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BOOLEAN ALGEBRAS WITH FEW ENDOMORPHISMS

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ABSTRACT. Using diamond for $\aleph_1$ we construct a Boolean algebra in $\aleph_1$, whose only endomorphisms are those definable using finitely many elements and ultrafilters. We also generalize Rubin's construction to higher cardinals.

0. Introduction. The aim we state in the name of the paper can be interpreted in two ways: every endomorphism is "simply defined", or the number of endomorphisms is small. However for every ultrafilter F on a Boolean algebra (or, equivalently maximal ideal $I = B - F$) we can define an endomorphism T:

$$T_b = \begin{cases} 1, & b \in F, \\ 0, & b \in I. \end{cases}$$

So "defined" should be interpreted as definable using maximal ideals. This is done in Theorem 1.4 (2) (saying there are indecomposable complicated Boolean algebras, see Definition 1.1, 2, 3) and Theorem 1.8(2) (saying any such algebra has only simply definable endomorphisms (see Definitions 1.5, 1.6)). Those theorems answer a question of Monk; for further historical background see [**R**].

As for the other interpretation (the number of endomorphisms should be small), Rubin [**R**] using $\diamondsuit_{\aleph_1}$ builds a Boolean algebra of power $\aleph_1$, which we call here 1-Rubin (or Rubin). Such algebras in particular, have only $\aleph_1$ ideals and subalgebras. (The exact definition speaks on any set of $\aleph_1$ elements having some properties.) We generalize this to sets of $\aleph_1$ n-tuples and get a Boolean algebra which is n-Rubin for each n (Theorem 2.5). In fact we do this for λ^+ instead $\aleph_1$, assuming $\diamondsuit_\lambda$ and $\diamondsuit_{\lambda^+}$, using [**Sh1**].

Variants of §1, in ZFC but for higher cardinals, will appear, as well as generalizing Theorem 2.5 to higher cardinals.

Further remarks. (1) We can generalize §1 using [**Sh1**], but it does not seem so interesting.

(2) We can ask in 2.5 for which Boolean algebras C and set K_0 of n-tuples from C, $K_0 = \{(b_i^0, \ldots, b_i^{n-1}): i < i_0 < \lambda\}$: for every set K of λ n-tuples from B (K_0, K_1 with disjoint sequences of distinct elements) we can find distinct $(a_i^0, \ldots, a_i^{n-1}) \in K$ $(i < i_0)$ such that the mapping $a_i^l \mapsto b_i^l$ induces an embedding of the subalgebra of C generated by $\{a_i^l: l < n, i < i_0\}$ into B. For $n = 1$, $\lambda = \aleph_1$ Rubin [**R**] answers this; for $n = 1$, $\lambda > \aleph_1$, the conditions

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are essentially the same. (No union of members of K_0 is 1, no intersection of members of K_0 is empty, and they can be well ordered so that no one is a Boolean combination of previous ones.) For $n > 0$, there seems to be no essential problem but it seems hard to phrase it intelligibly.

(3) It is open whether there are n-Rubin not $(n + 1)$-Rubin Boolean algebras (in particular this is hopeful for $n = 1, \lambda = \aleph_1$).

The results from [**R**] generalize easily from the case $\lambda = \aleph$ to other cases of §2 at least for $\lambda = \mu^+$, B μ-saturated, but the other questions were not checked (can $\mu^+ < \lambda$, can λ be singular, etc.).

Notation. Let m, n, k, l denote natural numbers, α, β, γ, i ordinals, λ an infinite cardinality. $P(A)$ is the family of subsets of A.

Let B denote a Boolean algebra, 0, 1, $\cap$, $\cup$, $-$, $\leqslant$ operations and relations of Boolean algebras.

We let $[a, x]^{\mathrm{if}(t)}$ be $a \subseteq x$ if t is true and $a \cap x = 0$ otherwise.

1. Complicated Boolean algebras and Boolean algebras with only definable endomorphisms.

1.1 DEFINITION. A Boolean algebra B is called *complicated* if for any candidate $\{(a_n, b_n): n < \omega\}$ for B (which means: $m \neq n$ implies $a_n \cap a_m = 0$, $b_n \cap b_m = 0$ and for each n, $b_n \not\subseteq a_n$ and of course $a_n, b_n \in B$) there is a witness set $S \subseteq \omega$ i.e. such that

(i) for some $x \in B$ for every n $[a_n, x]^{\mathrm{if}(n \in S)}$,

(ii) for no $x \in B$ for every n $[b_n, x]^{\mathrm{if}(n \in S)}$.

1.2 DEFINITION. An ideal I of a Boolean algebra B is called *indecomposable* if there are no nonprincipal disjoint ideals I_0, I_1 such that I is generated by them (so $I_0, I_1 \subseteq I$). Clearly such filter is not principal (otherwise I is called decomposable). (I is principal if for some a, $I = \{b \in B: b \leqslant a\}$.)

1.3 DEFINITION. A Boolean algebra B is called *indecomposable* if every nonprincipal maximal ideal of B is indecomposable.

1.4. THEOREM (1) (CH) *There is an atomless complicated Boolean algebra of cardinality* $\aleph_1$.

(2) ($\diamondsuit_{\aleph_1}$) *There is an atomless indecomposable complicated Boolean algebra of cardinality* $\aleph_1$.

(3) *In* (1) *we can assume that any maximal infinite antichain and chain in* B *is uncountable. In* (2) *we can assume this or that there are no uncountable chains and antichains in* B.

REMARK. (1) Instead of CH it suffices to assume in (1) that the union of $< 2^{\aleph_0}$ many nowhere dense subsets of $P(\omega)$ is not $P(\omega)$.

(2) We may want B to be atomic with $\aleph_0$ ($\aleph_1$) atoms. For this always enlarge $\{a_n^\alpha: n < \omega\}$ to a maximal antichain $\{a_\beta^\alpha: \beta < \beta_\alpha\}$ in the following proof.

PROOF. (1) We define by induction on $\alpha < \aleph_1$ an atomless Boolean algebra B_α and types p_β ($\beta < \alpha$) such that:

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(a) B_α ($\alpha < \omega_1$) is increasing and continuous, each B_α is countable and for convenience its set of elements is $\omega(1 + \alpha)$.

(b) $p_\alpha = \{[b_n^\alpha, x]^{\text{if}(n \in S(\alpha))}: n < \omega\}$ where $b_n^\alpha \in B_\alpha$, $b_n^\alpha \neq 0$, $n \neq m \Rightarrow b_n^\alpha \cap b_m^\alpha = 0$ and $S(\alpha) \subseteq \omega$ and $S(\alpha)$, $\omega - S(\alpha)$ are infinite.

(c) No element of B_α realizes any p_β ($\beta \leqslant \alpha$).

Our desired algebra will be $B = \bigcup_{\alpha<\omega_1} B_\alpha$.

As we use the continuum hypothesis, by standard methods we can assume that for each α we are given a candidate $\{(a_n^\alpha, b_n^\alpha): n < \omega\}$ for B_α, so that every candidate for B appears in the list.

Now it suffices to define $B_{\alpha+1}$, $S(\alpha)$, assuming B_α, p_β ($\beta < \alpha$) are given such that $S(\alpha)$ is a witness for the candidate $\{(a_n^\alpha, b_n^\alpha): n < \omega\}$ in $B_{\alpha+1}$, and $B_{\alpha+1}$ does not realize p_β ($\beta \leqslant \alpha$). Notice $S(\alpha)$ defines p_α (see (b)), and that for α zero or limit, there is no problem to define B_α.

After the choice of $S(\alpha)$ we will define $B_{\alpha+1}$ as follows: it is generated by B_α and x_α, freely except that x_α realizes $\{[a_n^\alpha, x]^{\text{if}(n \in S(\alpha))}: n < \omega\}$.

Now all demands on $B_{\alpha+1}$ are translated to demands on $S(\alpha)$. Now we should show that they are compatible; for this we shall show there are countably many demands (remembering $\alpha < \aleph_1$) each of them being: $S(\alpha)$ lies outside a subset of $P(\alpha)$ which is of the first category (or even is nowhere dense). By the Baire category theorem this is sufficient.

So let $\bar\sigma = (\sigma_1, \sigma_2)$ denote a pair of disjoint finite subsets of ω, $V_{\bar\sigma} = \{S \subseteq \omega: \sigma_1 \subseteq S, \sigma_2 \cap S = \varnothing\}$. Let $\bar\sigma \subseteq \bar\sigma'$ mean $\sigma_1 \subseteq \sigma_1'$, $\sigma_2 \subseteq \sigma_2'$ hence $V_{\bar\sigma'} \subseteq V_{\bar\sigma}$. So let us deal with the demand "$B_{\alpha+1}$ does not realize p_β" where $\beta \leqslant \alpha$. So for each partition b_1, b_2, b_3, b_4 of 1 in B_α, and $\bar\sigma$ we have to find $\bar\sigma'$, $\bar\sigma \subseteq \bar\sigma'$ which ensure that $y = b_1 \cup (b_2 \cap x) \cup (b_3 - x)$ does not realize p_β. (As there are countably many β's and such partitions, this is sufficient.) (Note y is a "general" element of $B_{\alpha+1}$.)

By inessential changes we can assure $\sigma_1 \cup \sigma_2 = \{0, \ldots, n-1\}$ and $a_0^\alpha \cup \cdots \cup a_{n-1}^\alpha \subseteq b_1 \cup b_4$.

First assume $\beta < \alpha$. If for some $m \geqslant n$ and k, $b_k^\beta \cap a_m^\alpha \neq 0$ then we can assure y does not realize p_β, by assuring not $[b_k^\beta, y]^{\text{if}(m \in S(\beta))}$ and this we do by adding m to σ_1 or σ_2 (and so getting $\bar\sigma'$). If there is no such k, but for some k, $b_k^\beta - b_1 - b_4 \neq 0$, the freeness of x_α assures our desire. In the remaining case, if y realizes p_β (for some choice of $S(\alpha) \in V_{\bar\sigma}$) then b_1 will do as well, contradiction.

So remains the case $\beta = \alpha$. If for some $m \geqslant n$, $b_1 \cap b_m^\alpha \neq 0$ we adjoin m to σ_2 (to get $\bar\sigma'$), and if for some $m \geqslant n$, $b_4 \cap b_m^\alpha \neq 0$ we adjoin m to σ_1, and this is clearly sufficient. If for some $m \neq l \geqslant n$, $(b_m - a_m) \cap a_l \neq 0$ we adjoin m to σ_1 and l to σ_2. In the remaining case the freeness of x_2 ensures the result.

Now it is trivial to check $B = \bigcup_{\alpha<\omega_1} B_\alpha$ is as required.

(2) We repeat the proof of the first part: but the enumeration of candidates is only for successor ordinals and for such α's we act as before. (Also for limit β we have two p_β's and they are defined a little differently.) For limit $\alpha < \omega_1$,

we are given I_0^α, $I_1^\alpha \subseteq B_\alpha$, such that eventually for every I_0, $I_1 \subseteq B$, $\{\alpha: I_1^\alpha = I_1 \cap B_\alpha,\ I_0^\alpha = I_0^\alpha \cap B_\alpha\}$ is a stationary set of limit ordinals (hence nonempty). So if I_0, I_1 were disjoint nonprincipal ideals in B, whose union generates a maximal ideal in B, so are $I_1 \cap B_\alpha$, $I_0 \cap B_\alpha$ in B_α for a closed unbounded set of α's. So some α satisfies both demands, so it suffices in defining $B_{\alpha+1}, p_\alpha$ to assure no such I_0, I_1 will exist.

So suppose I_0^α, I_1^α are disjoint nonprincipal ideals in B_α whose union generates a maximal ideal. We can find pairwise disjoint $a_n^\alpha \in I_0^\alpha \cup I_1^\alpha$, such that $\{a_{2n}: n < \omega\}$, $\{a_{2n+1}: n < \omega\}$ generate I_0^α, I_1^α resp. We shall choose $S(\alpha) \subseteq \omega$, so that there are infinitely many odds and even, in $S(\alpha)$ and in its complement, and define $B_{\alpha+1}$ as in (1).

We now let for $l = 0, 1$, $p_\beta^l = \{[a_n, x]^{\text{if}[n \in S(\alpha)]}: n = l \bmod 2\}$. We have to define $S(\alpha)$ so that again all types up to and including this stage are omitted, and this is done in the same way (using the properties of I_0^α, I_1^α).

(3) is left to the reader, we may have to change the last phrase of (b). (To make B_α atomless we may sometimes add freely an x.)

1.5 Definition (1) A scheme of an endomorphism of B consists of a partition $a_0, a_1, b_2, \ldots, b_{n-1}, c_0, \ldots, c_{m-1}$ of 1, maximal nonprincipal ideal I_l below b_l for $l < n$, nonprincipal disjoint ideals I_l^0, I_l^1 below c_l which generates a maximal ideal below c_l for $l < m$, a number $k < n$, and a partition $b_0^*, \ldots, b_{n-1}^*, c_0^*, \ldots, c_{m-1}^*$ of $a_0 \cup b_0 \cup \cdots \cup b_{k-1}$. We assume also that $k + m > 0 \Rightarrow a_0 = 0$, $(n-k) + m = 0 \Rightarrow a_1 = 0$ and except in those cases there are no zero elements in the partition.

(2) the scheme is simple if $m = 0$.

(3) The endomorphism of the scheme is the unique endomorphism (see 1.7) $T: B \to B$ such that:

(i) $Tx = 0$ when $x \leqslant a_0$ or $x \in I_l$, $l < k$, or $x \in I_l^0$, $l < m$.

(ii) $Tx = x$ when $x \leqslant a_1$ or $x \in I_l$, $k \leqslant l < n$ or $x \in I_l^1$, $l < m$.

(iii) $T(b_l) = b_l^*$ when $l < k$.

(iv) $T(b_l) = b_l \cup b_l^*$ when $k \leqslant l < n$.

(v) $T(c_l) = c_l \cup c_l^*$ when $l < m$.

1.6 Definition. An endomorphism of the Boolean algebra B is (simply) definable if there is a (simple) scheme which defines it.

1.7 Claim. Any scheme of an endomorphism of B defines uniquely an endomorphism of B.

Remark. The representation is almost unique. We can interchange elements among $\bigcup_{l<k} I_l \cup \bigcup_{l<m} I_l^0$ and among $\bigcup_{l \geqslant k} I_l \cup \bigcup_{l<m} I_l^1$.

Proof. Easy.

1.8 Theorem. (1) *Every endomorphism of a complicated Boolean algebra is definable.*

(2) *If in addition the Boolean algebra is indecomposable, then every endomorphism is simply definable.*

PROOF. Part (2) follows easily from (1) and Definition 1.3 (for if I_l^0, I_l^1 are as described in Definition 1.5, adjoin $1 - c_l$ to I_l^1, and get a contradiction to the indecomposability of B). So let us prove (1).

So let $T: B \to B$ be an endomorphism.

Stage (i). Define

$$I_0 = \{b \in B: \text{for every } a \leqslant b,\ Ta = 0\},$$
$$I_1 = \{b \in B: \text{for every } a \leqslant b,\ Ta = a\}.$$

Note that as $a \leqslant b \Rightarrow Ta \leqslant Tb$, clearly $Ta = 0 \Rightarrow a \in I_0$, but clearly not necessarily $Ta = a \Rightarrow a \in I_1$.

Clearly I_0, I_1 are disjoint ideals of B, and I is the ideal they generate.

Stage (ii). In B there are no disjoint elements a_n $(n < \omega)$ such that $Ta_n \nleqslant a_n$.

Clearly $m \neq n \Rightarrow Ta_m \cap Ta_n = 0$ (as it is $T(a_m \cap a_n) = T(0) = 0$), so $\{(a_n, Ta_n): n < \omega\}$ is a candidate (see Definition 1.1), so (as B is complicated) there is $S \subseteq \omega$ and $c \in B$ realizing $\{[a_n, x]^{\text{if}(n \in S)}: n \leqslant \omega\}$, but no $c' \in b$ realizes $\{[Ta_n, x]^{\text{if}(n \in S)}: n < \omega\}$, but Tc realizes it, contradiction.

Stage (iii). If $b \in B$, and $c \leqslant b \Rightarrow Tc \leqslant c$ then $b \in I$.

Clearly for every $c \leqslant b$, $T(c - Tc) \leqslant c - Tc$ by the hypothesis of the stage, and $T(c - Tc) \leqslant Tc$ as $c - Tc \leqslant c$, hence $T(c - Tc) = 0$. We can conclude that for $c \leqslant b$, $c - Tc \in I_0$, and $Tc = T(Tc)$.

Let $c_0 = Tb$, so $c_0 \leqslant b$, $Tc_0 = c_0$ and suppose $c \leqslant c_0$, $Tc \neq c$. Then $Tc < c$, and let $d = c - Tc \neq 0$ (so $d \leqslant c \leqslant c_0$, $d \in I_0$) now $Tc_0 = T((c_0 - d) \cup d) = T(c_0 - d) \cup T(d) = T(c_0 - d) \cup 0 = T(c_0 - d) \leqslant c_0 - d$ so $Tc_0 < c_0$ (as $d \neq 0$) but $Tc_0 = c_0$ as $c_0 = Tb$, contradiction. So $c \leqslant c_0 \Rightarrow Tc = c$, i.e. $c_0 \in I_1$, but $b = (b - Tb) \cup Tb = (b - Tb) \cup c_0$, $b - Tb \in I_0$, $c_0 \in I_1$, so $b \in I$, as desired.

Stage (iv). B/I is finite.

Otherwise there are pairwise disjoint nonzero a_n/I $(n < \omega)$. By replacing a_n by $a_n - \bigcup_{l<n} a_l$, we can assure the a_n's are pairwise disjoint, and of course $a_n \notin I$. If for every n there is $a_n' \leqslant a_n$, $Ta_n' \nleqslant a_n'$, we get a contradiction by (ii), but if $c \leqslant a_n \Rightarrow Tc \leqslant c$ then by (iii) $a_n \in I$, again a contradiction. So necessarily B/I is finite.

Stage (v). T is definable.

Let b_l/I $(l < n)$ be the atoms of B/I, and w.l.o.g. b_l $(l < n)$ is a partition of 1. Clearly the restriction of I to each b_l is a maximal filter. The rest is easy checking.

2. On Rubin Boolean algebras.

2.1 DEFINITION. (1) Let B be a Boolean algebra. A formal n-interval is: a partition $b_0, \ldots, b_{m-1}$ of 1 in B, $b_k \neq 0$ and $\tau_k^l \in \{0, b_k, x_k, b_l - x_k\}$ $(l < n$, $k < m)$ and elements c_k^0, c_k^1, where $0 \leqslant c_k^0 < c_k^1 \leqslant b_k$ (for not necessarily atomless B we may want to demand $c_k^1 - c_k^0$ is infinite). We name a formal

n-interval ν, and then write n_ν, m_ν, b_l^ν, $\tau_k^{\nu,l}$, $c_k^{\nu,l}$ when the identity of ν is not clear.

(2) The formal n-interval is simple if $c_k^0 = 0$, $c_k^1 = b_k$. We say ν is a formal n-subinterval of ν^* if $c_k^{\nu^*,0} \leqslant c_k^{\nu,0} < c_k^{\nu,1} \leqslant c_k^{\nu^*,1}$ and $b_l^\nu = b_l^{\nu^*}$, $\tau_k^{\nu,l} = \tau_k^{\nu^*,l}$.

(3) Let $\varphi^\nu(\bar{x}, \bar{y})$ be $\bigwedge_{l<n;\, k<m} y_l \cap b_k = \tau_k^l\ (\bigwedge\bigwedge_k c_k^0 \leqslant x_k \leqslant c_k^1)$ and $\psi^\nu(\bar{y}) = (\exists \bar{x})\varphi^\nu(\bar{x}, \bar{y})$.

(4) We say $\bar{a} = \langle a_0, \ldots, a_{n-1}\rangle$ realizes ν if $\varphi^\nu(\bar{a})$ is satisfied.

We now consider a generalization. Let λ, μ be fixed infinite cardinals in this section.

2.2 Definition. (1) Let B be a Boolean algebra, γ an ordinal $< \mu$. A formal γ-interval is: nonzero elements b_i $(i < i_1)$, $i_0 < i_1$, and ordinals i^α $(\alpha < \gamma)$ such that $\sum_{\alpha<\gamma} i^\alpha < \mu$, $i < j < i_0 \Rightarrow b_i \cap b_j = 0$, $i < i_0 \leqslant j < i_1 \Rightarrow [b_i \subseteq b_j$ or $b_i \cap b_j = 0]$, terms τ_i^α $(\alpha < \gamma, i < i_0)$ such that for $i < i_0$, $\tau_i^\alpha \in \{0, b_i, x_i, b_i - x_i\}$ and for $i_0 \leqslant i < i_1$, formulas σ_i of the form: the intersection of b_i with finitely many y_α's, $(1 - y_\alpha)$'s is empty (but no two have the same form) and we let for $i < i_0$, $\sigma_i^\alpha = [y_\alpha \cap b_i = \tau_i^\alpha] \wedge 0 < x_i < b_i$ and the set of σ's is finitely satisfiable in B.

We name formal γ-intervals by ν, and then write γ^ν instead of γ, i_1^ν, b_i^ν, etc. When the identity of ν is clear we omit it.

(2) If $\gamma = 1$ we can assume $i_1 = i_0 + 2$, $\tau_i = x_i$ for $i < i_0$, and all b_i's are pairwise disjoint and $\sigma_{i_0} = [y_0 \cap b_{i_0} = 0]$, $\sigma_{i_1} = [(1 - y_0) \cap b_{i_0+1} = 0]$.

(3) We say ν is a formal γ-subinterval of ν^* if $\gamma^\nu = \gamma^{\nu^*}$, $i_0^\nu = i_0^{\nu^*}$, $\tau_i^{\nu,\alpha} = \tau_i^{\nu^*,\alpha}$, $b_i^\nu \leqslant b_i^{\nu^*}$ for $i < i_0$, $\sigma_i^\nu = \sigma_i^{\nu^*}$ when $i_0^\nu \leqslant i < i_1^{\nu^*}$, $i_1^{\nu^*} \leqslant i_1^\nu$. If $n = 1$, by assumption, we can demand $i_0^{\nu^*} = i_0^\nu$, $i_1^\nu = i_1^{\nu^*}$, and $b_i^\nu \leqslant b_i^{\nu^*}$ for $i < i_0$, $b_{i_0}^\nu \geqslant b_{i_0}^{\nu^*}$, $b_{i_0+1}^\nu \geqslant b_{i_0+1}^{\nu^*}$ and $b_i^{\nu^*} \leqslant (b_i^\nu \cup b_{i_0}^\nu \cup b_{i_0+1}^\nu)$ for $i < i_0$.

(4) Let $\psi^\nu(\bar{x}, \bar{y})$ be the conjunction of the formulas $\tau_i^\alpha = y_\alpha \cap b_i$ $(i < i_0)$, σ_i^α $(i_0 \leqslant i < i_1)$ $(\alpha < \gamma_0)$.

Let $\varphi^\nu(\bar{y}) = (\exists \bar{x})\psi^\nu(\bar{x}, \bar{y})$.

(5) We say $\bar{a} = \langle a_\alpha \colon \alpha < \gamma\rangle$ realizes ν if $\varphi^\nu(\bar{a})$ holds.

Notation. K will be a family of sequences of a fixed length from a Boolean algebra.

2.3 Definition. Let B be a Boolean algebra, $\gamma < \mu$ and K a family of γ-tuples of elements of B. We call *B-small* (λ-small) if for any formal γ-interval there is a formal γ-subinterval such that no (such that $< \lambda$) γ-tuple from K realizes it.

2.4 Definition. (1) An atomless Boolean algebra B of power $\geqslant \lambda$ is called γ-Rubin [γ-*Rubin] ($\gamma < \lambda$), if any λ-small (any small) family of disjoint γ-tuples from B has power $< \lambda$. We demand also that below each nonzero element there are λ elements (otherwise there are $< \lambda$ exceptions, so this is a technical demand).

(2) If $n = 1$ we omit it, and if it holds for every $\gamma < \mu$ we omit it and write strongly.

So Rubin [R] proves (assuming $\diamondsuit_{\aleph_1}$, when $\lambda = \aleph_1$, $\mu = \aleph_0$) the existence of a

Rubin Boolean algebra of cardinality $\aleph_1$. The demand λ-small instead of small is equivalent.

2.5 THEOREM. *Let $\lambda = \mu^+$, $S = \{\alpha < \lambda\colon \operatorname{cf} \alpha = \mu\}$, μ regular and assume $\diamondsuit_\mu$, hence $\mu = \mu^{<\mu}$ and $\diamondsuit_S$ holds. There is a strongly Rubin Boolean algebra which is μ-saturated.*

REMARK. The proof gives a little more then μ-saturation.

PROOF. We use the construction of **[Sh1]** on omitting types in $L(Q)$. Our theory is the first-order theory of atomless Boolean algebras together with the axiom $(\forall x)\,[x \neq 0 \to (Qy)y < x]$, and it is easy to check it is complete and has elimination of quantifiers.

Now we build by induction on α atomless Boolean algebras B_α $(\alpha < \lambda)$ and γ_β-types p_β $(\beta < \alpha)$ over B_β such that

(1) B_α $(\alpha < \lambda)$ is increasing and continuous, each $B_{\alpha+1}$ is saturated of cardinality μ, and for convenience its set of elements is $\mu(1 + \alpha)$.

(2) p_β is an γ_β-type over B_β $(\gamma_\beta < \mu)$ which has no support (see **[Sh1]**) over each B_α, $\alpha \geqslant \beta$, hence each B_α does not realize it.

By **[Sh1]** we just have to define for each given B_α a type p_β with parameters over B_α which has no support over it. Now by $\diamondsuit_S$ for each $\alpha \in S$ we will be given a $\gamma_\alpha < \mu$ and a family K_α of γ_α-tuples of elements from B_α, such that for each $\gamma < \mu$ and family K of γ-tuples from $B = \bigcup_{\alpha<\gamma} B_\alpha$, $\{\alpha\colon K \cap B_\alpha^\gamma = K_\alpha\}$ is stationary. Note that for each such small K, $\{\alpha \in S\colon K \cap B_\alpha^\gamma$ is small and for each formal γ-interval ν over B_α, ν is realized in K_α iff it is realized in K, and is realized by λ members of K iff for every $\beta < \alpha$, $(B_\alpha - B_{\beta+1})^\gamma \cap K \neq 0\}$ is a closed unbounded set of α's, hence for each such K there is an $\alpha \in S$ for which $K \cap B_\alpha^\gamma$ is small and $K \cap B_\alpha^\gamma = K_\alpha$, $\gamma = \gamma_\alpha$. Now we shall try to let

$$p_\alpha = \Big\{ \neg\, \varphi^\nu(\bar{y})\colon \text{no } \gamma_\alpha\text{-tuple from } K \text{ realizes } \nu, \quad \text{a simple formal } n_\alpha\text{-interval from } B_\alpha \Big\}$$

$$\cup \Big\{ \bigvee_{i<\gamma_\alpha} x_i \neq a_i\colon a_0, \ldots \in B_\alpha \Big\}$$

(notice p_α consists of negations of conjunctions of first order formulas) by the μ-saturation we can get rid of the $\exists \bar{x}$. If p_α is not realized in B, we see that no n-tuple from K is disjoint from B_α, hence K has cardinality $\leqslant \mu$ or is not disjoint so we shall finish. We only have to prove p_α has no support over B_α. So suppose

$$Qx_1 \exists x_1' Qx_2 \exists x_2' \cdots \exists y_0, \ldots, y_{\gamma-1} \varphi(x_1, \ldots, y_0, \ldots, y_{\gamma-1}, b_0, \ldots)$$

is a support $(b_0, \ldots \in B_\alpha$, $|\varphi| < \mu)$. We can represent φ as a disjunction of conjunctions each conjunction containing complete information as to which Boolean combination of the x's, y's and b's is empty. We can then replace it w.l.o.g. by one of the disjunctions. Then by using more parameters, we can add information to φ, so it is still a support, but φ is of the form ψ^ν for some ν, so we finish.

ADDED IN PROOF. 1. Monk improved 1.4(1) by getting a complicated Boolean algebra of power $2^{\aleph_0}$ without assuming CH.

2. It seems that the authors proved 1.4(2) assuming CH only.

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