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JONSSON ALGEBRAS IN SUCCESSOR CARDINALS

BY

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ABSTRACT

We shall show here that in many successor cardinals λ, there is a Jonsson algebra (in other words $\mathrm{Jn}(\lambda)$, or λ is not a Jonsson cardinal). In connection with this we show that, e.g., for every ultrafilter D over ω, in $(\omega_\omega, <)^\omega/D$ there is no increasing sequence of length $\aleph_{(2^{\aleph_0})^+}$. On Jonsson algebras see e.g. [1]; for successor $\lambda^+ = 2^\lambda$ there is a Jonsson algebra, $\mathrm{Jn}(\lambda) \Rightarrow \mathrm{Jn}(\lambda^+)$ (due to Chang, Erdös and Hajnal) and even in $2^{\aleph_\alpha} = \aleph_{\alpha+n}$ ([3]). We give here a method to prove, e.g., $\mathrm{Jn}(\aleph_{\omega+1})$ when $2^{\aleph_0} \leqq \aleph_{\omega+1}$ and $\mathrm{Jn}(2^{\aleph_0})$ when $2^{\aleph_0} = \aleph_{\alpha+1}$, $\alpha < \omega_1$; and similar results for higher cardinals.

QUESTIONS. (1) Does $\mathrm{Jn}(\aleph_{\omega+1})$ always hold?

(2) Does $\mathrm{Jn}(\lambda^+)$ always hold, or at least when $(\lambda^+)^{\aleph_0} = \lambda^+$?

(3) Does always $\aleph_{\omega+1} \in P\,\mathrm{cf}\langle \aleph_n : n < \omega \rangle$?

DEFINITION 1. (A) A Jonsson algebra is an algebra M, with countably many operations (finitary, of course), which has no proper subalgebra of the same cardinality. A Jonsson model is a model with countably many relations and operations which has no proper elementary submodel of the same cardinality.

(B) $\mathrm{Jn}(\lambda)$, or λ is not a Jonsson cardinal if there is a Jonsson algebra of cardinality λ. This is equivalent to the existence of a Jonsson model (expand by Skolem functions).

CONVENTION 2. (A) We do not distinguish between a model and its universe; and unless stated otherwise a model has only countably many operations and relations.

(B) For simplicity we restrict ourselves to models of the form M_λ, where M_λ^1 will be $(H(\lambda^*), \in)$ for $\lambda^* > \lambda$ (e.g. $(2^\lambda)^+$) ($H(\lambda^*)$ is the family of sets whose

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transitive closure has cardinality $<\lambda^*$); let M^2_λ be an elementary submodel of M^1_λ of cardinality λ, $\lambda+1\subseteq M^2_\lambda$, and $M_\lambda=(M^2_\lambda,\in,F)$ where F is a one-to-one function from λ onto M^2_λ. So M will denote some M_λ.

Notice that $\mathrm{Jn}(\lambda)$ implies that any M_λ is a Jonsson model (proof as for 4A). If there is a Jonsson algebra $\mathfrak{A}=(\lambda,f_i)_{i\in\omega}$ then $\mathfrak{A}\in M^1_\lambda$, thus $M^1_\lambda\models$ "there is a Jonsson algebra on λ". By way of contradiction, assume there is a $N<M_\lambda$, $N\neq M_\lambda$, $\|N\|=\lambda$. Clearly (since λ is definable in M_λ as $\sup\mathrm{Dom}\,F$) $\lambda\in N$ and $N\models$ "there is a Jonsson algebra on λ". Let $\mathscr{B}$ be such an algebra but $\mathscr{B}\cap N<\mathscr{B}$, $\mathscr{B}\cap N\neq\mathscr{B}$ (for $\lambda\not\subseteq N$) and $\|\mathscr{B}\cap N\|=\|\lambda\cap N\|=\lambda$. This is a contradiction to $\mathscr{B}$ being Jonsson.

Definition 3. (A) For sets S_1,S_2 of cardinals, and a cardinal (or ordinal) μ, $S_1\to S_2[\mu]$ means that for every M (as in 2B) and $N<M$, *if*

(i) $\mu+1\subseteq N$ (for $\mu=\aleph_0$ this is empty),

(ii) for every $\lambda\in S_1$, $|\lambda\cap N|=\lambda$,

(iii) $S_1\subseteq N$ (if each $\lambda\in S_1$ is a successor, this follows by (ii)),

(iv) $S_1,S_2\in N$,

then for some $\lambda\in S_2$, $|\lambda\cap N|=\lambda$ and $\lambda\in N$. (The interesting case is $\operatorname{Sup}S_1\geqq\operatorname{Sup}S_2+\mu$.)

(B) When $S_l=\{\lambda\}$ we write λ instead of S_l, and instead of $S^1_l\cup S^2_l$ we write S^1_l,S^2_l. Note that in 3(A) we can replace S_l by a sequence, and nothing changes.

For Notational simplicity let $\operatorname{Sup}S=\bigcup\{\lambda+1\colon\lambda\in S\}$.

Observation 4. (A) $S_1\to S_2[\mu]$ iff (*) iff (**), where

(*) There is a model N_0, $\operatorname{Sup}S_1\subseteq N_0$, N_0 has $\leqq|\mu|$ operations and relations and if $N<N_0$, $|N\cap\lambda|=\lambda$, $\lambda\in N$ for each $\lambda\in S_1$ then $|N\cap\lambda|=\lambda$, $\lambda\in N$ for some $\lambda\in S_2$.

(**) There is a model N_0 as in (*) with universe $\operatorname{Sup}S_1$.

(B) In Definition 3A(i) we can demand only $\mu\subseteq N$ or even $|\mu|\subseteq N$ for μ ordinal.

(C) In Definition 3A we can demand M to vary only on $M_\lambda<H(\lambda^*)$ where $\lambda=\operatorname{Sup}S_1$ and $\lambda^*>\lambda$ is a constant, and demand some specific elements $\in M_\lambda$.

Proof. $S_1\to S_2[\mu]\Rightarrow$ (*): take $\lambda=\operatorname{Sup}S_1$, $N_0=(M_\lambda,S_1,S_2,i)_{i\leqq\mu}$.

(*) $\Rightarrow$ (**): take N_0 as in (*). Since any $N_1<N_0$ s.t. $\operatorname{Sup}S_1\subseteq N_1$ satisfies (*) we can assume $\|N_0\|=\operatorname{Sup}S_1$. Add Skolem functions to N_0 and add a name to each formula, getting a model N_1 satisfying (*). Take $N_2=N_1\restriction\operatorname{Sup}S_1$. We show N_2 satisfies (**). Let $N'_2<N_2$ such that $(\forall\lambda\in S_1)(\lambda\in N'_2\wedge|\lambda\cap N'_2|=\lambda)$; take N'_0—the Skolem closure of N'_2 in N_0. By (*) for N_0 there is $\lambda\in S_2$ s.t. $\lambda\in N'_0$ and

$|\lambda \cap N'_0| = \lambda$. Since $|N'_0| \cap \operatorname{Sup} S_1 = |N'_2|$ we have $\lambda \in S_2$ s.t. $\lambda \in N'_2$ and $|\lambda \cap N'_2| = \lambda$.

$(**) \Rightarrow S_1 \to S_2[\mu]$. Suppose N_0 is as in $(**)$, and with minimal μ (for the given S_1, S_2); hence $\mu \in M_\lambda$. Suppose $N < M_\lambda$, as in 3(A). Now $N_0 \in M^1_\lambda$, but as $M^2_\lambda < M^1_\lambda$, w.l.o.g. $N_0 \in M^2_\lambda$, and even $N_0 \in N$. So N^*_0, the submodel of N_0 with universe $N_0 \cap N = \{a : N \models \text{“}a \in N_0\text{”}\}$, has universe $N \cap \operatorname{Sup} S_1$ and $N^*_0 < N_0$.

By the hypothesis of 3(A), the hypothesis of $(*)$ holds, so for some $\lambda \in S_2$, $\lambda \cap N^*_0| = \lambda \in N^*_0$ hence $|\lambda \cap N| = \lambda \in N$, so we finish.

(B), (C) Easy from (A).

The basis of our proof is the following

Observation 5. (A) If $\lambda \to \mu^+[\aleph_0]$ for every $\mu < \lambda$, then $\mathrm{Jn}(\lambda)$.

(B) If $\aleph_\alpha \to \mu^+[\mu]$ for every $\mu < \aleph_\alpha$ and $\alpha \subseteq N < M_{\aleph_\alpha}, \|N\| = \aleph_\alpha$ then $N = M_{\aleph_\alpha}$.

(C) If $N < M_\lambda, \|N\| = \lambda$, and for each $\mu \in N, \mu < \lambda, |N \cap \mu^+| = \mu^+$ *then* $N = M_\lambda$.

(D) If $\mathrm{Jn}(\lambda)$, then $\lambda \to \kappa[\aleph_0]$ for every $\kappa \leqq \lambda$.

Proof. (A) By (C); let $N < M, \|N\| = \lambda$, now $\mu \in N$ implies $\mu^+ \in N$, so by a hypothesis $|N \cap \mu^+| = \mu^+$.

(B) Like (C), as for $\mu < \lambda, \mu = \aleph_\beta$ for some $\beta < \alpha$ hence $\mu \in N$.

(C) Because of the function F it suffices to prove $\lambda \subseteq N$, and we know $|N \cap \lambda| = \lambda$.

Let μ be a maximal cardinality for which $\mu \subseteq N$. If $\mu = \lambda$ we finish, and if $\mu \in N$ then by a hypothesis $|N \cap \mu^+| = \mu^+$, but then $\mu^+ \subseteq N$ (there is $f = f^\mu \in N$, such that for every $\beta < \mu^+, x \mapsto f(\beta, x)$ is a map from μ onto β; so for each $\alpha < \mu^+$, there is $\beta \in N, \alpha < \beta < \mu^+$, so for some $\gamma < \mu, f(\beta, \gamma) = \alpha$, hence $\alpha \in N$). So $\mu \notin N$. Choose a minimal $\alpha, \mu \leqq \alpha \in N$; as $|\alpha| \in N, \alpha$ is a cardinal. Clearly $\alpha < \lambda$ (as $\|N\| = \lambda$, and by F) so $|\alpha|^+ \in N$, hence $|N \cap |\alpha|^+| = \alpha|^+$, so for some $\gamma \in N, \alpha < \gamma < |\alpha|^+, |N \cap \gamma| = |\alpha| > \mu$, using $f^{|\alpha|}(\gamma, x)$ we get a contradiction.

(D) By 4$(*)$.

Lemma 6. (A) *If* $S_0 \to S_1[\mu]$, *and for each* $\kappa \in S_1$ $S_0, \kappa \to S_2[\mu]$ *then* $S_0 \to S_2[\mu]$.

(B) *If* λ_i $(i \leqq \alpha)$ *is an increasing sequence of cardinals, and* $\lambda_i \to \{\lambda_j : j < i\}[\mu]$ *then* $\lambda_\alpha \to \lambda_0[\mu]$ *(we can replace the assumption by: for every i for some nonempty* $S_i \subseteq \{\lambda_i : j < i\}, \lambda_i \to S_i[\mu]$).

(C) *The relation* $S_1 \to S_2[\mu]$ *is preserved under increasing* S_1, S_2 *and* μ.

PROOF. (A) By 4(*) there is a model on $\lambda = \operatorname{Sup} S_0$ with $\leqq \mu$ relations demonstrating that $S_0 \to S_1[\mu]$. Add to this model μ relations demonstrating for every $\kappa \in S_1$: $S_0, \kappa \to S_2[\mu]$. The resulting model shows $S_0 \to S_2[\mu]$.

(B), (C) Similar proofs.

By 5 and 6(B), in order to prove the existence of Jonsson algebras it suffices to prove enough cases of the form $\lambda \to S[\aleph_0]$.

LEMMA 7. (A) $\lambda^+ \to \lambda[\aleph_0]$ (*hence by* 6(A) $\aleph_{\alpha+n} \to \aleph_\alpha[\aleph_0]$).

(B) $\lambda \to \operatorname{cf}\lambda[\aleph_0]$.

(C) $2^\lambda \to \lambda[\aleph_0]$ *when* $2^\mu < 2^\lambda$ *for every* $\mu < \lambda$.

(D) $\lambda \to \{\lambda_i : i < \delta\}[\delta]$ *if* $\lambda_i < \lambda, \lambda \in P\operatorname{cf}\langle \lambda_i : i < \delta\rangle$ (*see below*). *If* $\lambda \in P\operatorname{Sc}_D\langle \lambda_i : i < \delta\rangle$, *we can strengthen the demand in* 3(A) *to* $\{i : |N \cap \lambda_i| \neq \emptyset$ mod D.

DEFINITION 8. (A) $\lambda \in P\operatorname{Sc}_D \bar{\lambda}$ (λ is a possible scale for $\bar{\lambda}$), where $\bar{\lambda} = \langle \lambda_i : i < \delta\rangle, D$ a filter over $\delta, D \supseteq D(\delta) = \{A \subseteq \delta : \delta - A$ bounded$\}$, *if* λ, λ_i are regular cardinals or 1 and there are functions $f_\alpha (\alpha < \lambda)$ exemplifying it, i.e.

(a) $f_\alpha(i) < \lambda_i$ for $i < \delta$, and $\operatorname{Dom} f_\alpha = \delta$ (that is $f_\alpha \in \Pi_{i<\delta}\lambda_i$),

(b) $f_\alpha \leqq_D f_\beta$ for $\alpha < \beta$ (this means that $\{i : f_\alpha(i) \leqq f_\beta(i)\} \in D$),

(c) we cannot define f_λ satisfying (a) and (b).

(B) $\lambda \in P\operatorname{cf}\bar{\lambda}$ iff $\lambda \in P\operatorname{Sc}_D \bar{\lambda}$ for some ultrafilter D over δ.

(C) $\lambda \in P\operatorname{Sc}\bar{\lambda}$ if $\lambda \in P\operatorname{Sc}_{D(\delta)}\bar{\lambda}$

(D) $\bar{\lambda}$ is D trivial if $\{i : \lambda_i = 1\} \in D$; we always assume $\bar{\lambda}$ is not D-trivial.

OBSERVATION 9. (A) If $\lambda \in P\operatorname{Sc}_D \bar{\lambda}, \bar{\lambda} = \langle \lambda_i : i < \delta\rangle, 2^{|\delta|} < \lambda$, then $\lambda \in P\operatorname{cf}\bar{\lambda}$.

(B) $\lambda \in P\operatorname{Sc}_D\langle \lambda_i : i < \delta\rangle$ is equivalent to $\lambda = \operatorname{cf}[\Pi_{i<\delta}\lambda_i/D]$, for D an ultrafilter.

(C) Suppose $h: \delta^1 \to \delta^2, h_1: \delta^2 \to \delta^1, D_l$ a filter over δ^l,

$$\{i < \delta^1 : \lambda_i \geqq \mu_{h(i)}\} \in D_1, A \in D_2 \Rightarrow \{i : h(i) \in A\} \in D_1,$$

$$\{j : hh_1(j) = j, \lambda_{h_1(j)} = \mu_j\} \in D_2$$

and $\delta^2 - A \notin D_2 \Rightarrow \delta^1 - \{h_1(j) : j \in A\} \notin D_1$. Then $\mu \in P\operatorname{Sc}_{D_2}\langle \mu_j : j < \delta^2\rangle$ implies $\mu \in P\operatorname{Sc}_{D_1}\langle \lambda_i : i < \delta^1\rangle$.

(D) $\lambda \to \{\lambda_i : i < \delta\}[\delta]$ if $\lambda \in P\operatorname{Sc}_D\langle \lambda_i : i < \delta\rangle$.

PROOF OF LEMMA 7. (A), (B), (C). Immediate.

(D) Let M, N be as in Definition 3 (so $\lambda, \{\lambda_i : i < \delta\} \in N, \delta + 1 \subseteq N$). W.l.o.g. $\langle \lambda_i : i < \delta\rangle \in N, D \in N$ (by 4C); so there is $\langle f_\alpha : \alpha < \lambda\rangle \in N$ exemplifying $\lambda \in P\operatorname{Sc}_D\langle \lambda_i : i < \delta\rangle$. As $\delta + 1 \subseteq N, \lambda_i \in N$ for each i. If for each i $|N \cap \lambda_i| < \lambda_i$, then $A_i = \{f_\alpha(i)\colon \alpha \in N \cap \lambda\}$ is a subset of λ_i of cardinality $< \lambda_i$, so by λ_i's regularity it has an upper bound $< \lambda_i$ which we call $f_\lambda(\alpha)$. It follows that for $\alpha \in N\, f_\alpha <_{D(\delta)} f_\lambda$ hence $f_\alpha <_D f_\lambda$: as $|N \cap \lambda| = \lambda$, and $<_D$ is transitive $f_\alpha <_D f_\lambda$ for each $\alpha < \lambda$; a contradiction.

Now we shall prove some cases of $\lambda \in P\,\mathrm{Sc}\,\bar{\lambda}$.

LEMMA 10. (A) *Let λ_i $(i < \delta)$ be increasing, $\delta < \lambda_* = \Sigma_{i<\delta}\lambda_i$, each λ_i a successor (at least for i limit or for an unbounded set of i's), then for any $f_\alpha \in \Pi_{i<\delta}\lambda_i$ $(\alpha < \lambda_*)$ there is an upper bound in $\Pi_{i<\delta}\lambda_i/D(\delta)$. Hence $\lambda \in P\,\mathrm{cf}_D\,\langle \lambda_i \colon i < \delta\rangle$ implies $\lambda > \lambda_*$.*

(B) *$\lambda \in P\,\mathrm{cf}_D\,\langle \lambda_i \colon i < \delta\rangle$ implies $\lambda \leqq \Pi_{i<\delta}\lambda_i$ (as cardinals).*

(C) *For every $\bar{\lambda}, D$, for some λ, $\lambda \in P\,\mathrm{Sc}_D\,\langle \lambda_i \colon i < \delta\rangle$.*

(D) *If $|\Pi_{i<\delta}\lambda_i/D| = \lambda_*^+$, $D \supseteq D(\delta)$ and the assumption of* (A) *holds then $\lambda_*^+ \in P\,\mathrm{Sc}_D\,\bar{\lambda}$.*

PROOF. Immediate (in (A) choose f such that $|\alpha|^+ < \lambda_i$ implies $f_\alpha(i) < f(i)$).

LEMMA 11. *Suppose $\bar{\lambda} = \langle \lambda_i \colon i < \kappa\rangle$, κ regular $< \lambda_* = \Sigma_{i<\kappa}\lambda_i$, λ_i is increasing.*

(A) *If $\lambda \in P\,\mathrm{Sc}_D\bar{\lambda}$, $\lambda_* < \mu < \lambda$, μ regular, D $\aleph_1$-complete or $2^\kappa < \mu$* then *$\mu \in P\,\mathrm{Sc}_D\,\langle \lambda'_i \colon i < \kappa\rangle$ for some $\lambda'_i \leqq \lambda_i$, $\langle \lambda'_i \colon i < \kappa\rangle$ is not D-trivial.*

(B) *In* (A), *instead of $\lambda \in P\,\mathrm{Sc}_D\,\bar{\lambda}$ it suffices to assume: in $\Pi_{i<\kappa}\lambda_i/D$ there is a $<_D$-increasing sequence of length μ (or even $\leqq_D$-increasing, if it is not eventually constant by $=_D$).*

(C) *Note that in* (A) *and* (B) *if $\lambda_i^\kappa < \lambda_* \leqq \mu$ (for every i) then $\Sigma_{i<\kappa}\lambda'_i = \lambda_*$.*

(D) *If $\kappa > \aleph_0$ or $2^{\aleph_0} \leqq \lambda_*$ then $\mu = \lambda_*^+$ satisfies the requirement on μ in* (A) *for $D = D(\delta)$. (In the first case D is $\aleph_1$-complete and in the second $2^\kappa < \mu$.)*

PROOF. (A) follows from (B).

(B) Let f_α $(\alpha < \mu)$ be $<_D$-increasing (in $\Pi_{i<\kappa}\lambda_i/D$) s.t. $(\forall \alpha < \mu)$ $(\exists \beta < \mu)$ $(\alpha < \beta \wedge \neg f_\alpha =_D f_\beta)$. If they would exemplify $\mu \in P\,\mathrm{Sc}_D\,\bar{\lambda}$, we finish. Otherwise we shall show that

(*) there is $f \in \Pi_{i<\kappa}\lambda_i/D$ such that $f_\alpha \leqq_D f$, for $\alpha < \mu$, but for no g is $f_\alpha \leqq_D g <_D f$ for every $\alpha < \mu$.

Now (*) is sufficient, for let $\lambda'_i = \mathrm{cf}\, f(i)$, $A_i \subseteq f(i)$ a close unbounded set of order-type $\mathrm{cf}\, f(i)$, $A_i = \{\alpha(i,j)\colon j < \lambda'_i\}$ ($\alpha(i,j)$ increasing with j) (if $f(i)$ is a successor ordinal $\lambda'_i = 1$).

Let $f'_\alpha(i) = \min\{j \colon \alpha(i,j) \geqq f_\alpha(i)\}$, then f'_α $(\alpha < \mu)$ exemplify $\mu \in P\,\mathrm{Sc}_D\,\langle \lambda'_i \colon i < \kappa\rangle$, $\langle \lambda'_i \colon i < \kappa\rangle$ is not D-trivial, as otherwise we find g contradicting (*).

Let us prove (*).

Case (i). D is $\aleph_1$-complete.

In this case $<_D$ is well-founded, as we assume there is $f \in \Pi_{i<\kappa}\lambda_i/D$, $f_\alpha \leqq_D f$ for every $\alpha < \mu$, there is one as required.

Case (ii). $2^\kappa < \mu$.

It is well known that there is no decreasing sequence of length $(2^\kappa)^+$ in $<_D$. So define by induction on γ $f^\gamma \in \Pi_{i<\kappa}\lambda_i$, such that $\beta < \gamma \Rightarrow f^\gamma <_D f^\beta$, and $\alpha < \mu \Rightarrow f_\alpha \leqq_D f^\gamma$. Now f^0 exists by an assumption in the beginning of the proof. So there is a first γ_0 for which f^{γ_0} is not defined. We shall now prove γ_0 is a successor so f^{γ_0-1} is as required. As mentioned above $\gamma_0 < (2^\kappa)^+$. Let $P_i = \{f^\gamma(i);\ \gamma < \gamma_0\} \subseteq \lambda_i$, so $|P_i| \leqq 2^\kappa$. Let $(\Pi_{i<\kappa}\lambda_i/D, \leqq, P) = \Pi_i(\lambda_i, \leqq_D, P_i)/D$ so $|P| \leqq \Pi_{i<\kappa}|P_i| \leqq 2^\kappa$. Now $2^\kappa < \mu$, μ regular so for some $\alpha_0 < \mu$, for every $a \in P$, and $\alpha_0 \leqq \alpha < \mu$, $f_{\alpha_0} \leqq_D a \Leftrightarrow f_\alpha \leqq_D a$. Now

$$(\lambda_i, \leqq, P_i) \models (\forall x)[(\exists z)(P(z) \wedge x \leqq z) \rightarrow (\exists y)[(P(y) \wedge x \leqq y) \wedge (\forall z)(P(z) \wedge x \leqq z \rightarrow y \leqq z)]].$$

This is a Horn sentence, so $(\Pi_{i<\kappa}\lambda_i/D, \leqq_D, P)$ satisfies it, so taking f_{α_0} for x the antecedent holds ($z = f^0$) so we get f for y. So $f_{\alpha_0} \leqq_D f$ hence for every α $f_\alpha \leqq_D f$ by the choice of f_{α_0}; also $f \leqq_D f^\gamma$ as $(\Pi_{i<\kappa}\lambda_i/D, \leqq_D, P) \models P(f^\gamma) \wedge f_{\alpha(0)} \leqq_D f^\gamma$. Clearly f is as required.

(C), (D) left to the reader.

Conclusion 12. For $\aleph_\delta$ singular, D an ultrafilter over $\mathrm{cf}\,\delta$, in $(\omega_\delta, <)^{\mathrm{cf}\,\delta}/D$ there is no increasing sequence of length $\aleph_\gamma$ where $\gamma = (|\delta|^{\mathrm{cf}\,\delta}/D)^+$.

Proof. Otherwise for every $\beta < \gamma$, β successor, $\beta > \delta$ there are $\alpha(\beta, i) < \delta$ ($i < \mathrm{cf}\,\delta$) such that $\mathrm{cf}[\Pi_{i<\delta}(\omega_{\alpha(\beta,i)}, <)/D] = \aleph_\beta$ (by 11A, 9A) but the number of possible $\langle \alpha(\beta, i): i < \mathrm{cf}\,\delta \rangle$ is $\leqq |\delta|^{\mathrm{cf}\,\delta}/D$, contradiction.

This has relation to Galvin and Hajnal [2], but 12 is applicable when $\mathrm{cf}\,\delta = \aleph_0$ too. In fact

Claim 13. If $\aleph_\delta$ is singular, $\mathrm{cf}\,\delta > \aleph_0$, $\mu \leqq \aleph_\delta^{\mathrm{cf}\,\delta}$ regular, $(\forall \alpha < \delta)(\forall k < \mathrm{cf}\,\delta)\aleph_\alpha^k < \aleph_\delta$ then for some $\alpha(i) < \delta$, $\mu \in P\,\mathrm{Sc}\langle \aleph_{\alpha(i)}: i < \delta \rangle$.

If $\beta(i)$ ($i < \mathrm{cf}\,\delta$) are increasing and continuous with limit δ, for $\mu = \aleph_{\delta+1}$ we can choose $\alpha(i) = \beta(i) + 1$ provided that $\Pi_{i<j}\aleph_{\alpha(i)} \leqq \aleph_{\alpha(j)}$.

We can now apply our theorems.

Conclusions 14. (A) $\mathrm{Jn}(\aleph_{\omega+1})$ if $2^{\aleph_0} \leqq \aleph_{\omega+1}$.

(B) If $(\forall\lambda)\ (\mathrm{cf}\,\lambda > \aleph_0 \rightarrow \lambda^{\aleph_0} = \lambda)$ and there is no weakly inaccessible cardinal then $(\forall\lambda)\ \mathrm{Jn}(\lambda^+)$.

PROOF. (A) First note that for any non-principal ultrafilter D over ω, $\aleph_{\omega+1} \in P\,\mathrm{Sc}_D\langle \aleph_{n(k)}: k < \omega\rangle$ (for some $n(k) < \omega$) (if $2^{\aleph_0} = \aleph_{\omega+1}$, by 10(D), otherwise for some λ, $\lambda \in P\,\mathrm{Sc}_D\langle \aleph_n: n < \omega\rangle$; by 10(A) $\lambda > \aleph_\omega$, by 11A $\aleph_{\omega+1} \in P\,\mathrm{Sc}_D\langle \aleph_{n(k)}: k < \omega\rangle$ for some $n(k)$). For a given $m < \omega$, we can assume $n(k) \geqq m$ (as $\{k: n(k) < m\} \notin D$), by 7(D) $\aleph_{\omega+1} \rightarrow \{\aleph_{n(k)}: k < \omega\}[\aleph_0]$. As $\aleph_n \rightarrow \aleph_m[\aleph_0]$ for $n \geqq m$ (by 7A), by 6(A) $\aleph_{\omega+1} \rightarrow \aleph_m[\aleph_0]$. So by 5(A) $\mathrm{Jn}(\aleph_{\omega+1})$.

(B) Left to the reader.

CONCLUSION 15. $\mathrm{Jn}(2^{\aleph_0})$ if $2^{\aleph_0} = \aleph_{\alpha+1}$, $\alpha < \omega_1$.

PROOF. Let $\beta \leqq \alpha$ and we shall prove $\aleph_{\alpha+1} \rightarrow \aleph_{\beta+1}[\aleph_0]$ (this is sufficient by 5A). We define increasing $\beta(i) \leqq \alpha + 1$, and $S_i \subseteq \{\aleph_{\beta(j)}: j < i\}$, $\beta(0) = \beta + 1$, each $\beta(i)$ is a successor, to satisfy 6(B). For $i = 0, \beta(0) = \beta + 1, \beta(i+1) = \beta(i) + 1, S_{i+1} = \{\aleph_{\beta(i)}\}$. For i limit of cofinality ω let $i_n < i$ be increasing with limit i, $S_i = \{\aleph_{\beta(i_n)}: n < \omega\}$, and we choose a successor $\beta(i) > \bigcup_n \beta(i_n), \beta(i) \leqq \alpha + 1$ such that $\aleph_{\beta(i)} \rightarrow S_i[\aleph_0]$; we can do it by 10C and 10A, B. By 6B $\aleph_{\alpha+1} \rightarrow \aleph_{\beta+1}[\aleph_0]$, thus we finish.

LEMMA 16. *If* $\lambda \rightarrow \mu^+[\aleph_0]$ *for every* $\mu, \lambda_0 \leqq \mu < \lambda$ *and* $N < M_\lambda, \|N\| = \lambda$ *then*:

(A) *If* $\lambda_0 \leqq \mu \leqq \lambda$ *then* $\mu \in N$ *and* $|\mu \cap N| = \mu$ (*so* $\lambda \rightarrow \mu[\aleph_0]$).

(B) *For every* $a \in \lambda$ *there is* b *such that* $a \in b \in N$, *and* $|b| < \lambda_0$.

(C) *If* $\lambda^{\lambda_0} = \lambda$ *then* $\mathrm{Jn}(\lambda)$.

PROOF. (A) Like 5(A) (notice we can assume λ_0 is minimal with such properties, hence definable in M_λ).

(B) Let μ be a minimal cardinal such that for some $b_\mu, |b_\mu| \leqq \mu, a \in b_\mu \in N$. Now $\mu \leqq \lambda$ as we can choose $b_\lambda = \lambda$.

Let us prove $\mu < \lambda_0$; otherwise as $b_\mu \in N$ also $\mu = |b_\mu| \in N$, so in N there is a function f from μ onto b_μ. We know by 15(A) that $|\mu \cap N| = \mu$, so $N \cap \mu$ is unbounded in μ, so there is $\alpha < \mu, \alpha \in N$ such that $a \in \{f(\beta); \beta < \alpha\}$. Now $b' = \{f(\beta): \beta < \alpha\} \in N$ contradicts μ's minimality.

(C) It suffices to prove $\lambda \subseteq N$, so let $a \in \lambda$. By 15(B) there is $b \in N, |b| \leqq \lambda_0, a \in b$, and as $\lambda_0 \in N$ we can assume $|b| = \lambda_0$. As $|N \cap \lambda| = \lambda$ there is a set $A \subseteq \lambda \cap N, |A| = \lambda_0$ and necessarily $A \in M^1_\lambda$ but possibly $A \notin N$. Let $F^* \in N$ be a function from λ onto $\{B \subseteq \lambda: |B| = \lambda_0\}$; so for some $i, j < \lambda, F^*(i) = A, F^*(j) = b$. By 15(A) there is $C \in N, |C| \leqq \lambda_0$ such that $i, j \in C$. $\{F^*(\alpha): \alpha \in C\}$ is a family of $\leqq \lambda_0$ sets each of power exactly λ_0. So there is a function $g \in N$, $\mathrm{Dom}\, g = \bigcup_{\alpha \in C} F^*(\alpha)$, such that for every $\alpha \in C$, $\{g(x): x \in F^*(\alpha)\} = \mathrm{Dom}\, g$ (clearly $|\mathrm{Dom}\, g| = \lambda_0$).

This holds for $\alpha = i$, but $g \in N$, $A = F^*(i) \subseteq N$; so $\mathrm{Dom}\, g \subseteq N$, but $a \in b = F^*(j)$, $j \in C$ so $a \in N$.

Conclusion 17. Suppose $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma+1}$, then $\mathrm{Jn}(2^{\aleph_\alpha})$ if (A) or (B) or (C):

(A) $\gamma < \omega_1$,

(B) $2^{\aleph_\alpha} \to \mu[\aleph_0]$ for every $\mu \leqq |\gamma|$,

(C) $\beta < \alpha \Rightarrow 2^{\aleph_\beta} < 2^{\aleph_\alpha}$, and $\mathrm{Jn}(\aleph_\alpha)$ and $\gamma < \aleph_{\alpha+1}$.

Proof. Similar to 14.

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