{\gamma_i: i < \omega_\tau\}$ and letting $p(0)(\gamma_i) = q_0(0)(\gamma_i) \cup q_1(0)(\gamma_i) \cup \{\langle\tau, i\rangle\}$. For $\beta > 0$, let $p(\beta) = q_0(\beta) \cup q_1(\beta)$.

1.3. THEOREM. *Suppose λ is strongly inaccessible and $\mu > \lambda$. Then there is a λ-directed complete poset* $\mathbf{P}$ *so that in $V^{\mathbf{P}}$, λ is still strongly inaccessible and the cardinals of V are preserved. Further there exists (in $V^{\mathbf{P}}$) a tree-like sequence $\langle A_\nu: \nu < \mu\rangle$ of subsets of λ so that, for all ν and $\alpha < \lambda$, $|A_\nu \cap (\beth_\alpha, \beth_{\alpha+1})| = 1$,* Unif$\langle A_\nu: \nu < \mu\rangle$ *holds and, in $V^{\mathbf{P}}$, $2^\lambda = \mu^+$.*

PROOF. In view of the lemmas above it is enough to choose the $\langle \tilde{f}^i_\nu: \nu < \mu\rangle$ $(i < \lambda^\mu)$ so that if $\langle \tilde{f}_\nu: \nu < \mu\rangle$ is a sequence of $\mathbf{P}_{\lambda^\mu}$-names for functions from $\tilde{A}_\nu$ to λ, then, for some i and all $\nu < \mu$, $\Vdash_{\mathbf{P}_{\lambda^\mu}} \tilde{f}^i_\nu = \tilde{f}_\nu$. But this is a routine enumeration.

1.3A. REMARK. In the above theorem we could replace "$2^\lambda = \mu^+$" by "$2^\lambda = \chi$" if V satisfies $\chi^\mu = \chi$.

1.4. THEOREM. *In Theorem* 1.3 *above if λ is supercompact, we can demand that λ be supercompact in $V^{\mathbf{P}}$ as well.*

PROOF. By [L] we can assume λ remains supercompact in any generic extension by a λ-directed complete poset.

1.4A. *Generalization.* We are also interested in having the A_ν's continuous at many ordinals. (We say $\langle A_\nu: \nu < \mu\rangle$ is *continuous at* α if whenever $A_\nu \cap \beth_\alpha = A_{\nu'} \cap \beth_\alpha$ then $A_\nu \cap (\beth_\alpha, \beth_{\alpha+1}) = A_{\nu'} \cap (\beth_\alpha, \beth_{\alpha+1})$.) To achieve continuity at all limit ordinals in a stationary costationary set S, we can vary the definition of the A_ν's. Namely we let $A_\nu = \{g_\nu \upharpoonright \alpha: \alpha \in S\}$. The rest of the argument goes as before. (We use the fact that in condition (**) of Lemma 1.1 we can choose τ so that $\tau \notin S$.)

1.5. THEOREM. *Suppose it is consistent that a supercompact cardinal λ exists and $\mu > \lambda$. Then it is consistent that λ is supercompact, μ is a cardinal and there exists a tree-like sequence $\langle A_\alpha: \alpha < \mu\rangle$ of unbounded subsets of λ which is continuous at every limit ordinal of cofinality $\neq \omega$ such that, for all α and i, $|A_i \cap (\beth_\alpha, \beth_{\alpha+1})| = 1$ and* Unif$\langle A_\alpha: \alpha < \mu\rangle$ *holds.*

1.6. REMARK. If we wanted continuity at every limit ordinal, then we could not have full uniformization. If we had a continuous sequence, then it would be impossible to uniformize the functions which predict the next element of the set (cf. the remark after Theorem 3.2). But we can restrict the c_i's and hope for full continuity (see Theorem 2.5).

§2. Uniformization at regular cardinals. If we want to extend the methods of the last section to arbitrary regular cardinals it is necessary that for each $\alpha < \lambda$, $|\{g_i \upharpoonright \alpha: i < \mu\}| \leq \lambda$ (this is used in the proof of the λ^+-c.c.). That is, the collection $\{g_i \upharpoonright \alpha: \alpha < \lambda, i < \mu\}$ forms a weak Kurepa tree. If we have this condition we can regard each A_i as a subset of λ. Since it is no more difficult we shall construct a forcing extension in which $\{g_i \upharpoonright \alpha: \alpha < \lambda, i < \mu\}$ actually forms a Kurepa tree. First we recall a few definitions.

DEFINITION. Suppose $\{g_i: i < \mu\}$ is a collection of distinct functions from λ to λ. Then $\{g_i \upharpoonright \alpha: i < \mu, \alpha < \lambda\}$ forms a *Kurepa tree* if $\mu > \lambda$ and for all α, $|\{g_i \upharpoonright \alpha: i < \mu\}| \leq |\alpha| + \aleph_0$.

DEFINITION. Suppose κ is a cardinal and $\mathbf{P}$ is a partial order. $\mathbf{P}$ is *κ-strategically complete* if for all $\alpha < \kappa$, Player II has a winning strategy in the following game of length α. Players I and II alternately choose an increasing sequence p_β $(\beta < \alpha)$ of elements of $\mathbf{P}$, where Player I chooses at all the even ordinals and Player II at the odd ordinals. Player I wins if for some $\beta < \alpha$ there is no legal play or if the sequence p_β $(\beta < \alpha)$ has no upper bound.

For many purposes κ-strategically complete posets have the same properties as κ-complete posets. For example if $\mathbf{P}$ is κ-strategically complete and G is $\mathbf{P}$-generic, then $V[G]$ has no new sets of ordinals of cardinality $<\kappa$. Also if we iterate κ-strategically complete forcings with supports closed under the union of fewer than κ sets, then the resulting poset is κ-strategically complete.

Suppose λ is a regular cardinal. We will modify the forcing of Theorem 1.3 to prove the desired result by changing $\mathbf{Q}_0$. A condition $p \in \mathbf{Q}_0$ is a pair $\langle f, \alpha \rangle$, where $\alpha < \lambda$ and f is a function with domain μ. Further $|\{i: f(i) \neq 0, i < \mu\}| < \lambda$; if $f(i) \neq 0$ then $f(i)$ is a function from α to λ; and for all $\beta \leq \alpha$, $|\{f(i) \restriction \beta: f(i) \neq 0\}| \leq |\beta| + \aleph_0$. Next we let $\langle f, \alpha \rangle \leq \langle g, \beta \rangle$, if $\alpha \leq \beta$, $f(i) \subseteq g(i)$ $(i < \mu)$ and for all i such that $g(i) \neq 0$ for some j, $g(i) \restriction \alpha = f(j)$. Assume for the moment that $\mathbf{Q}_0$ preserves cardinals and G is $\mathbf{Q}_0$-generic. If we let $g_i = \bigcup_{\langle f, \alpha \rangle \in G} f(i)$, then $\{g_i \restriction \alpha: \alpha < \lambda, i < \mu\}$ forms a Kurepa tree.

2.1. PROPOSITION. *$\mathbf{Q}_0$ is λ-strategically complete.*

PROOF. We define a winning strategy for Player II in the game of length $\alpha < \lambda$. If for some β, Player I has played $p_\beta = \langle f_\beta, \gamma_\beta \rangle$ and $|\{i: f_\beta(i) \neq 0\}| = \kappa$, then Player II chooses $\langle f_{\beta+1}, \gamma_{\beta+1} \rangle > \langle f_\beta, \gamma_\beta \rangle$ so that $|\gamma_{\beta+1}| \geq \kappa$ and $\gamma_{\beta+1} > \gamma_\beta$. Suppose σ is a limit ordinal $\leq \alpha$. In order to see that the strategy above really is a winning strategy for Player II, it suffices to see that $\langle f_\beta, \gamma_\beta \rangle$ $(\beta < \sigma)$ has an upper bound. But

$$|\{i: \text{for some } \beta < \sigma,\ f_\beta(i) \neq 0\}| \leq \left|\sup_{\beta < \sigma} \gamma_\beta\right|.$$

So $\langle g, \gamma \rangle$ is the desired condition, where $g(i) = \bigcup f(i)$ for $i < \mu$ and $\gamma = \sup \gamma_\beta$.

It is possible to change the definition of $\mathbf{Q}_0$ so that $\mathbf{Q}_0$ is λ-complete. However, $\mathbf{Q}_0$ cannot be made λ-directed complete if λ is inaccessible (and $\mu > \lambda$). The rest of the proof follows the same lines as that of Theorem 1.3. So we can establish the following.

2.2. THEOREM. *Suppose $\lambda^{<\lambda} = \lambda$ and $\mu > \lambda$. Then there is a λ-strategically complete poset $\mathbf{P}$ with the λ^+-c.c. such that if g is $\mathbf{P}$-generic then in $V[G]$ there is a sequence $\langle g_i: i < \mu \rangle$ of functions from λ to λ so that $\{\{g_i \restriction \alpha + 1: \alpha < \lambda\}: i < \mu\}$ forms a Kurepa tree and* $\mathrm{Unif}\langle \{g_i \restriction \alpha + 1: \alpha < \lambda\}: i < \mu \rangle$ *holds.*

REMARK. Since forcing with $\mathbf{P}$ preserves cofinalities and adds no subsets of λ of cardinality $<\lambda$, λ is (strongly) inaccessible in V $[G]$ iff λ is (strongly) inaccessible in V. However if λ is supercompact in V, when we add a Kurepa tree we destroy the supercompactness of λ. In fact κ will not be subtle (cf. [KM]). We can hope to show the consistency of λ being supercompact with the existence of a tree with at most $|\alpha|^+$ elements of height α $(\omega \leq \alpha < \lambda)$, where $\mu = \lambda^+$.

Define a poset $\mathbf{Q}_0$ as follows. A condition $p \in \mathbf{Q}_0$ is a pair $\langle f, \alpha \rangle$ such that: $\alpha < \lambda$; f is a function with domain λ^+; $|\{i: f(i) \neq 0, i < \lambda^+\}| < \lambda$; if $f(i) \neq 0$, then $f(i)$ is a function from α to λ; and for all $\beta \leq \alpha$,

$$|\{f(i) \restriction \beta : f(i) \neq 0\}| \leq |\beta|^+ + \aleph_0.$$

Let $\langle f, \alpha \rangle \leq \langle g, \beta \rangle$ if $\alpha \leq \beta$, $f(i) \subseteq g(i)$ $(i < \lambda^+)$, and for all i such that $g(i) \neq 0$, for some j, $g(i) \restriction \alpha = f(j)$. The only change we have made in our definition of $\mathbf{Q}_0$ is to relax the cardinality restriction on $\{f(i) \restriction \beta : f(i) \neq 0\}$. Just as before we can show $\mathbf{Q}_0$ is λ-strategically complete. If $\mathbf{P}$ is defined as before, then $\mathbf{P}$ is λ-strategically complete and has the λ^+-c.c.

To prove our next theorem we need the following proposition.

2.3. PROPOSITION. *Assume it is consistent that λ is a supercompact cardinal. Then it is consistent that λ is supercompact and if* $\mathbf{P}$ *is as described above, then forcing with* $\mathbf{P}$ *preserves the supercompactness of λ.*

The proof of Proposition 2.3 is a modification of Laver's argument in [L]. We will give a sketch of this proof later. First we state the theorem which is an immediate consequence.

2.4. THEOREM. *Assume it is consistent that λ is a supercompact cardinal. Then it is consistent that λ is supercompact and there exists a tree-like sequence $\langle A_i : i < \lambda^+ \rangle$ of subsets of λ such that for all α and i, $|A_i \cap (\aleph_\alpha, \aleph_{\alpha+1})| = 1$ and* $\mathrm{Unif}\langle A_i : i < \lambda^+ \rangle$ *holds.*

In this result we cannot replace λ^+ by an arbitrary cardinal μ without making other changes. For example, if we want to replace λ^+ by λ^{++}, we can only ask that $|A_i \cap (\aleph_\alpha, \aleph_{\alpha+2})| = 1$.

PROOF OF PROPOSITION 2.3. Laver shows there is a function $f: \lambda \to H(\lambda)$ (the sets of hereditary cardinality λ) so that for any x and $\mu > |\mathrm{TC}(x)|$, there is an elementary embedding $j: V \to M$ so that ${}^\mu M = M$, $j(\lambda) > \mu$ and $jf(\lambda) = x$. (Here $\mathrm{TC}(x)$ denotes the transitive closure of x.) He uses f to define an iterated forcing and cardinals λ_α $(\alpha < \lambda)$ as follows. The forcings $\mathbf{R}_\alpha$ have as support Easton sets of ordinals (i.e. if $p \in \mathbf{R}_\alpha$ then $p(\beta) \neq 0$ implies β is a regular cardinal and, for all regular cardinals $\kappa \leq \alpha$, $|\{\beta < \alpha : p(\beta) \neq 0\}| < \kappa$). Suppose $\mathbf{R}_\alpha$ and λ_β $(\beta < \alpha)$ have been defined. Then $\tilde{S}_\alpha = \{0\}$ and $\lambda_\alpha = \alpha$ unless α is an inaccessible cardinal, $\alpha \geq \sup_{\beta < \alpha} \lambda_\beta$, $f(\alpha) = (\tilde{S}, \kappa)$, where $\tilde{S}$ is an $\mathbf{R}_\alpha$-name and $\Vdash_{\mathbf{R}_\alpha}$ "$\tilde{S}$ is α-strategically complete". (This last clause is the only place our definition varies from Laver's. He demands that $\Vdash_{\mathbf{R}_\alpha}$ "$\tilde{S}$ is α-directed complete".) In this case we let $\tilde{S}_\alpha = \tilde{S}$, $\lambda_\alpha = \kappa$ and $\mathbf{R}_{\alpha+1} = \mathbf{R}_\alpha * \tilde{S}_\alpha$.

Assume H is $\mathbf{R}_\lambda$-generic, $\mathbf{P}$ is the poset described above in $V[H]$ and G is $\mathbf{P}$-generic. Let $\tilde{P}$ be an $\mathbf{R}_\lambda$-name for $\mathbf{P}$. Consider $\gamma > \lambda$. To show λ is γ-supercompact in $V[H][G]$, one chooses $\mu > \gamma^+$ and $j: V \to M$ so that $jf(\lambda) = \langle \tilde{P}, \gamma^+ \rangle$ and ${}^\mu M \subseteq M$. So, in the construction of $j(\mathbf{R}_\lambda)$, $\tilde{S}_\lambda = \tilde{P}$ and $\lambda_\lambda = \gamma^+$. All that remains to prove is that, in $V[H][G][K]$ (where $H * G * K$ is $j(\mathbf{R}_\lambda)$-generic), $\{jp : p \in G\}$ has an upper bound in $j(\mathbf{P})$ (we abuse notation and let $j(\mathbf{P})$ denote the interpretation of $j(\tilde{P})$). The rest of the proof is the same as in [L]. The key point is that $\langle f, \lambda \rangle$ where f is a function with domain $j(\lambda^+)$ and

$$f(i) = \begin{cases} g_\nu, & \text{if } i = j(\nu), \\ 0, & \text{otherwise}, \end{cases}$$

is a condition for $j(\mathbf{Q}_0)$.

Finally we turn our attention to uniformization of continuous sequences.

2.5. Theorem. *Suppose λ is a strongly inaccessible cardinal, $S \subseteq \lambda$ is stationary, $h: S \to \lambda$, $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$. There is a poset* $\mathbf{P}$ *so that if G is* $\mathbf{P}$*-generic, then in $V[G]$ there is a sequence $\langle g_i : i < \lambda^+ \rangle$ of functions from λ to λ such that if we let $A_i = \{g_i \restriction \alpha : \alpha < \lambda\}$ then* $\mathrm{Unif}_\varphi\langle A_i : i < \lambda^+\rangle$, *where for all $\delta \in S$ and $i < \lambda^+$, $\varphi(g_i \restriction \delta) \le h(\delta)$. Further if λ is supercompact, then λ is supercompact in $V[G]$.*

Proof. We let $\mathbf{Q}_0$ be our initial forcing as before (i.e. in Lemma 1.1), except this time we do not put any restriction on the value of the functions. So $\mathbf{Q}_0$ is the set of functions $q: \lambda^+ \to {}^{<\lambda}\lambda$ such that for all but $< \lambda$ ν's, $q(\nu) = 0$. Suppose $\langle g_\nu : \nu < \lambda^+ \rangle$ is $\mathbf{Q}_0$-generic. For $\nu < \lambda^+$, let $A_\nu = \{g_\nu \restriction \alpha : \alpha < \lambda\}$. (This change guarantees that the A_ν's will form a continuous sequence.) For $0 < \alpha < \lambda^{++}$, we impose an additional requirement on the sequence $\langle \tilde{f}^\alpha_\xi : \xi < \lambda^+ \rangle$. As well as requiring $\Vdash_{\mathbf{P}_\alpha} \tilde{f}^\alpha_\xi : \tilde{A}_\xi \to \lambda$, we demand that, for all $\nu \in S$, $\Vdash_{\mathbf{P}_\alpha} \tilde{f}^\alpha_\xi(\tilde{g}_\xi \restriction \nu) < h(\nu)$. As before, $\mathbf{P}_\alpha$ is λ-directed complete for all $\alpha \le \lambda^{++}$. The only new difficulty is showing $\mathbf{P}_\alpha$ has the λ^+-c.c.

We first note that since $\mathbf{P}_\alpha$ is λ-complete every condition p has an extension r satisfying the following properties for some cardinal $\delta \in S$:

(i) if $r(0)(\nu) \ne 0$, then $\operatorname{dom} r(0)(\nu) = \delta$;

(ii) $|\{\nu : r(0)(\nu) \ne 0\}| = \delta$;

(iii) if $r(0)(\nu), r(0)(\nu') \ne 0$, then $r(0)(\nu) \ne r(0)(\nu')$;

(iv) for all $0 < \beta < \alpha$, if $\nu \in \operatorname{dom} r(\beta)$ then $r(0)(\nu) \ne 0$;

(v) for $0 < \beta < \alpha$, if $\nu \in \operatorname{dom} r(\beta)$ then $r \restriction \beta$ determines $\tilde{f}^\beta_\nu$ on $\{\tilde{g}_\nu \restriction \gamma : \gamma < \delta\}$;

(vi) $|\{\beta : r(\beta) \ne 0\}| \le \delta$.

Suppose now $\{q_i : i < \lambda^+\} \subseteq \mathbf{P}_\alpha$. We can assume that, for some cardinal $\delta \in S$, each q_i satisfies (i)–(vi) above. As in Lemma 1.2 we can use the Δ-lemma to produce X, X_0, T_i and T. Here $X \subseteq \alpha$ is such that for $i \ne j$,

$$\{\beta : q_i(\beta) \ne 0 \ne q_j(\beta)\} = X.$$

For $i \ne j$,

$$X_0 = \{\gamma : q_i(0)(\gamma) \ne 0 \ne q_j(0)(\gamma)\}.$$

Next we let

$$T_i = \{q_i(0)(\gamma) \restriction \rho : \rho < \delta \text{ and } q_i(0)(\gamma) \ne 0\}.$$

Then we demand that the T_i's form a Δ-system with root T. Finally we can assume there is a set F of functions from δ to λ such that for $i \ne j$

$$\{q_i(0)(\nu) : \nu < \lambda^+\} \cap \{q_j(0)(\nu) : \nu < \lambda^+\} = F \cup \{0\}.$$

For each i and $\beta \in X$, let $f_{i\beta}$ denote the function $f_{i\beta} : T_i \to \lambda$ such that

$$q_i \restriction \beta \Vdash \text{“for all } q_i(\beta)(\gamma) \le \rho \le \delta\ \tilde{f}^\beta_\gamma(q_i(0)(\gamma) \restriction \rho) = f_{i\beta}(q_i(0)(\gamma) \restriction \rho)\text{”},$$

whenever $q_i(0)(\gamma) \ne 0$.

Since $\lambda^{<\lambda} = \lambda$, we can assume for all $\beta \in X$, $\gamma \in X_0$, $\eta \in T$ and $i, j < \lambda^+$ that $q_i(\beta)(\gamma) = q_j(\beta)(\gamma)$, $q_i(0)(\gamma) = q_j(0)(\gamma)$ and $f_{i\beta}(\eta) = f_{j\beta}(\eta)$. Note that, by clauses (ii) and (vi), $|X|, |F| \le \delta$. Let $\chi = (h(\delta)^\delta)^+$. We will inductively define a refining sequence of equivalence relations E_β ($\beta \le \alpha$) on $\{q_i : i < \chi\}$ and conditions $p_{\beta b}$ for each block b of (the partition associated with) E_β so that $p_{\beta b} \ge q_i \restriction \beta$ for all $q_i \in b$ and if $b \subset b'$ and $\beta > \beta'$ then $p_{\beta b} \restriction \beta' \ge p_{\beta' b'}$. To begin we let E_0 and E_1 be the trivial equivalence relation (i.e. having only one block). Of course $p_0 = 0$, the unique condition in $\mathbf{P}_0$. Let $p_1 \in \mathbf{P}_1$ be a condition extending the restrictions of each of the q_i's ($i < \chi$) so that for all $\nu \neq \gamma$ if $p_1(\nu) \neq 0 \neq p_1(\gamma)$, then, for some $\tau \leq \delta$, $p_1(\nu)(\tau) \neq p_1(\gamma)(\tau)$. (To get p_1 we have extended each branch of the $q_i(0)$'s and made them differ as soon as possible.) Note that to define p_1 we only use the fact that there are χ possible values for $g_i(\delta)$.

Suppose now that E_β and $p_{\beta b}$ have been defined for $\beta < \alpha$ and b a block of E_β. There are three cases to consider in defining $E_{\beta+1}$ and $p_{\beta+1b'}$ for all blocks $b' \subseteq b$.

Case 1. For all $q_i \in b$, $q_i(\beta) = 0$. Then we define $E_{\beta+1}$ to have a unique block $\subseteq b$, namely b itself, and we let $p_{\beta+1b} = p_{\beta b} \cup \{\langle \beta, 0 \rangle\}$.

Case 2. There is a unique $q_i \in b$ so that $q_i(\beta) \neq 0$. Again we let $E_{\beta+1}$ have b as a block. But we let $p_{\beta+1b}$ be defined by

$$p_{\beta+1b}(\nu) = \begin{cases} p_{\beta b}(\nu), & \text{if } \nu < \beta, \\ q_i(\beta), & \text{if } \nu = \beta. \end{cases}$$

Case 3. $\beta \in X$. Choose $p_{\beta b} \leq p \in \mathbf{P}_\beta$ so that for all $q_i \in b$, if $q_i(0)(\nu) \in F$ then p determines $\tilde{f}^\beta_\nu(\tilde{g}_\nu \restriction \delta)$. Now define $E_{\beta+1}$ by letting $q_i, q_j \in b$ be in the same block of $E_{\beta+1}$ iff whenever $q_i(0)(\nu) = q_j(0)(\gamma) \in F$ then

$$p \Vdash \tilde{f}^\beta_\nu(\tilde{g}_\nu \restriction \delta) = \tilde{f}^\beta_\gamma(\tilde{g}_\gamma \restriction \delta).$$

Any block of E_β is split into at most $h(\delta)^\delta$ blocks. If $b' \subseteq b$ is a block of $E_{\beta+1}$, then let $p_{\beta+1b'}$ be defined by

$$p_{\beta+1b'}(\tau) = \begin{cases} p(\tau), & \text{if } \tau < \beta, \\ \bigcup_{q_i \in b'} q_i(\beta), & \text{if } \tau = \beta. \end{cases}$$

If β is a limit ordinal we let $E_\beta = \bigcap_{\nu<\beta} E_\nu$. Suppose b is a block of E_β. Then for each $\nu < \beta$ there is a block b_ν so that $b_\nu \supseteq b$. Consider $\langle p_{\nu b_\nu} : \nu < \beta \rangle$. We can let $p_{\beta b} = \bigcup_{\nu<\beta} p_{\nu b_\nu}$.

To complete the proof, we calculate the number of blocks for E_α. Since Case 3 above occurs at most δ times and each block splits into at most $h(\delta)^\delta$ blocks, E_α has at most $(h(\delta)^\delta)^\delta = h(\delta)^\delta$ blocks. So some block b has at least two elements. By construction $p_{\alpha b}$ is an upper bound to the members of b.

As we noted above, the proof still works if we demand $g_i(\delta) < (h(\delta)^\delta)^+$ for $\delta \in S$. So we can also prove the following theorem.

2.6. THEOREM. *Suppose it is consistent that a strongly inaccessible cardinal λ exists, $S \subseteq \lambda$ is stationary, $h: S \to \lambda$, $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$. There is a poset $\mathbf{P}$ and a sequence of ordinals $\langle \alpha_\nu : \nu < \lambda \rangle$ so that if G is $\mathbf{P}$-generic, then in $V[G]$ there is a tree-like continuous sequence $\langle A_i : i < \lambda^+ \rangle$ of subsets of λ satisfying: for all $\nu < \lambda$, $|A_i \cap [\alpha_\nu, \alpha_{\nu+1})| = 1$; and if for all $\delta \in S$ and $\gamma \in [\alpha_\delta, \alpha_{\delta+1})$ we have $\varphi(\gamma) \leq h(\delta)$, then* $\mathrm{Unif}_\varphi\langle A_i : i < \lambda^+ \rangle$ *holds. Further, if λ is supercompact, then λ is supercompact in* $V[G]$.

§3. Changing cofinalities. In this section we will investigate how much of the uniformization properties created in the first section can be preserved if we change the cofinality of λ. First we review Prikry's [P] forcing for changing the cofinality of a measurable cardinal to ω without collapsing cardinals. Suppose λ is a measurable cardinal and D is a normal ultrafilter on λ. Then by $\mathbf{Q}(D)$ we denote the poset whose elements are pairs $\langle u, A\rangle$ where u is a finite subset of λ and $A \in D$. Then $\langle u, A\rangle \leq \langle v, B\rangle$, if u is an initial segment of v, $B \subseteq A$ and $v\backslash u \subseteq A$. We will use the following facts about $\mathbf{Q}(D)$.

3.1. *Facts.* (1) Suppose $\tilde{\alpha}$ is a $\mathbf{Q}(D)$-name of an ordinal less than λ and, for some β, $\Vdash_{\mathbf{Q}(D)} \tilde{\alpha} < \beta$. Then for all finite $u \subseteq \lambda$ there is $\gamma < \beta$ and $B \in D$ so that $\langle u, B\rangle \Vdash \tilde{\alpha} = \gamma$.

(2) Let w^* denote a Prikry sequence (i.e. w^* is the $\bigcup_{\langle u, B\rangle \in G} u$ for some generic G). Then, for all $B \in D$, $w^* \subseteq^* B$ (i.e. for some β, $w^*\backslash\beta \subseteq B$).

3.2. THEOREM. *Assume it is consistent that a supercompact cardinal exists. Then it is consistent that there is a cardinal λ, a sequence $\langle A_i : i < \lambda^+\rangle$ of inaccessible cardinals λ_n $(n < \omega)$ cofinal in λ and each $A_i \subseteq \lambda$ such that* $\mathrm{Unif}_h\langle A_i : i < \lambda^+\rangle$ *holds. Here $h(\alpha) = \min\{\lambda_n : \lambda_n > \alpha\}$. Further the A_i's are tree-like; for all $i < \lambda^+$, $A_i = \bigcup_{n<\omega} A_i \cap (\lambda_n, 2^{\lambda_n})$; and for all $i < \lambda^+$ and $n < \omega$, $|A_i \cap (\lambda_n, 2^{\lambda_n})| = 1$.*

PROOF. We can assume λ is supercompact and $\langle A_i' : i < \lambda^+\rangle$ is as constructed in §1. Let D be a normal measurable ultrafilter on λ. Suppose $\langle \lambda_n : n < \omega\rangle$ is a Prikry sequence for $\mathbf{Q}(D)$. Since D concentrates on inaccessible cardinals, we can assume each λ_n is inaccessible. Let

$$A_i = \bigcup_{n<\omega} A_i' \cap (\lambda_n, 2^{\lambda_n}).$$

We will show, in $V[\langle \lambda_n : n < \omega\rangle]$, that $\mathrm{Unif}_h\langle A_i : i < \lambda^+\rangle$ holds.

Suppose $\langle \tilde{f}_i : i < \lambda^+\rangle$ is a sequence of $\mathbf{Q}(D)$ names such that for all i, $\Vdash \tilde{f}_i : A_i' \to \lambda$ and for all $\alpha \in A_i$, $\tilde{f}_i(\alpha) < \tilde{h}(\alpha)$. Here $\tilde{h}$ is the obvious $\mathbf{Q}(D)$ name for h. Note: it clearly suffices to show that $\mathrm{Unif}_h\langle A_i' : i < \lambda^+\rangle$ holds.

3.3. *Claim. For each $i < \lambda^+$, there is $C_i \in D$ such that if $\alpha \in A_i'$ and u is a finite subset of $\alpha + 1$, then $\langle u, C_i\rangle \Vdash \tilde{f}_i(\alpha) < \min(C_i\backslash\alpha + 1)$.*

We postpone the proof of this claim.

3.4. *Claim. For each i, there is $B_i \in D$ such that if $\alpha \in A_i'$ and u is a finite subset of $\alpha + 1$, then there is some $\gamma < \lambda$ such that*

$$\langle u, B_i\backslash\alpha + 1\rangle \Vdash \tilde{f}_i(\alpha) = \gamma.$$

Proof of Claim 3.4. By Fact 3.1(1), for each $\alpha \in A_i'$ we can choose $B_{\alpha i} \subseteq C_i\backslash\alpha + 1$ so that if $u \subseteq \alpha + 1$ is finite then, for some γ, $\langle u, B_{\alpha i}\rangle \Vdash \tilde{f}_i(\alpha) = \gamma$. Now let B_i be the diagonal intersection of the $B_{\alpha i}$'s (i.e. $B_i = \{v : v \in B_{\alpha i}$ for all $\alpha < v\}$). Since $B_i\backslash\alpha + 1 \subseteq B_{\alpha i}$, B_i is as desired.

Proof of 3.2, *continued.* For $i < \lambda^+$ define a function $g_i : A_i' \to \lambda$ (in V) by

$$g_i(\alpha) = \{\langle u, \gamma\rangle : u \subseteq \alpha + 1 \text{ and } \langle u, B_i\backslash\alpha + 1\rangle \Vdash \tilde{f}_i(\alpha) = \gamma\}.$$

Choose $g : \lambda \to \lambda$ which uniformizes $\langle g_i : i < \lambda^+\rangle$. We view g as a (partial) function from $\lambda \times {}^{<\omega}\lambda \to \lambda$. In $V[\langle \lambda_n : n < \omega\rangle]$ define f by

$$f(\alpha) = g(\alpha)(\{\lambda_n : n < \omega\} \cap \alpha + 1).$$

Consider $i < \lambda^+$. Choose $v < \lambda$ so that $\{\lambda_n : n < \omega\}\backslash v \subseteq B_i$ (see Fact 3.1(2)). For $\alpha \in A_i'\backslash v$, $\langle \alpha + 1 \cap \{\lambda_n : n < \omega\}, B_i\backslash\alpha + 1\rangle$ is in the $\mathbf{Q}(D)$-generic set determined by $\{\lambda_n : n < \omega\}$. So $f(\alpha) = f_i(\alpha)$.

It remains to prove Claim 3.3.

Proof of Claim 3.3. For each $\alpha, \beta < \lambda$ and finite $u \subseteq \alpha + 1$, let $C^u_{\alpha\beta} = \{v : v \subseteq (\alpha, \lambda),$

v is finite and there is $A = A^{uv}_{\alpha\beta} \in D$ such that $\langle u \cup v, A\rangle \Vdash \tilde{f}_i(\alpha) = \beta\}$. Since D is a normal ultrafilter, for each $u \subseteq \alpha + 1$ and $\beta < \lambda$ there is $B^u_{\alpha\beta} \in D$ such that for all finite $v \subseteq B^u_{\alpha\beta}$ the truth value of $v \in C^u_{\alpha\beta}$ depends only on $|v|$. We can assume $B^u_{\alpha\beta} \subseteq (\alpha, \lambda) \cap (\beta, \lambda)$. In addition we can assume if $v \subseteq B^u_{\alpha\beta}$ and $v \in C^u_{\alpha\beta}$ then $A^{uv}_{\alpha\beta} \supseteq B^u_{\alpha\beta}\backslash\max(v) + 1$. (This last assumption can be achieved by taking a diagonal intersection.) Let $B^u_\alpha = \{\delta: \alpha < \delta < \lambda$ and for all $\beta < \delta$, $\delta \in B^u_{\alpha\beta}\}$. Since this last set is a diagonal intersection, $B^u_\alpha \in D$.

Now define a function $e^u_1\colon {}^{<\omega}\lambda \to 2$ by

$$e^u_1(v) = \begin{cases} 0, & \text{if } v \in C^u_{\alpha\beta} \text{ for some } \beta, \\ 1, & \text{otherwise.} \end{cases}$$

Without loss of generality we can assume B^u_α is homogeneous for e^u_1 (i.e. for all finite $v \subseteq B^u_\alpha$ the value of $e^u_1(v)$ depends only on $|v|$). If β as above exists, we denote it as $\beta^u_\alpha(v)$. For some $v \subseteq B^u_\alpha, e^u_1(v) = 0$. Otherwise $\langle u, B^u_\alpha\rangle \Vdash$ "$\tilde{f}_i(\alpha)$ is not defined". Further note that for all $v, \beta = \beta^u_\alpha(v) < \min(v)$. This is so, since $\langle u \cup v, A^{uv}_{\alpha\beta}\rangle \Vdash \tilde{h}(\alpha) = \min(v)$. Now we can find $B^u_\alpha \supseteq B' \in D$ and β so that for $v \subseteq B'$ if $\beta^u_\alpha(v)$ is defined, then $\beta^u_\alpha(v) = \beta$. To save notation we assume $B^u_\alpha = B'$. Next we let $B_\alpha = \bigcap B^u_\alpha$ (u a finite subset of α). Finally we let C_i be the diagonal intersection of B_α for $\alpha \in A_i$.

3.5. REMARK. The uniformization we have achieved is close to the optimal result. Suppose $g(\alpha) = 2^{\lambda_n}$, where $\lambda_n = h(\alpha)$. Then $\mathrm{Unif}_g\langle A_i: i < \lambda^+\rangle$ does not hold. Otherwise we could let $f_i(\alpha) = A_i \cap (\lambda_n, 2^{\lambda_n})$ (recall that $\lambda_n = h(\alpha)$). If there were f which uniformized the f_i, we could find $\beta < \lambda$ and $i \neq j$ such that $f_i \restriction (A_i\backslash\beta)$, $f_j(A_j\backslash\beta) \subseteq f$ and $A_i \cap \beta = A_j \cap \beta$. But these conditions imply that $A_i = A_j$.

Although we have chosen a set to be uniformized of cardinality λ^+, this is purely for notational convenience. We could repeat the argument for any regular $\mu > \lambda$.

We used the supercompactness of λ to guarantee that we could add the $\langle A'_i: i < \lambda^+\rangle$ while preserving the supercompactness and hence the measurability of λ. However we can hope that if we work a bit harder we can make do with the assumption that λ is a suitable hypermeasurable in the ground model.

§4. Collapsing cardinals. In this section we improve on the results of the last section and get a uniformization for subsets of $\aleph_\omega$. In our model $\aleph_\omega$ will be a strong limit cardinal. As before we begin with a supercompact cardinal λ and a family $\langle A'_i: i < \lambda^+\rangle$ of subsets of λ satisfying $\mathrm{Unif}\langle A'_i: i < \lambda^+\rangle$. Following [M1] we will collapse λ to $\aleph_\omega$. In the resulting model we will show that $2^{\aleph_n} = \aleph_{n+2}$ if $n \equiv 1 \pmod 3$, and $2^{\aleph_n} = \aleph_{n+1}$ otherwise; λ becomes $\aleph_\omega$; all cardinals $\geq \lambda$ are preserved; and $\mathrm{Unif}_h\langle A_i: i < \lambda^+\rangle$ holds. Here for each $\alpha < \lambda^+$,

$$A_i = \bigcup_{\substack{n \equiv 1 \,(\mathrm{mod}\, 3)\\ n < \omega}} A'_i \cap (\aleph_n, \aleph_{n+1})$$

and $h(\beta) = \aleph_n$, where n is the least integer $\equiv 1 \pmod 3$ such that $\beta < \aleph_n$. In this section our notation is taken from [M1]. We refer the reader there for any unexplained notation. Since it would be impractical to summarize [M1], we assume the reader will refer to that paper while reading this section.

4.1. LEMMA. *There are $B \in U$, $B \subseteq D$, and F a function on B so that for all $i < \lambda^+$, $\alpha \in A'_i$, $g_0, \ldots, g_n$, $P_1, \ldots, P_n \in D$ above $\langle P_1, \ldots, P_n, g_0, \ldots, g_n, B(\alpha, i), F\rangle$ the value of $\tilde{f}(\alpha)$ is determined by the extensions of $g_0, \ldots, g_n$. Here $B(\alpha, i) = \{Q \in B: \alpha, i \in Q\}$.*

PROOF. Using the methods of the proof of Theorem 2.6 of [M1], it suffices to prove the following claim.

Claim. Fix $i < \lambda^+$ *and* $\alpha \in A'_i$ *and a condition* $\pi = \langle P_1, \ldots, P_n, g_0, \ldots, g_n, B, G\rangle$. *Then there is* B' *and* G' *so that* $\pi \leq \pi' = \langle P_1, \ldots, P_n, g_0, \ldots, g_n, B', G'\rangle$ *and above* π' *the value of* $\tilde{f}_i$ *depends only on the extensions of* $g_0, \ldots, g_n$.

Proof of the claim. By Lemma 2.9 of [M1] we can choose B'' and G'' so that

$$\pi \leq \langle P_1, \ldots, P_n, g_0, \ldots, g_n, B'', G''\rangle = \pi''$$

and if $\langle P_1, \ldots, P_n, Q_0, \ldots, Q_m, g'_0, \ldots, g'_n, h_0, \ldots, h_m, B''', G'''\rangle = \pi''' \geq \pi''$ and $\pi''' \Vdash \tilde{f}_i = \beta$, then

$$\langle P_1, \ldots, P_n, Q_0, \ldots, Q_m, g'_0, \ldots, g'_n, G''(Q_0), \ldots, G''(Q_m), B'', G''\rangle \Vdash \tilde{f}_i(\alpha) = \beta.$$

We will choose $B' \subseteq B''$.

For any $P \in B$, define a partition of $B''(P)^{<\omega}$ into $P \cap \lambda$ pieces by $\langle Q_0, \ldots, Q_m\rangle \equiv \langle Q'_0, \ldots, Q'_m\rangle$ iff for any $\langle g'_0, \ldots, g'_n\rangle$ extending $\langle g_0, \ldots, g_n\rangle$ (where $\langle P_1, \ldots, P_n, P, g'_0, \ldots, g'_n, G''(P), B'', G''\rangle$ is a condition), if $\langle P_1, \ldots, P_n, P, Q_0, \ldots, Q_m, g'_0, \ldots, g'_n, G''(P), G''(Q_0), \ldots, G''(Q_m), B'', G''\rangle \Vdash \tilde{f}_i(\alpha) = \beta$, then so does the condition with $Q'_0, \ldots, Q'_m$ and vice versa. (Also if one sequence does not yield a condition then neither does the other.) Let C_P be homogeneous for this partition. Fix $\eta = \langle g'_0, \ldots, g'_n\rangle$, an extension of $\langle g_0, \ldots, g_n\rangle$. For $P \in B''$ define $\varphi(P) = \beta$ if for some $Q_0, \ldots, Q_m \in C_P$

$$\langle P_1, \ldots, P_n, P, Q_0, \ldots, Q_m, g'_0, \ldots, g'_n, G''(P), \\ G''(Q_0), \ldots, G''(Q_m), B'', G''\rangle \Vdash \tilde{f}_i(\alpha) = \beta.$$

Otherwise let $\varphi(P) = -1$. Note φ is a well-defined function and $\varphi(P) < P \cap \lambda$. So there is $B_\eta \subseteq B''$ with $B_\eta \in U$ so that φ is constant on B_η.

Finally let $B' \subseteq B''$, $B' \in U$, be such that if $P \subsetneq Q \in B'$ then $Q \in C_P$, and for all $P \in B'$ and $\eta = \langle g'_0, \ldots, g'_n\rangle$ extending $\langle g_0, \ldots, g_n\rangle$, if $g'_n \in \mathrm{Col}((P_n \cap \lambda)^{++}, P \cap \lambda)$, then $P \in B_\eta$. Let $G' = G''$. By our definition of the C_P's and B_η's it is easy to see that B' and F' are as required.

Next we want to follow the strategy of §3 and define in the ground model functions from the A'_i's to λ which can be uniformized. There are two problems, both of which can be avoided. First note that if $\langle P_1, \ldots, P_n, g_0, \ldots, g_n, B(\alpha, i), G\rangle \Vdash \tilde{f}_i(\alpha) = \beta$, $P'_j \cap \lambda = P_j \cap \lambda$ $(1 \leq j \leq n)$, and $P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$, then

$$\langle P'_1, \ldots, P'_n, g_0, \ldots, g_n, B(\alpha, i), G\rangle \Vdash \tilde{f}_i(\alpha) = \beta.$$

For each i, α, $\lambda_1, \ldots, \lambda_n \leq \alpha$, $g_0, \ldots, g_{n-1}$ and β where $g_j \in \mathrm{Col}(\lambda_{j-1}^{++}, \lambda_j)$, choose a maximal antichain $X_{i,\alpha,\bar{\lambda},\bar{g},\beta}$ in $\mathrm{Col}(\lambda_n^{++}, \lambda)$ amongst the g's such that there is $P_1, \ldots, P_n$ satisfying $P_j \cap \lambda = \lambda_j$ $(1 \leq j < n)$ and

$$\langle P_1, \ldots, P_n, g_0, \ldots, g_n, g, B(\alpha, i), G\rangle \Vdash \tilde{f}_i(\alpha) = \beta.$$

Define

$$\varphi_i(\alpha) = \{\langle \lambda_1, \ldots, \lambda_n, g_0, \ldots, g_{n-1}, g, \beta\rangle : g \in X_{\lambda_1, \ldots, \lambda_n, g_0, \ldots, g_{n-1}, \beta}\}.$$

Note that, for fixed $\lambda_1, \ldots, \lambda_n$ and $g_0, \ldots, g_{n-1}$,

$$\bigcup_{\beta < \lambda} X_{\lambda_1, \ldots, \lambda_n, g_0, \ldots, g_{n-1}, \beta}$$

is an antichain in $\mathrm{Col}(\lambda_n^{++},\lambda)$. As $\mathrm{Col}(\lambda_n^{++},\lambda)$ satisfies the λ-c.c., this set has cardinality $<\lambda$. Since there are at most $<\lambda$ possibilities for $\lambda_1,\ldots,\lambda_n$, $g_0,\ldots,g_{n-1}$, there are at most λ possible values for any $\varphi_i(\alpha)$.

4.2. THEOREM. $\mathrm{Unif}_h\langle A_i': i<\lambda^+\rangle$ *holds after the forcing.*

PROOF. Let $\tilde{f}_i$, φ_i $(i<\lambda^+)$, B and G be as above. Choose φ which uniformizes the $\langle\varphi_i: i<\lambda^+\rangle$ (we assume that for all α there is i such that $\varphi(\alpha)=\varphi_i(\alpha)$). Assume H is $\mathbf{P}$-generic with $\langle B,G\rangle\in H$. Define f by $f(\alpha)=\beta$ iff there is $\langle P_1,\ldots,P_n,g_0,\ldots,g_n,C,F\rangle\in H$ such that $P\cap\lambda=\lambda_j$ for $1\le j\le n$ and $\langle\lambda_1,\ldots,\lambda_n,g_0,\ldots,g_n,\beta\rangle\in\varphi(\alpha)$ and $\lambda_{n+1}>\alpha$. If no such condition exists let $f(\alpha)=0$. We must check that f is well defined. Suppose $\langle\lambda_1,\ldots,\lambda_n,g_0,\ldots,g_n,\beta\rangle$ and $\langle\lambda_1,\ldots,\lambda_n,h_0,\ldots,h_n,\gamma\rangle\in\varphi(\alpha)=\varphi_i(\alpha)$ for some i. Further suppose the hypotheses require $f(\alpha)=\beta$ and $f(\alpha)=\gamma$. Then by the compatibility of H and the fact that $\tilde{f}_i$ is a name for a function in the submodel for some $P_1,\ldots,P_n$,

$$\langle P_1,\ldots,P_n,h_0\cup g_0,\ldots,h_n\cup g_n,B(\alpha,i),G\rangle\Vdash\gamma=\tilde{f}_i(\alpha)=\beta.$$

So $\gamma=\beta$. Note that f is in the submodel.

It remains to show that f uniformizes the f_i's. Fix i and consider a condition $\langle P_1,\ldots,P_n,g_0,\ldots,g_n,C,F\rangle\in H$ so that $B(i)\supseteq C$. By weakening the condition we can take $F=G$. We claim that for all $\alpha\ge\lambda_n$ $(=P_n\cap\lambda)$ if $\varphi(\alpha)=\varphi_i(\alpha)$ then $f(\alpha)=f_i(\alpha)$. Suppose α is as above and $\pi=\langle P_1,\ldots,P_n,P_{n+1},\ldots,P_m,P_{m+1},h_0,\ldots,h_{m+1},C',F'\rangle\in H$, $\pi\Vdash\tilde{f}_i(\alpha)=\beta$, $P_m\cap\lambda=\lambda_m\le\alpha<\lambda_{m+1}=P_{m+1}\cap\lambda$, and $C'\subseteq B(i)$. Then

$$\pi\ge\langle P_1,\ldots,P_m,h_0,\ldots,h_m,B(\alpha,i),G\rangle=\pi'.$$

So by the choice of B and G, $\pi'\Vdash\tilde{f}_i(\alpha)=\beta$. As $\pi'\in H$, $f(\alpha)=\beta$.

§5. Black box. There are various diamond-like principles due to Shelah which are provable in ZFC. These have become known as black boxes (see for example [CG]). The simplest of these principles was introduced in [S1] as a paradigm for the entire family. This principle is sufficient to establish that any stable nonsuperstable theory has λ^+ models in power λ for many λ. For $X\subseteq{}^{\omega}\mu$ and $\lambda\le\mu$ define $D(X,\lambda)$ to hold iff

> there is a family of functions $\langle g_\eta:\eta\in X\rangle$, where each $g_\eta:\{\eta\restriction n: n<\omega\}\to\lambda$, such that if $f:{}^{<\omega}\mu\to\lambda$ is a function then $f\supseteq g_\eta$ for some $\eta\in X$.

5.1. PROPOSITION [S1]. *$D({}^{\omega}\mu,\mu)$ holds.*

In this section we investigate the ideal of "small" sets, i.e. the collection $\mathscr{I}=\{X\subseteq{}^{\omega}\mu: D(X,\mu)$ does not hold$\}$. It is easy to see that $\mathscr{I}$ is an ideal. (Our next proposition establishes this by considering the case where λ is finite.) The question remains, "How complete is $\mathscr{I}$?" In contrast to the situation presented here, for μ a regular cardinal the ideal of subsets of μ for which $\diamondsuit$ does not hold is a normal μ-complete ideal [DS]. The results in this section place some limit on how nearly the black boxes can resemble diamond principles.

5.2. PROPOSITION. *Let $\mathscr{I}$ be the ideal of small subsets of ${}^{\omega}\mu$. If $\mu^\lambda=\mu$, then $\mathscr{I}$ is λ^+-complete (i.e. the union of λ elements of $\mathscr{I}$ is in $\mathscr{I}$).*

PROOF. Suppose $X\notin\mathscr{I}$ and $X=\bigcup_{\alpha<\lambda}X_\alpha$, where the X_α's are disjoint. Let φ: $\mu\to{}^{\lambda}\mu$ be a bijection and let $\langle g_\eta:\eta\in X\rangle$ witness that $D(X,\mu)$ holds. For $\alpha<\lambda$ and

$\eta \in X_\alpha$ define h_η by $h_\eta(\eta \restriction n) = \varphi(g_\eta(\eta \restriction n))(\alpha)$. Suppose that for $\alpha < \lambda$ the f_α are functions which witness that $X_\alpha \in \mathscr{I}$ with respect to $\langle h_\eta : \eta \in X_\alpha \rangle$. Define f by $f(\nu) = \varphi^{-1}(\langle f_\alpha(\nu) : \alpha < \lambda \rangle)$. Now if $f \supseteq g_\eta$ and $\eta \in X_\alpha$, then $f_\alpha \supseteq h_\eta$. So we have a contradiction.

We will show it is consistent that for some $\lambda < \mu$, the ideal of small sets is not λ^+-complete.

5.3. THEOREM. *Fix uncountable cardinals $\lambda < \mu$ such that $\lambda^{<\lambda} = \lambda$. It is consistent that ${}^\omega\mu = \bigcup_{\alpha<\lambda} A_\alpha$ and, for all $\alpha < \lambda$, $D(A_\alpha, \lambda)$ fails. In fact for $\alpha < \lambda$ if $\langle g_\eta : \eta \in A_\alpha \rangle$ is a family of functions where $g_\eta : \{\eta \restriction n : n < \lambda\} \to \lambda$, then there is a function $g : {}^{<\omega}\mu \to \lambda$ which almost contains each g_η.*

PROOF. It is enough to prove the last assertion. Given such a function g, choose a function f which disagrees with g everywhere. Then f shows that $\langle g_\eta : \eta \in A_\alpha \rangle$ does not have the property demanded by $D(A_\alpha, \lambda)$.

We define an iterated forcing with $<\lambda$ support. The generic set for our first poset will yield the A_α's and guarantee that each A_α is λ-free (i.e. if $X \subseteq A_\alpha$ and $|X| < \lambda$, then to each $\eta \in X$ we can assign $n(\eta) \in \omega$ so that for $\eta \neq \nu \in X$ the sets $\{\eta \restriction m : m \geq n(\eta)\}$ and $\{\nu \restriction m : m \geq n(\nu)\}$ are disjoint). Since this forcing will introduce no new subsets of μ of cardinality $<\lambda$, we can afford to confuse ${}^\omega\mu$ in the generic extension with ${}^\omega\mu$ in the ground model.

The elements of $\mathbf{Q}_0$ are sequences $\langle B_\alpha : \alpha < \lambda \rangle$ of disjoint sets, where $|\{\alpha : B_\alpha \neq 0\}| < \lambda$; for each α, $|B_\alpha| < \lambda$; for each α, $B_\alpha \subseteq {}^\omega\mu$; and for each α, there is $h : B_\alpha \to \omega$ such that the sets $\{\eta \restriction m : m > h(\eta)\}$ $(\eta \in B_\alpha)$ are disjoint (i.e. h witnesses B_α is free). We define $\langle B_\alpha : \alpha < \lambda \rangle \leq \langle C_\alpha : \alpha < \lambda \rangle$ if, for all α, $B_\alpha \subseteq C_\alpha$ and $(C_\alpha \backslash B_\alpha) \cap \mathrm{cl}(B_\alpha) = 0$. Here $\mathrm{cl}(B_\alpha) = \{\eta :$ for all n there is $\nu \in B_\alpha$ so that $\eta \restriction n = \nu \restriction n\}$. Note that if $\langle B_\alpha : \alpha < \lambda \rangle \leq \langle C_\alpha : \alpha < \lambda \rangle$ and $h : B_\alpha \to \omega$ is as above, then h can be extended to $g : C_\alpha \to \omega$ which witnesses C_α is free. To see this choose functions $h_1, h_2 : C_\alpha \to \omega$, where h_1 witnesses that C_α is free and h_2 witnesses $(C_\alpha \backslash B_\alpha) \cap \mathrm{cl}(B_\alpha) = 0$. Define $g : C_\alpha \to \omega$ by

$$g(\eta) = \begin{cases} h(\eta), & \text{if } \eta \in B_\alpha, \\ \max\{h_1(\eta), h_2(\eta)\}, & \text{otherwise.} \end{cases}$$

It is not clear whether the poset we have defined is λ-complete or λ-strategically complete (according to our definition). However, it satisfies another closure condition.

DEFINITION. Let $\mathbf{P}$ be a poset. An increasing sequence $\langle p_\beta : \beta < \alpha \rangle$ is *continuous* if for every limit ordinal γ, $p_\gamma = \sup\{p_\beta : \beta < \gamma\}$. We say $\mathbf{P}$ is *λ-continuous complete* if for all $\alpha < \lambda$ and increasing continuous sequence $\langle p_\beta : \beta < \alpha \rangle$ there is $p = \sup\{p_\beta : \beta < \alpha\} \in \mathbf{P}$.

It is easy to see that if we iterate λ-continuous complete posets with $<\lambda$ support the result will be λ-continuous complete. Also, forcing with a λ-continuous complete poset will add no new sets of ordinals of cardinality $<\lambda$.

5.4. PROPOSITION. *Let $\mathbf{Q}_0$ be the poset defined above. Then $\mathbf{Q}_0$ is λ-continuous complete and satisfies the λ^+-chain condition.*

PROOF. First we see $\mathbf{Q}_0$ is λ-continuous complete. Assume $\langle \langle B_{\alpha i} : \alpha < \lambda \rangle : i < \beta \rangle$ is an increasing continuous sequence and $\beta < \lambda$. Define $B_\alpha = \bigcup B_{\alpha i}$. We claim $\langle B_\alpha : \alpha < \lambda \rangle \in \mathbf{Q}_0$. For this it is enough to see that B_α is free. But B_α is the union of a continuous chain $B_{\alpha i}$ $(i < \beta)$, each $B_{\alpha i}$ is free and $B_{\alpha i+1}$ is free over $B_{\alpha i}$.

Suppose now that $\{\langle B_{\alpha i}: \alpha < \lambda\rangle : i < \lambda^+\}$ is a collection of elements of $\mathbf{Q}_0$. Applying the Δ-lemma we can assume there is a set $X \subseteq \lambda$ of cardinality $< \lambda$ such that for $i \neq j$ if $B_{\alpha i} \neq 0 \neq B_{\alpha j}$ then $\alpha \in X$. For $\alpha \in X$ and $i < \lambda^+$, let $C_{\alpha i} = \{\eta \restriction n : n < \omega \text{ and } \eta \in B_{\alpha i}\}$. Again applying the Δ-lemma, we can assume there is $\langle C_\alpha : \alpha \in X\rangle$ such that for $i \neq j$, $C_{\alpha i} \cap C_{\alpha j} = C_\alpha$. For any α, $|\mathrm{cl}(C_\alpha)| \leq \lambda$. Since $\lambda^{<\lambda} = \lambda$, we can assume that $B_{\alpha i} \cap \mathrm{cl}(C_\alpha) = B_{\alpha j} \cap \mathrm{cl}(C_\alpha)$ for $i \neq j$ and $\alpha \in X$. Under these assumptions

$$\langle B_{\alpha 0} \cup B_{\alpha 1} : \alpha < \lambda\rangle \in \mathbf{Q}_0.$$

Suppose G is $\mathbf{Q}_0$-generic. For $\alpha < \lambda$, let $\tilde{A}_\alpha$ be a name for $\bigcup\{B_\alpha : \langle B_\beta : \beta < \lambda\rangle \in G\}$. To finish the proof of the theorem we choose appropriately $\{\langle \tilde{H}_{\alpha i} : \alpha < \lambda\rangle : i < 2^{\lambda^+}\}$, where

$$\Vdash_{\mathbf{P}_i} \text{“}\tilde{H}_{\alpha i} = \langle g_\eta : \eta \in \tilde{A}_\alpha\rangle \text{ and for } \eta \in \tilde{A}_\alpha\ h_\eta : \{\eta \restriction n : n < \omega\} \to \lambda\text{”}.$$

Then we define Q_i to be the set of sequences $\langle f_\alpha : \alpha < \lambda\rangle$ such that: $|\{\alpha : f_\alpha \neq 0\}| < \lambda$; for all α, f_α is a partial function from ${}^\omega\mu$ to ω; and for all α, $|\mathrm{dom}\, f_\alpha| < \lambda$. For $i \leq 2^{\lambda^+}$ the elements of $\mathbf{P}_i$ will be functions p with domain i such that: $p(0) \in \mathbf{Q}_0$; for all $j \neq 0$, $p(j) \in Q_j \cup \{0\}$; and $|\{j : p(j) \neq 0\}| < \lambda$. Further suppose $p(0) = \langle B_\alpha : \alpha < \lambda\rangle$, $j \neq 0$ and $p(j) \neq 0$. Then $p(j) = \langle f_\alpha : \alpha < \lambda\rangle$ is such that for all α, $\mathrm{dom}\, f_\alpha = B_\alpha$ and for η, $\nu \in \mathrm{dom}\, f_\alpha$ and $m \geq f_\alpha(\eta), f_\alpha(\nu)$

$$p \restriction j \Vdash \text{“if } \eta \restriction m = \nu \restriction m \text{ then } g_\eta(\eta \restriction m) = g_\nu(\nu \restriction m)\text{”}.$$

Choose χ so that $\chi^\mu = \chi$. The proof that each $\mathbf{P}_i$ $(i \leq \chi)$ is λ-continuous complete is straightforward. The only difficulties come in showing that $\mathbf{P}_i$ has the λ^+-c.c. and that, for all j, $\{p \in \mathbf{P}_i : p(j) \neq 0\}$ is dense. Since the two proofs are similar we will only give the density proof. Suppose $p \in \mathbf{P}_i$ and $p(j) = 0$. Suppose, as well, that $p(0) = \langle B_\alpha : \alpha < \lambda\rangle$. Let $\langle h_\alpha : \alpha < \lambda\rangle$ witness that each B_α is free. Define q by $q(l) = p(l)$ if $l \neq j$ and $q(j) = \langle h_\alpha : \alpha < \lambda\rangle$. By the choice of χ and a routine enumeration we can take $\mathbf{P}_\chi$ to be the desired poset.

§6. Applications. In this section we will study two applications of the uniformization results proved in §§3 and 4. Rather than require a knowledge of those sections, we will state the hypothesis we will use in this section.

Hypothesis. There is a cardinal λ, an increasing sequence $\langle \lambda_n : n < \omega\rangle$ cofinal in λ, and a tree-like sequence $\langle A_i : i < \lambda^+\rangle$ of subsets of λ such that: for all n, $n^{\prod_{k<n}\lambda_k} < \lambda_n$; for all n and $i < \lambda^+$, $|A_i \cap (\lambda_n, 2^{\lambda_n})| = 1$; and $\mathrm{Unif}_h\langle A_i : i < \lambda^+\rangle$ holds, where $h(\alpha) = \min\{\lambda_n : \lambda_n < \alpha, n < \omega\}$.

For $\lambda = \aleph_0$ the consistency of this hypothesis is proved in [S3]. In §3 we have shown the hypothesis is consistent for some uncountable λ. Further, in these two cases λ^+ can be replaced by any $\lambda < \mu < 2^\lambda$. In §4 we showed the hypothesis is consistent for $\lambda = \aleph_\omega$. For λ uncountable, we can assume all the λ_n's are infinite. So the cardinal arithmetic condition simplifies to $2^{\lambda_n} < \lambda_{n+1}$. For uncountable λ the consistency results were proved relative to the consistency of some large cardinal.

Before beginning our applications, we will note a few consequences of our hypothesis.

6.1. PROPOSITION. *Let λ, $\langle \lambda_n : n < \omega \rangle$ and $\langle A_i : i < \lambda^+ \rangle$ be as in the hypothesis. Then $2^\lambda = 2^{\lambda^+}$.*

PROOF. For each $X \subseteq \lambda^+$, define functions $f_i^X (i < \lambda^+)$ by

$$f_i^X(\alpha) = \begin{cases} 0, & i \in X, \\ 1, & \text{otherwise.} \end{cases}$$

If $X, Y \subseteq \lambda^+$ and g^X and g^Y uniformize $\langle f_i^X : i < \lambda^+ \rangle$ and $\langle f_i^Y : i < \lambda^+ \rangle$ respectively, then $g^X \neq g^Y$.

6.2. PROPOSITION. *Let λ, $\langle \lambda_n : n < \omega \rangle$ and $\langle A_i : i < \lambda^+ \rangle$ be as in the hypothesis. Then for any $X \subseteq \lambda^+$ with $|X| = \lambda$ there is $\langle A_i^* : i < X \rangle$ such that: the A_i^* are pairwise disjoint; for all i, $A_i^* \subseteq A_i$; and for all i, $|A_i \backslash A_i^*| < \aleph_0$.*

PROOF. List the elements of X as $\{i_\nu : \nu < \lambda\}$. Define $f_{i_\nu} : A_{i_\nu} \to \lambda$ by

$$f_{i_\nu}(\alpha) = \begin{cases} \nu, & \text{if } \nu < h(\alpha), \\ \alpha, & \text{otherwise.} \end{cases}$$

Suppose g uniformizes $\{f_i : i \in X\}$. Define $A_{i_\nu}^* = \{\alpha \in A_{i_\nu} : g(\alpha) = \nu\}$.

Our first applications concern abelian group theory. First we make a remark about Whitehead groups. Already as a consequence of results in [S2] it is consistent that $\aleph_\omega$ is a strong limit cardinal and there is a nonfree Whitehead group which is $\aleph_{\omega+1}$-free of cardinality $\aleph_{\omega+1}$. The group constructed there is necessarily strongly $\aleph_{\omega+1}$-free. Using our hypothesis we can prove a different theorem.

6.3. THEOREM. *Assume it is consistent that a supercompact cardinal exists. Then it is consistent that there is an $\aleph_{\omega+1}$-free not strongly $\aleph_{\omega+1}$-free group which is a Whitehead group.*

PROOF. This theorem follows from the consistency of our hypothesis for $\lambda = \aleph_\omega$ together with the methods of [S3].

REMARK. We can prove an analogous result for Crawley groups (cf. [MS]).

An application we will spend more time on is an answer to the following question: "Which separable abelian p-groups are determined by their socles (as valuated vector spaces)?" To rephrase the question, fix B a separable reduced torsion complete abelian p-group (where p is a prime). (A separable reduced torsion complete abelian p-group is the torsion subgroup of the closure of a direct sum of cyclic p-groups. Here the basic neighborhoods of 0 are the elements which are divisible by p^n for $n < \omega$.) Suppose $G \subseteq B$ is a pure subgroup. Does $G[p] = \{g \in G : pg = 0\}$ determine G up to isomorphism? That is, if $H \subseteq B$ is pure and $H[p] = G[p]$, is $H \cong G$? If G is not a direct sum of cyclic groups or torsion complete, then the answer is usually no. There is no loss of generality in assuming $|G| = \min_{n<\omega} |p^n G|$. It can be shown that the answer is no, provided that there is no pure subgroup A of G so that $|A| < |G|$ and $G/cl(A)$ is torsion complete. Here cl(A) denotes the closure of A inside of G. (Again the basic neighborhoods of 0 are the elements which are divisible by p^n for $n < \omega$.) Even the exceptional case is decided by GCH. In fact if GCH holds and G is a pure subgroup of B which is determined by $G[p]$, then G is a direct sum of cyclic groups or torsion complete. For proofs of these results, see [S4]. We will show that our hypothesis implies the existence of a nontrivial G which is determined by its socle.

6.4. THEOREM. *It is consistent that there is a separable abelian p-group G which is neither the direct sum of cyclic groups nor torsion complete but is determined by* $G[p]$. *Further, if* λ *is as in the hypothesis, then the essential cardinality of G is* λ^+ *and the density character of G is* λ.

PROOF. Assume λ, $\langle \lambda_n : n < \omega \rangle$ and $\langle A_i : i < \lambda^+ \rangle$ are as in the hypothesis. Let $T_n = \prod_{k<n} 2^{\lambda_k}$ and $T = \bigcup T_n$. For $i < \lambda^+$, define η_i by $\eta_i(n) = A_i \cap (\lambda_n, 2^{\lambda_n})$. Let B be the torsion complete p-group with basis $t_\eta (\eta \in T)$, where the order of t_η is $p^{l(\eta)+1}$. Here $l(\eta)$ denotes the length of η. As usual, we identify B with the set of formal sums of the form $\sum_{\eta \in T} a_\eta t_\eta$ such that: each $a_\eta \in \mathbf{Z}$; for all n, $\{\eta : l(\eta) = n \text{ and } a_\eta \neq 0\}$ is finite; and there is m such that for all η, $p^m a_\eta t_\eta = 0$.

Define a group $G \subseteq B$ to be the group generated by $\{t_\eta : \eta \in T\} \cup \{y_i^m : i < \lambda^+, m < \omega\}$. Here for $i < \lambda^+$ and $m < \omega$, $y_i^m = \sum_{n \geq m} p^{n-m} t_{\eta_i \restriction n}$. Suppose H is a pure subgroup of B and $H[p] = G[p]$. For $\eta \in T$, there is $s_\eta \in H$ so that $p^{l(\eta)} s_\eta = p^{l(\eta)} t_\eta$ (since $G[p] = H[p]$). Now $\{s_\eta : \eta \in T\}$ is a basis for B. So there is an automorphism of B which takes each s_η to t_η and fixes $H[p]$. So we can assume $s_\eta = t_\eta$ (i.e. $H \supseteq \{t_\eta : \eta \in T\}$).

Claim. For each i, there is $\{\alpha_{\eta i} \in \mathbf{Z} : \eta \in T\}$ *and* $z_i^m \in H$ *such that*

$$z_i^m = \sum_{n \geq m} p^{n-m} t_{\eta_i \restriction n} + \sum \alpha_{\eta i} p^{l(\eta)-m+1} t_\eta$$

(*where the second sum is over* $\{\eta \in T : l(\eta) \geq m\}$) *and for each n*, $\{\alpha_{\eta i} : \alpha_{\eta i} \neq 0, \eta \in T_n\}$ *is finite.*

Proof. First we set some notation. For any $x \in B$ and $n < \omega$, let

$$\varphi_n(x) = \sum_{l(\eta) \leq n} b_\eta t_\eta, \quad \text{where } x = \sum_{\eta \in T} b_\eta t_\eta.$$

By our assumptions on H, for any $x \in B$ and $n < \omega$, $\varphi_n(x) \in H$. We define z_i^m and $a_{\eta i}^m$ by induction on m. We let $z_i^0 = \sum_{n \geq 0} p^n t_{\eta_i \restriction n} (\in H[p])$. So $a_{\eta i}^0 = 0$, for all $\eta \in T$. In general if z_i^m and the $a_{\eta i}^m$ have been chosen, choose z_i^{m+1} so that $p z_i^{m+1} = z_i^m - \varphi_m(z_i^m)$ and $\varphi_m(z_i^{m+1}) = 0$. Then

$$z_i^{m+1} = \sum_{n \geq m+1} p^{n-m} t_{\eta_i \restriction n} + \sum_{l(\eta) \geq m+1} a_{\eta i}^{m+1} p^{l(\eta)-m+1} t_\eta.$$

It is clear that if $l(\eta) = n$, then for all $m < n$

$$p^{n-m}(a_{\eta i}^n p t_\eta) = a_{\eta i}^m p^{n-m+1} t_\eta.$$

So we can let $a_{\eta i} = a_{\eta i}^{l(\eta)}$.

For $i < \lambda^+$, define functions g_i with dom $g_i = \{t_{\eta \restriction n} : n < \omega\}$ by

$$g_i(t_{\eta_i \restriction n}) = \sum_{\nu \in T_n} a_{\nu i} p t_\nu.$$

So the g_i's have at most $|T_{l(\eta)}| + \aleph_0$ possible values on t_η for uncountable λ and $p^{\prod_{m<l(\eta)} \lambda_m}$ values on t_η for $\lambda = \aleph_0$. By our hypothesis and the choice of the η_i there is a g which uniformizes the g_i. For $i < \lambda^+$, choose n_i so that, for all $n \geq n_i$, $g(t_{\eta_i \restriction n}) = g_i(t_{\eta_i \restriction n})$. For $\eta \in T$, define $t_\eta + g(t_\eta) = u_\eta$. Note that each $u_\eta \in H$ and the u_η's form a basis for B (in fact they generate the same basic subgroup). For $m \geq n(i)$ and $i < \lambda^+$,

$$z_i^m = \sum_{n \geq m} p^{n-m} u_{\eta_i \restriction n}.$$

So the automorphism ψ of B induced by taking u_η to t_η for all $\eta \in T$, takes z_i^m to y_i^m $(i < \lambda^+, n(i) \leq m)$. So ψ restricts to an isomorphism of G with the pure subgroup H_1 of H generated by $\{t_\eta : \eta \in T\} \cup \{z_i^m : i < \lambda^+ \text{ and } m \geq n(i)\}$. But $H_1[p] = H[p]$. So $H_1 = H$.

Our third application is to set-theoretic topology. Rather than stating which properties we are interested in, we shall give a general construction of a topological space. Then we will state various conditions which imply our space has certain properties. Finally we will summarize our construction as a consistency result.

Suppose λ and μ are cardinals and $\langle A_i : i < \mu \rangle$ is a sequence of subsets of λ with no last element. Define a space X with points $\lambda \cup \{A_i : i < \mu\}$. Put a topology on X by letting each $\alpha \in \lambda$ be isolated and a neighborhood base for any A_i is $\{A_i \setminus \gamma : \gamma \in A_i\}$. We assume, for $i \neq j$, that $A_i \cap A_j$ is bounded in A_i.

6.5. *Fact. X is a Hausdorff space of cardinality $\lambda + \mu$. The density character of X is λ. Further, X is locally $\leq \sup\{|A_i| : i < \mu\}$.*

6.6. *Fact. For any cardinal κ, X is κ-collectionwise Hausdorff iff for any $Y \subseteq \mu$ if $|Y| \leq \kappa$ then there are $\{\gamma_i : i \in Y\}$ such that the $A_i \setminus \gamma_i (i \in Y)$ are disjoint.*

Our space X is at most λ-collectionwise Hausdorff. Further, if λ is regular and $\{\sup A_i : i < \mu\}$ is stationary in λ, then X is not λ-collectionwise Hausdorff.

6.7. *Fact.* (1) *The space X is normal iff $\langle A_i : i < \mu \rangle$ has the uniformization property for any sequence of functions $\langle f_i : i < \mu \rangle$ where* $\operatorname{dom} f_i = A_i$ *and each f_i is constantly* 0 *or* 1.

(2) *For any cardinal κ, X is κ-collectionwise normal iff $\langle A_i : i < \mu \rangle$ has the uniformization property for any sequence of functions $\langle f_i : i < \mu \rangle$, where* $\operatorname{dom} f_i = A_i$ *and each f_i is constantly some value $< \kappa$.*

Our space X is at most λ-collectionwise Hausdorff. (This is a consequence of X having density character λ.) The uniformization results we have proved allow us to construct X with these parameters as generous as possible.

6.8 THEOREM. *Assume it is consistent that a supercompact cardinal exists. Then it is consistent that there is a first countable, locally countable Hausdorff space X of cardinality $\aleph_{\omega+1}$ such that X has density character $\aleph_\omega$, X is $\aleph_\omega$-collectionwise Hausdorff and X is $\aleph_\omega$-collectionwise normal.*

PROOF. It is enough to see that we have the necessary uniformization property where $\langle A_i : i < \aleph_{\omega+1} \rangle$ is as provided by the hypothesis. Suppose $\langle f_i : i < \aleph_{\omega+1} \rangle$ is a collection of functions where, for each i, $\operatorname{dom} f_i = A_i$ and f_i is constantly some value $< \aleph_\omega$. For each i, let α_i denote the constant value of f_i. For each i, define a function g_i with domain A_i by

$$g_i(\alpha) = \begin{cases} 0, & \text{if } h(\alpha) \leq \alpha_i, \\ \alpha_i, & \text{if } \alpha_i < h(\alpha). \end{cases}$$

Any function which uniformizes $\langle g_i : i < \aleph_{\omega+1} \rangle$ also uniformizes $\langle f_i : i < \aleph_{\omega+1} \rangle$.

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