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THE NUMBER OF PAIRWISE NON-ELEMENTARILY-EMBEDDABLE MODELS

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Abstract. We get consistency results on $I(\lambda, T_1, T)$ under the assumption that $D(T)$ has cardinality $> |T|$. We get positive results and consistency results on $IE(\lambda, T_1, T)$.

The interest is model-theoretic, but the content is mostly set-theoretic: in Theorems 1–3, combinatorial; in Theorems 4–7 and 11(2), to prove consistency of counterexamples we concentrate on forcing arguments; and in Theorems 8–10 and 11(1), combinatorics for counterexamples; the rest are discussion and problems. In particular:

(A) By Theorems 1 and 2, *if* $T \subseteq T_1$ are first order countable, T complete stable but $\aleph_0$-unstable, $\lambda > \aleph_0$, and $|D(T)| > \aleph_0$, *then* $IE(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, \beth_2\}$.

(B) By Theorems 4, 5, 6 of this paper, if e.g. $V = L$, then in some generic extension of V not collapsing cardinals, for some first order $T \subseteq T_1$, $|T| = \aleph_0$, $|T_1| = \aleph_1$, $|D(T)| = \aleph_2$ and $IE(\aleph_2, T_1, T) = 1$.

This paper (specifically the ZFC results) is continued in the very interesting work of Baldwin on diversity classes [Bl]. Some more advances can be found in the new version of [Sh300] (see Chapter III, mainly §7); they confirm 0.1, 0.2 and 14(1), 14(2).

Here we continue [ShA1, VIII, §1], improving results and showing complementary consistency results. We let $T \subseteq T_1$ be complete first order theories. We want to know what we can say about $I(\lambda, T_1, T)$ (see below) and $IE(\lambda, T_1, T)$ under various assumptions on T and on the cardinals, where:

$I(\lambda, T_1, T)$ is the number of models of T, up to isomorphism, of cardinality λ which are reducts of models of T_1.

$IE(\lambda, T_1, T) = \mathrm{Max}\{|K|: K$ a family of $L(T)$-reducts of models of T_1 of cardinality λ no one elementarily embeddable into another$\}$.

$IE(T_1, T)$ is defined similarly.

(If there is no maximal $|K|$, and the supremum is χ, we write $IE(\lambda, T_1, T) = \chi^-$, and say $\chi^- < \chi$ and $(\forall \theta < \chi)\theta < \chi^-$.)

$D_n(T) = \{p: p$ a complete type in $L(T)$ consistent with T in the variables $x_0, \ldots, x_{n-1}\}$.

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[2]I thank Rami Grossberg for many corrections, and in particular for rewriting the proof of Theorem 1; and also John Baldwin for adding explanations. The paper's volume has grown considerably under their pressure. I also thank Alice Leonhardt for the beautiful typing of the manuscript.

$D(T) = \bigcup_{n<\omega} D_n(T)$.

By Ehrenfeucht and Mostowski [EM], if $|D(T)| > |T_1|$ then $I(\lambda, T_1, T) \geq |D(T)|$. By Keisler [K], if $|T| = |T_1| = \aleph_0$ and $|D(T)| > \aleph_0$ (hence $= 2^{\aleph_0}$) then $I(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, 2^{\aleph_1}\}$. Trivially, $1 \leq IE(\lambda, T_1, T) \leq I(\lambda, T_1, T) \leq 2^\lambda$.

About the consistency results we were influenced by the related results on trees: Silver [Si], who proved the consistency of the Kurepa hypothesis, and Mitchell [M], who proved the consistency of "no $\aleph_2$-tree".

Let us review the relevant results and the open questions from [ShA1, VIII, §1], [Sh100], and [Sh135]. Our new results are mostly in Cases C and D.

Notation. We set $\kappa \stackrel{\text{def}}{=} |T|$ and $\mu = |T_1|$, where $T \subseteq T_1$ are complete first order theories.

Case A: T unstable. By [ShA1, VIII, 3.3], $I(\lambda, T_1, T) = 2^\lambda$ for $\lambda \geq |T_1| + \aleph_1$ (see also [Sh300, Chapter III, §3]).

By [Sh100] it is consistent that there is an expansion T_1 of the theory $T = \mathrm{Th}(Q, <)$ ($=$ the theory of dense linear order with no extremal points), such that $|T_1| = \aleph_1$ and $IE(\aleph_1, T_1, T) = 1$ holds.

By [Sh175, §1] and [Sh175a] (which replaces [Sh175, §2]), for $T = T_{\mathrm{ind}} \stackrel{\text{def}}{=}$ the model completion of the theory of graphs, for many λ it is consistent that for some $T_1, |T_1| = \lambda, IE(\lambda, T_1, T) = 1$.

By [ShA1, VIII, Theorem 2.2], for λ regular $> |T_1|$, $IE(\lambda, T_1, T) = 2^\lambda$, and for $\lambda > |T_1|$, $IE(\lambda, T_1, T) \geq 2^\theta$ for every regular $\theta \leq \lambda$.

If $\mu^{\aleph_0} < \lambda < 2^\mu$ and $|T_1| \leq \mu$, then, by [Sh136, Theorem 0.1], $IE(\lambda, T_1, T) = 2^\lambda$.

0.1. *Conjecture.* For $\lambda > |T_1|$, $IE(\lambda, T_1, T) = 2^\lambda$.

Case B: T stable unsuperstable. By [ShA1, VIII, §2], for $\lambda > |T_1|$ we have $I(\lambda, T_1, T) = 2^\lambda$, and $IE(\lambda, T_1, T) \geq 2^\mu$ when $\mu \leq \lambda$ is regular.

By [Sh100] it is consistent that there are T, T_1 such that $|T| = \aleph_0, |T_1| = \aleph_1$, and $I(\aleph_1, T_1, T) = 1$.

By [Sh136], if $|T_1| \leq \mu^{\aleph_0} < \lambda < 2^\mu$ then $IE(\lambda, T_1, T) = 2^\lambda$.

Here in Theorem 2, we prove that if $\kappa = \mu = \aleph_0$ and $\aleph_0 < \lambda \leq 2^{\aleph_0}$, then $IE(\lambda, T_1, T) = 2^\lambda$.

0.2. *Problem.* Is $IE(\lambda, T_1, T) = 2^\lambda$ when $\lambda > |T_1|$?

Case C: T superstable, $|D(T)| = \aleph_0$, $\kappa = \mu = \aleph_0$, and T $\aleph_0$-unstable. By Theorem 1 below, for $\lambda > \aleph_0$, $IE(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, \beth_2\}$. (This improves results from [ShA1, VIII, §1].)

Remember that *if* T is superstable and $|D(T)| \geq 2^{\aleph_0}$, *then* T is stable in θ iff $\theta \geq |D(T)|$.

Main Case D: $|D(T)| > |T_1|^+$. By Ehrenfeucht and Mostowski, $I(\lambda, T_1, T) \geq |D(T)|$. By Fact 3, below (improving [ShA1, VIII, 1.2(2)]), $IE(\lambda, T_1, T) \geq |D(T)|^{\aleph_0}$.

We have a number of results which show the relative consistency of certain values for $I(\lambda, T, T)$ and $IE(\lambda, T_1, T)$. The technical lemma underlying these results is Lemma 4. We prove there that in an appropriate forcing extension of the set-theoretic universe the category of members of $PC(T_1, T)$ and elementary embedding is described by the containment relation among subalgebras of an algebra N_0^* (for each cardinality). In the later results we force again to require this algebra to have the desired pattern of subalgebras.

Now the following are consistent.

1) By 8(2) and 4 it is consistent that $\kappa = \aleph_0$, $\mu = \kappa^+$, $|D(T)| = 2^{\aleph_0} > \mu$, and for every $\theta \geq \mu$ we have $I(\theta, T_1, T) = IE(\theta, T_1, T) = 2^{\aleph_0}$.

2) In $D(1)$ above, we can by Lemma 7 replace $\kappa = \aleph_0$ and $\mu = \aleph_1$ by $\aleph_0 < \kappa = \kappa^{<\kappa}$, $\mu = \kappa$, and $|D(T)| = 2^\kappa$ *is* an arbitrarily large cardinal.

3) We then can change cardinal arithmetic with no change in the value of $IE(\lambda, T_1, T)$ (see 5A).

4) We can have $\kappa = \kappa^{<\kappa}$, 2^κ arbitrarily large, and, for every $|T| = \kappa$ and $|T_1| < 2^\kappa$, if $|D(T)| > \kappa^+$ then ($|D(T)| = 2^\kappa$ and) $I(\lambda, T_2, T) = IE(\lambda, T_1, T) = \mathrm{Min}\{2^\lambda, 2^{2^\kappa}\}$. [Just add many (i.e. $\geq (2^\kappa)^V$) (Cohen) subsets to κ.]

5) Previously cardinal arithmetic was bounded by the covering lemma. Starting with supercompacts, we get (see 11(2)) consistency of: κ strong limit singular of uncountable cofinality, and, for some T and T_1 with $|T| = \kappa$ and $|T_1| = \kappa^+$, $\lambda = \kappa^+$, $|D(T)| = \kappa^{++}$ and $IE(\lambda, T_1, T) = 1$. Also others.

6) If $\kappa = |T| = |T_1|$ is strong limit of cofinality $\aleph_0$, $|D(T)| > |T|$, under some set-theoretic assumptions (large filters on arbitrarily large $\chi < \kappa$) we get $(\kappa^{\aleph_0}, 0)$-freedom (see [ShA1, Chapter VIII, §1]); hence $IE(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, 2^{2^\kappa}\}$ for $\lambda \geq \kappa$. Whether this can be proved in ZFC is open (see 13).

Case E: $|D(T)| = |T_1|^+$. Note that by Case D (see 6) there), this is sometimes impossible. Now it is consistent that $IE(\aleph_2, T_1, T) = 1$, $I(\aleph_2, T_1, T) = \aleph_2$, $|T| = \aleph_0$, $|T_1| = \aleph_1$, and $2^{\aleph_0} = \aleph_2 = |D(T)|$ (see Conclusion 6 (based on Lemma 4, with $\kappa = \aleph_0$ and $\lambda = \aleph_2$)). Also for regular $\kappa > \aleph_0$ it is consistent that $IE(\lambda, T_1, T) = 1$, $I(\lambda, T_1, T) = \kappa^+$, $|T| = \kappa = \kappa^{<\kappa} = |T_1|$, and $|D(T)| = \kappa^+$ (by Lemma 7).

In fact (by 11(1)), if κ is strong limit singular of uncountable cofinality, there are always suitable T and T_1 such that $|T| = |T_1| = \kappa$, $|D(T)| = \kappa^+$, and $IE(\kappa^+, T_1, T) = 1$, i.e. we prove examples exist, rather than merely proving they may exist. Adding many Cohen subsets restrict our freedom for regulars (see (4) of Case D above).

But results from Case C apply here. Also see D(5).

Case F: There is a family of κ independent formulas in T. See 10(2)(B). By [ShA1, VIII, 1.10], *if* $\kappa = |T_1| = |T|$ and there is a κ-tree with μ branches *then* we have (μ, T_1, T)-freedom; hence for $\chi \geq \kappa$

$$IE(\chi, T_1, T) \geq \mathrm{Min}\{2^\chi, 2^\mu\}$$

(see also Remark 12).

Notation. Standard; remember that $|A|$ is the cardinality of A, but $|N|$ is the universe of a model N and $\|N\|$ is its cardinality.

* * *

THEOREM 1. *Suppose $T \subseteq T_1$, T_1 countable, and T complete, superstable but $\aleph_0$-unstable. Then, for $\lambda > \aleph_0$,*

$$IE(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, \beth_2\}.$$

REMARK. 1) As this supersedes [ShA1, VIII, 1.8], we give a complete proof not based on it.

2) We can replace "$|T_1| = \aleph_0$" by "$\mathrm{MA}_{|T_1|}$" or even "$\mathbf{R}$ is not the union of $|T_1|$ nowhere-dense subsets", but then we should demand $\lambda > |T_1|$ and $|S_\xi| = |T_1|^+$.

3) In Chapter VIII of [ShA1], we proved that for pairs of theories (T, T_1)

satisfying the hypothesis of Theorem 1 we have $I(\lambda, T_1, T) \geq \min\{2^\lambda, \beth_2\}$. The proof proceeded by two different cases: VIII, 1.7(2) for $\lambda > 2^{\aleph_0}$ and VIII, 1.8 for $\lambda \leq 2^{\aleph_0}$. The second argument in fact yielded $IE(\lambda, T_1, T) \geq 2^{\aleph_0}$, while the first did not. To improve the result for IE, we have redesigned the proof from [ShA1, VII] and "souped up" the one from [ShA1, VIII]. The added energy comes from the trees U_η defined in Fact 1.B.

PROOF OF THEOREM 1. The assumption that the theory is superstable and not totally transcendental is used to obtain $m_a, m_b < \omega$ and a countable set of definable (without parameters) equivalence relations $\{E_n(\bar{x}; \bar{y}): n < \omega\} \subseteq L(T)$ such that:

(i) $\lg(\bar{x}) = \lg(\bar{y}) = m_a + m_b$,

(ii) if M is a model of T and $\bar{a} \in {}^{m_a}|M|$, then the set $\{\bar{a}^\frown \bar{b}/E_n: \bar{b} \in {}^{m_b}|M|\}$ is finite,

(iii) if, for $e = 1, 2$, $\lg(\bar{a}_e) = n_a$, $\lg(\bar{b}_e) = n_b$, and $\bar{a}_1{}^\frown \bar{b}_1 E_n \bar{a}_2{}^\frown \bar{b}_2$, then $\bar{a}_1 = \bar{a}_2$,

(iv) E_{n+1} refines E_n, i.e., for every $n < \omega$, $\bar{x} E_{n+1} \bar{y}$ implies $\bar{x} E_n \bar{y}$, and

(v) there are (in some model M of T) $\bar{c}_\eta$ for $\eta \in {}^{\omega>}2$ such that $[\lg(\eta) \geq n$ and $\lg(\nu) \geq n$ imply $\bar{c}_\eta E_n \bar{c}_\nu \Leftrightarrow \eta \restriction n = \nu \restriction n]$, $\bar{c}_\eta \restriction m_a = \bar{c}_\nu \restriction m_a$, and $\lg(\bar{c}_\eta) = m_a + m_b$.

The existence of this set of equivalence relations was proved in III, 5.1–5.3 of [ShA1].

Clearly without loss of generality we may expand the theory T_1. Let $\{\bar{c}_\eta: \eta \in {}^{\omega>}2\}$ be new constants in T_1, and suppose

$$\begin{aligned} T_1 \supseteq \{&E_n(\bar{c}_\eta, \bar{c}_\nu): \eta \restriction n = \nu \restriction n, \lg(\eta), \lg(\nu) \geq n\} \\ &\cup \{\neg E_n(\bar{c}_\eta, \bar{c}_\nu): \eta \restriction n \neq \nu \restriction n, \lg(\eta), \lg(\nu) \geq n\}. \end{aligned}$$

Also without loss of generality suppose that T_1 has Skolem functions (and the axioms saying it has Skolem functions belong to T_1).

We will use the following fact [for a sequence $\bar{\eta}$ let $\bar{\eta} = \langle \bar{\eta}[l]: l < \lg(\bar{\eta})\rangle$ and $\bar{a}_{\bar{\eta}} = \bar{a}_{\bar{\eta}[0]}{}^\frown \bar{a}_{\bar{\eta}[1]}{}^\frown \bar{a}_{\bar{\eta}[2]} \cdots$].

Fact 1.A. *There exists a model $M \models T_1$, and there exists $\{\bar{a}_\eta: \eta \in {}^\omega 2\} \subseteq |M|$, $\bar{a}_\eta \restriction m_a = \bar{a}_\nu \restriction m_a$, $\lg(\bar{a}_\eta) = m_a + m_b$ and*

$$\lg(\eta) \geq n \;\&\; \lg(\nu) \geq n \Rightarrow [\eta \restriction n = \nu \restriction n \Leftrightarrow E_n(\bar{a}_\eta, \bar{a}_\nu)]$$

*such that: for every sequence of terms $\bar{\tau}(\bar{x}) \in L(T_1)$, **if** $m \times (m_a + m_b) = \lg(\bar{x})$, $m_a + m_b = \lg(\bar{\tau})$, $\bar{\tau}(\bar{x}) \restriction m_a = (\bar{\tau} \restriction m_a)(\bar{x} \restriction m_d)$, and $m_d = m_e \times (m_a + m_b)$ [i.e. for $\bar{\eta} \in {}^m({}^\omega 2)$, $\bar{\tau}(\bar{a}_{\bar{\eta}}) \restriction m_a = (\bar{\tau} \restriction m_a)(\bar{a}_{\bar{\eta} \restriction m_e})$], **then** there exists $n_{\bar{\tau}} < \omega$ such that the following two requirements are met*:

(1) *For $n \geq n_{\bar{\tau}}$ and $\bar{\eta}$, $\bar{\nu} \in {}^m({}^\omega 2)$ with no repetitions, $\bar{\eta} \restriction m_e = \bar{\nu} \restriction m_e$, **if** $l \neq k \Rightarrow \bar{\eta}[l] \restriction n \neq \bar{\eta}[k] \restriction n$ and $(\forall l < m)[\bar{\eta}[l] \restriction n = \bar{\nu}[l] \restriction n]$, **then**, for every $\bar{\rho} \in {}^m({}^\omega 2)$, $\bar{\rho} \restriction m_e = \bar{\eta} \restriction m_e$ implies*

$$E_n(\bar{\tau}(\bar{a}_{\bar{\eta}}), \bar{\tau}(\bar{a}_{\bar{\rho}})) \Leftrightarrow E_n(\bar{\tau}(\bar{a}_{\bar{\nu}}), \bar{\tau}(\bar{a}_{\bar{\rho}})).$$

(2) *For $n \geq n_{\bar{\tau}}$ and $\bar{\eta}$, $\bar{\nu} \in {}^m({}^n 2)$, each with no repetition, $\bar{\eta} \restriction m_e = \bar{\nu} \restriction m_e$, **if** there are $k \geq n$ and $\bar{\eta}_1$, $\bar{\nu}_1 \in {}^m({}^\omega 2)$ such that $\neg E_k(\bar{\tau}(\bar{a}_{\bar{\eta}_1}), \bar{\tau}(\bar{a}_{\bar{\nu}_1}))$, for $l < m$, $\bar{\eta}_1[l] \restriction n = \bar{\eta}[l]$, $\bar{\nu}_1[l] \restriction n = \bar{\nu}[l]$, and*

$$(\forall l, i < m)[\bar{\eta}_1[l] = \bar{\nu}_1[i] \Leftrightarrow \bar{\eta}[l] = \bar{\nu}[i]],$$

***then** for every $\bar{\eta}^*, \bar{\nu}^* \in {}^m({}^\omega 2)$ satisfying $\bar{\eta}^*[l] \restriction n = \bar{\eta}[l]$, $\bar{\nu}^*[l] \restriction n = \bar{\nu}[l]$ (for each l*

$l < m$), and

$$(\forall l, i < m)[\bar{\eta}^*[l] = \bar{\nu}^*[i] \Leftrightarrow \bar{\eta}[l] = \bar{\nu}[i]]$$

we have

$$\neg E_n(\bar{\tau}(\bar{a}_{\bar{\eta}^*}), \bar{\tau}(\bar{a}_{\bar{\nu}^*})).$$

REMARK. This is really the only place where we use countability.

PROOF. Use Theorem [ShA1, VII, 3.7] to satisfy requirement (1) by letting $\varphi_n^l(\bar{x}^\frown\bar{z}) \stackrel{\text{def}}{=} E_n(\bar{x}^\frown\bar{z}, F_l(\bar{z})^\frown\bar{z})$ for $l < l_n^* < \omega$, where the F_l are such that $\{F_l(\bar{z}): l < l_n^*\}$ is a complete set of representatives for $\{\bar{x}^\frown\bar{z}/E_n: \bar{x}\}$, possibly with repetition. (Remember T_1 has Skolem functions, and by compactness there is l_n^* which does not depend on $\bar{z}$.) Requirement (2) is fulfilled by trimming the perfect tree and renaming.

We will use the following combinatorial fact, which is slightly stronger than Sierpiński's lemma on almost disjoint sets of integers:

Fact 1.B. *There are $W(*) \subseteq \omega$, $\{W_\eta \subseteq \omega: \eta \in {}^\omega 2\}$, and $\{U_\eta: \eta \in {}^\omega 2\}$ such that for all $\eta \in {}^\omega 2$ the following requirements are met*:

(0) *$W(*)$ and W_η are infinite subsets of ω.*

(1) *U_η is a perfect tree, i.e. $U_\eta \subseteq {}^{\omega>}2$ is downward closed, $\langle\ \rangle \in U_\eta$, and*

$$\forall \rho \in U_\eta\, \exists \nu \in U_\eta[\rho = \nu \restriction \lg(\nu) \wedge \nu^\frown\langle 0\rangle \in U_\eta \wedge \nu^\frown\langle 1\rangle \in U_\eta].$$

(2) *$\rho, \nu \in U_\eta$, $\rho \neq \nu$ and $\lg(\rho) = l(\nu) \Rightarrow h(\rho, \nu) \in W_\eta$, where $h(\rho, \nu)$ is the length of the largest common initial segment of ρ and ν, i.e.*

$$h(\rho, \nu) \stackrel{\text{def}}{=} \operatorname{Max}\{n < \omega: \rho \restriction n = \nu \restriction n\}.$$

(3) *For all $\eta_1 \neq \eta_2 \in {}^\omega 2$ and every $\rho \in U_{\eta_1}$ and $\nu \in U_{\eta_2}$ there are three possibilities*: (a) *$h(\rho, \nu) \in W_{\eta_1} \cap W_{\eta_2}$, or* (b) *$h(\rho, \nu) \in W(*)$, or* (c) *$\rho \leq \nu$ or $\nu < \rho$.*

(4) *$W(*) \cap W_\eta = \varnothing$.*

(5) *For distinct η, ν from ${}^\omega 2$, $W_\eta \cap W_\nu$ is finite.*

Proof. By induction on n define $k(n) = k_n < \omega$ and the set $W_n(*) \subseteq k(n)$, and for $\eta \in {}^n 2$ the sets $U_\eta \subseteq {}^{k(n)\geq}2$ and $W_\eta \subseteq k(n)$, such that in the end (this imposes natural restrictions on them)

$$[\eta \in {}^\omega 2 \Rightarrow W_\eta \cap k_n = W_{\eta\restriction n}, U_\eta \cap {}^{k(n)\geq}2 = U_{\eta\restriction n}, W(*) \cap k(n) = W_n(*)].$$

For the induction step, choose $k^1(n) = k(n) + n$, and for $\eta \in {}^n 2$ let

$$U_\eta^1 = U_\eta \cup \{\nu^\frown(\eta \restriction l): \nu \in U_\eta \cap {}^{k(n)}2, l \leq n\};$$

thus

$$(\forall \nu \in {}^{k(n)}2 \cap U_\eta)(\exists! \rho \in {}^{k^1(n)}2 \cap U_\eta^1)[\rho > \nu].$$

Define $W_{n+1}(*) = W_n(*) \cup [k(n), k^1(n)]$. Fix an enumeration $\{\eta_k: k < 2^n\}$ of ${}^n 2$. Let $k(n+1) \stackrel{\text{def}}{=} k^1(n) + 2^{n+1}$. For $\eta \in {}^{n+1}2$, there are unique $k < 2^n$ and $i < 2$ such that $\eta = \eta_k{}^\frown\langle i\rangle$. Let

$$\begin{aligned} U_\eta \stackrel{\text{def}}{=} U_{\eta_k}^1 \cup \{\nu: \nu \in {}^{k(n+1)\geq}2,\ & \nu \restriction k^1(n) \in U_{\eta_k}^1, \\ & [k^1(n) + l < \lg(\nu) \wedge (l \neq 2k+1) \Rightarrow \nu(k^1(n) + l) = 0], \\ & [k^1(n) + l < \lg(\nu) \wedge (l = 2k+1) \Rightarrow \nu(k^1(n) + l) = i]\,\} \end{aligned}$$

and $W_\eta = W_{\eta_k} \cup \{k^1(n) + 2k + 1\}$. It is easy to verify that this construction provides a family of sets as required.

Continuation of the Proof of Theorem 1. Assume $\lambda \leq 2^{\aleph_0}$, and fix an enumeration $\{\eta_\xi: \xi < 2^{\aleph_0}\} = {}^\omega 2$. Let the set

$$S_\xi \subseteq \{\rho \in {}^\omega 2: (\forall n < \omega)[\rho \restriction n \in U_{\eta_\xi}]\}$$

be of cardinality $|T_1|^+$. Fix $\{\rho_i^\xi: i < |T_1|^+\} = S_\xi$. We call $S \subseteq {}^\omega 2$ *large* if for every $n < \omega$ and $\nu \in S$ we have $|\{\rho \in S: \rho \restriction n = \nu \restriction n\}| > |T_1|$. Note that, for every $S \subseteq {}^\omega 2$ of cardinality $> |T_1|$, for some $S_1 \subseteq S$ we have that $|S_1| \leq |T_1|$ and $S - S_1$ is large.

For $X \subseteq \lambda$ let M_X^1 be the Skolem hull of $\{\bar{a}_\eta: \eta \in \bigcup_{\xi \in X} S_\xi\}$, and put $M_X \overset{\text{def}}{=} M_X^1 \restriction L(T)$. In order to prove the theorem it is enough to assume X, $Y \subseteq \lambda$ and $X \not\subseteq Y$, and show there does not exist an elementary embedding f from M_X into M_Y. Let $\xi \in X - Y$. For the sake of contradiction suppose $f: M_X \to M_Y$ is an elementary embedding. For $\nu \in S_\xi$ let $f(\bar{a}_\nu) = \bar{\tau}_{\bar{\nu}}(\bar{\eta}_{\bar{\nu}})$. So there are $S^* \subseteq S_\xi$ which is large, and $\bar{\tau}$, and an integer n_0 such that $[\nu \in S^* \Rightarrow \bar{\tau}_\nu = \bar{\tau} \wedge \lg(\bar{\eta}_\nu) = n_0]$ and, without loss of generality,

$$\bar{\tau}(\bar{\eta}_\nu) \restriction m_a = f(\bar{a}_\nu \restriction m_a) = (\bar{\tau} \restriction m_a)(\bar{\eta}_{\bar{\nu} \restriction m_e}).$$

Notation. For $\eta \in \bigcup_{\zeta \in Y} S_\zeta$ let $\zeta(\eta)$ be the unique element of Y such that $\eta \in S_{\zeta(\eta)}$ (this element is unique by Fact 1.B, (1) and (2), and the choice of the S_ζ).

Fact 1.C. *We can find a large $S^{**} \subseteq S^*$, $k_0 < \omega$, and $\bar{\eta}_0 \in {}^{n_0}({}^\omega 2)$ with the following properties*:

(0) $\eta \neq \nu \in S^{**} \Rightarrow h(\eta, \nu) > k_0$.

(1) *For* $\nu \in S^{**}$, $(\forall l < n_0)[\bar{\eta}_\nu[l] \restriction k_0 = \bar{\eta}_0[l] \restriction k_0]$ *and* $\{\bar{\eta}_\nu[l] \restriction k_0: l < n_0\}$ *are pairwise distinct.*

(2) $k_0 > n_{\bar{\tau}}$.

(3) *For each* $l < n_0$ *either* $\{\bar{\eta}_\nu[l]: \nu \in S^{**}\} = \{\bar{\eta}_0[l]\}$ *or the elements* $\{\bar{\eta}_\nu[l]: \nu \in S^{**}\}$ *are pairwise distinct.*

(4) $W_\xi \cap (W(*) \cup \bigcup_{l < n_0} W_{\zeta(\bar{\eta}_0[l])}) \subseteq k_0$.

(5) *For each* $l < n_0$, *either* $\{\zeta(\bar{\eta}_\nu[l]): \nu \in S^{**}\} = \{\zeta(\bar{\eta}_0[l])\}$ *or the elements* $\{\zeta(\bar{\eta}_\nu[l]): \nu \in S^{**}\}$ *are pairwise distinct.*

Proof of 1.C. Left to the reader.

MAIN LEMMA 1.D. *If* $\nu \neq \rho \in S^{**}$, *then there is an* l $(< n_0)$ *such that* $h(\nu, \rho) = h(\bar{\eta}_\nu[l], \bar{\eta}_\rho[l])$.

This lemma was proved in the course of the proof of Theorem 1.8 of VIII of [ShA1], but for the convenience of the reader we will prove it below. But first we use it to conclude the proof of Theorem 1.

Since S^{**} is large, clearly we can choose $\{\nu_\alpha \in S^{**}: \alpha \leq \omega\}$ such that for all $n < \omega$ we have $\nu_n \neq \nu_\omega$ and $\nu_\omega \restriction n = \nu_n \restriction n$ (fix first ν_ω, and pick the other elements by largeness of the set).

Applying Lemma 1.D, for $n < \omega$ let $l_n < n_0$ be such that

$$h(\nu_\omega, \nu_n) = h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]).$$

Since $[\nu_\alpha \in S^{**} \subseteq S_\xi]$, clearly by 1.B(2) $h(\nu_\omega, \nu_n) \in W_\xi$, hence also

$$h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]) \in W_\xi;$$

and by 1.C(1) $k_0 < h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n])$. From this it follows by 1.C(4) that

$$h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]) \notin W(*)$$

and that

$$h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]) \notin W_{\zeta(\bar{\eta}_0[l_n])}.$$

By 1.B(2), as $[\eta \in S_\xi \Rightarrow \bigwedge_n \eta \restriction n \in U_\zeta]$ we know $\{\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]\}$ is not a subset of $S_{\zeta(\bar{\eta}_0[l_n])}$. Hence, by 1.C(5), $\zeta(\bar{\eta}_{\nu_\omega}[l_n]) \neq \zeta(\bar{\eta}_{\nu_n}[l_n])$. So by 1.B(3) (applied to $(\bar{\eta}_{\nu_\omega}[l_n]) \restriction k$ and $(\bar{\eta}_{\nu_n}[l_n]) \restriction k$, k large enough) we have

$$h(\bar{\eta}_{\nu_\omega}[l_n], \bar{\eta}_{\nu_n}[l_n]) \in W_{\zeta(l_n)},$$

where $\zeta(l_n) = \zeta(\bar{\eta}_{\nu_\omega}[l_n])$ (as (b) there is discarded above, (c) is trivially false; so (a) there holds and we get the statement above).

Let $A \subseteq \omega$ be unbounded and let $l^* < n_0$ be such that $n \in A \Rightarrow l_n = l^*$; now for every $n \in A$ we have $h(\nu_\omega, \nu_n) \in W_\xi \cap W_{\zeta(l^*)}$ (combine three facts from the previous paragraph; remember that $\zeta(l^*) = \zeta(\eta_{\nu_\omega}[l_{l^*}])$). But, since $\xi \notin Y$, $W_\xi \cap W_{\zeta(l^*)}$ is finite, which contradicts the choice of $\{\nu_\alpha : \alpha \leq \omega\}$ as satisfying $h(\nu_\omega, \nu_n) \geq n$.

Proof of Lemma 1.D. We have to show that for every $\rho \neq \nu \in S^{**} (\subseteq S_\xi)$ there exists $l < n_0$ such that

$$h(\rho, \nu) = h(\bar{\eta}_\nu[l], \bar{\eta}_\rho[l]).$$

Suppose $n = h(\rho, \nu)$. Hence $E_n(\bar{a}_\rho, \bar{a}_\nu) \wedge \neg E_{n+1}(\bar{a}_\rho, \bar{a}_\nu)$. For didactic reasons we first suppose, for the sake of contradiction, that for every $l < n_0$ we have

$$\bar{\eta}_\nu[l] \neq \bar{\eta}_\rho[l] \Rightarrow h(\bar{\eta}_\nu[l], \bar{\eta}_\rho[l]) < n.$$

Since f is elementary, $\neg E_{n+1}(\bar{\tau}(\bar{a}_{\bar{\eta}_\rho}), \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}))$; now we can deduce by Facts 1.A(2) and 1.C(0), (2) that

$$\neg E_n(\bar{\tau}(\bar{a}_{\bar{\eta}_\rho}), \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}));$$

again as f is elementary, $\neg E_n(\bar{a}_\rho, \bar{a}_\nu)$, in contradiction to $E_n(\bar{a}_\rho, \bar{a}_\nu)$. Now we deal with the general case, i.e. we assume

$$(*) \qquad (\forall l < n_0) h(\bar{\eta}_\nu[l], \bar{\eta}_\rho[l]) \neq n.$$

We shall derive a contradiction.

Define $\bar{\eta} \in {}^{n_0}({}^\omega 2)$

$$\bar{\eta}[l] = \begin{cases} \bar{\eta}_\rho[l] & \textit{if } \bar{\eta}_\nu[l] \restriction n \neq \bar{\eta}_\rho[l] \restriction n, \\ \bar{\eta}_\nu[l] & \textit{otherwise.} \end{cases}$$

Clearly $\bar{\tau}(\bar{a}_\eta) \restriction m_a = \bar{\tau}(\bar{a}_{\eta_\rho}) \restriction m_a = \bar{\tau}(\bar{a}_{\eta_\nu}) \restriction m_a$ and $\bar{\eta} \restriction m_e = \bar{\eta}_\nu \restriction m_e = \bar{\eta}_\rho \restriction m_e$, and also $\bar{\eta}$ is with no repetition and $\langle \bar{\eta}[l] \restriction n : l < n_0 \rangle$ are pairwise distinct.

Since, by the definition of $\bar{\eta}$, for each l we have $\bar{\eta}[l] \restriction n = \bar{\eta}_\rho[l] \restriction n$, using $(*)$ we obtain $\eta[l] \restriction (n+1) = \eta_\rho[l] \restriction (n+1)$. Let $\bar{b} = \bar{\tau}(\bar{a}_{\bar{\eta}})$. By reflexivity of the equivalence relation we have $E_{n+1}(\bar{\tau}(\bar{a}_{\bar{\eta}_\rho}), \bar{\tau}(\bar{a}_{\bar{\eta}_\rho}))$; by Fact 1.A(1), $E_{n+1}(\bar{\tau}(\bar{a}_{\bar{\eta}}), \bar{\tau}(\bar{a}_{\bar{\eta}_\rho}))$, i.e. $E_{n+1}(\bar{b}, \bar{\tau}(\bar{a}_{\bar{\eta}_\rho}))$. Finally (as $\neg E_{n+1}(\bar{\tau}(\bar{a}_{\bar{\eta}_\nu}), \bar{\tau}(\bar{a}_{\bar{\eta}_\rho}))$), using transitivity of the equivalence relation, we have $\neg E_{n+1}(\bar{b}, \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}))$.

By the definition of $\bar{\eta}$, for every $l < n_0$ we have $\bar{\eta}[l] = \bar{\eta}_\nu[l]$ or $h(\bar{\eta}[l], \eta_\nu[l]) < n$.

But, since $n > k_0$,

$$|\{\bar{\eta}[l] \restriction k_0 : l < n_0\}| = n_0 \quad \text{and} \quad |\{\bar{\eta}_\nu[l] \restriction k_0 : l < n_0\}| = n_0.$$

So, by Fact 1.A(2), as $\neg E_{n+1}(\bar{b}, \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}))$ (see above) we have $\neg E_n(\bar{b}, \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}))$. But $E_n(\bar{b}, \bar{\tau}(\bar{a}_{\bar{\eta}_\rho}))$ (see above) and $E_n(\bar{\tau}(\bar{a}_{\bar{\eta}_\rho}), \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}))$, contradiction.

So the proof of Theorem 1 for the case $\lambda \leq 2^{\aleph_0}$ is completed. How do we deal with the case $\lambda > 2^{\aleph_0}$? We just need to revise Fact 1.A. Add to $L(T_1)$ countable many new constants $\{d_n : n < \omega\}$. Now prove the following variation of Fact 1.A:

Fact 1.E. *There exists a model $M \models T_1$, and there exists $\{\bar{a}_\eta : \eta \in {}^\omega 2\} \subseteq |M|$, such that, for $\eta, \nu \in {}^\omega 2$,*

$$\bar{a}_\eta \restriction m_a = \bar{a}_\nu \restriction m_a, \qquad [\eta \restriction n = \nu \restriction n \Leftrightarrow E_n(\bar{a}_\eta, \bar{a}_\nu)],$$

and $\langle d_n : n < \omega \rangle$ is an indiscernible sequence over $\{\bar{a}_\eta : \eta \in {}^\omega 2\}$ of distinct elements such that for every sequence of terms $\bar{\tau}(\bar{x}) \in L(T_1 \cup \{d_n : n < \omega\})$, $m = l(\bar{x})$ (with m_c, m_d, m_e as in Fact 1.A) *there exists $n_{\bar{\tau}} < \omega$ such that the following requirements are met*:

(1) *For $n \geq n_{\bar{\tau}}$ and $\bar{\eta}$, $\bar{\nu} \in {}^m({}^\omega 2)$ with no repetitions, **if** $\bar{\eta} \restriction m_e = \bar{\nu} \restriction m_e$, $[l \neq k \Rightarrow \eta[l] \restriction n \neq \eta[k] \restriction n]$ and $(\forall l < m)\ \bar{\eta}[l] \restriction n = \bar{\nu}[l] \restriction n$, **then**, for $\bar{\rho} \in {}^m({}^\omega 2)$, $\bar{\rho} \restriction m_e = \bar{\eta} \restriction m_e$ implies $E_n(\bar{\tau}(\bar{a}_{\bar{\eta}}), \bar{\tau}(\bar{a}_{\bar{\rho}})) \Leftrightarrow E_n(\bar{\tau}(\bar{a}_{\bar{\nu}}), \bar{\tau}(\bar{a}_{\bar{\rho}}))$.*

(2) *For $n \geq n_\tau$ and $\bar{\eta}$, $\bar{\nu} \in {}^m({}^n 2)$, each with no repetitions, **if** $\bar{\eta} \restriction m_e = \bar{\nu} \restriction m_e$ and there are $k \geq n$ and $\eta_1, \nu_1 \in {}^m({}^\omega 2)$ such that $\neg E_k(\tau(\bar{a}_{\bar{\eta}_1}), \tau(\bar{a}_{\nu_1}))$, for $l < m$*

$$\bar{\eta}_1[l] \restriction n = \bar{\eta}[l], \qquad \bar{\nu}_1[l] \restriction n = \bar{\nu}[l],$$

*and $(\forall l, i < m)(\bar{\eta}_1[l] = \bar{\nu}_1[i] \Leftrightarrow \bar{\eta}[i] = \bar{\nu}[i])$, **then** for every $\bar{\eta}^*, \bar{\nu}^* \in {}^m({}^\omega 2)$ satisfying $\bar{\eta}^*[l] \restriction n = \eta[l]$ and $\bar{\nu}^*[l] \restriction n = \nu[l]$ (for each $l < m$) and*

$$(\forall l, i < m)[\bar{\eta}^*[l] = \nu^*[i] \Leftrightarrow \bar{\eta}[l] = \bar{\nu}[i]]$$

we have

$$\neg E_n(\bar{\tau}(\bar{a}_{\bar{\eta}^*}), \bar{\tau}(\bar{a}_{\bar{\nu}^*})).$$

The proof of the fact is done similarly to the proof of Fact 1.A, but in the place of Theorem VII.3.7, use Exercise VII.3.1 of [ShA1]. Now we can blow up the models by extending the sequence of indiscernibles $\{d_n : n < \omega\}$. So we have $\beth_2$ models in power λ, as required. □ Theorem 1

THEOREM 2. *Suppose $T \subseteq T_1$ are countable and complete, T is stable but not superstable, and $\lambda > |T_1|$. Then $IE(\lambda, T_1, T) \geq \mathrm{Min}\{2^\lambda, \beth_2\}$.*

REMARK 2.A. This gives new information only when λ is singular $\leq 2^{\lambda_0}$ and $2^{\aleph_0} < 2^\lambda$ (see [ShA1, VIII, 2.2]).

PROOF. We combine the proof of Theorem 1 and VIII, 1.11.

For notational simplicity assume $\lambda \leq 2^{\aleph_0}$, and assume $\lambda > \aleph_1$ (see 2.A). Let Φ be as in VIII, 2.2 (or see VII, 3.6(2)), and let $M = EM({}^{\omega\geq}\lambda, \Phi)$, $p_\eta = \mathrm{tp}(\bar{a}_\eta, \bigcup_{l<\omega} \bar{a}_{\eta \restriction l})$ (for $\eta \in {}^\omega\lambda$).

As in [ShA1, VII] we write M^1, EM^1, etc. for $L(T^1)$-structures, and M, EM, etc. for their reducts to L.

Fact 2.B. *Without loss of generality, the following conditions can be assumed to hold*:

(i) *p_η is stationary.*

(ii) *If* $\eta \in {}^{\omega}\lambda$, $\{\eta \restriction l: l < \omega\} \subseteq J \subseteq {}^{\omega}\lambda$ *and* $\eta \notin J$, ***then***

$$\mathrm{tp}_{L(T)}(\bar{a}_\eta, \bigcup\{\bar{a}_\nu: \nu \in J\}) \textit{ does not fork over } \bigcup_{l<\omega} \bar{a}_{\eta\restriction l}.$$

(iii) $\bar{a}_{\eta\restriction m} \subseteq \bar{a}_\eta$ *for* $\eta \in {}^{\omega>}\lambda$ *and* $m < \lg(\eta)$.

(iv) *If* $\eta_1, \ldots, \eta_{n(1)}$ *and* $\nu_1, \ldots, \nu_{n(2)}$ *are distinct members of* ${}^{\omega\geq}\lambda$, $\nu_1, \ldots, \nu_{n(2)} \in {}^{\omega}\lambda$, *and* $\bar{\tau} = \tau(\bar{x}_1, \ldots, \bar{x}_{n(1)}, \bar{y}_1, \ldots, \bar{y}_{n(2)})$ *is a sequence of terms from* $L(T_1)$, *then*

$$\mathrm{tp}_{L(T)}(\bar{\tau}(\bar{a}_{\eta_1}, \ldots, \bar{a}_{\eta_{n(1)}}, \bar{a}_{\nu_1}, \ldots, \bar{a}_{\nu_{n(2)}}), EM({}^{\omega\geq}\lambda - \{\nu_1, \ldots, \nu_{n(2)}\}, \Phi))$$

is finitely satisfiable in

$$EM(\{\eta_i \restriction l: i = 1, \ldots, n(1), l \leq \lg(\eta)\} \cup \{\nu_i \restriction l: i = 1, \ldots, n(2), l < \omega\}, \Phi).$$

(v) *For* $\eta \in {}^{\omega}\lambda$, $\mathrm{tp}_{L(T)}(\bar{a}_\eta, EM({}^{\omega>}\lambda - \{\eta\}, \Phi))$ *does not fork over* $\bigcup_{k<\omega} \bar{a}_{\eta\ k}$.

Proof of 2.B. By the unsuperstability of T there are formulas $\varphi_n(\bar{x}, \bar{y}_n) \in L(T)$ and $\bar{a}_\eta \in M$, M a model of T, such that ($\lg(\bar{a}_\eta) = 1$ if $\eta \in {}^{\omega}\lambda$ and) the following conditions (α)(1)–(4) hold:

(α) (1) If $\eta \in {}^{\omega}\lambda$ and $\nu \in {}^{n}\lambda$, then $M^1 \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu]$ iff $\eta \restriction n = \nu$.

(2) If $\eta \in {}^{\omega}\lambda$, then $\mathrm{tp}[\bar{a}_\eta, \bigcup\{\bar{a}_\nu: \nu \in {}^{\omega>}\lambda\}]$ does not fork over $\bigcup_{n<\omega} \bar{a}_{\eta\restriction n}$.

(3) For all $\nu \in {}^{\omega}\lambda$, $\bar{a}_{\nu\restriction 0} {}^\wedge \bar{a}_{\nu\restriction 1} {}^\wedge \cdots {}^\wedge \bar{a}_{\nu\restriction n} {}^\wedge \bar{a}_\nu$ realize the same type in M.

Without loss of generality we can add

(4) $\bar{a}_{n\restriction l} \subseteq \bar{a}_\eta$ for $\eta \in {}^{n}\lambda$ and $l < n < \omega$

(as T is not superstable but is stable: see [ShA1, III, 3.3]).

Now we can find Φ proper for $({}^{\omega\geq}\lambda, T_1)$ (see the definition in [ShA1, VIII]) such that:

(β) For any $\varphi(\bar{x}_1, \ldots, \bar{x}_n) \in L(T^1)$ and $\nu_1, \ldots, \nu_n \in {}^{\omega\geq}\omega$, there are $\rho_1, \ldots, \rho_n \in {}^{\omega\geq}\lambda$ such that

(a) $\langle \nu_1, \ldots, \nu_n \rangle$ and $\langle \rho_1, \ldots, \rho_n \rangle$ are similar, and

(b) $EM^1({}^{\omega\geq}\omega, \Phi) \models \varphi[\bar{a}_{\nu_1}, \ldots, \bar{a}_{\nu_n}]$ iff $M^1 \models \varphi[\bar{a}_{\rho_1}, \ldots, \bar{a}_{\rho_n}]$.

This holds by the proof of [ShA1, VII, 3.6].

Let $M^1 = EM^1({}^{\omega\geq}\lambda, \Phi)$, $\lambda > \aleph_0$. For $n < \omega$ and $\eta \in {}^{n}\lambda$, let

$$I_\eta = \{\nu \in {}^{\omega\geq}\lambda: \text{for every } l < n,\ \nu(l) \in \{j_1, \eta(j_2) + j_3: j_1, j_2, j_3 < \lg(\eta)\}$$
$$\text{and, for } l \geq n \text{ (but } < \omega),\ \nu(l) = 0 \text{ and } \lg(\nu) \leq \lg(\eta) \text{ or } \lg(\nu) = \omega\}$$

and for $\eta \in {}^{\omega}\lambda$ let $I_\eta = \{\nu \in {}^{\omega}\lambda: \nu \restriction n \in I_{\eta\ n}$ for $n < \omega$, and for every large enough $l < \omega$, $\nu(l) = \eta(l)\}$.

For notation simplicity let h be a one-to-one function from $\{\delta < \lambda: \delta \text{ limit}\}$ onto ${}^{\omega>}\lambda$. Let $I = \{\eta \in {}^{\omega\geq}\lambda$, and for every $n < \lg(\eta)$, $h(\eta(n)) = \eta \restriction n\}$.

Let $\{\bar{\tau}^i_l(x^i): l < \omega\}$ list the $L(M^1)$-terms with $\bar{x}^i = \langle x_m: m < i\rangle$. Let us define $\bar{a}'_\eta$ for $\eta \in I$.

If $\lg(\eta) < \omega$, then $\bar{a}'_\eta$ is the concatenation of sequences $\bar{\tau}^i_l(\bar{a}_{\nu_1}, \ldots, \bar{a}_{\nu_k})$ for which: $l \leq \lg(\eta)$, $i \leq \lg(\eta)$ and for each $m \in \{1, \ldots, k\}$, $\nu_m \in I_\eta$ (in some natural ordering). If $\lg(\eta) = \omega$, $\bar{a}'_\eta = a_\eta$.

Easily, $\langle \bar{a}'_\eta: \eta \in I\rangle$ generates $M^2 \prec M^1$, and without loss of generality it is indiscernible in M^1, and for appropriate Φ', (i)–(iv) of 2.B hold; so without loss of generality they hold for Φ.

Now by (β), in $EM({}^{\omega\geq}\lambda, \Phi)$, for $\eta \in {}^{\omega}\omega$, $\mathrm{tp}_{L(T)}(\bar{a}_\eta, \bigcup\{\bar{a}_\nu: \nu \in {}^{\omega\geq}\lambda - \{\eta\}\})$ does not fork over $\bigcup_{n<\omega} \bar{a}_{\eta\ n}$. Hence $\{\bar{a}_\eta: \eta \in {}^{\omega}\omega\}$ is independent over $\bigcup\{\bar{a}_\nu: \nu \in {}^{\omega>}\omega\}$. Hence, for some countable $S \subseteq {}^{\omega}\omega$, $\bigcup\{\bar{a}_\eta \in {}^{\omega}\omega - S\}$ is independent over $EM({}^{\omega\geq}\omega, \Phi)$; hence, by the indiscernibility of $\langle \bar{a}_\eta : \eta \in {}^{\omega\geq}\omega\rangle$, $\{\bar{a}_\eta : \eta \in {}^{\omega}\omega\}$ is independent over $EM({}^{\omega>}\omega, \Phi)$. Similarly $\operatorname{tp}(\bar{a}_\eta, EM({}^{\omega\geq}\lambda - \{\eta\}, \Phi))$ does not fork over $EM({}^{\omega>}\lambda, \Phi)$, hence over $\{\bar{a}_{\eta\restriction l} : l < \omega\}$. So 2.B(i)–(v) all hold.

Fact 2.C. ***If*** ${}^{\omega>}2 \subseteq S^* \subseteq {}^{\omega\geq}2$, $\bar{\tau}(\bar{x}, \bar{y})$ *is a finite sequence of terms of* $L(T_1)$, $\bar{\eta} \in {}^{\omega>}(S^*)$, *and* $\bar{v} \in {}^{\omega>}({}^{\omega}2 - S^*)$, ***then*** $\operatorname{tp}_{L(T)}[\bar{\tau}(\bar{a}_{\bar{\eta}}, \bar{a}_{\bar{v}}), EM(S^*, \Phi)]$ *does not fork over* $EM({}^{\omega>}2 \cup \{\bar{\eta}[l] : l < \lg(\bar{\eta})\}, \Phi)$.

Proof of 2.C. By 2.B(iv), $\operatorname{tp}_{L(T)}(\bar{\tau}(\bar{a}_{\bar{\eta}}, \bar{a}_{\bar{v}}), EM(S^*, \Phi))$ is finitely satisfiable in $EM({}^{\omega>}2 \cup \{\bar{\eta}[l] : l < \lg(\bar{\eta})\}, \Phi)$. Now apply [ShA1, III, 0.1].

We continue the proof of Theorem 2.

We can choose Δ_n $(n < \omega)$ such that the set $L(T)$ of formulas of T, is $\bigcup_{n<\omega} \Delta_n$, Δ_n finite and increasing, and for $\eta, v \in {}^{\omega}\lambda$

$$\operatorname{tp}_{\Delta_n}(\bar{a}_\eta, EM({}^{\omega>}\lambda, \Phi)) = \operatorname{tp}_{\Delta_n}(\bar{a}_v, EM({}^{\omega>}\lambda, \Phi)) \quad \text{iff} \quad \eta\restriction n = v\restriction n.$$

[Why? Let $\{\psi_l(x, \bar{y}^l) : l < \omega\}$ list the formulas of $L(T)$ with $\bar{y}^l = \langle y_i : i < n_l\rangle$. For each l, for some $k_l < \omega$,

$$\operatorname{tp}_{\psi_l}\left[\bar{a}_\eta, \bigcup_{n<\omega} \bar{a}_{\eta\restriction n}\right] \quad \text{and} \quad \operatorname{tp}_{\psi_l}\left[\bar{a}_\eta, \bigcup_{m<k_l} \bar{a}_{\eta\restriction m}\right]$$

have the same $R(-, \psi_l, 2)$-rank, k_l minimal. (Remember that $\lg(\bar{a}_\eta) = 1$ when $\eta \in {}^{\omega}\lambda$.) Let $\Delta_n = \{\psi_l : l < n, k_l < n\} \cup \{\varphi_n\}$. For some infinite set $W \subseteq \omega$, $\langle \Delta_n : n \in W\rangle$ is strictly increasing, and by renaming, etc., we get the conclusion.]

Next, by induction on n, define $k(n)$, $k_1(n)$ and $g_n : {}^{n}2 \to {}^{k(n)}2$, with $k(n) < k_1(n) < k(n+1) < \omega$, $g_n(\eta\restriction(n-1)) \lhd g_n(\eta)$ (for $\eta \in {}^{n}2$), g_n one-to-one, and $[\eta \in {}^{n}2 \Rightarrow g_{n+1}(\eta^\wedge\langle 0\rangle)\restriction k_1(n) = g_{n+1}(\eta^\wedge\langle 1\rangle)\restriction k_1(n)]$, and (like $(*)$ (2) from the proof of VIII, 1.8) such that:

$(*)$ For a term $\bar{\tau}(\bar{x}_0, \ldots, \bar{x}_{n-1})$ there is $m_{\bar{\tau}}$ such that (suppressing in $\bar{\tau}$ the sequences $\eta \in {}^{\omega>}2$ viewed as part of $\bar{\tau}$) ***if*** $m \geq m_l$, $\bar{\eta}$ a sequence of members of $\operatorname{Range}(g_m)$ (without repetition) of length $n-1$, $n(*) < n$, and similarly $\bar{v}$ and there are sequences $\bar{\eta}_1$ and $\bar{v}_1$ with $\bar{\eta}[l] \lhd \bar{\eta}_1[l] \in {}^{\omega}2$ and $\bar{v}[l] \lhd \bar{v}_1[l] \in {}^{\omega}2$ $[\bar{\eta}[l] = \bar{v}[l_2] \Rightarrow \bar{\eta}_1[l_1] = \bar{v}_1[l_2]]$ and

$$\begin{aligned}&\operatorname{tp}_{\Delta_{k(n)}}(\bar{\tau}(\bar{a}_{\bar{\eta}_1}), EM({}^{k_1(n)>}\lambda \cup \{\bar{\eta}_1[l] : l < n(*)\} \cup \{\bar{v}_1[l] : l < n(*)\}, \Phi))\\ &\qquad \neq \operatorname{tp}_{\Delta_{k(n)}}(\bar{\tau}(\bar{a}_{\bar{v}_1}), EM({}^{k_1(n)>}\lambda \cup \{\bar{\eta}_1[l] : l < n(*)\} \cup \{\bar{v}_1[l] : l < n(*)\}, \Phi)),\end{aligned}$$

then, for any such $\bar{\eta}_1$ and $\bar{v}_1$,

$$\begin{aligned}&\operatorname{tp}_{\Delta_{k(n)}}(\bar{\tau}(\bar{a}_{\bar{\eta}_1}), EM({}^{k_1(n)\geq}2 \cup \{\bar{\eta}_1[l], v_1[l] : l < n(*)\}, \Phi))\\ &\qquad \neq \operatorname{tp}_{\Delta_{k(n)}}(\bar{\tau}(\bar{a}_{\bar{v}_1}), EM({}^{k_1(n)\geq}2 \cup \{\bar{\eta}_1[l], \bar{v}_1[l] : l < n(*)\}, \Phi)).\end{aligned}$$

Now we define $W(*)$, W_η, U_η, and S_ζ as in the proof of Theorem 1. Then for $u \subseteq \lambda$ let M^1_u be the Skolem hull of

$$J_u = \{\bar{a}_\eta : \eta \in {}^{\omega>}2\} \cup \left\{a_\eta : \eta \in \bigcup_{\zeta \in u} S_\zeta\right\},$$

and M_u its L-reduct. It suffices to get a contradiction from the following: f is an elementary embedding of $M_{u(1)}$ into $M_{u(2)}$, where $u(1), u(2) \subseteq \lambda$ and $\xi \in u(1) - u(2)$, $|u(1)| = |u(2)| = \lambda$. For $\eta \in S_\xi$, let $f(\bar{a}_v) = \bar{\tau}_v(\bar{a}_{\bar{\eta}_{\bar{v},0}})$ and $\bar{\eta}_{v,0} \in {}^{\omega>}(J_{u(2)})$. By the Δ-system lemma there are $S^1 \subseteq S_\xi$ of cardinality $|T_1|^+$ and $n(0)$, w, $\bar{\tau}$, $\bar{\eta}$ such that

(a) for $v \in S^1$, $\bar{\tau}_v = \bar{\tau}$ and $\lg(\eta_{v,0}) = n(0)$;

(b) for $l \in w$ and $\nu \in S^1$, $\bar{\eta}_{\nu,0}[l] = \bar{\eta}[l]$; and

(c) for $l < n(0)$, $l \notin w$, the set $\{\bar{\eta}_{\nu,0}[l]: \nu \in S^1\}$ has no repetitions and is disjoint to

$$\{\bar{\eta}_{\nu,0}[l']: \nu \in S^1, l' < n(*), l' \neq l\},$$

[since, without loss of generality, $\bar{\eta}_{\nu,0}$ can be taken to have no repetitions].

By renaming, we can assume that $\bar{\eta}$, $\bar{\eta}_\nu$ $(\nu \in S^1)$ are pairwise disjoint sequences from ${}^{\omega>}(J_{u(2)})$, and for $\nu \in S^1$

$$f(\bar{a}_\nu) = \bar{\tau}(\bar{a}_{\bar{\eta}_\nu}, \bar{a}_{\bar{\eta}}).$$

Let $S^* \subseteq {}^{\omega>}2 \bigcup_{\zeta \in u(2)} S_\zeta$ be a set of cardinality $\leq |T_1|$ such that ${}^{\omega>}2 \subseteq S^*$, $\bar{\eta} \in {}^{\omega>}(S^*)$ and $f(M_\varnothing) \subseteq EM(S^*, \Phi)$. Now $\{\bar{a}_\eta: \eta \in S^1\}$ is independent over $M({}^{\omega>}2)$ in $M_{u(2)}$; hence $\{f(\bar{a}_\eta): \eta \in S^1\}$ is independent over $S^2 \subseteq S^1, |S^2| > |T_1|$, and $\{f(\bar{a}_\eta): \eta \in S^2\}$ is independent over $(EM(S^*, \Phi), f(M({}^{\omega>}2)))$ in $M_{u(2)}$. Also, without loss of generality, for $\nu \in S^2$, $\bar{\eta}_\nu$ is disjoint to S^*. So by 2.B(v), $\{f(\bar{a}_\eta): \eta \in S^2\}$ is independent over

$$(EM(S^*, \Phi), EM({}^{\omega>}2 \cup \{\bar{\eta}[l]: l < \lg(\bar{\eta})\}, \Phi)).$$

As those sets are models:

(**) For $\eta_1, \eta_2 \in S^2$ and $n < \omega$ we have $\eta_1 \restriction n = \eta_2 \restriction n$ iff

$$\begin{aligned} &\operatorname{tp}_{\Delta_n}(\bar{\tau}(\bar{a}_{\bar{\nu}_{\eta_1}}, \bar{a}_{\bar{\eta}}), EM({}^{\omega>}2 \cup \{\bar{\eta}[l]: l < \lg(\bar{\eta})\}, \Phi)) \\ &\quad = \operatorname{tp}_{\Delta_n}(\bar{\tau}(\bar{a}_{\bar{\nu}_{\eta_2}}), EM({}^{\omega>}2 \cup \{\bar{\eta}[l]: l < \lg(\bar{\eta})\}, \Phi)). \end{aligned}$$

The rest is as in the proof of Theorem 1 (using (*) instead of 1.A(2)).

Fact 3. *If $T \subseteq T_1$, T complete, and $|D(T)| > |T_1|^+$, then, for every $\lambda \geq |T_1|$, $IE(\lambda, T_1, T) \geq |D(T)|^{\aleph_0}$.*

REMARK 3.A. Of course, if $|D(T)| = |T_1|^+$, still $I(\lambda, T_1, T) \geq |D(T)|$ for $\lambda \geq |T_1|$.

Proof. Let M be a model of T_1, let $\bar{a}_i \in |M|$ (for $i < |D(T)|$) realize distinct types from $D(T)$, and $\{z_i: i < \lambda\}$ an indiscernible sequence over $\{\bar{a}_i: i < |D(T)|\}$ that is not trivial (i.e. $z_0 \neq z_1$; and without loss of generality T_1 has Skolem functions, of course). For $w \subseteq |D(T)|$, let M^1_w be the Skolem hull of $\{\bar{a}_i: i \in w\} \cup \{z_i: i < \lambda\}$. For w a subset of $|D(T)|$, let D_w be the set of types $p \in D(T)$ realized in M^1_w, i.e. $D_w = \{\operatorname{tp}_{L(T)}(\bar{b}, \varnothing): \bar{b} \in M_w\}$. So it suffices to find a family $\{w_i: i < |D(T)|^{\aleph_0}\}$ of subsets of $|D(T)|$, $|w_i| \leq \lambda$, and $D_{w_i} \nsubseteq D_{w_j}$ for $i \neq j$ (equivalently, for some $\alpha \in w_i$, $\operatorname{tp}_L(\bar{a}_\alpha, \varnothing) \notin D_{w_j}$). As $|D_w| \leq |T_1| + |w|$, if $|D(T)|^{\aleph_0} = |D(T)|$ this follows by Hajnal's free subset theorem. So we assume $|D(T)|^{\aleph_0} > |D(T)|$. We can choose cardinals μ_n such that $\prod_{n<\omega} \mu_n = |D(T)|^{\aleph_0}$, $|T_1|^+ < \mu_n \leq \mu_{n+1} \leq |D(T)|$, and each μ_n is regular. [If $|D(T)| \leq |T_1|^{\aleph_0}$ let $\mu_n = |T_1|^{++}$; as $|D(T)| > |T_1|^+$, clearly $\mu_n \leq |D(T)|$, and the rest is easy. If $|D(T)| \geq |T_1|^{\aleph_0}$ (hence $|D(T)| > |T_1|^{\aleph_0}$), we let $\mu = \operatorname{Min}\{\kappa: \kappa^{\aleph_0} > |D(T)|\}$, so clearly $\mu^{\aleph_0} = |D(T)|^{\aleph_0}$ and $\mu > |T_1|^{\aleph_0}$ and $(\forall \kappa < \mu)\kappa^{\aleph_0} < \mu$, and $\operatorname{cf}(\mu) = \aleph_0$; now we can choose $\mu_n < \mu$ as required.]

Let E_n be the filter on μ_n generated by the closed unbounded subsets of μ_n and the set $\{\delta < \mu_n: \operatorname{cf} \delta > |T_1|\}$. For $\alpha \leq \omega$ and $\eta \in \prod_{n<\alpha} \mu_n$ let $D_\eta \stackrel{\text{def}}{=} D_{\{\eta(i):\, i < \lg(\eta)\}}$ and $M^1_\eta \stackrel{\text{def}}{=} M^1_{\{\eta(i):\, i<\lg(\eta)\}}$. Let $S_0 = \bigcup_{m<\omega} \prod_{i<m} \mu_i$. By induction on $n < \omega$ we now define S_n such that:

(i) $S_{n+1} \subseteq S_n$;

(ii) $\langle\ \rangle \in S_n$;

(iii) $\{\eta \in S_n: \lg(\eta) \leq n\} = \{\eta \in S_{n+1}: \lg(\eta) \leq n\}$;

(iv) S_n is closed under initial segments;

(v) for $\eta \in S_n$, $\{\gamma : \eta{}^\wedge\langle\gamma\rangle \in S_n\} \neq \emptyset \bmod E_{\lg(\eta)}$;
(vi) if $\eta \in S_n$, $\lg(\eta) = n$, $\eta{}^\wedge\langle\gamma\rangle \in S_{n+1}$, and $\mathrm{cf}(\gamma) > |T_1|$, then, for some $\alpha_{\eta,\gamma} < \gamma$,

$$(\forall \nu \in S_{n+1})[\nu \upharpoonright (n+1) = \eta{}^\wedge\langle\gamma\rangle \wedge \alpha_{\eta,\gamma} < \beta < \gamma \Rightarrow \mathrm{tp}_L(\bar{a}_\beta, \emptyset) \notin D_\nu];$$

(vii) if $\eta{}^\wedge\langle\gamma_1\rangle$ and $\eta{}^\wedge\langle\gamma_2\rangle \in S_{n+1}$, then $\alpha_{\eta,\gamma_1} = \alpha_{\eta,\gamma_2}$, and hence $\gamma_1 > \alpha_{\eta,\gamma_2}$.

There is no problem in the definition. For $n = 0$, S_0 is given, (i), (iii), (vi), and (vii) require nothing, and (ii), (iv), and (v) are easily checked.

For $n+1$, first for each $\eta{}^\wedge\langle\gamma\rangle \in S_n$, $\lg(\eta) = n$, $\mathrm{cf}(\gamma) > |T_1|$, we choose $S_{\eta,\gamma} \subseteq \{\nu \in S_n : \nu, \eta{}^\wedge\langle\gamma\rangle$ are comparable$\}$ satisfying:
(a) $S_{\eta,\gamma}$ is closed under nonempty initial segments;
(b) if $\nu \in S_{\eta,\gamma}$ and $\lg(\nu) > n$, then

$$\{\beta : \eta{}^\wedge\langle\beta\rangle \in S_{\eta,\gamma}\} \neq \emptyset \mod E_{\lg(\nu)};$$

(c) for some $\alpha_{\eta,\gamma} < \gamma$, for every $\nu \in S_{\eta,\gamma}$ we have

$$[\alpha_{\eta,\gamma} < \beta < \gamma \Rightarrow \mathrm{tp}_L(\bar{a}_\beta, \emptyset) \notin D_\nu].$$

For each ν there is such an $\alpha_{\eta,\nu}$ (as $|D_\nu| \leq |T_1|$, $\mathrm{cf}(\gamma) > |T_1|$), and any $\alpha'_{\eta,\gamma}$, $\alpha_{\eta,\gamma} \leq \alpha'_{\eta,\gamma} < \gamma$, can serve. As $\mathrm{cf}(\gamma) > |T_1| \geq \aleph_0$, by Rubin and Shelah [RSh117] (or see [ShA2, Chapter XI, Lemma 3.5, p. 362]) $\alpha'_{\eta,\gamma} S_{\eta,\gamma}$ exists.

Now as $E_{\lg(\eta)}$ is normal, for some α_η and $A_\eta \subseteq \{\gamma : \eta{}^\wedge\langle\gamma\rangle \in S_n\}$ we have $A_\eta \neq \emptyset \bmod E_{\lg(\eta)}$ and $(\forall \gamma \in A_\eta)\alpha_{\eta,\gamma} = \alpha_\eta$. We let

$$S_{n+1} = \bigcup\{S_{\eta,\gamma} : \eta \in S_n, \lg(\eta) = n, \gamma \in A_\eta\}.$$

Clearly (i)–(vii) hold.

So we have carried the induction on n. Let $S_\omega = \bigcap_{n<\omega} S_n$. Clearly S_ω satisfies (i) (i.e. $S_\omega \subseteq S_n$), (ii), (iv), (v), (vi)$'$, and (vii).

(vi)$'$ If $\eta \in S_\omega$, $\lg(\eta) = n$, and $\eta{}^\wedge\langle\gamma\rangle \in S_\omega$, then $\mathrm{cf}(\gamma) > |T_1|$ and, for some $\alpha_{\eta,\gamma} < \gamma$,

$$(\forall \nu \in S_\omega)[\nu \upharpoonright (n+1) = \eta{}^\wedge\langle\gamma\rangle \wedge \alpha_{\eta,\gamma} < \beta < \gamma \Rightarrow \mathrm{tp}_L(\bar{a}_\beta, \emptyset) \notin D_\nu].$$

Let $\mathrm{Lim}\, S_\omega = \{\eta : \lg(\eta) = \omega, \eta \upharpoonright n \in S_\omega$ for $n < \omega\}$. Clearly

$$|\mathrm{Lim}\, S_\omega| = \prod_n \mu_n = |D(T)|^{\aleph_0},$$

and $\{M^1_\eta : \eta \in \mathrm{Lim}\, S_\omega\}$ are as required. Alternatively: let F_0 be a one-to-one function from $\Gamma = \bigcup_n \prod_{m<n} \mu_m$ to $|D(T)|$, and for $\eta \in \Gamma$ let

$$F_1(\eta) = \{\alpha : a_\alpha \text{ belong to the Skolem hull of } \{F_0(\eta \upharpoonright l) : l \leq \lg(\eta)\} \cup \{d_n : n < \omega\}\},$$
$$F_2(\eta) = \{F_1(\eta) - F_0(\eta \upharpoonright l) : l \leq \lg(\eta)\}.$$

By [RSh, Theorem 2] there is a $\Gamma_1 \leq \Gamma$ such that, for $\alpha, \eta \neq \nu \in \Gamma_1$, $\eta \notin F_2(\nu)$. So we can easily finish.

M is called an *algebra* if $L(M)$ has no predicates (only functions).

REMARK 3.B. Also, if $|D(T)| > (|T_1|^\theta)^+$ and $\theta \leq |T_1|$, then $IE(\lambda, T_1, T) \geq |D(T)|^\theta$.

In fact, as above it suffices to prove (use with $\mu = |T_1|$ and $|D(T)| = \|M\|$):

$(*)$ ***If*** M is an algebra with μ functions, $\mu \geq \theta$ and $\|M\| > (\mu^\theta)^+$, ***then*** M has $\geq \|M\|^\theta$ subalgebras, no one a subalgebra of another.

Clearly $(*)$ holds: let $\{\langle a^\alpha_i : i < i_\alpha \leq \theta\rangle : \alpha < \|M\|^\theta\}$ list the sequences of length $\leq \theta$ from $|M|$, let N_α be the subalgebra of M which $\{a^\alpha_i : i < i_\alpha\}$ generates, and let $F(\alpha)$

$= \{\beta: \beta \neq \alpha, N_\beta \subseteq N_\alpha\}$; then $|N_\alpha| \leq \mu + \theta = \mu$, and now for each α

$$|\{\beta: N_\beta \subseteq N_\alpha\}| < (\mu^\theta)^+ < \|M\|,$$

so by the Hajnal free subset theorem we finish.

* * *

Now we turn to consistency results.

LEMMA 4. *Let $\kappa = \kappa^{<\kappa}$, $\lambda = \lambda^\kappa$, N^0 an algebra with universe λ and $\leq \kappa^+$ functions, $N^1 \subseteq N^0$ a subalgebra, and $|N^0 - N^1| = \lambda$. Then for some κ-complete forcing Q of power λ, satisfying the κ^+-c.c. (hence not collapsing cardinals nor changing cofinalities) it is forced that:*

For some complete $T \subseteq T_1$, $|T| = \kappa$, $|T_1| = \kappa^+$, $\lambda = |D(T)|$, T superstable, and for some algebra N^* with universe λ the following holds:

(*) *There is a function H from $PC(T_1, T)$ onto $\{M: M \subseteq N^*\}$ (in the universe after the forcing) such that, for $M_1, M_2 \in PC(T_1, T)$:*

(i) *$M_1 \cong M_2$ iff $H(M_1) = H(M_2) \wedge \|M_1\| = \|M_2\|$;*

(ii) *M_1 is elementarily embeddable into M_2 iff $H(M_1) \subseteq H(M_2) \wedge \|M_1\| \leq \|M_2\|$;*

(iii) (a) *$\{|N|: N \subseteq N^*\} \subseteq \{|N|: N \subseteq N^0\}$ and*

(b) *$\{|N|: N \subseteq N^*$ and $|N| \subseteq N^1\} = \{|N|: N \subseteq N^1\}$; and*

(iv) *Suppose for simplicity that there is no Erdös cardinal in K (the core model; this holds if $\neg 0^\#$). Then any $N \subseteq N^*$, $|N| \nsubseteq |N^1|$ (in V^Q), is the union of $\aleph_0$ submodels of N^1 from V (and if N^0 contains the functions definable in $(K_\lambda, \in)$, this holds for $N \subseteq N^1$ too) [and if $\kappa > \aleph_0$ the union of $\aleph_0$ models from V is from V].*

REMARKS. 1) Part (iv) of (*) is usually not used, e.g. in Conclusion 6 (below); in this case one can discard part of the proof.

2) Our aim is to show that $I(\lambda, T_1, T)$ and $IE(\lambda, T_1, T)$ may be small, even though $|D(T)| > |T_1|$. This lemma produces appropriate T and T_1 (in an appropriate extension of our universe V) such that we have a strong a priori control over the set of isomorphism types of models in $PC(T_1, T)$: they correspond to submodels of N^* which necessarily are submodels of N^0.

Note that (iii) of (*) gives, e.g. upper ((a)) and lower ((b)) bounds to $I(\lambda, T_1, T)$, which in applications coincide.

3) If we assume only "in the core model K there is no λ, $\lambda \to (\mu)_2^{<\omega}$", then in (iv) of (*) we should replace "$\aleph_0$-models" by "$(<\mu)$-models".

PROOF. Let $J = {}^{\kappa>}2$ in the sense of V and $\mu = \lambda$ (just to later simplify reading this proof as a proof of Lemma 7). So N^0 (and N^*) have universe μ. For $I \subseteq {}^{\kappa>}2$ let

$$\mathrm{Br}_\alpha(I) = \{\eta \in {}^\alpha 2: (\forall \beta < \alpha)\eta \restriction \beta \in I\}$$

(so it depends on the universe of set theory). Let $L(T) = \{P_\eta: \eta \in J\}$, P_η a monadic predicate, and define

$$\begin{aligned} T_0 = {} & \{(\forall x)P_{\langle\ \rangle}(x)\} \cup \{(\forall x)[P_\eta(x) \to P_\nu(x)]: \nu \lessdot \eta \in J\} \\ & \cup \{\neg(\exists x)[P_\eta(x) \wedge P_\nu(x)]: \eta \neq \nu \text{ are } \lessdot\text{-incomparable}\} \\ & \cup \{(\forall x)[P_\eta(x) \equiv P_{\eta\frown\langle 0\rangle}(x) \vee P_{\eta\frown\langle 1\rangle}(x)]: \eta \in J\} \\ & \cup \{(\exists x)P_\eta(x): \eta \in J\}. \end{aligned}$$

T_0 is a complete theory with elimination of quantifiers.

In the new universe $V^1 = V^Q$ we shall define a family $\mathscr{P}$; each $f \in \mathscr{P}$ is a partial $n(f)$-place function from ${}^\kappa 2$ to ${}^\kappa 2$ such that:

$(*)_1$ ***for every*** $f \in \mathscr{P}$ ($n(f)$-place) and $\eta, \eta_1, \ldots, \eta_{n(f)} \in {}^\kappa 2$ such that $\eta = f(\eta_1, \ldots, \eta_{n(f)})$ and every $\alpha < \kappa$ ***there exists*** β, $\alpha < \beta < \kappa$, such that ***if*** $\nu_1, \ldots, \nu_{n(f)} \in {}^\kappa 2$ and for every $1 \le l \le n(f)$ we have $\nu_l \restriction \beta = \eta_l \restriction \beta$ and $f(\nu_1, \ldots, \nu_{n(f)})$ is defined, ***then***

$$f(\eta_1, \ldots, \eta_{n(f)}) \restriction \alpha = f(\nu_1, \ldots, \nu_{n(f)}) \restriction \alpha.$$

We then let $T_1 = T_0 \cup T_{1,0} \cup T_{1,1} \cup T_{1,2}$, where:

$$T_{1,0} = \{(\forall xy)[P_\eta(x) \equiv P_\eta(g(x,y))]: \eta \in J\} \cup \{(\forall x, y_1, y_2)[y_1 \ne y_2 \to g(x, y_1) \ne g(x, y_2)]\},$$

$$T_{1,1} = \Bigg\{(\forall x_1, \ldots, x_n)\left[\bigwedge_{i=1}^{n} P_{\rho_i}(x_i) \to P_\rho(f(x_1, \ldots, x_n))\right]:$$

$$\rho_1, \ldots, \rho_n, \rho \in J,\ f \in \mathscr{P},\ n = n(f) \text{ for some } \eta_1, \ldots, \eta_n \in {}^\kappa 2:$$

$$\rho \lessdot f(\eta_1, \ldots, \eta_n), \text{ and}$$

$$(\forall \eta_1, \ldots, \eta_{n(f)})\left[\bigwedge_{i=1}^{n} \rho_i \lessdot \eta_i \in {}^\kappa 2 \Rightarrow \rho \lessdot f(\eta_1, \ldots, \eta_n) \text{ (if defined)}\right]\Bigg\},$$

$$T_{1,2} = \{P_{\eta\restriction\beta}(c_\eta): \eta \text{ a sequence of zeros and ones of length a limit ordinal} < \kappa,\ (\forall \alpha < \lg(\eta))[\eta \restriction \alpha \in J] \text{ and } \beta < \lg(\eta)\}$$

(note that $T_{1,2}$ can be waived now, but will be used in some later variations).

Now L_1 is the vocabulary of T_1, and $T = T_1 \cap L$. Clearly T is equal to T_0, T superstable, and $|D(T)|$ will be $\ge |\bigcup_{\delta \le \kappa} \mathrm{Br}_\delta(J)|$ (in fact, equality holds).

In the forcing we shall also construct a function H_a, one-to-one, from $\mathrm{Br}_\kappa(J)$ (in V^1) onto λ such that:

(A) If j is in the subalgebra of N^0 generated by $\{j_1, \ldots, j_n\}$ ($\subseteq \lambda$), then, for some $f \in \mathscr{P}$,

$$H_a^{-1}(j) = f(H_a^{-1}(j_1), \ldots, H_a^{-1}(j_n)).$$

(B) If $j, j_1, \ldots, j_n \in N^1$ (note: N^1, not N^0!) and

$$(\exists f \in \mathscr{P})[H_a^{-1}(j) = f(H_a^{-1}(j_1), \ldots)],$$

then j is in the subalgebra of N^1 generated by $\{j_1, \ldots, j_n\}$.

(C) Definition. Now for $M \in PC(T_1, T)$ and $a \in M$ we can define

$$H_b(a, M) = \bigcup\{\eta \in J: M \models P_\eta(a)\}, \qquad H_c(M) = \{H_b(a, M): a \in M\},$$

$$H(M) = \{H_a(H_b(a, M)): a \in M,\ H_b(a, M) \in \mathrm{Br}_\kappa(J)\}.$$

Note that $H_b(a, M)$ belongs to $\mathrm{Br}(J) = \bigcup_{\alpha \le \kappa} \mathrm{Br}_\alpha(J) - J$, $H_c(M) \subseteq \mathrm{Br}(J)$, and $H(M) \subseteq |N^0| = \mu$.

We have to define the model N^*.

(D) Definition. For $f \in \mathscr{P}$, we define F_f^* as a complete $n(f)$-place function from μ to μ:

$$F_f^*(j_1, \ldots, j_{n(f)}) = \begin{cases} H_a(f(H_a^{-1}(j_1), \ldots, H_a^{-1}(j_{n(f)}))) & \text{if defined,} \\ j_1 & \text{otherwise.} \end{cases}$$

Lastly, N^* is N^0 expanded by the functions F_f^* $(f \in \mathscr{P})$ (or we could use N^* with partial functions).

Let us check that T, T_1, N^* will be as required.

Note that

$(*)_2$ $H(M)$ is a submodel of N^* of *power* $\leq \|M\|$ when $M \in PC(T_1, T)$. [Why? By (D) and $(*)_1$ and T_{11}, above.]

$(*)_3$ If $N \subseteq N^*$ is a submodel and $\lambda \geq \|N\|$, then for some $M \in PC(T_1, T)$, $\|M\| = \lambda$, $H(M) = N$. [Why? Check.]

Note, however, that we shall use freely

$$(\oplus) \qquad H(M_1) = H(M_2) \quad \text{iff} \quad H_c(M_1) = H_c(M_2).$$

(This follows from $(*)_1$, (A), and (B), as

$$H(M_l) = \{H_a(x) \colon x \in H_c(M_l),\ x \in {}^{\kappa}2\}$$

and if $x \in \bigcup_{\delta<\kappa} \mathrm{Br}_\delta(J) - J$ then always $x \in H_c(M_l)$ (using $T_{1,2}$).)

Now we check $(*)$ of Lemma 4.

Claim 4.A. If in V^Q there are $\mathscr{P}$ and H_a satisfying $(*)_1$, (A), and (B), then $(*)$ of Lemma 4 holds (N^* is defined in (D) and H is defined in (C)) and $\tau_1 = \kappa + |\mathscr{P}|$.

Proof of (i). Let $M_l \in PC(T_1, T)$. The implication "$M_1 \cong M_2 \Rightarrow H(M_1) = H(M_2) \wedge \|M_1\| = \|M_2\|$" is totally trivial. Now suppose $N \overset{\text{def}}{=} H(M_1) = H(M_2)$ and $\|M_1\| = \|M_2\|$. Hence, by $(\oplus)$ above, $H_c(M_1) = H_c(M_2)$; call it Y. Then, for each $l = 1, 2$, $x \in Y$; as $Y = H_c(M_l)$, the set $\{a \in M_l \colon H_b(a, M_l) = x\}$ has cardinality $\|M_l\|$ (use $T_{1,0} \subseteq T_1$, i.e. the function g). As $\|M_1\| = \|M_2\|$, there is a one-to-one function F_x from $A_x^1 = \{a \in M_1 \colon H_b(a, M_1) = x\}$ onto $A_x^2 = \{a \in M_2 \colon H_b(a, M_2) = x\}$.

However, $|M_l|$ is the disjoint union of $\{A_x^l \colon x \in H_c(M_l)\}$, so $F \overset{\text{def}}{=} \bigcup_{x \in Y} F_x$ is a one-to-one function from M_1 onto M_2. It is easy to check that it is an isomorphism.

Proof of (ii). Similar to (i) (remembering T has easy elimination of quantifiers).

Proof of (iii)(a). Suppose $N \subseteq N^*$ (in V^Q). We want to prove that $|N|$ is closed under the functions of N^0. So suppose j is $F(j_1, \ldots, j_n)$, F a function of N^0, and assume that $j_1, \ldots, j_n \in |N|$. By (A) above, for some $f \in \mathscr{P}$, $F_f(j_1, \ldots, j_n) = j$ (see (A) and definition of F_f). As F_f is one of the functions of N^*, and $N \subseteq N^*$, clearly $j \in |N|$ so we are finished.

Proof of (iii)(b). Use (B) above and the choice of N^*.

Proof of (iv). Note that (iv) is a direct consequence of Magidor's covering theorem (by [Mg2]; see the Appendix to the present paper for an explanation) *provided that* we expand N^0 by partial functions (not changing the set of submodels of N^0 included in N^1) such that, for every $c \in N^0 - N^1$, in (N^1, c) there are terms for all functions definable in $(K_\lambda, \in)$. (Remember, if $0^\#$ does not exist then $K = L$.)

The only thing which remains to be done is to force, i.e.

Claim 4.B. *There is a forcing Q satisfying* $(*)_1$, (A), (B), *and* $|\mathscr{P}| = \kappa$.

Proof. Let $\lambda = \bigcup_{i<\kappa^+} A_i$, the A_i's being pairwise disjoint, with $|A_i| = \lambda$ for $i \geq 0$, such that $|N^1| \subseteq A_0$ (remember, we assume in Lemma 4 that $|N^0 - N^1| = \lambda$). The forcing Q is P_{κ^+}, where $\langle P_i, \mathbf{Q}_j \colon i \leq \kappa^+, j < \kappa^+ \rangle$ is a $(<\kappa)$-support iteration.

We let Q_0 be $\{f \colon f$ a partial function from λ to $\{0, 1\}$, $|\mathrm{Dom}\, f| < \kappa\}$, ordered by inclusion. Clearly in V^{Q_0} the cardinality of $\mathrm{Br}_\kappa(J)^{V^{Q_0}}$ is λ (remember that $\lambda \neq \lambda^\kappa$), so let $\langle B_i \colon i < \lambda \rangle$ be a partition of it into λ pairwise disjoint sets each of cardinality λ.

Next we define (in V^{Q_0}) H_0, a one-to-one function from B_0 onto A_0.

We now define, by induction on i, a function H_i and $Q_i \in V^{P_i}$ such that:

(α) $H_i \in V^{P_i}$ is a one-to-one function such that

$$\operatorname{Range} H_i = \bigcup_{j \le i} A_j, \qquad \bigcup_{j<i} \operatorname{Br}_\kappa(J)^{V^{P_j}} - \bigcup_{i<j<\lambda} B_j = \operatorname{Dom} H_i$$

and H_i extends H_γ when $0 \le \gamma < i$ [in our present proof we can somewhat simplify].

Let $N^0 = (|N^0|, F_\beta)_{\beta<\beta(*)}$, where $\beta(*) \le \kappa^+$. Let F_β be an n_β-place function, and let $F_{\beta,i}$ be the partial n_β-place function from $\operatorname{Br}_\kappa(J)^{V^{P_i}}$ into itself, defined by

$(*)_4$ $F_{\beta,i}(\nu_1, \dots, \nu_{n_\beta}) = \nu$ iff

$$\nu_1, \dots, \nu_{n_\beta}, \nu \in \operatorname{Dom} H_i \quad \text{and} \quad F_\beta(H_i(\nu_1), \dots, H_i(\nu_{n_\beta})) = H_i(\nu).$$

Now we have

(β) For some $\beta_i = \beta(i) < \beta(*)$, Q_i is the family of pairs $(f, \bar{g})$, where f is a partial function from

$$\{(\nu_1, \dots, \nu_{n(\beta)}, \nu) \colon F_{\beta_i,i}(\nu_1, \dots, \nu_{n(\beta)}) = \nu\}$$

into κ, $\bar{g} = \langle g_\zeta \colon \zeta < \zeta(0)\rangle$, $\zeta(0) < \kappa$, and g_ζ is an $n(\beta_i)$-place function from J to J (remember that $J = ({}^{\kappa>}2)^V$) such that:

(a) each of $f, g_\zeta (\zeta < \zeta(0))$ has domain of cardinality $<\kappa$;

(b) ***if*** $f(\nu_1, \dots, \nu_{n(\beta(i))}, \nu) = \zeta$, $g_\zeta(\rho_1, \dots, \rho_{n(\beta(i))}) = \rho$, and $\bigwedge_{l=1}^{n(\beta(i))} \rho_1 <'' \nu_l$, ***then*** $\rho \lhd \nu$, and

(c) suppose $g_\zeta(\rho_1^l, \dots, \rho_{n(\beta(i))}^l) = \rho^l$ for $l = 1, 2$; then

(i) $\lg(\rho_1^l) = \lg(\rho_2^l) = \cdots = \lg(\rho_{n(\beta(i))}) = \lg(\rho^l)$, and

(ii) if $\bigwedge_{i=1}^{n(\beta(i))} \rho_i^1 \lhd \rho_i^2$, then $\rho^1 \lhd \rho^2$.

The order is natural.

There is no problem to carrying out the definition, and each Q_i is κ-complete and satisfies the κ^+-c.c. in a strong sense, e.g. $(*)$ of [Sh80, p. 297] with κ replacing $\aleph_1$.

Clearly for $i \le \kappa^+$, P_i is κ-complete; it also satisfies the κ^+-c.c. (prove directly, or quote [Sh80] or the proof of Baumgartner's axiom). Hence

$$\operatorname{Br}_\kappa(J)^{V^{P_{\kappa^+}}} = \bigcup_{i<\kappa^+} \operatorname{Br}_\kappa(J)^{V^{P_i}}.$$

So $H_a \stackrel{\text{def}}{=} \bigcup_{i<\kappa^+} H_i$ is a one-to-one function from $\operatorname{Br}_\kappa(J)^{V^{P_{\kappa^+}}}$ onto λ.

By any reasonable bookkeeping we can choose the β_i ($i < \kappa^+$) such that, for every $\beta < \beta(*)$, for κ^+ ordinals i, $\beta_i = \beta$.

We now define for each $i < \kappa^+$ and $\zeta < \kappa$ a partial $n(\beta_i)$-place function $F_i^{0,\zeta}$, in V^{P_κ}, from $\operatorname{Br}(J)^{V^{P_\kappa}}$ to $\operatorname{Br}(J)^{V^{P_\kappa}}$: $F_i^{0,\zeta}(\nu_1, \dots, \nu_{n(\beta(i))}) = \nu$ ***iff*** $\nu_1, \dots, \nu_{n(\beta(i))}, \nu \in \operatorname{Br}_\kappa(J)^{V^{P_\kappa}}$ and for every $\alpha < \kappa$ for some $\beta < \kappa$ and $(f, \bar{g}) \in G_{Q_i}$, $\nu \restriction \alpha$ is an initial segment of $\bar{g}_\zeta(\nu_1 \restriction \beta, \dots, \nu_{n(\beta(i))} \restriction \beta)$.

Let F_i^ζ be a (complete) $n(\beta_i)$-function from $\operatorname{Br}(J)^{V^{P_\kappa}}$ to $\operatorname{Br}(J)^{V^{P_\kappa}}$:

$$F_i^\zeta(\nu_1, \dots, \nu_{n(\beta(i))}) = \begin{cases} F_i^{0,\zeta}(\nu_1, \dots, \nu_{n(\beta(i))}) & \text{if defined,} \\ \nu_1 & \text{otherwise.} \end{cases}$$

It is easy to see that F_i^ζ is well defined, $F_i^{0,\zeta}$ satisfies the requirement $(*)_1$ (on members of $\mathscr{P}$), and that (see $(*)_4$ for the definition of $F_{\beta_i,i}$):

$(*)_5$ $F_{\beta_i,i}(\nu_1, \dots, \nu_{n(\beta(i))}) = \nu \Rightarrow \bigvee_{\zeta<\kappa} F_i^\zeta(\nu_1, \dots, \nu_{n(\beta(i))}) = \nu$ and

$(*)_6$ if $F_i^{0,\zeta}(v_1, \ldots, v_{n(\beta(i))}) = v$ and $v_1, \ldots, v_{n(v(i))} \in \mathrm{Br}_\kappa(J)^{VP_i}$ then

$$F_{\beta_i, i}(v_1, \ldots, v_{n(\beta(i))}) = v \quad \text{and} \quad (v \in \mathrm{Br}_\kappa(J)^{VP_i})$$

(by the genericity of $\mathbf{G}_{Q_i}$).

We let $\mathscr{P} = \{F_i^{0,\zeta}: i < \kappa^+, \zeta < \kappa\}$. So demand (A) holds by $(*)_5$. It is easy to check condition (B) too (remember that $|N^1| \subseteq B_0$) (the values of F_i^ζ not in $F_{\beta_i, i}$ give as a result a new branch).

So we finish the proof of Lemma 4.

Claim 5. (1) *If R is a forcing notion (in V^Q) which does not increase* $\mathrm{Br}_\kappa(J)$, *and*

$$\Vdash_R \text{“}\left|\bigcup_{\alpha<\kappa} \mathrm{Br}_\alpha(J)\right| \leq \kappa^+\text{”},$$

then the conclusion of 4 *still holds in V^{Q*R} (and only $T_{1,2}$ increases).*

(2) *If, for example, $R = R_0 * R_1$, R_0 is adding $\aleph_1$ Cohen reals, and R_1 is $\aleph_1$-complete, the assumption of* 5(1) *holds.*

REMARK 5.A. This claim enables us to get various situations. For example, start with $V = L$; we choose $\kappa = \aleph_{14}$ and $\lambda = \aleph_{25}$, we force as in Lemma 4, and then we let (in V^Q) R_0 be adding $\aleph_1$ Cohen reals, and R_1 be (in V^{Q*R_0}) adding $\aleph_{32}$ subsets to $\aleph_3$:

$$R_1 = \{f: f \text{ a partial function from } \aleph_{32} \text{ to } \{0,1\} \text{ of cardinality } < \aleph_3\},$$

ordered by inclusion. Now in $V_1 = ((V^Q)^{R_0})^{R_1}$ no cardinal of L is collapsed, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ if $\alpha < 3$ or $\alpha \geq 31$, $2^{\aleph_\alpha} = \aleph_{32}$ if $3 \leq \alpha \leq 31$ and for some complete first order $T \subseteq T_1$, T is superstable,

$$|T| = \aleph_{14}, \qquad |T_1| = \aleph_{15}, \qquad |D(T)| = \aleph_{25}.$$

Assume we have started with N^0 equal to some $(\lambda, f_n, \alpha)_{\alpha<\kappa^+, n<\omega}$, where $\{f_n: n < \omega\}$ is a list of the functions with finite arity, from λ $(=\aleph_{25})$ to λ, definable in $(L_{\lambda^+}, \in)$, and N^1 is the closure of κ^+ under f_n $(n < \omega)$. Then $\aleph_{15} \leq \lambda \Rightarrow I(\lambda, T_1, T) = \aleph_{25}$ (remember N^0 in L is a Jónsson algebra, and the forcing adds no new subalgebra (see 4$(*)$(iv) and its proof).

Of course also $IE(\lambda, T_1, T) = \aleph_{25}$ (see Fact 3). If we had started with $\lambda = \aleph_{16}$, we would have gotten $IE(\lambda, T_1, T) = 1$.

Proof of Claim 5. (1) Look at the proof of 4 and notice that in our proof of (i)–(iv) of $(*)$ of 4, all that we used holds in V^{Q*R}; i.e. in the proof of (i)–(iv) of $(*)$ of 4 we need only $\oplus$, which holds by our assumptions (using $T_{1,2} \subseteq T_1$).

2) Well known.

Conclusion 6. *If in Lemma* 4 *(or Claim* 5*) $\lambda = \kappa^{++}$, we can get that $I(\lambda, T_1, T) = \lambda$ and $IE(\lambda, T_1, T) = 1$ (in fact $\{M/\cong: M \in PC(T_1, T), \|M\| = \lambda\}$ is linearly ordered by elementary embeddability, and has order type λ).*

We shall return to the proof shortly.

REMARK 6.A. Really the T and T_1 we got in Conclusion 6 satisfies $IE(T_1, T) = \aleph_0^-$; i.e. there is any finite but no infinite family of models in $PC(T_1, T)$, no one elementarily embeddable into another. (In fact the order type of the class $PC(T_1, T)$ under elementary embeddability is $\lambda \times$ cardinals, i.e. to $\{(i, \theta): i \text{ an ordinal} < \lambda, \theta \text{ a cardinal}\}$, with the partial order: $(i_1, \theta_1) \leq (i_2, \theta_2)$ iff $i_1 \leq i_2$ & $\theta_1 \leq \theta_2$.)

REMARK 6.B. $|D(T)| > |T_1| \Rightarrow IE(T_1, T) \geq \aleph_0^-$.

Proof of 6.B. Let $n < \omega$. We choose by induction on $m \leq n$ a model $M_m \in PC(T_1, T)$ of cardinality $|T_1|^{+(n-m)}$, realizing $\leq |T_1|$ types from $D(T)$, such that one of them, p_m, is not realized in $M_0, \ldots, M_{m-1}$; so $IE(T_1, T) \geq n$ for each n. Moreover, if $|D(T)| > |T_1|$ and $\lambda \leq |D(T)|$, then for $\mu \geq |D(T)|^{+\lambda}$ there are $M_i^\mu \in PC(T_1, T)$, $\|M_i^\mu\| = \mu$ for $i \leq \lambda$, such that $[i \leq j \wedge \mu(1) \leq \mu(2) \Leftrightarrow M_i^{\mu(1)}$ can be elementarily embedded into $M_j^\mu]$. (Use Ehrenfeucht-Mostowski models.)

Proof of 6. We shall apply Lemma 4 with $N^0 = (\lambda, F, i)_{i<\kappa^+}$, F a two-place function from λ to λ such that, for each $\beta < \lambda^+$, $\{\alpha: \alpha < \beta\} \subseteq \{F(\beta, i): i < \kappa^+\}$, and N^1 is N^0 restricted to κ^+ (without loss of generality, F maps κ^+ to κ^+). So we get $T \subseteq T_1$, complete first order theories, with $|T| = \kappa$, $|T_1| = \kappa^+$, and T superstable.

We want to see what is $K = \{M_1 \restriction L: \|M_1\| = \lambda, M_1 \models T_1\}$. We really are interested in isomorphism types, i.e. $K' = \{M/\cong: M \in K\}$; let $M_1 \leq_{\text{em}} M_2$ if M_1 is elementarily embeddable into M_2. Now, by 4(*)(i), (ii), $(K', \leq_{\text{em}})$ is isomorphic to $(\{|N|: N \subseteq N^*\}, \subseteq)$, and, by 4(*)(iii)(a), $\{|N|: N \subseteq N^*\}$ is a subset of $\{\alpha: \alpha \leq \lambda\}$. However we know that it is closed under increasing unions (by the definition of $N \subseteq N^*$); hence, for some closed subset C of λ, $\{|N|: N \subseteq N^*\}$ is $C \cup \{\lambda\}$. By the "elementary submodels existence", C is unbounded in λ. So K' is isomorphic to $(C \cup \{\lambda\}, \leq)$ (where C is a closed unbounded subset of λ), which is isomorphic to $(\lambda + 1, \leq)$. So $I(\lambda, T_1, T) = \lambda$, $IE(\lambda, T_1, T) = 1$, and we can prove the rest as well.

LEMMA 7. *If $\kappa > \aleph_0$ and $\kappa \leq \mu \leq \lambda$, then in Lemma* 4 (*and Claim* 5) [*i.e. we assume κ, λ, and N^1 satisfy the assumptions of* 4], *allowing Q to be only strategically κ-complete, we can have $N^* = N^1 = N^0$ have universe μ, and $|T_1| = \kappa$.*

REMARK 7.A. The main improvement in 7 is "$|T_1| = \kappa$" rather than "$|T_1| = \kappa^+$" from 4. Also we can more easily control $|T|$, $|T_1|$, $|D(T)|$, and 2^κ (and cardinal arithmetic in general).

However, the weakening of κ-completeness means that e.g. preservation of supercompactness is more delicate.

PROOF. First we shall force (by Q_0, defined below) a subtree of ${}^{\kappa>}2$ with μ branches (in the well known way). For $A \subseteq {}^{\kappa>}2$, let us define

$$\delta(A) \stackrel{\text{def}}{=} \bigcup\{\lg(\eta): \eta \in A\}$$

and

$$\begin{aligned} Q_0^a = \{(A, B, Y):\ & A \subseteq {}^{\kappa>}2, B \subseteq {}^{\kappa>}2, A \cap B = \emptyset, |A| < \kappa, |B| < \kappa, \\ & A \text{ closed under initial segments}, \\ & Y \text{ a partial two-place function from } \mu \times \kappa \text{ to } A \text{ of power} < \kappa, \\ & Y(\alpha, \zeta) \lessdot Y(\alpha, \xi) \text{ when (both are defined and) } \zeta < \xi, \\ & Y(\alpha, \zeta) \in {}^{\zeta}2, \text{and if } Y(\alpha, \zeta) \text{ is defined then } Y(\alpha, \xi) \text{ is defined for} \\ & \text{every } \xi \leq \delta(A), \\ & \qquad \eta \in A \Rightarrow \textstyle\bigwedge_{l=0,1} \eta^\frown\langle l\rangle \in A \text{ and } (\forall \eta \in B)\lg(\eta) < \delta(A), \text{ and} \\ & \qquad \forall \eta \in A \exists \nu \in A \Big[\eta \trianglelefteq \nu \wedge \nu \in {}^{\delta(A)}2 \wedge \bigwedge_{\gamma < \delta(A)} \nu \restriction \gamma \in A\Big], \\ & \delta(A) \textit{ is a successor ordinal}\} \end{aligned}$$

and the order of Q_0 is

$$(A_1, B_1, Y_1) \leq (A_2, B_2, Y_2) \quad \text{iff} \quad A_1 \subseteq A_2, B_1 \subseteq B_2, Y_1 \subseteq Y_2, A_1 = A_2 \cap {}^{\delta(A)\geq}2.$$

Let $Q_0^b = \{f: f \text{ a partial function from } \lambda \text{ to } \{0, 1\}, |\text{Dom } f| < \kappa\}$, ordered by inclusion.

It is easy to see that $Q_0 \stackrel{\text{def}}{=} Q_0^a \times Q_0^b$ is strategically κ-complete of power λ, and $\Vdash_Q$"$2^\kappa = \lambda$".

If $G_0^a \subseteq Q_0^a$ is generic over V let

$$J[G_0^a] = \bigcup\{A\colon (\exists B, Y)[(A, B, Y) \in G_0^a]\}.$$

For $\alpha < \mu$ and $\eta \in {}^\kappa 2$ (from $V[G_0^a]$) let $H_0(\eta) = \alpha$ if

$$(\forall \zeta < \kappa)(\exists A, B, Y)[(A, B, Y) \in G_0^a \wedge Y(\alpha, \zeta) = \eta \restriction \zeta].$$

Easily $\operatorname{Dom} H_0 = \operatorname{Br}_\kappa(J[G_0^a])$ in $V[G_0^a]$, and forcing with Q_0^b adds no branch to $J[G_0^a]$, i.e.

$$\operatorname{Br}_\kappa(J[G_0^a])^{V[G_0^a * G_0^b]} = \operatorname{Br}(J[G_0^a])^{V[G_0^a]}.$$

We proceed as in Lemma 4 with $H_a = H_0$. Now P_{κ^+}/P_1 is κ-complete and satisfies the stronger κ^+-c.c.

* * *

The following trivial fact is contained in the proof of 4:

Observation 8. 1) *Suppose I is a tree (i.e., for $x \in I$, $\{y\colon I \models y < x\}$ is well ordered), $\delta \leq \kappa$ a limit ordinal, $|I| = \kappa$, $B_\delta \stackrel{\text{def}}{=} \operatorname{Br}_\delta(I)$ has power μ, $\mu \geq \lambda$, and for every $i < \delta$, $|\operatorname{Br}_i(I)| < \kappa$. Then, for some complete first order T with $|T| = \kappa$, $|D(T)| = |D_1(T)| = \mu$ (in fact, if I is a tree with κ nodes and μ branches,* cf *$\mu > \kappa$, then there is such a T).*

2) *Suppose further that $\mathscr{P}$ is a family of partial functions from $B \stackrel{\text{def}}{=} B_\delta$ to B which are continuous, i.e., if $F \in \mathscr{P}$ is an n-place function, $\eta, \eta_1, \ldots, \eta_n \in B$, $\alpha < \delta$ and $\eta = F(\eta_1, \ldots, \eta_n)$, then, for some $\beta < \delta$,*

$$\left[\nu, \nu_1, \ldots, \nu_n \in B_\delta, \nu = F(\nu_1, \ldots, \nu_n), \bigwedge_{l=1}^{n} \nu_l \restriction \beta = \eta_l \restriction \beta \Rightarrow \nu \restriction \alpha = \eta \restriction \alpha\right].$$

Then, for some first order complete T_1, $T \subseteq T_1$, $|T_1| = \kappa + |\mathscr{P}|$, and for $\chi \geq |T_1|$:

a) *$I(\chi, T_1, T) = \operatorname{Sb}_{\leq\chi}(B, f)_{f \in \mathscr{P}}$, where $\operatorname{Sb}_{\leq\chi}(B, f)_{f \in \mathscr{P}} \stackrel{\text{def}}{=}$ the number of submodels of $(B, f)_{f \in \mathscr{P}}$ (i.e. subsets of B closed under each $f \in \mathscr{P}$) of cardinality $\leq \chi$.*

b) *$IE(\chi, T_1, T) = \operatorname{Sb} E(B, f)_{f \in \mathscr{P}}$, where $\operatorname{Sb} E(B, f)_{f \in \mathscr{P}} \stackrel{\text{def}}{=}$ the maximal number of submodels of $(B, f)_{f \in \mathscr{P}}$ with no one a submodel of another.*

Proof. Read the proof of 4.

Discussion 9. Using Lemmas 4, 5, and 7, we can get many examples contradicting conjectures of the forms, "if $|D(T)| > |T_1|$, then $I(\lambda, T_1, T)$ and $IE(\lambda, T_1, T)$ are large". However they all can be obtained starting with L, hence cannot deal with cases in which cardinal arithmetic contradicts the covering theorem.

Suppose we want $|T| = \kappa$, κ strong limit singular and $|T_1|$ is κ or κ^+. By 10 below (together with 8) we can get some results.

Fact 10. (1) *Suppose $\theta = \operatorname{cf} \kappa > \aleph_0$, $(\forall \sigma < \kappa)[\sigma^{<\theta} < \kappa]$, $2^\theta < \kappa$ and $\kappa < \chi \leq \kappa^\theta$. Then there is a tree I with $|I| = \kappa$, $|\operatorname{Br}_\theta(I)| = \chi$, and $|\operatorname{Br}_\alpha(I)| < \kappa$ for $\alpha < \kappa$.*

(2A) *If $\chi = \kappa^+$ and $\forall \sigma < \kappa[\sigma^\theta < \kappa]$, then, for some $\mathscr{P}$ as in 8(2), $|\mathscr{P}| = \kappa$ and* $\operatorname{Sb} E(B, f)_{f \in \mathscr{P}} = 1$.

(2B) *Suppose that $\kappa < \chi < \kappa^\theta$, χ regular, $\langle \lambda_i\colon i < \theta\rangle$ is increasingly continuous with limit κ, $\chi = \operatorname{cf}(\prod_{i<\theta} \lambda_i^{+\alpha_i})/D$, D a normal filter on κ, $\lambda^{+\alpha_i} < \kappa$ is regular (see* [Sh111] *and* [Sh355] *on existence*: *for each regular such* χ *there are such* D *and* $\lambda_i^{+\alpha(i)}$; *and note that Magidor forcing for failure of* $2^\kappa > \kappa^+$ *gives this for* $\chi = \kappa^{+\alpha}$ *for* $\alpha_i = \alpha$), *and, finally*, $\forall\sigma < \kappa[\sigma^\theta < \kappa]$. ***Then*** *there are* $T \subseteq T_1$, $|T| = |T_1| = \kappa$, $|D(T)| = \chi$, *and there is no* $w \subseteq D(T)$, $|w| = |\alpha|^+$, *and models* M_u^1 *of* T_1 *for* $u \subseteq w$ *such that for* $p \in w$, M_u^1 *realizes* p *iff* $p \in u$.

(2C) *If* κ, $\langle\lambda_i: i < \theta\rangle$, D, χ *are as above, and* $\chi = \kappa^{+\alpha}$ *and* $\aleph_\alpha < \theta = \operatorname{cf}\kappa$, *then, for some* $\mathscr{P}$ *as in* 8(2), $|\mathscr{P}| = \kappa$ *and* $\operatorname{Sb}(B, f)_{f\in\mathscr{P}} = \chi$.

REMARK. These statements are in fact just variants of Galvin and Hajnal's results [GH]. See also [Sh111], [Sh282], [Sh345, §4] and [Sh355].

Proof. (1) Without loss of generality, $\operatorname{cf}\chi > \kappa$. For a filter E on θ and $h \in {}^\theta\mathrm{Ord}$ (i.e., a function from θ to ordinals), let $\mathscr{F}_E(h) = \{\mathscr{P}: \mathscr{P}$ a family of functions from θ to Ord such that $(\forall f \in \mathscr{P}) f <_E h$ (i.e., $\{i < \kappa: f(i) < h(i)\} \in E$), and $(\forall f_1, f_2 \in \mathscr{P})(\forall\alpha < \theta)(f_1(\alpha) = f_2(\alpha) \to f_1 \restriction \alpha = f_2 \restriction \alpha)\}$.

Suppose E is θ-complete extending the filter of all cobounded subsets of θ. There is $h \in {}^\theta\kappa$ and $\mathscr{P} \in F_E(h)$ such that $|\mathscr{P}| = \kappa^\theta$. [If $\kappa = \sum_{i<\theta}\lambda_i$, $\lambda_i < \kappa$, let $h(i) = \prod_{j<i}\lambda_j$ and let H_i be a one-to-one function from $\prod_{j<i}\lambda_j$ onto $|\prod_{j<i}\lambda_j|$. Now for $\eta \in \prod_{i<\theta}\lambda_i$ let $f_\eta: \theta \to \kappa$ be defined by $f_\eta(i) = H_i(\eta \restriction i)$.] So there are $h^* \in {}^\theta\kappa$ and $\mathscr{P}^* \in \mathscr{F}_E(h^*)$, $|\mathscr{P}^*| \geq \chi$, such that h^* is $<_E$-minimal with this property, i.e.,

$$(\forall h)[h <_E h^* \wedge \mathscr{P} \in \mathscr{F}_E(h) \to |\mathscr{P}| < \chi]$$

(this holds since $<_E$ is well founded, because E is θ-complete hence $\aleph_1$-complete).

As $\operatorname{cf}\chi > \kappa \geq 2^\theta$, there is $A \in E$ such that $\mathscr{P}' = \{f \in \mathscr{P}^*: (\forall\alpha \in A) f(\alpha) < h^*(\alpha)\}$ has power χ. Let $\mathscr{P}$ be a maximal family satisfying (for our already chosen h^* and A):

(a) $\mathscr{P}' \subseteq \mathscr{P} \subseteq {}^\theta\kappa$;

(b) $(\forall f \in \mathscr{P})(\forall\alpha \in A) f(\alpha) < h^*(\alpha)$;

(c) if $f_1, f_2 \in \mathscr{P}$, $\alpha < \theta$, and $f_1(\alpha) = f_2(\alpha)$, then $f_1 \restriction \alpha = f_2 \restriction \alpha$.

As $\mathscr{P}'$ satisfies (a), (b), (c), and as the conditions are finitary, there is such a maximal $\mathscr{P}$.

Now $|\mathscr{P}| = \chi$ (by (a), $|\mathscr{P}| \geq |\mathscr{P}'| = \chi$; if $|\mathscr{P}| > \chi$, for $f \in \mathscr{P}$ let $\mathscr{P}_f = \{g \in \mathscr{P}: g <_E f\}$. Now for $f \in \mathscr{P}$ we have $|\mathscr{P}_f| < \chi$ [otherwise f contradicts the choice of h^*], so by Hajnal's free subset theorem there are distinct $f_i \in \mathscr{P}$ (for $i \leq |\mathscr{P}|$), with $[i \neq j \Rightarrow f_j \notin \mathscr{P}_{f_i}]$, i.e., $[i \neq j \Rightarrow \neg(f_i <_E f_j)]$. As $|\mathscr{P}| > (2^\theta)^+$ we get a contradiction easily (using the Erdös-Rado theorem; see [Sh111, 2.2])).

Let $I = \{f \restriction i: f \in \mathscr{P}, i < \theta\}$, ordered by inclusion. Then I is a tree with level i of power $\leq |h^*(\operatorname{Min}(A - i))| < \kappa$ (use (b) and (c) for the first inequality and (b) for the second). Note that for limit $\delta < \theta$,

$$|\operatorname{Br}_\delta(I)| \leq \prod_{j<\delta}(1 + h^*(j)) < \kappa.$$

Let us now define $B \stackrel{\text{def}}{=} \operatorname{Br}_\theta(I)$; we claim it is exactly $\{b_f: f \in \mathscr{P}\}$, where $b_f = \{f \restriction i: i < \theta\}$. Clearly, for $f \in \mathscr{P}$, $b_f \in \operatorname{Br}_\kappa(i)$. Suppose $b = \{g_i: i < \theta\} \in \operatorname{Br}_\theta(I)$, g_i in the level i of I, and $[j < i \Rightarrow g_i \subseteq g_j]$. Then $g_i = f_i \restriction i$, $f_i \in \mathscr{P}$. Let $g = \bigcup_{i<\theta} g_i$. Clearly $\mathscr{P} \cup \{g\}$ satisfies (a), (b), and (c), contradicting $\mathscr{P}$'s maximality except when $g \in \mathscr{P}$; so also $\operatorname{Br}_\theta(I) \subseteq \{b_f: f \in \mathscr{P}\}$.

(2A) In the proof of (1) choose $\langle\lambda_i: i < \theta\rangle$ increasing and continuous $[\lambda = \bigcup_{i<\theta}\lambda_i]$, E any normal filter on θ. By induction on $\alpha < \kappa^+$ we can choose

$$f_\alpha \in \prod_{i<\theta}\lambda_i^+, \qquad (\forall\beta < \alpha) f_\beta <_E f_\alpha.$$

So we could let $h^*(i) = \lambda_i^+$ [as, by the above,

$$(\exists \mathscr{P} \in \mathscr{F}_E(h^*))[|\mathscr{P}| \geq \chi = \kappa^+],$$

h^* is minimal by Galvin and Hajnal [GH]—and really this is proved below]. Then choose $\mathscr{P}$ and I as in the proof of 8(1) (without loss of generality we can take $A = \theta$). Note: this $\mathscr{P}$ is not the one for 8(2)! Choose for every $i < \theta$ and $\alpha < \lambda_i^+$ a sequence $\langle g_i(\alpha, \beta): \beta < \alpha \rangle$ such that $g_i(\alpha, \beta) < \lambda_i$ and $[\beta_1 < \beta_2 < \alpha \Rightarrow g_i(\alpha, \beta) \neq g_i(\alpha, \beta_2)]$. Now for every set $B \subseteq \theta$, $B \neq \varnothing$ mod E, ordinal $j < \theta$ and ordinals $\gamma_i < \lambda_j$ for $i \in B$ (not $\gamma_i < \lambda_i$!), letting $\bar{\gamma} = \langle \gamma_i : i \in B \rangle$, we define a function $G = G_{B,j,\bar{\gamma}}$. It is a partial one-place function from $\mathscr{P}$ to $\mathscr{P}$ (equivalently, from $\mathrm{Br}_\theta(I)$ to $\mathrm{Br}_\theta(I)$) satisfying

(α) $G(f^a) = f^b$ iff, for every $i \in B$, $f^a(i) < f^b(i)$ and $g_i[f^a(i), f^b(i)] = \gamma_i$.

[Note: in (β) below we prove also that G is a function, i.e. single valued.]

Now we note the following fact:

(β) $G = G_{B,j,\bar{\gamma}}$ is such that if $G(f^c) = f^d$, $G(f^a) = f^b$, $\zeta \in B$ and $f^c \restriction (\zeta + 1) = f^a \restriction (\zeta + 1)$, then $f^d \restriction (\zeta + 1) = f^b \restriction (\zeta + 1)$.

[Why? By (c) in the proof of 10(1) it is enough to prove that $f^d(\zeta) = f^b(\zeta)$. By the choice of g_i it is enough to prove that for some $\alpha < \lambda_\zeta^+$ we have $f^d(\zeta) < \alpha$, $f^b(\zeta) < \alpha$ and $g_\zeta(\alpha, f^d(\zeta)) = g_\zeta(\alpha, f^b(\zeta))$. As $G(f^c) = f^d$, we know that $f^d(\zeta) < f^c(\zeta)$ and $g_i[f^c(\zeta), f^d(\zeta)] = \gamma_i$. Similarly, as $G(f^a) = f^b$, we know that $f^b(\zeta) < f^a(\zeta)$ and $g_i[f^a(\zeta), f^b(\zeta)] = \gamma_i$. So by a previous sentence it is enough to show that $f^a(\zeta) = f^c(\zeta)$, which holds by an assumption of (β).]

The next fact to notice is:

(γ) If $f^a, f^b \in \mathscr{P}$, then, for some $B, j, \bar{\gamma}$ (as above),

$$G_{B,j,\bar{\gamma}}(f^a) = f^b \quad \text{or} \quad G_{B,j,\bar{\gamma}}(f^b) = f^a.$$

[Why? We know by (c) of the proof of 2(1) that $\{i < G: f^a(i) = f^b(i)\} = \varnothing$ mod E. So, possibly interchanging f^a and f^b, we have

$$B_1 = \{i < \theta: f^a(i) > f^b(i)\} \neq \varnothing \bmod E.$$

Let us define $g^*: B_1 \to \theta$:

$$g^*(i) = \mathrm{Min}\{j < \theta: g_i(f^a(i), f^b(i)) < \lambda_j\}.$$

For every limit $i \in B_1$, $g^*(i) < i$ (as $\lambda_i = \bigcup_{j<i} \lambda_j$); so, by the normality of E, for some j

$$B \overset{\text{def}}{=} \{i \in B_1 : g^*(i) = j\} \neq \varnothing \bmod E.$$

Lastly let $\gamma_i = g_i(f^a(i), f^b(i))$ for $i \in B$. Now $B, j, \bar{\gamma}$ are as required.]

The last fact we need to note is

(δ) The number of functions in $\{G_{B,j,\bar{\gamma}}: B, j, \bar{\gamma}$ as above$\}$ is $\leq \kappa$.

[Proof: trivial.]

By (β), (γ) and (δ) we are finished.

(2B) Similar.

(2C) Use in addition the ideas in Baumgartner's proof for the existence of small clubs of $\mathscr{P}_{<\aleph_m}(\aleph_n)$ (see in [Mg2]).

Conclusion 11. (1) *Suppose κ is a strong limit of uncountable cofinality, or just as in* 10(2A). *Then, for some complete first order $T \subseteq T_1$ of cardinality κ,*

$$I(\lambda, T_1, T) = \kappa^+ = |D(T)|, \qquad IE(T_1, T) = \aleph_0^-.$$

Also, if cf $\kappa > \aleph_1$ *for some* $T \subseteq T_1$, $|T_1| = \kappa$, *then*

$$IE(\kappa^{++}, T_1, T) = \kappa^{++} = |D(T)|.$$

(2) *Suppose* κ *is supercompact. Without loss of generality* (*by Laver* [Lv]), κ *is still supercompact even if we force by any* κ*-directed complete forcing notion* (*and slightly more; see Note* (2) *at the end of the paper*) *and still* $(\forall \chi \geq \kappa) 2^\chi = \kappa^+$.

Let Q_0 *be any* κ*-directed complete forcing notion, and suppose that in* $V_1 = V^{Q_0}$:

(*) $\aleph_0 < \theta = \mathrm{cf}\, \theta < \kappa \leq \mu_0 < \mu_1 \leq \lambda \leq \mu_2$, $\mu_2^\kappa = \mu_2$, *and* $\lambda^\kappa = \lambda$ (*the most interesting case is* $\kappa = \mu_0$). N *is an algebra with universe* μ_1 *with* μ_0 *functions* (*each with finite number of places*) *and such that no* ($< \kappa$)*-strategically complete* (*see Note* (3) *at the end of the paper*) *forcing with the* κ^+*-c.c change the number of subalgebras, even if we further force by rg.*

***Then**, in some generic extension* V_1^Q *of* $V_1 = V^{Q_0}$:

(a) *cardinals are not collapsed; the only change in the power set function is that* 2^κ *becomes* λ;

(b) cf κ *becomes* θ, *and no larger regular cardinal changes its cofinality*; *and*

(c) *there are first order complete* $T \subseteq T_1$, $|T| = \kappa$, $|T_1| = \mu_0$, $|D(T)| = \mu_1$ (*T has only monadic predicates*), *with* $I(\chi, T_1, T) = \mu_2$ *for* $\chi \leq \mu_1$, *and* $\mu_3 \leq IE(\chi, T_1, T) \leq (\mu_3^\kappa)^{V^{Q_0}}$, *where* $\mu_3 = (\mathrm{Sb}(N))^{V^{Q_0 * Q_2}}$ *for some* ($<\kappa$)*-strategically complete forcing with* κ^+*-c.c.*

(3) *In* (2) *we can allow* $\theta = \aleph_0$ *if we demand* $\kappa < \mu_0$ *and* $|T| = \kappa^+$.

REMARK 11.A. We can use $|T|$ larger—any regular cardinal $\leq \mu_0$.

PROOF. (1) We have the first possibility by 8(2) and 10(2A) (so T has monadic predicates only). We have the second possibility by 8(2) and 10(2C).

(2) Apply Lemma 7 (for N and κ) and get Q, V_2, V_1^Q, T_1, and T. We have to observe that Laver's argument [Lv] works (see Note (2) at the end of the paper). Then we can apply Magidor forcing Mg [Mg 1] to shoot a club to κ of order type θ, with no cardinal collapse and the power set function preserved. Clearly Mg satisfies κ^+-c.c.

So we just need:

Observation 11.B. *Magidor forcing* Mg *from* [Mg 1] (*changing the cofinality of* κ *to* θ, *where* $\theta = \mathrm{cf}\, \theta > \aleph_0$) *adds no* κ *branches to trees from* V^Q.

For the proof, see Note (1) at the end of the paper.

We finish by noting that in V_2^{Mg} any subalgebra of N is a limit of θ old subalgebras. So the old T and T_1 work. Thus we finish the proof of Conclusion 11.

REMARK 12. In [ShA1, Chapter VIII, Theorem 1.10] we get results on $I(\lambda, T_1, T)$ and $IE(\lambda, T_1, T)$ under the following assumptions:

(i) $|T_1| = |T| = \mu$.

(ii) In T there is an independent family of μ formulas (i.e., $\psi_i(x) \in L(T)$ for $i < \lambda$, and

$$(\exists x)\left(\bigwedge_{i \in u} \psi_i(x) \wedge \bigwedge_{j \in v} \neg \psi_j(x)\right) \in T$$

for any finite disjoint u, v).

(iii) There is a μ-Kurepa tree with at least χ μ-branches (i.e. a tree of power μ, with μ levels, each level of power $<\mu$, and χ μ-branches), and λ is regular or strong limit. Note that, by the previous independent results, we *cannot* prove a similar theorem if we assume just the following natural weakened version of (ii):

(*) $\mu = 2^{<\mu}$ and there are $\varphi_\eta(x) \in L(T)$ for $\eta \in 2^{<\mu}$ such that, for every $\eta \in {}^{\mu}2$, $\{\varphi_{\eta \restriction \alpha}(x)^{\text{if}(\eta(\alpha)=0)}: \alpha < \mu\}$ is consistent.

PROBLEM 12.A. Is the assumption on the Kurepa tree necessary?

PROBLEM 13. Fact 9, particularly 9.2(2B), leaves the following question open:

(*) If λ is strong limit of cofinality $\aleph_0$, $T \subseteq T_1$, $|T| = |T_1| = \lambda$, and $|D(T)| > \lambda$ (hence $|D(T)| = 2^\lambda$), *is there* a $w \subseteq D(T)$, $|w| = |D(T)|$, and, for $u \subseteq \lambda$ and $\mu \geq |u| + \lambda$, a model $M^1_{u,\mu}$ of cardinality μ such that, for $p \in w$, $M^1_{u,\mu}$ realizes p iff $p \in u$?

For this a partition theorem on trees suffices.

$(*)_2$ ***If*** $\lambda = \sum_{n<\omega} \lambda_n$ and, for $i < \lambda_0$,

$$T_i = \{\eta: \eta \in {}^{\omega>}\lambda, \eta(0) = i, \eta(l+1) < \lambda_l\},$$

f an m-place function from $\bigcup_{i<\lambda_0} T_i$ into λ_0, ***then*** there are $T^1_i \subseteq T_i$, closed under initial segments, and an infinite $w \subseteq \omega$ and $\langle \mu_k: k \in w\rangle$ increasing, $\lambda = \sum_k \mu_k$,

$$(\forall i < \lambda_0)\forall \eta \in T^1_i [\lg(\eta) \in w \rightarrow |\{\alpha: \eta\char`^\langle\alpha\rangle \in T^1_i\}| = \mu_{\lg(\eta)}]$$

such that for $i_1 < \cdots < i_m < \lambda_0$, $k < \omega$, and $\eta_l, \nu_l \in T_{i_l}$, η_l and ν_l having length k, the following equality holds:

$$f(\eta_1, \ldots, \eta_m) = f(\nu_1, \ldots, \nu_m).$$

PROBLEM 14. Let $T \subseteq T_1$ be complete first order theories.

(1) If $\lambda > |T_1|$ and T is not superstable, is $IE(\lambda, T_1, T) = 2^\lambda$?

A weaker version:

(2) If $\lambda > |T_1|$ and T is stable but not unsuperstable, then $IE(\lambda, T_1, T) = 2^\lambda$.

Note that by [Sh136] the open cases are when

(a) λ is singular, and

(b) $|T_1|^{\aleph_0} \geq \lambda$ or $(\exists \kappa < \lambda < 2^\kappa)$ [κ strong limit of cofinality $\aleph_0$].

On the case $\lambda = 2^{\aleph_0}$ and the black box, see [Sh300, III, §§4, 5, 6].

PROBLEM 15. Can you have (χ, μ)-freedom, $\chi = \mu^+ + |T|$, and $IE(\chi, T_1, T) < 2^\chi$?

Appendix.

1. DEFINITION. We say $(V, W, \lambda, \chi, \theta, M)$ is a *Magidor witness* if the following conditions hold:

(a) $W \subseteq V$ are universes of set theory, W a (transitive) class of V. (So they have the same ordinals.)

(b) $M \in W$ is a model with universe λ and χ functions.

(c) For every subalgebra $M' \in V$ of M, M' is the union of $<\theta$ subalgebras of M which belong to W.

REMARK. So if $(\text{ord}^{<\theta})^V \subset W$ (i.e. every set of ordinals of cardinality $<\theta$ in V belongs to W), then every subalgebra of M from V belongs to W.

2. DEFINITION. 1) We say (W, V) *satisfies* (λ, χ, θ)-*Magidor covering* if, for some N, $(V, W, \lambda, \chi, \theta, N)$ is a Magidor witness.

2) If $\theta = \chi$, we omit it.

3. MAGIDOR COVERING THEOREM. *If in K there is no Erdös cardinal (i.e. $\lambda \not\rightarrow (\omega_1)^{<\omega}_2$ for every λ), then every submodel N (in V) of K_μ which is closed under all primitive recursive functions ($N \prec L_\mu$ is more than enough) is the union of countably many such constructible models.*

Final notes.

Note (1). *Proof of* 11.B. Straightforward from [Mg]. Let the tree be I. For every condition q let $h^{[q]}$ be its a-pure part. For every pure condition q and finite function h let $q^{[h]}$ be the a-pure extension q' of q with $h^{[q']} = h$ (maybe not defined). For simplicity, without loss of generality we let $\boldsymbol{\eta}$ be a Mg-name of a κ-branch, and p a pure condition. For some pure extension q of p:

($*$) For every $\alpha < \theta$ and $\delta < \lambda$, if $q^{[\{\langle\alpha,\delta\rangle\}]}$ is defined then:

(i) $\boldsymbol{\eta} \restriction \delta$ depends, above the condition $q^{[\{\langle\alpha,\delta\rangle\}]}$, only on the forcing "below $\langle\alpha,\delta\rangle$" (i.e. on the function restricted to α).

(ii) Moreover, for some finite $w[\alpha,\delta] \subseteq \alpha$, it depends only on the function restricted to $w[\alpha,\delta]$.

(iii) In fact, $w[\alpha,\delta] = w[\alpha]$ when defined [this is straightforward by Magidor's proofs in [Mg1]].

So by Fodor's lemma, for some stationary $S \subseteq \theta$, for every $\alpha \in S$, $w[\alpha] = w$. Let h be a function from w to λ such that $q \leq q^{[h]} \in$ Mg. So for every $\alpha \in S$ (such that α $>$ Max Dom h) for a set $A_\alpha \subseteq \kappa$ unbounded in κ (really belongs to D_α, the αth filter Magidor used) for every $\delta \in A_\alpha$, $q^{[h \cup \{\langle\alpha,\delta\rangle\}]} \in$ Mg is $\geq q^{[h]}$; hence it forces a value $\eta_{\alpha,\delta}$ to $\boldsymbol{\eta} \restriction \delta$.

Now any η_{α_1,δ_1} and η_{α_2,δ_2} are comparable in the tree—as we can choose $\alpha_3 < \theta$ and $\delta_3 < \kappa$ large enough so that

$$q^{[h \cup \{\langle\alpha_1,\delta_1\rangle,\langle\alpha_3,\delta_3\rangle\}]} \in \text{Mg}, \qquad q^{[h \cup \{\langle\alpha_2,\delta_2\rangle,\langle\alpha_3,\delta_3\rangle\}]} \in \text{Mg}$$

are well defined; so as $\delta_1, \delta_2 < \delta_3$ and $\alpha_1, \alpha_2 < \delta_3$ necessarily $\eta_{\alpha_1,\delta_1} < \eta_{\alpha_3,\delta_3}$ and $\eta_{\alpha_2,\delta_2} < \eta_{\alpha_3,\delta_3}$. So $q^{[h]}$ already determines the branch.

Note (2). On this see Baumgartner's work on squares above a supercompact.

Let $h: \kappa \to H(\kappa)$ be a Laver diamond (see [Lv]) and define an iteration $\langle P_i, \mathbf{Q}_j : i \leq \kappa, j < \kappa\rangle$ with Easton support, $[i < \kappa \Rightarrow |P_i| < \kappa]$, and: *if* (e.g.) i is strongly inaccessible, Mahlo $|P_i| \leq i$, and $h(i)$ is a P_i-name of a forcing which is α-strategically complete (see note (3), below) for every $\alpha < i$, *then* $\mathbf{Q}_i = h(i)$; ***if not***, $\mathbf{Q}_i$ is trivial.

Now in $V^{P_\kappa} = V[G_\kappa]$ for a forcing notion Q to preserve supercompactness it suffices that Q is α-strategically complete for $\alpha < \kappa$, and ***if*** $j: V \to M$ is an elementary embedding, $(j(h))(\kappa) = \mathbf{Q}$, $M^\sigma \subseteq M$, where $\sigma > 2^{|Q|}$, and $G \subseteq Q$ is generic over $V[G_\kappa]$, ***then*** $\{j(p): p \in G\}$ has an upper bound in $j(Q)$ in the universe $M^{j(P_\kappa)}$ (or at least there is $G' \subseteq j(P_\kappa)$ generic over μ, extending $G_\kappa \cup \{j(p): p \in G\}$).

Note (3). Q is α-strategically closed if for each $r \in Q$ in the following game player I wins: the play last α moves, and in the αth move I chooses $P_\alpha \in Q$ such that $r \leq p_\alpha \wedge \bigwedge_{\beta<\alpha} q_\beta \leq p_\alpha$, and then player II chooses $q_\alpha \in Q$ such that $p_\alpha \leq q_\alpha$. Player I wins if he always has a legal move.

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