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THE CONSISTENCY WITH CH OF SOME CONSEQUENCES OF MARTIN'S AXIOM PLUS $2^{\aleph_0} > \aleph_1$

BY

U. AVRAHAM, K. J. DEVLIN AND S. SHELAH[†]

ABSTRACT

We present here a (weak) axiom which implies some of the consequences of MA, but is consistent with GCH. We use the method of Jensen in his proof of consis (ZFC + GCH + SH).

§1. Introduction

The aim of this paper is to find a combinatorial principle which is consistent with ZFC + GCH, and which implies some of the consequences of Martin's Axiom (plus $2^{\aleph_0} > \aleph_1$). Many mathematical statements, P, have been proved independent of ZFC set theory by showing that $\diamondsuit$ implies P and the MA + $2^{\aleph_0} > \aleph_1$ implies $\neg P$. In such cases, the question arises as to whether the continuum hypothesis can be used instead of $\diamondsuit$. It would be helpful, therefore, to find a combinatorial principle consistent with ZFC + GCH, which has as its consequences many of the consequences of MA + $2^{\aleph_0} > \aleph_1$. Now, Solovay and Tennenbaum in 1965 established (see [6]), by means of an iterated forcing argument, the consistency with ZFC of the Souslin Hypothesis, and after examining the proof, Martin and, independently, Rowbottom formulated the principle MA and made the slight changes to the Solvay–Tennenbaum argument in order to prove the consistency of MA + $2^{\aleph_0} > \aleph_1$. Later, in 1969, Jensen obtained (see [2]) the consistency with ZFC + GCH of the Souslin Hypothesis. The question arises: Can we extract from the Jensen argument an MA-like principle, provably consistent with GCH by a Jensen style argument? In other words, we want to solve the following "equation" for x:

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$$\frac{\text{MA}}{\text{Solovay–Tennenbaum}} = \frac{x}{\text{Jensen}}.$$

This paper presents a partial solution to this equation. Our solution is not strong enough to yield "Every Aronszajn tree is special", but will hopefully have many consequences other than the three we present here. These three applications are:

(1) the solution to a problem of Hajnal and Mate about chromatic numbers of graphs (see [4]);

(2) a result about the weak uniformisation of colouring of ladder systems;

(3) a strengthening of (2).

In order to complete the (present) picture, one should read [3], which shows why we could not obtain stronger results in one direction, and [5], which describes an entirely different approach to the problem of proving the consistency with GCH of known consequences of $\text{MA} + 2^{\aleph_0} > \aleph_1$.

The history of the present paper is as follows. In [1], Avraham and Shelah formulated and proved the consistency with GCH (by a Jensen style argument) of an axiom which solved (1) above (together with some variations of (1)). In 1976, Devlin, while dealing with a question of Shelah, showed that one cannot go too far in this direction, by proving that if the CH holds, then every ladder system on ω_1 possesses a colouring which is not uniformisable. (See [3] for details.) However, Devlin was able to establish the consistency with GCH of the statement that every colouring of every ladder system is weakly uniformisable: this is result (2) above. Finally, Avraham and Shelah formulated the combinatorial principle presented in this paper, and proved its consistency with GCH. The proof is along the lines of Jensen's proof (in [2]), but is simpler because there is no need for the closed unbounded set forcing required there. Result (3) is due to Shelah. Section 4, where we show that the combinatorial principle does not imply the Souslin Hypothesis, is due to Avraham and Shelah.

Apart from the Jensen iteration lemma, which we quote here without proof, the paper can be read without knowledge of [2]. Our notation and terminology is standard. We use Ω to denote the set of all countable limit ordinals. κ^λ denotes $\{f \mid f: \lambda \to \kappa\}$, and $\kappa^{\underset{\smile}{\lambda}} = \bigcup_{\alpha<\lambda} \kappa^\alpha$.

§2. Formulation and applications of the principle

A *tree* is a poset $\boldsymbol{T} = \langle T, \leqq \rangle$ such that for every x in T, $\hat{x} = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$, the order-type of $\hat{x}$ being the *height* of x, $\text{ht}(x)$. The α'th level of $\boldsymbol{T}$ is the set $T_\alpha = \{x \in T \mid \text{ht}(x) = \alpha\}$. We set $T \mid \alpha = \bigcup_{\beta<\alpha} T_\beta$. The *height* of T is the least λ such that $T_\lambda = \emptyset$, and is denoted by $\mathrm{ht}(T)$. *A branch* of T is a maximal, totally ordered subset of T; if α is its order-type it is called an *α-branch*. A tree T is *normal* iff:

(i) $\mathrm{ht}(T)$ is a limit ordinal;

(ii) for all $\alpha < \beta < \mathrm{ht}(T)$ and all $x \in T_\alpha$ there is a $y \in T_\beta$ with $x <_T y$;

(iii) if $\alpha < \mathrm{ht}(T)$ and $\lim(\alpha)$, and if $x, y \in T_\alpha$, then $\hat{x} = \hat{y}$ implies $x = y$.

If T is a tree and $S \subseteq T$, we say S is a *subtree* of T iff S is an initial section of T (i.e. $t <_T s \in S$ implies $t \in S$) which meets every non-empty level of T. If T is a normal tree, a subtree S of T is *normal* iff S is (with the relativised ordering) a normal tree.

An array of filters is a collection $D = \{D_{\alpha,f} \mid \alpha \in \Omega \ \& \ f \in \omega^{\underline{\alpha}}\}$ such that $D_{\alpha,f}$ is a countably complete filter on ω^α. Now let T be a normal tree of height ω_1 such that T_α consists of elements of ω^α and the ordering of T is inclusion. We say T is *appropriate* for the array of filters D iff:

(i) if $\alpha \in \Omega$ and $f \in T \mid \alpha$, there is a set $A \in D_{\alpha,f}$ such that whenever $h \in A$ is such that $f \subseteq h$ and $(\forall \xi < \alpha)(h \mid \xi \in T)$, then $h \in T$;

(ii) if $\alpha \in \Omega$ and $W \subseteq T \mid \alpha$ is a normal subtree of $T \mid \alpha$ which is closed under immediate successors in T (i.e. if $a \in W \cap T_\gamma$ and $a <_T b \in T_{\gamma+1}$, then $b \in W$), then for any $f \in W$ and any set $A \in D_{\alpha,f}$ there is $h \in A$ such that $f \subseteq h$ and $(\forall \xi < \alpha)(h \mid \xi \in W)$.

Let SAD (for Shelah–Avraham–Devlin) denote the conjunction of the following statements:

(i) GCH;

(ii) every constructible cardinal is a cardinal;

(iii) for every cardinal κ, $\mathrm{cf}(\kappa) = \mathrm{cf}^L(\kappa)$;

(iv) every countable sequence of ordinals is constructible;

(v) if D is a constructible array of filters, then every tree which is appropriate for D has an ω_1-branch.

In §3, we prove that SAD is consistent. (Indeed, we prove a somewhat stronger result.) In the remainder of this section, we give two applications of SAD.

Chromatic number of graphs

Our first application is in graph theory. A *graph* on a set X is given by a set, E, of two-element subsets of X. The members of X are called the *vertices* of the graph, the members of E are the edges of the graph. If $\{x_1, x_2\} \in E$, we say x_1 and x_2 are *connected* in the graph. We call the pair $g = \langle X, E \rangle$ a *graph*. The chromatic numbers of the graph g is the least cardinal κ for which there is a mapping

$f\colon X \to \kappa$ such that $\{x_1, x_2\} \in E \to f(x_1) \neq f(x_2)$. A *Hajnal–Mate graph* is a graph $g = \langle \omega_1, E\rangle$ on ω_1 such that for every $\alpha \in \omega_1$, $\{\beta \in \alpha \mid \{\beta, \alpha\} \in E\}$ is either finite or else is an ω-sequence cofinal in α. In [4], Hajnal and Mate show that if $\diamondsuit$ holds, then there is a Hajnal–Mate graph of chromatic number $\aleph_1$, and if $MA + 2^{\aleph_0} > \aleph_1$ holds, then every Hajnal–Mate graph has countable chromatic number. They ask the question as to what effect CH has upon the chromatic number of Hajnal–Mate graphs and suggest Jensen's method. The next result answers this question.

2.1. THEOREM. *Assume SAD. Then every Hajnal–Mate graph has countable chromatic number.*

PROOF. Let $g = \langle \omega_1, E\rangle$ be a Hajnal–Mate graph. We may assume that g has infinite chromatic number. If $\{\beta \in \alpha \mid \{\beta, \alpha\} \in E\}$ is infinite, let $\{\eta_\alpha(n)\}_{n<\omega}$ enumerate this set in canonical ordering: thus η_α is increasing and cofinal in α. Let T_α be the set of all functions $f\colon \alpha \to \omega$ such that:

(i) if $\gamma < \delta < \alpha$ and $\{\gamma, \delta\} \in E$, then $f(\gamma) \neq f(\delta)$;

(ii) if $\lim(\gamma)$, $\gamma \leqq \alpha$, and $\{\beta \in \gamma \mid \{\beta, \gamma\} \in E\}$ is infinite, then $\omega - \{f(\eta_\gamma(n)) \mid n < \omega\}$ is infinite.

Let $T = \bigcup_{\alpha<\omega_1} T_\alpha$. Under inclusion, T is clearly a tree of height ω_1. (In particular, if $f \in T_\alpha$ and $\beta < \alpha$, then $f \mid \beta \in T_\beta$.) And it is easily seen that T is normal.

Placing ourself inside L, the constructible universe, now, we define an array of filters. Let $\alpha \in \Omega$, $f \in \omega^\alpha$. Let Q_α be the set of all increasing ω-sequences cofinal in α, and for $p \in Q_\alpha$, let

$$A_p = \{f \in \omega^\alpha \mid \omega - \{f(p(n)) \mid n < \omega\} \text{ is infinite}\}.$$

Let $D_{\alpha,f}$ be the countably complete filter generated by $\{A_p \mid p \in Q_\alpha\}$. (Of course, we must check that the family $\{A_p \mid p \in Q_\alpha\}$ has the countable intersection property: if $p^i \in Q_\alpha$, $i < \omega$, we must have a $p \in Q_\alpha$ such that $A_p \subseteq \bigcap_{i<\omega} A_{p^i}$. But an easy diagonalisation construction gives a $p \in Q_\alpha$ such that $\operatorname{ran}(p^i) - \operatorname{ran}(p)$ is finite for all i, and such a p suffices.) Let $D = \{D_{\alpha,f} \mid \alpha \in \Omega \;\&\; f \in \omega^\alpha\}$. We now leave L and return to the real world. We show that T is appropriate for D. Let $\alpha \in \Omega$, $f \in T \mid \alpha$. If $\{\beta \in \alpha \mid \{\beta, \alpha\} \in E\}$ is finite, then for any $A \in D_{\alpha,f}$ it is the case that $h \in A$ and $h \supseteq f$ and $(\forall \xi < \alpha)(h \mid \xi \in T)$ implies $h \in T$. Now suppose $\{\beta \in \alpha \mid \{\beta, \alpha\} \in E\}$ is infinite. Since every countable sequence of ordinals is constructible, $\eta_\alpha \in Q_\alpha$. But then $A_{\eta_\alpha} \in D_{\alpha,f}$ and for any $h \in A_{\eta_\alpha}$, if $(\forall \xi < \alpha)$ $(h \mid \xi \in T)$, then $h \in T$. This verifies property (i) of the definition of appropriate. We check (ii). Let $\alpha \in \Omega$, and let $W \subseteq T \mid \alpha$ be a normal subtree of $T \mid \alpha$ closed under immediate successors. Let $f \in W$, $A \in D_{\alpha,f}$. We seek an $h \in A$, $h \supseteq f$, such that $(\forall \xi < \alpha)(h \mid \xi \in W)$. Pick $\eta \in Q_\alpha$ such that $A_\eta \subseteq A$. We construct an $h \in \omega^\alpha$ such that $h \supseteq f$, $(\forall \xi < \alpha)(h \mid \xi \in W)$, and $\omega - \{h(\eta(n)) \mid n < \omega\}$ is infinite (whence $h \in A_\eta \subseteq A$, of course). Pick $n < \omega$ with $\eta(n) > \mathrm{ht}(f)$. We construct, by induction, functions f_i and sets F_i, $n \leqq i < \omega$, such that:

(i) $f \subseteq f_n \subseteq f_{n+1} \subseteq \cdots \subseteq f_i \subseteq \cdots$;

(ii) $f_i \in W \cap T_{\eta(i)+1}$;

(iii) F_i is a set of i integers;

(iv) $F_n \subset F_{n+1} \subset \cdots \subset F_i \subset \cdots$;

(v) $F_i \cap \{f_i(\eta(k)) \mid k \leqq i\} = \emptyset$.

To commence, pick any $f_n \supseteq f$ in $W \cap T_{\eta(n)+1}$ and let F_n consist of any n elements of $\omega - \{f_n(\eta(k)) \mid k \leqq n\}$. Since W is a normal subtree of $T \mid \alpha$, this is always possible. Suppose now that f_i, F_i have been defined. Pick $f'_i \supseteq f_i$ in $W \cap T_{\eta(i+1)}$. Now, the tree T clearly has the property that every element has infinitely many immediate successors. Hence as W is closed under immediate successors in $T \mid \alpha$, f'_i has infinitely many successors in $W \cap T_{\eta(i+1)+1}$. So we can find f_{i+1} in $W \cap T_{\eta(i+1)+1}$ such that $f_{i+1} \supseteq f'_i$ and $f_{i+1}(\eta(i+1)) \not\in F_i$. Let F_{i+1} consist of F_i together with one new element chosen from $\omega - \{f_{i+1}(\eta(k)) \mid k \leqq i+1\}$. Set $h = \bigcup_{i<\omega} f_i$. Then $h \in \omega^\alpha$ and $h \supseteq f$. Moreover, for $\eta \leqq i < \omega$, $h \mid \eta(i)+1 = f_i \in W$, so $(\forall \xi < \alpha)(h \mid \xi \in W)$. Finally, since $\{h(\eta(k)) \mid k < \omega\} \cap (\bigcup_{i<\omega} F_i) = \emptyset$, $\omega - \{h(\eta \mid (k)) \mid k < \omega\}$ is infinite. So we are done.

We may now apply SAD to conclude that T has an ω_1-branch. Let b be an ω_1-branch of T, and set $l = \cup b$. Then $l \in \omega^{\omega_1}$ and whenever $\{\alpha, \beta\} \in E$, $l(\alpha) \neq l(\beta)$. Hence g has countable chromatic number. □

Weak uniformization of ladder systems

Our second application of SAD concerns ladder systems. Let $\alpha \in \Omega$. A *ladder* on α is an increasing ω-sequence cofinal in α. A *ladder system* is a sequence $\langle \eta_\alpha \mid \alpha \in \Omega \rangle$ such that η_α is a ladder on α for each $\alpha \in \Omega$. An *ω-colouring* of a ladder system $\langle \eta_\alpha \mid \alpha \in \Omega \rangle$ is a sequence $\langle k_\alpha \mid \alpha \in \Omega \rangle$ such that $k_\alpha \in \omega^\omega$ for each $\alpha \in \Omega$ (the idea being that $k_\alpha(n)$ is the colour assigned to $\eta_\alpha(n)$ in the ladder η_α). A *uniformisation* of a colouring $\langle k_\alpha \mid \alpha \in \Omega \rangle$ of a ladder system $\langle \eta_\alpha \mid \alpha \in \Omega \rangle$ is a function $h: \omega_1 \to \omega$ such that for every $\alpha \in \Omega$, $k_\alpha(n) = h(\eta_\alpha(n))$ for all but finitely many n. The basic question is: does every colouring of a ladder system have a uniformisation? Shelah introduced these notions and proved that if $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$ is assumed, the answer is "yes" (for any ladder system), and if $\diamond$ is assumed, the answer is "no".

But the question does not provide our application of SAD, for Devlin proved that the assumption of $\diamond$ above can be weakened to CH, or even $2^{\aleph_0} < 2^{\aleph_1}$. (Details of all of this can be found in [3].) So Devlin considered the following weaker notion. A *weak uniformisation* of a colouring, k, of a ladder system, η, is a function $h = \omega_1 \to \omega$ such that for every $\alpha \in \Omega$, $k_\alpha(n) = h(\eta_\alpha(n))$ for infinitely many n. The Shelah proof shows that if $\diamond$ holds, then every ladder system has a colouring which is not weakly uniformisable. Devlin was able to prove that in this case, the assumption of $\diamond$ cannot be weakened to CH. We prove this here by deducing from SAD the fact that every colouring of every ladder system is weakly uniformisable.

2.2. THEOREM. *Assume SAD. Let $\eta = \langle \eta_\alpha \mid \alpha \in \Omega \rangle$ be a ladder system. Let $k = \langle k_\alpha \mid \alpha \in \Omega \rangle$ be a colouring of η. Then k is weakly uniformisable.*

PROOF. Let $T_\alpha = \{f \in \omega^\alpha \mid (\forall \gamma \in \Omega \cap (\alpha + 1))\{n \in \omega \mid f(\eta_\gamma(n)) = k_\gamma(n)\}$ is infinite$\}$. Clearly, $T = \bigcup_{\alpha < \omega_1} T_\alpha$ is, under inclusion, a tree of height ω_1. Indeed, T is normal, as is easily seen. Placing ourselves in L, we define an array of filters. Let $\alpha \in \Omega$, $f \in \omega^{\underset{\sim}{\alpha}}$. Let Q_α be the set of all increasing ω-sequences cofinal in α. For all $i \in \omega^\omega$ and all $p \in Q_\alpha$, set

$$A_{i,p} = \{g \in \omega^\alpha \mid \{n \in \omega \mid g(p(n)) = i(n)\} \text{ is infinite}\}.$$

Let $D_{\alpha,f}$ be the countably complete filter generated by the set $\{A_{i,p} \mid i \in \omega^\omega \ \& \ p \in Q_\alpha\}$. To see that T is appropriate for the array defined, observe that if $g \in T_\alpha$ then for any $n < \omega$, $g \cup \{(\alpha, n)\} \in T_{\alpha+1}$; hence if $W \subseteq T \mid \gamma$, $\gamma \in \Omega$, is a normal subtree closed under successors, $f \in W$, $i \in \omega^\omega$ and $p : \omega \to \alpha$ is increasing and cofinal in α (assume height $(f) < p(0)$), then for some g, $f \subseteq g$, $g : \gamma \to \omega$, $(\forall \xi < \gamma)(g \mid \xi) \in W$ and $g(p(n)) = i(n)$ for *all* $n < \omega$. So by SAD, T has an ω_1-branch, say b. Clearly, $f = \bigcup b$ is a weak uniformisation of k. □

A generalization

We present here a generalization of the preceding section. Let E_α, $\alpha \in \Omega$ be non-principal filters $E_\alpha \subseteq P(\omega)$ such that every set of positive measure in E_α can be decomposed into $\aleph_1$ almost disjoint subsets with positive measure.

Let $\langle \eta_\alpha \mid \alpha \in \Omega \rangle$ be a ladder system and $\langle C_\alpha \mid \alpha \in \Omega \rangle$ a colouring. $F : \omega_1 \to \omega$ is a weak uniformisation (relative to $\langle E_\alpha \mid \alpha \in \Omega \rangle$) iff for each $\alpha \in \Omega$ $\{n \mid F(\eta_\alpha(n)) = C_\alpha(n)\}$ is of positive measure in E_α.

2.3. THEOREM. *Assume SAD. Let $\langle \eta_\alpha \mid \alpha \in \Omega \rangle$ be a ladder system, $\langle C_\alpha \mid \alpha \in \Omega \rangle$ be its colouring and $\langle E_\alpha \mid \alpha \in \Omega \rangle$ a sequence* in L *of filters with the properties above. Then there is a weak uniformisation relative to $\langle E_\alpha \mid \alpha \in \Omega \rangle$.*

PROOF. Construct the tree $T \subseteq \omega^{\omega_1 >}$ as follows:

$$T_\alpha = \{f: \alpha \to \omega \mid \text{for any limit } \gamma \leqq \alpha, \{n \mid f(\eta_\gamma(n)) = C_\gamma(n)\}$$

$$\textit{has positive measure in } E_\gamma\}.$$

Our tree T is $\bigcup_{\alpha \in \omega_1} T_\alpha$. Now we define in L the array of filters which will make T appropriate. For $\alpha \in \Omega$, $c: \omega \to \omega$ and $r: \omega \to \alpha$ (r is increasing and cofinal in α) we define $A_{c,r} = \{f: \alpha \to \omega \mid \{n \mid f(r(n)) = c(n)\}$ has positive E_α measure$\}$. To see that for each $\alpha \in \Omega$ the filter generated by $\{A_{c,r} \mid c: \omega \to \omega,\ r: \omega \to \alpha$ is increasing and cofinal$\}$ is countably complete and to check condition (ii) in the definition of appropriateness we need the following lemma:

1.4. LEMMA. *If E_n, $n < \omega$ is a countable set of non-principal filters on ω such that for any filter E_n, any A with positive measure ($-A \notin E_n$) can be split into $\aleph_1$ almost disjoint sets of positive measure (i.e. $A = \bigcup_{\alpha < \omega_1} A_\alpha$, $A_\alpha \cap A_\beta$ is finite and A_α is of positive E_n measure) then there are disjoint A_n, $n < \omega$ such that A_n is of positive measure in E_n.*

PROOF. Define by induction on $n < \omega$, $A_n \subseteq \omega$ such that A_n is of positive E_n measure and for no $l < \omega$ is $\bigcup_{k \leq n} A_k \in E_l$.

§3. The consistency of SAD

Assume $V = L$. We construct a complete boolean algebra, B, such that:
(i) B has cardinality $\aleph_2$;
(ii) B satisfies the C.C.C.;
(iii) B is (ω, ∞)-distributive;
(iv) $\|\mathrm{SAD}\|^B = 1$.
By virtue of (i)–(iii) and standard facts about forcing, the only part of (iv) which we shall need to check is the crucial clause concerning trees appropriate for arrays in $V(=L)$.

The construction of B is by an inductive procedure. We construct an iteration sequence $\langle B_\nu \mid \nu < \omega_2 \rangle$ of complete boolean algebras such that:
(v) $B_0 = \mathscr{D}$;
(vi) B_ν has cardinality $\aleph_1$;
(vii) B_ν satisfies the C.C.C.;
(viii) B_ν is (ω, ∞)-distrubitive.
Then if we set $B = \lim_{\nu \to \omega_2} B_\nu$, conditions (i)–(iii) follow immediately from (vi)–(viii). And, of course, the idea is to construct the iteration sequence so that B will satisfy (iv). Now, in constructing our iteration sequence there are two

different cases which arise. At successor stages we shall arrange for the limit algebra to satisfy (iv). At limit stages we shall just try to keep the iteration going, preserving conditions (vi)–(viii) in particular. The problem as to what is required to ensure that the limit construction is possible was solved by Jensen some years ago. We describe briefly what is needed. The details can be found in [2].

Recall that a *Souslin tree* is a normal tree, T, of height ω_1, such that any pairwise incomparable subset of T is countable. An algebra B_ν will satisfy conditions (vi)–(viii) iff there is a set $T \subseteq B_\nu$, $0 \notin T$, such that, under $\geqq_B$, T is a Souslin tree. In this case, $B \cong \mathrm{BA}(T)$, where $\mathrm{BA}(T)$ denotes the complete boolean algebra determined by the poset T in the usual manner. (We usually assume $\mathrm{BA}(T)$ isomorphed so that T is a dense subset of it.) An algebra B_ν satisfying (vi)–(viii) is thus called a *Souslin algebra*, and a subset $T \subseteq B_\nu$ of the above kind a *Souslinisation* of B_ν. Any Souslin algebra has essentially only one Souslinisation, in the sense that if T, T' are Soulinisations of B_ν, then for some club set $C \subseteq \omega_1$, $\alpha \in C \rightarrow T_\alpha = T'_\alpha$. Now, normally, when one constructs an iteration sequence $\langle B_\nu \mid \nu < \omega_2 \rangle$, is suffices that B_ν is a complete subalgebra of B_τ whenever $\nu < \tau$. But in order for our construction to proceed, we require a stronger notion. Suppose $\nu < \tau$ and B_ν is a complete subalgebra of B_τ. The *canonical projection* $h_{\tau\nu}: B_\tau \rightarrow B_\nu$ is defined by $h_{\tau\nu}(b) = \inf_\nu\{d \in B_\nu \mid b \leqq_\tau d\}$ Now let T^ν, T^τ be Souslinisations of B_ν, B_τ. We say B_ν is a *nice subalgebra* of B_τ iff there is a club set $C \subseteq \omega_1$ such that for any $\alpha \in C$, $h_{\tau\nu}[T^\tau_\alpha] = T^\nu_\alpha$. The following result is proved in [2], page 86. (Recall that we are assuming $V = L$ here. Some known consequences of $V = L$ are required for this result to hold.)

3.1. Lemma (Iteration Lemma). *Let σ be a function such that $\sigma(\varnothing)$ is a Souslin algebra, and such that whenever $\langle B_\nu \mid 0 < \nu \leqq \tau \rangle$ is a nicely increasing sequence of Souslin algebras, $\tau < \omega_2$, $\sigma(\langle B_\nu \mid \nu \leqq \tau \rangle)$ is a Souslin algebra of which B_τ is a nice subalgebra. Then there is a sequence $\langle B_\nu \mid \nu < \omega_2 \rangle$ such that*

(i) $B_0 = \sigma(\varnothing)$;

(ii) $0 < \nu < \omega_2 \rightarrow B_\nu$ *is a Souslin algebra*;

(iii) $B_{\nu+1} = \sigma(\langle B_\tau \mid \tau \leqq \nu \rangle)$;

(iv) $\nu < \tau < \omega_2 \rightarrow B_\nu$ *is a nice subalgebra of* B_τ. □

With 3.1 at our disposal, we now turn to the problem of what to do at successor stages in the iteration: in other words, we "define" the function σ which we shall eventually plug into 3.1. The following lemma summarises what we require. We need the following notion:

An array of filters, D, is said to be *principal* if each filter in D is a principal filter.

3.2. Lemma ($\diamondsuit$). *Let B be a Souslin algebra. Let D be a principal array of filters. Let $\mathring{T} \in V^{(B)}$ be such that*

$$\| \mathring{T} \textit{ is a normal } \omega_1\textit{-tree which is appropriate for } \check{D} \|^{B} = 1.$$

Then there is a Souslin algebra $\tilde{B}$ such that B is a nice subalgebra of $\tilde{B}$ and

$$\| \mathring{T} \textit{ has an } \omega_1\textit{-branch} \|^{\tilde{B}} = 1. \qquad \square$$

Before we prove 3.2, let us see how this gives the desired consistency result. Fix some bijection $\theta: \omega_2 \times \omega_2 \times \omega_2 \leftrightarrow \omega_2$ such that $\theta(\alpha, \beta, \gamma) \geqq \alpha, \beta, \gamma$ for all α, β, γ. Let i, j, k be the inverse functions. By GCH, there are $\aleph_2$ principal filter arrays, so let $\langle D_\alpha \mid \alpha < \omega_2 \rangle$ enumerate them. Suppose $\nu < \omega_2$ and that $\langle B_\tau \mid 0 < \tau \leqq \nu \rangle$ is a nicely increasing sequence of Souslin algebras. Set $\alpha = i(\nu)$, $\beta = j(\nu)$, $\gamma = k(\nu)$. Let

$$X_\alpha = \{ \mathring{T} \in V^{(B_\alpha)} \mid \| \mathring{T} \text{ is a normal } \omega_1\text{-tree} \|^{B_\alpha} = 1 \}.$$

Assuming (as usual) that V^{B_α} is normalised with regards to boolean equality, $|X_\alpha| = \aleph_2$. Let $\langle \mathring{T}^{\alpha,\xi} \mid \xi < \omega_2 \rangle$ be some enumeration of X_α as a one-one ω_2-sequence.

If $\| \mathring{T}^{\alpha,\gamma}$ is a normal ω_1-tree which is appropriate for $\check{D}_\beta \|^{B_\nu} = 1$, we may obtain a Souslin algebra $B = \sigma(\langle B_\tau \mid \tau \leqq \nu \rangle)$ from B_ν, $\mathring{T}^{\alpha,\gamma}$, D_β as in 3.2, so that B_ν is a nice subalgebra of B and $\| \mathring{T}^{\alpha,\gamma}$ has an ω_1-branch$\|^{B} = 1$. Letting $\sigma(\emptyset)$ be any Souslin algebra, we thus obtain a function satisfying the requirements of 3.1. Let $\langle B_\nu \mid \nu < \omega_2 \rangle$ be as guaranteed by 3.1. Let $B = \lim_{\nu \to \omega_2} B_\nu$. By the C.C.C., $B = \bigcup_{\nu < \omega_2} B_\nu$. We show that B is as required. By our previous remarks, we know that it suffices to prove the following lemma:

3.3. Lemma. $\| \mathrm{SAD} \|^{B} = 1$.

Proof. Suppose not. Then, by the maximum principle we can find a $D \in V$ and a $\mathring{T} \in V^{B}$ such that:

(i) D is a filter array in V;

(ii) $\| \mathring{T}$ is a normal ω_1-tree which is appropriate for $\check{D} \|^{B} = 1$;

(iii) $\| \mathring{T}$ has an ω_1-branch$\|^{B} < 1$.

Since $\| \mathring{T} \subseteq \omega^{\underset{\smile}{\omega_1}} \|^{B} = 1$ and $\| \check{\omega}^{\underset{\smile}{\check{\omega}_1}} = (\omega^{\underset{\smile}{\omega_1}})^{\vee} \|^{B} = 1$, we can find a $\delta < \omega_2$ such that $\mathring{T} \in V^{(B_\delta)}$.

Let $D = \{ D_{\alpha,f} \mid \alpha \in \Omega \ \& \ f \in \omega^{\underset{\smile}{\alpha}} \}$. For each pair (α, f), well-order $D_{\alpha,f}$ as $\langle A_\xi^{\alpha,f} \mid \xi < \omega_2 \rangle$.

Let $E_{\alpha,f}$ be the set of all $\xi < \omega_2$ such that

$$\|\text{if } \check{f} \in \mathring{T} \text{ and if } \check{f} \subseteq h \in (A^{\alpha,f}_{\xi})^{\vee} \text{ and if } (\forall \eta < \check{\alpha})(h \mid \eta \in \mathring{T}),$$

$$\text{then } h \in \mathring{T}; \text{ and } \check{\xi} \text{ is least for which this occurs}\|^{B_\delta} > 0.$$

Since B_δ satisfies the C.C.C., $E_{\alpha,f}$ is clearly countable. Let $A^{\alpha,f} = \bigcap_{\xi \in E_{\alpha,f}} A^{\alpha,f}_{\xi}$. Since $D_{\alpha,f}$ is countably complete, $A^{\alpha,f} \in D_{\alpha,f}$.

Let $D'_{\alpha,f}$ be the principal filter generated by $A^{\alpha,f}$. Let

$$D' = \{D'_{\alpha,f} \mid \alpha \in \Omega \ \& \ f \in \omega^{\underline{\alpha}}\}.$$

D' is thus a principal filter array. It is easily seen that

$$\|\mathring{T} \text{ is appropriate for } \check{D}'\|^{B_\delta} = 1.$$

For some $\beta < \omega_2$, $D' = D_\beta$. For some $\gamma < \omega_2$, $\mathring{T} = \mathring{T}^{\delta,\gamma}$. Let $\nu = \theta(\delta, \beta, \gamma)$. By construction of $B_{\nu+1}$,

$$\|\mathring{T} \textit{ has an } \omega_1\text{-branch}\|^{B_{\nu+1}} = 1.$$

Hence $\|\mathring{T}$ has an ω_1-branch$\|^{B} = 1$, a contradiction. The proof is complete. □

It remains only to prove 3.2. We have the following situation: D is a principal array of filters, $D = \{D_{\alpha,f} \mid \alpha \in \Omega \ \& \ f \in \omega^{\underline{\alpha}}\}$, $A_{\alpha,f}$ generates $D_{\alpha,f}$ for each α, f, B is a Souslin algebra, and $\mathring{T} \in V^{(B)}$ is such that

$$\|\mathring{T} \text{ is a normal } \omega_1\text{-tree which is appropriate for } \check{D}\|^{B} = 1.$$

We must show that there is a Souslin algebra $\tilde{B}$ such that

$$\|\mathring{T} \text{ has an } \omega_1\text{-branch}\|^{\tilde{B}} = 1.$$

What we shall in fact do is construct a Souslin tree $\tilde{T}$ such that $\tilde{B} = \mathrm{BA}(\tilde{T})$ is as required.

Let T be a Souslinisation of B. (We take care not to confuse T and $\mathring{T}$!) We construct our Souslin tree by recursion on the levels, simultaneously constructing a stricly increasing, continuous function $\gamma: \omega_1 \to \omega_1$. The elements of $\tilde{T}_\alpha$ will be pairs (x, f) such that $x \in T_{\gamma(\alpha)}$, $f \in \omega^\alpha$, and $x \Vdash_B$ "$\check{f} \in \mathring{T}$". The ordering of $\tilde{T}$ will be

$$(x, f) \leqq_{\tilde{T}} (x', f') \text{ iff } x \leqq_T x' \text{ and } f \subseteq f'.$$

(Recall, however, that $\leqq_T = \geqq_B \cap T^2$, with a reversal of the orderings here!)

We shall carry out the construction to preserve the following conditions:

(I) If $(x, f) \in \tilde{T}_\beta$ and $\alpha > \beta$ and $x' \in T_{\gamma(\alpha)}$ and $x' >_T x$, then for some $f' \supseteq f$, $(x', f') \in \tilde{T}_\alpha$.

(II) If $(x, f) \in \tilde{T}_\alpha$ and $x' \in T_{\gamma(\alpha+1)}$, $x <_T x'$, then for each $n \in \omega$, either $x' \Vdash_B$ "$\check{f} \cup \{(\check{\alpha}, \check{n})\} \in \mathring{T}$" or else $x' \Vdash_B$ "$\check{f} \cup \{(\check{\alpha}, \check{n})\} \notin \mathring{T}$".

(III) If $x \in T_{\gamma(\alpha)}$ and $W = \{f \mid (\exists x' \leqq_T x)[(x', f) \in \tilde{T}]\}$, then $x \Vdash_B$ "$\check{W}$ is a normal subtree of $\dot{T} \mid \check{\alpha} + 1$ which is closed under immediate successors".

Defining $h\colon \tilde{T} \to T$ by $h(x, f) = x$, h induces (in a natural manner, described fully in [2]) a complete embedding, e, of B into $\tilde{B} = \mathrm{BA}(\tilde{T})$ for which h is the restriction to $\tilde{T}$ of the canonical projection. (This uses condition (I) above.) It follows at once that, up to isomorphism, $\tilde{B}$ is a nice extension of B.

We shall not dwell any further on the details, but shall commence at once with the construction of $\tilde{T}$. We need a version of the combinatorial principle $\diamond$. As usual, H_{ω_1} denotes the collection of the hereditarily countable sets. For $\alpha < \omega_1$, we set $H_\alpha = H_{\omega_1} \cap V_\alpha$. By $V = L$, there is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that each S_α is a countable subset of H_α and, whenever $A \subseteq H_{\omega_1}$ and $\alpha < \omega_1 \to |A \cap H_\alpha| \leqq \aleph_0$, then $\{\alpha \in \omega_1 \mid A \cap H_\alpha = S_\alpha\}$ is stationary in ω_1. We fix this sequence from now on.

To commence the construction we set

$$\tilde{T} = \{(x, \varnothing) \mid x \in T_0\};$$

$$\gamma(0) = 0.$$

Now suppose $\tilde{T}_\alpha$, $\gamma(\alpha)$ are defined and we wish to define $\tilde{T}_{\alpha+1}$, $\gamma(\alpha+1)$. For each $(x, f) \in \tilde{T}_\alpha$ and each $n \in \omega$, let $E_{x,f,n}$ be a maximal, pairwise incomparable set of extensions x' of x in T such that either $x' \Vdash_B$ "$\check{f} \cup \{(\check{\alpha}, \check{n})\} \in \dot{T}$" or $x' \Vdash_B$ "$\check{f} \cup \{(\check{\alpha}, \check{n})\} \notin \dot{T}$". Since T is Souslin, each $E_{x,f,n}$ is countable. Hence we may define $\gamma(\alpha+1)$ to be the least γ such that $\gamma > \gamma(\alpha)$ and for all x, f, n as above, $E_{x,f,n} \subseteq T \mid \gamma$. For each $(x, f) \in \tilde{T}_\alpha$, and each $x' \in T_{\gamma(\alpha+1)}$ with $x <_T x'$, and each $n \in \omega$ such that $x' \Vdash_B$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \in \dot{T}$", put $(x', f \cup \{(n, \alpha)\})$ into $\tilde{T}_{\alpha+1}$ (to extend (x, f)). Clearly, this definition preserves conditions (I)–(III).

Suppose finally that $\lim(\alpha)$ and we have defined $\tilde{T} \mid \alpha, \gamma \mid \alpha$. Set $\gamma(\alpha) = \sup_{\beta<\alpha} \gamma(\beta)$. Let $x_0 \in T_{\gamma(\alpha)}$. Set

$$W_{x_0} = \{f \mid (\exists x <_T x_0)[(x, f) \in \tilde{T} \mid \alpha]\}.$$

By condition III,

(a) $x_0 \Vdash_B$ "$\check{W}_{x_0}$ is a normal subtree of $\dot{T} \mid \check{\alpha}$ which is closed under immediate successors". Let $Y_{x_0} = \{(x, f) \mid x <_T x_0 \ \& \ (x, f) \in \tilde{T} \mid \alpha\}$.

Suppose first that $S_\alpha \cap Y_{x_0}$ is not cofinal in Y_{x_0} (under $\leqq_{\tilde{T}}$). Let $(x, f) \in \tilde{T} \mid \alpha, x <_T x_0$. Then $x_0 \Vdash_B$ "$\check{f} \in \dot{T}$". Hence, as

$$\| \dot{T} \text{ is appropriate for } \check{D} \|^B = 1,$$

we have

(b) $x_0 \Vdash$ "if $h \supseteq \check{f}$ and $h \in \check{A}_{\alpha,f}$ and $(\forall \xi < \check{\alpha})(h \mid \xi \in \mathring{T})$, then $h \in \mathring{T}$". But (a) and the fact that $\| \mathring{T}$ is appropriate for $\check{D} \|^B = 1$ also gives:

(c) $x_0 \Vdash$ "there is $h \supseteq \check{f}$ such that $h \in \check{A}_{\alpha,f}$ and $(\forall \xi < \check{\alpha})(h \mid \xi \in \check{W}_{x_0})$". By (c) we must be able to find an $h \supseteq f$, $h \in A_{\alpha,f}$, such that $(\forall \xi < \alpha)(h \mid \xi \in W_{x_0})$. By (a) and (b), we have

$$x_0 \Vdash \text{"}\check{h} \in \mathring{T}\text{"}.$$

Put (x_0, h) into $\tilde{T}_\alpha$ (to extend (x, f)). Add one such extension for each pair $(x, f) \in \tilde{T} \mid \alpha$ with $x <_T x_0$.

Now suppose $S_\alpha \cap Y_{x_0}$ is cofinal in Y_{x_0}. In this case, add (as above) an extension (x_0, h) of each pair $(x, f) \in \tilde{T} \mid \alpha$ with $x <_T x_0$, such that (x, f) extends some member of $S_\alpha \cap Y_{x_0}$.

It is easily seen that this defines $\tilde{T}_\alpha$ so as to preserve (I)–(III). That completes the construction. The following two lemmas complete the proof of 3.2.

3.4. Lemma. *$\| \mathring{T}$ has an $\check{\omega}_1$-branch $\|^B = 1$.*

Proof. We give an intuitive, forcing proof. Let $\tilde{b}$ be a generic branch of $\tilde{T}$. Let $\mathring{b} = \{f \mid (\exists x)[(x, f) \in \tilde{b}]\}$. Then $\mathring{b}$ is a generic branch of $\mathring{T}$. □

3.5. Lemma. *$\tilde{T}$ is a Souslin tree.*

Proof. Let A be a maximal, pairwise incomparable subset of T. Let $\bar{A} = \{u \in \tilde{T} \mid (\exists a \in A)(a \leqq_{\tilde{T}} u)\}$. then $\bar{A}$ is cofinal in $\tilde{T}$. For each $(x, f) \in \tilde{T}$, now, let $E_{x,f}$ be a maximal, pairwise incomparable subset of the set

$$\{x' \in T \mid (x <_T x') \ \& \ (\exists f' \supseteq f)[(x', f') \in \bar{A}]\}.$$

Since T is Souslin, $E_{x,f}$ is countable. Let

$$C = \{\alpha \in \omega_1 \mid \gamma(\alpha) = \alpha \ \& \ \tilde{T} \mid \alpha = \tilde{T} \cap V_\alpha \ \& \ (\forall (x, f) \in \tilde{T} \mid \alpha)(E_{x,f} \subseteq T \mid \alpha)\}.$$

Clearly, C is closed and unbounded in ω_1. So we can pick an $\alpha \in C$ with $S_\alpha = \bar{A} \cap \tilde{T} \mid \alpha$. Let $x_0 \in T_\alpha$ now. Suppose $(x, f) \in Y_{x_0}$. Pick $f_0 \supset f$ with $(x_0, f_0) \in \tilde{T}_\alpha$. Since $\bar{A}$ is cofinal in $\tilde{T}$ there is an $(x', f') \in \bar{A}$ such that $(x_0, f_0) \leqq_{\tilde{T}} (x', f')$. But $E_{x,f} \subseteq T \mid \alpha$, so we cannot have $x' \in E_{x,f}$. So, as $(x', f') \geqq_{\tilde{T}} (x, f)$, it must be the case that for some $x'' <_T x_0, x'' \in E_{x,f}$. Pick $f'' \supset f$ so that $(x'', f'') \in \bar{A}$. Then $(x'', f'') \geqq_{\tilde{T}} (x, f)$ and $(x'', f'') \in S_\alpha$ and $(x'', f'') \in Y_{x_0}$. Hence $S_\alpha \cap Y_{x_0}$ is cofinal in Y_{x_0}. Thus, by construction, every element of $\tilde{T}$ of the form (x_0, f_0) for some f_0 extends a member of $\bar{A}$. But $x_0 \in T_\alpha$ was arbitrary. Hence every member of $\tilde{T}_\alpha$

extends a member of $\bar{A}$, hence of A. Thus as A is pairwise incomparable, $A = A \cap \tilde{T} \mid \alpha$. Hence A is countable. The proof is complete. □

By comparison with the development in [2], it is easily seen that a simple modification of the above consistency proof will yield the following more general result:

3.6. Theorem. *Assume* GCH. *There is a generic extension V^* of V such that*:

(i) $V^* \models$ GCH;

(ii) *V and V^* have the same cardinals and cofinality function*;

(iii) *V and V^* have the same countable sequences of ordinals*;

(iv) *If D is an array of filters in V, then*

$$V^* \models \text{"Every tree which is appropriate for } D \text{ has an } \omega_1\text{-branch"}. \qquad \square$$

§4. Consistency of SAD + there exists a Souslin tree

We show now that our principle does not imply the Souslin hypothesis.

Definition. For $\alpha \in \Omega, \langle V, S, A \rangle$ is a *possible situation* iff V and S are normal trees of height α, A is a function Dom $A = S$, and for $s \in S$, $A(s) \subseteq V$ is a (possibly void) subset of pairwise incompatible elements in V such that:

(1) $s < s' \Rightarrow A(s) \subseteq A(s')$,

(2) if $A(s) \neq \emptyset$ and $u \in V$ then for some $s' \geqq s$, $A(s')$ contains an element compatible with u.

At first, using a $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ we build (in L) a Souslin tree U having the following property: For all $\alpha \in \Omega$, if $S_\alpha = \langle U \mid \alpha, S, A \rangle$ is a possible situation, then for any $u \in U_\alpha$ and $s \in S$ such that $A(s) \neq \emptyset$ there is $s' > s$ such that u is above some element of $A(s')$.

Next we show that the construction of the boolean algebras $\langle B_\nu \mid \nu < \omega_2 \rangle$ can be carried out in such a way that $\| U$ is a Souslin tree$\|^{B_\alpha} = 1$ for all $\alpha < \omega_2$, hence getting the desired situation.

We shall deal here only in the case of the successor stage; the reader familiar with Jensen's iteration lemma will then have no problem in providing the argument for the limit stages.

So we have to strengthen Lemma 3.2, adding $\| U$ is Souslin$\|^B = 1$ to the hypothesis and getting $\| U$ is Souslin$\|^{\tilde{B}} = 1$ in the conclusion.

Let T be a Souslin tree, a Souslinisation of B, as in the proof of Lemma 3.2, D a principal array. We build $\tilde{T}$ as in the proof, but with the following change. In case α is a limit ordinal, $\gamma_\alpha = \alpha$, we add a further possibility: Suppose

$S_\alpha = \langle U \mid \alpha, \tilde{T} \mid \alpha, A \rangle$ is a possible situation such that the following hold: Let $x_0 \in T_{\gamma(\alpha)}$ and let $\langle U_n \mid n < \omega \rangle$ enumerate U_α. Then there exists an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ of limit ordinals, cofinal in α, such that for any n and $(x,f) \in Y_{x_0}, (x,f) \in \tilde{T} \mid \alpha_n$, if $A((x',f')) \neq \emptyset$ for some extension (x',f') of (x,f) in Y_{x_0} then there is such an extension in $\tilde{T}_{\alpha_n}$, with U_n above an element of $A((x',f'))$. In this case, we construct a normal subtree $W^*_{x_0}$ of W_{x_0}, closed under successors, with the additional property:

For any $(x',f') \in Y_{x_0} \cap \tilde{T}_{\alpha_n}$ if $f' \in W^*_{x_0}$ and $A((x',f')) \neq \emptyset$ then U_n is above some element in $A((x',f'))$. As before for each pair $(x,f) \in \tilde{T} \mid \alpha, x < x_0$ we find $h \supseteq f, h \in A_{\alpha,f}$ such that $(\forall \xi < \alpha)(h \mid \xi \in W^*_{x_0})$ (this time) and put (x_0, h) into $\tilde{T}_\alpha$. It follows in this case that if $A((x,f)) \neq \emptyset$ then $\bigcup_{s < (x_0,h)} A(s)$ is a maximal anti-chain in U!

The proof of 3.5 shows that $\tilde{T}$ is a Souslin tree. To prove that in forcing over $\tilde{T}$, U remains Souslin, suppose that A is a name in $V^{B(\tilde{T})}$ and $(x,f) \Vdash_{\tilde{T}}$ "A is a maximal anti-chain of $\check{U}$ and $\check{a} \in A$".

Define for $(\bar{x},\bar{f}) \geqq (x,f)$, $A((\bar{x},\bar{f})) = \{u \in U \mid (\bar{x},\bar{f}) \Vdash \check{u} \in A\}$ a countable pairwise imcompatible subset of U. (U is Souslin.) For $(\bar{x},\bar{f}) \ngeqq (x,f)$ define $A((\bar{x},\bar{f})) = \emptyset$.

Now construct an increasing and continuous chain of elementary submodels $M_\alpha \preccurlyeq H_{\omega_2}, \alpha < \omega_1$, such that $M_\alpha \in M_{\alpha+1}$ and $T, U, \tilde{T}, A \in M_\alpha$. M_α is collapsed to $\bar{M}_\alpha$ which is transitive and ω_1 is collapsed to δ_α. $\{\alpha \mid \delta_\alpha = \alpha\}$ is closed unbounded so we find $\alpha < \omega_1$ such that $\delta_\alpha = \alpha$, $\gamma_\alpha = \alpha$ and $S_\alpha = \langle U \mid \alpha, \tilde{T} \mid \alpha, A \mid (\tilde{T} \mid \alpha) \rangle$ is a possible situation. $U \mid \alpha, T \mid \alpha$ are Souslin trees in $\bar{M}_\alpha$ such that in $\bar{M}_\alpha$, $\| U \mid \alpha$ is Souslin$\|^{BA(T|\alpha)} = 1$. So $(U \mid \alpha) \times (T \mid \alpha)$ satisfies the C.C.C. in $\bar{M}_\alpha$, hence $\| T \mid \alpha$ is Souslin$\|^{BA(U|\alpha)} = 1$ in $\bar{M}_\alpha$. For any $u \in U_\alpha$, $\check{u} = \{a \mid a \in U \ \& \ a < u\}$ is an $\bar{M}_\alpha$-generic branch of $U \mid \alpha$, and in the generic extension $\bar{M}_\alpha[\hat{u}]$ the following holds because of the property of U: $\{(\bar{x},\bar{f}) \in \tilde{T} \mid \alpha \mid \hat{u} \cap A((\bar{x},\bar{f})) \neq \emptyset\} \in \bar{M}_\alpha[\hat{u}]$ is a dense subset of $\tilde{T} \mid \alpha$ above (x,f). Now let $x_0 \in T_\alpha, x < x_0$. As $T \mid \alpha$ is Souslin in $\bar{M}_\alpha[\hat{u}]$, $\hat{x}_0 = \{x \in T \mid \alpha \mid x < x_0\}$ is $\bar{M}_\alpha[\hat{u}]$-generic and we find that in $\bar{M}_\alpha[\hat{u}][\hat{x}_0]$ every $(x',f') \in \tilde{T} \mid \alpha$ such that $x' < x_0$ and $A((x',f')) \neq \emptyset$ has an extension $(x'',f'') \geqq (x',f')$, $x'' < x_0$ such that $A((x'',f'')) \cap \hat{u} \neq \emptyset$. It follows that the special case mentioned before holds and that we put $(x_0,h) \in \tilde{T}_\alpha$, $(x_0,h) > (x,f)$, such that $(x_0,h) \Vdash A \subseteq U \mid \alpha$ (hence is countable).

So we have proved $\| U$ is Souslin$\|^{BA(\tilde{T})} = 1$.

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THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL

UNIVERSITY OF LANCASTER
LANCASTER, ENGLAND

AND

THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL