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MINIMAL BOUNDED INDEX SUBGROUP FOR DEPENDENT THEORIES

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ABSTRACT. For a dependent theory T, in $\mathfrak{C}_T$ for every type definable group G, the intersection of type definable subgroups with bounded index is a type definable subgroup with bounded index.

§0. INTRODUCTION

Assume that T is a dependent (complete first-order) theory, $\mathfrak{C}$ is a $\bar{\kappa}$-saturated model of T (a monster), G is a type definable (in $\mathfrak{C}$) group in $\mathfrak{C}$ (of course we consider only types of cardinality $< \bar{\kappa}$).

A type definable subgroup H of G is call bounded if the index $(G : H)$ is $< \bar{\kappa}$. We prove that there is a minimal bounded definable subgroup. The first theorem on this line for T stable is due to Baldwin and Saxl [BaSx76].

Recently Hrushovski, Peterzil and Pillay [HPP0x] investigated definable groups, o-minimality and measure. In an earlier work on definable subgroups in o-minimal T in Berarducci, Otero, Peterzil and Pillay [BOPP05] the minimal type-definable bounded index theorem and more results are proved for o-minimal theories.

Hrushovski, in a lecture at the Hebrew University, mentioned that he, Peterzil and Pillay had observed the main result of the current paper, but assuming in addition the existence of an invariant measure on the group in question, and Hrushovski asked if the measure assumption could be removed. So we answer it positively. The current version of their paper [HPP0x] includes an exposition of our proof.

Recent works of the author on dependent theories are [Sh:783] (see §3,§4 on groups) [Sh:863] (e.g., the first-order theory of the p-adics is strongly^1 dependent but not strongly^2 dependent, see end of §1; on strongly^2 dependent fields, see §5) and [Sh:F705]. This work is continued in [Sh:F753] (getting mainly a parallel result for G abelian and $\mathbb{L}_{\infty,\bar{\kappa}}$-definable subgroups).

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§1

1.1 Theorem. *Assume T is a dependent (complete first-order) theory, $\mathfrak{C}$ a $\bar{\kappa}$-saturated model for T. We consider types of cardinality $< \bar{\kappa}$.*

1) *If $\circledast$ below holds, then:*

(α) *$q(\mathfrak{C})$ is a subgroup of $p(\mathfrak{C})$,*
(β) *$q(\mathfrak{C})$ is of index $< \bar{\kappa}$,*
(γ) *essentially $q(x) \backslash p(x)$ is of cardinality $\leq \lambda := |T|^{\aleph_0}$ (i.e., for some $q'(x) \subseteq q(x)$ of cardinality $\leq \lambda, q(x)$ is equivalent to $p(x) \cup q'(x)$),*
(δ) *we can strengthen $(\alpha), (\beta)$ to $q(\mathfrak{C})$ is a subgroup of index $\leq 2^\lambda$,*

where

$\circledast$ (*a*) *$p(x)$ is a type such that $p(\mathfrak{C})$ is a group which we call G (with some definable operations xy, x^{-1} and the identity e_G which is constant here),*
(*b*) *$q(x) = p(x) \cup \bigcup\{r(x) : r(x) \in \mathbf{R}\}$, where*
(*c*) *$\mathbf{R} = \{r(x) : r(x)$ a type such that $(p \cup r)(\mathfrak{C})$ is a subgroup of $p(\mathfrak{C})$ of index $< \bar{\kappa}\}$.*

2) *There exists some $q' \subseteq q$ over* $\mathrm{Dom}(p)$, *equivalent to q and such that $|q'| \leq |T| + |\mathrm{Dom}(p)|$. So $(p(\mathfrak{C}) : q(\mathfrak{C})) \leq 2^{|\mathrm{Dom}(p)|+|T|}$.*

3) *If $r_i(x) \in \mathbf{R}$ for $i < \lambda^+$, then for some $\alpha < (|T|^{\aleph_0})^+$ we have $(p(x) \cup \bigcup\{r_i(x) : i < \alpha\})(\mathfrak{C}) = (p(x) \cup \bigcup\{r_i(x) : i < \lambda^+\})(\mathfrak{C})$.*

Proof. 1) Note

$\circledast_1$ $\mathbf{R}$ is closed under unions of $< \bar{\kappa}$.

$\circledast_2$ (*a*) If $r(x) \in \mathbf{R}, r'(x) \subseteq r(x)$ is countable, then there is a countable $r''(x) \subseteq r(x)$ including $r'(x)$ which belongs to $\mathbf{R}$.
(*b*) If $p(x) \subseteq r(x) \in \mathbf{R}$ and $r(x)$ is closed under conjunctions and $r'(x) \subseteq r(x)$ is countable, then we can find $\psi_n(x, \bar{b}_n)$ for $n < \omega$ such that
(α) $\psi_n(x, \bar{b}_n) \in r(x)$,
(β) $\psi_{n+1}(x, \bar{b}_{n+1}) \vdash \psi_n(x, \bar{b}_n)$,
(γ) $\psi_{n+1}(x, \bar{b}_{n+1}), \psi_{n+1}(y, \bar{b}_{n+1}) \vdash \psi_n(x^{-1}y, \bar{b}_n) \wedge \psi_n(x^{-1}, \bar{b}_n) \wedge \psi_n(xy, \bar{b}_n)$,
(δ) $\psi_n(x, \bar{b}_{m+1}) \vdash \varphi_n(x, \bar{a}_n)$, where $\{\varphi_n(x, \bar{a}_n) \colon n < \omega\}$ list r',
(ε) $\mathfrak{C} \models \psi_n(e_G, \bar{b}_n)$, actually follows from clause (α).
[Why? Let $r'(x) = \{\varphi_n(x, \bar{a}_n) : n < \omega\}$ (can use $\varphi_n = (x = x)$). Without loss of generality, $r(x)$ is closed under conjunctions and also $r'(x)$ is. Now we choose $\psi_n(x, \bar{b}_n)$ by induction on $n < \omega$ such that $\psi_{n+1}(x, \bar{b}_{n+1}) \wedge \psi_{n+1}(y, \bar{b}_{n+1}) \vdash \psi_n(xy^{-1}, \bar{b}_n) \wedge \varphi_n(x, \bar{a}_n) \wedge \psi_n(xy, \bar{b}_n)$; notice that trivially $e_G \in \varphi_n(\mathfrak{C}, \bar{a}_n) \cap \psi_n(\mathfrak{C}, \bar{b}_n)$. Such a formula exists as $(p(x) \cup r(x)) \cup (p(y) \cup r(y)) \vdash \psi_n(xy^{-1}, \bar{b}_n) \wedge \varphi_n(x, \bar{a}_n) \wedge \psi_n(xy, \bar{b}_n)$.
Now $r''(x) = \{\varphi_n(x, \bar{a}_n), \psi_n(x, \bar{b}_n) : n < \omega\}$ is as required in clause (a), $\langle \psi_n(x, \bar{b}_n) : n < \omega \rangle$ in (*b*).]

In the conclusion of Theorem 1.1, Clause (α) is obvious.

Assume toward a contradiction that the conclusion $(\beta) + (\gamma)$ fails. So we can choose (c_α, r_α) by induction on $\alpha < \lambda^+$ such that

$\circledast_3$ (*a*) $c_\alpha \in (p(x) \cup \bigcup\{r_\beta : \beta < \alpha\})(\mathfrak{C})$,
(*b*) $r_\alpha = \{\psi^\alpha_n(x, \bar{b}^\alpha_n) : n < \omega\} \subseteq q$ and $\bar{b}^\alpha_n \triangleleft \bar{b}^\alpha_{n+1}$,
(*c*) $r_\alpha \in \mathbf{R}$, and $\psi_{n+1}(x, \bar{b}^\alpha_{n+1}) \vdash \psi_n(x, \bar{b}^\alpha_n)$,
(*d*) c_α does not realize r_α, in fact $\mathfrak{C} \models \neg\psi^\alpha_0(c_\alpha, \bar{b}^\alpha_0)$, hence $c_\alpha \notin q(\mathfrak{C})$.

[Why? Arriving to α, we try to let $q_\alpha = p(x) \cup \bigcup\{r_\beta : \beta < \alpha\}$, so $q_\alpha \subseteq q$ and by cardinality considerations, $q_\alpha(\mathfrak{C})$ is a subgroup of G of index $< \bar\kappa$. So by our assumption toward a contradiction, $q_\alpha(x) \nVdash q(x)$; hence there is $r^*_\alpha(x) \in \mathbf{R}$ such that $q_\alpha(x) \nVdash r^*_\alpha$, so $q_\alpha(x) \nVdash \vartheta_\alpha(x, \bar d_\alpha)$ for some $\vartheta_\alpha(x, \bar d_\alpha) \in r^*_\alpha$. Let $r_\alpha(x) = \{\psi^\alpha_n(x, \bar b^\alpha_n) : n < \omega\} \subseteq r^*_\alpha(x)$ belong to $\mathbf{R}$ and be such that $\vartheta_\alpha(x, \bar d_\alpha) = \psi^\alpha_0(x, \bar b^\alpha_0)$; it exists by $\circledast_2(\alpha)$ above.

Without loss of generality, we can assume $\circledast_4 - \circledast_8$:

$\circledast_4$ e_G is an individual constant, $\bar y^\alpha_n \trianglelefteq \bar y^\alpha_{n+1}$ and $\psi^\alpha_{n+1}(x, \bar y^\alpha_{n+1}) \vdash \psi^\alpha_n(x, \bar y^\alpha_n)$ and $(\psi^\alpha_{n+1}(x_1, \bar y^\alpha_{n+1}) \wedge \psi^\alpha_{n+1}(x_2, \bar y^\alpha_{n+1})) \vdash (\psi^\alpha_n(e_G, \bar y^\alpha_n) \wedge \psi^\alpha_n(x_1 x_2^{-1}, \bar y^\alpha_n))$.

[Why? As in the proof of $\circledast_2$ above, i.e., during the induction in the proof of $\circledast_3$, we use $\circledast_2(\beta)$ and get $\psi^\alpha_n(x, \bar y^\alpha_n), \bar b^\alpha_n$ and without loss of generality $\bar b^\alpha_n \trianglelefteq \bar b^\alpha_{n+1}, \bar y^\alpha_n \triangleleft \bar y^\alpha_{n+1}$ (as we can change the order and name the free variable and add dummy variables). Now we define $\psi^{\alpha,*}_n(x, \bar y^\alpha_n)$ by induction on n by

$$\begin{aligned}\psi^{\alpha,*}_0(x, \bar y^\alpha_0) = \psi^\alpha_0(x, \bar y^\alpha_0), \psi^{\alpha,*}_{n+1}(x, \bar y^\alpha_{n+1}) &= \psi^\alpha_{n+1}(x, \bar y^\alpha_{n+1}) \\ &\wedge \bigwedge_{m \le n} (\forall z)[\psi^\alpha_{m+1}(z, \bar y^\alpha_{m+1}) \to \psi^\alpha_m(z, \bar y^\alpha_m)] \\ &\wedge \bigwedge_{m \le n} (\forall z_1, z_2)[\psi^\alpha_{m+1}(z_1, \bar y^\alpha_{m+1}) \wedge \psi^\alpha_{m+1}(z_2, \bar y^\alpha_{m+1}) \\ &\qquad \to \psi^\alpha_m(e_G, \bar y^\alpha_m) \wedge \psi^\alpha_m(z_1 z_2^{-1}, \bar y^\alpha_m)].\end{aligned}$$

So clearly $\langle \psi^{\alpha,*}_n(x, \bar y^\alpha_n) : n < \omega\rangle$ satisfies $\circledast_4$ and $\mathfrak{C} \models (\forall x)[\psi^{\alpha,*}_n(x, \bar b^\alpha_n) \equiv \psi^\alpha_n[x, \bar b^\alpha_n)]$. So renaming we are done.]

$\circledast_5$ $\psi^\alpha_n(x, \bar y^\alpha_n) = \psi_n(x, \bar y_n)$ and $\psi_{n+1}(x, \bar y_{n+1})) \vdash \psi_n(x, \bar y_n)$.

[Why? By the pigeon hole principle.]

$\circledast_6$ $\langle c_\alpha \bar{\mathbf{a}}_\alpha : \alpha < \lambda^+\rangle$ is an indiscernible sequence over $\mathrm{Dom}(p)$ where $\bar{\mathbf{a}}_\alpha = \cup\{b^{-\alpha}_n : n < \omega\}\bar b^\alpha_0{}^\frown \bar b^\alpha_1{}^\frown \bar b^\alpha_2{}^\frown \ldots$; note that by clause (b) of $\circledast_3$ we have $\bar b^\alpha_n = \bar{\mathbf{a}}_\alpha \restriction k_n$.

[Why? By the Ramsey theorem and compactness.]

$\circledast_7$ If $\alpha < \beta < \gamma$, then $c_\alpha c_\beta^{-1} \in r_\gamma(\mathfrak{C})$.

[Why? By the indiscernibility, without loss of generality, γ is infinite, so $\gamma \ge \omega$ and $\langle c_i : i < \gamma\rangle$ is an indiscernible sequence over $\mathrm{Dom}(p) \cup \bar b_\gamma$ of elements of $p(\mathfrak{C})$ pairwise nonequivalent modulo the subgroup $G_\gamma = (p \cup r_\gamma)(\mathfrak{C})$. Then we can extend it to $\langle \bar c_i : i < \bar\kappa\rangle$, an indiscernible sequence over $\mathrm{Dom}(p) \cup \bar b_\gamma$ and arrive at $\alpha < \beta \Rightarrow c_\alpha c_\beta^{-1} \notin G_\gamma \Rightarrow c_\beta c_\alpha^{-1} \notin G_\gamma$ so $\langle c_\alpha G_\gamma : \alpha < \bar\kappa\rangle$ are pairwise distinct (equivalently $\langle G_\gamma c_\alpha : \alpha < \bar\kappa\rangle$ are pairwise distinct), a contradiction.]

$\circledast_8$ $c_\alpha \in r_\beta(\mathfrak{C})$ iff $\alpha \ne \beta$.

[Why? Let

$$c^*_\alpha = c_{2\alpha+1} \cdot (c_{2\alpha})^{-1},$$
$$r^*_\alpha = r_{2\alpha}.$$

So:

(*i*) If $\beta < \alpha$, then $c^*_\alpha \in (p \cup r^*_\beta)(\mathfrak{C})$, because $c_{2\alpha+1}, c_{2\alpha}$ belong to the subgroup $(p \cup r_{2\beta})(\mathfrak{C})$ by clause (a) of $\circledast_3$.

(*ii*) If $\beta > \alpha$, then c^*_α belongs to $(p \cup r^*_\beta)(\mathfrak{C})$ by $\circledast_7$.

(*iii*) If $\beta = \alpha$, then c^*_α does not belong to $(p \cup r^*_\beta)(\mathfrak{C})$ because:

(α) it is a subgroup,
(β) $c_{2\alpha+1}$ belongs to it by clause (a) of $\circledast_3$ and
(γ) $c_{2\alpha}$ does not belong to it by clause (e) of $\circledast_3$.

Let $\bar{\mathbf{a}}^*_\alpha = \bar{\mathbf{a}}_{2\alpha+1}, \bar{b}^{\alpha,*}_n = \bar{b}^{2\alpha}_n$ retaining the same ψ's. So we have obtained an example as required in $\circledast_8$ (not losing the other demands).]

$\circledast_9$ For some $n < \omega$ for every α, we have if $d_1, d_2 \in (p \cup r_\alpha)(\mathfrak{C})$, then $d_1 c_\alpha d_2 \notin \psi_n(\mathfrak{C}, \bar{b}^\alpha_n)$; without loss of generality, $n = 1$.
[Why? Fix α. If this holds for some $\psi_n(-, \bar{b}^\alpha_n)$, by indiscernibility, renaming the φ_i's, this is O.K. Otherwise for each $n < \omega$ there are $d^n_1, d^n_2 \in (p \cup r_\alpha)(\mathfrak{C})$ such that $\mathfrak{C} \models \psi_n(d^n_1 c_\alpha d^n_2, \bar{b}^\alpha_n)$. By compactness for some $d^*_1, d^*_2 \in (p \cup r_\alpha)(\mathfrak{C})$ we have $\models \psi_n[d^*_1 c_\alpha d^*_2, \bar{b}^\alpha_n]$ for every $n < \omega$. So $d^*_1 c_\alpha d^*_2$ belongs to the subgroup $(p \cup r_\alpha)(\mathfrak{C})$, but also d^*_1, d^*_2 belongs to it; hence c_α belongs, a contradiction. Alternatively, note that $n = 2$ is O.K.: let $c' = d_1 c d_2$ and assume toward a contradiction that $c' \in \psi_2(\mathfrak{C}, \bar{b}_\alpha))$ and let $d'_1 = (d_1)^{-1}, d'_2 = (d_2)^{-1}$, so clearly $d_1, d'_2 \in (p \cup r_\alpha)(\mathfrak{C}) \subseteq \psi_2(\mathfrak{C}, \bar{b}^\alpha_2)$. Now by $\circledast_4$ as $d'_1, c' \in \psi_2(\mathfrak{C}, \bar{b}^\alpha_{\alpha_1})$ it follows that $d_2(c')^{-1} \in \psi_1(\mathfrak{C}, \bar{b}^\alpha_1)$. As $d_1, d'_2(c')^{-1} \in \psi_1(\mathfrak{C}, \bar{b}^\alpha_1)$ by $\circledast_4$ we have $d'_1(d_2(c')^{-1})^{-1} \in \psi_0(\mathfrak{C}, \bar{b}^\alpha_0)$. But $c' = d_1 c_\alpha d_2$; hence $c_\alpha = d'_1((d'_2)^{-1}(c^1)^{-1})^{-1} = d'_1(d_2(c')^{-1})^{-1}$, but $d'_1(d_2(c')^{-1})^{-1} \in \psi_0(\mathfrak{C}, \bar{b}^\alpha_0)$ by the previous sentence, whereas $c_\alpha \notin \psi_0(\mathfrak{C}, \bar{b}^\alpha_1)$ by $\circledast_3(e)$, a contradiction.]

$\circledast_{10}$ If $w = \{i_1, \ldots, i_n\}, i_1 < \ldots < i_n < \lambda^+$, and $d_w := c_{i_1} c_{i_2} \ldots c_{i_n} \in G$ and $\alpha < \lambda^+$, then $\models \varphi_1[d_w, \bar{b}^1_\alpha] \Leftrightarrow \alpha \notin w$.
[Why? If $\alpha \in w$, let k be such that $\alpha = i_k$, so $c_{i_1}, \ldots, c_{i_{k-1}} \in (p \cup r_\alpha)(\mathfrak{C})$ by $\circledast_8$ and similarly $c_{i_{k+1}} \ldots c_{i_n} \in (p \cup r_\alpha)(\mathfrak{C})$; hence

$$d_w = (c_{i_1} \ldots c_{i_k}) c_{i_k} (c_{i_{k+1}} \ldots c_{i_n}) \notin (p \cup r_\alpha)(\mathfrak{C})$$

by $\circledast_9$.
Second, if $\alpha \notin w$, this holds by $\circledast_8$ as $\{c_{i_\ell} : \ell < n\}$ is included in the subgroup $(p \cup r_\alpha)(\mathfrak{C})$.]

So we get a contradiction to "T is dependent"; hence clauses $(\beta), (\gamma)$ hold. Also clause (δ) follows by the following observation:

Observation: If $r(x) \in \mathbf{R}$ and $|r(x)| \leq \theta$, then $(p(\mathfrak{C}) : (p \cup r)(\mathfrak{C})) \leq 2^\theta$ (except for being just finite when θ is finite).

Proof. If θ is finite, then the proof follows by compactness. If θ is infinite, then without loss of generality, r is closed under conjunctions. Let $r = \{\varphi_i(x, \bar{\mathbf{b}}) : i < \theta\}$, where $\bar{\mathbf{b}}$ is possibly infinite.

Let u be a set of ordinals $(< \bar{\kappa})$ such that $\bar{\kappa} > |u| > (p(\mathfrak{C}) : (p \cup r)(\mathfrak{C}))$. Now for each $i < \theta$, let $\Gamma_{i,u} = \bigcup\{p(x_\alpha) : \alpha \in u\} \cup \{\neg\varphi_i(x_\alpha x^{-1}_\beta, \bar{\mathbf{b}}) : \alpha < \beta \text{ from } u\}$. So for some finite $u^*_i \subseteq u, \Gamma_{i,u^*_i}$ is contradictory, so Γ_{i,n_i} is contradictory, when $n_i = |u_i|$. It suffices to use $(2^\theta)^+ \rightarrow (\ldots n_i \ldots)_{i<\theta}$ (why? let $\langle c_\alpha : \alpha < (2^\theta)^+\rangle$ exemplify the failure and let $\zeta_{\alpha,\beta} = \text{ Min}\{i : \models \neg\varphi_i(c_\alpha c^{-1}_\beta, \bar{\mathbf{b}})\}$).

This finishes the proof of part (1). We still need to prove 2), 3).

2) Let $q'(x) \subseteq q(x)$ have cardinality $\leq |T|^{\aleph_0}$ and be such that $q(\mathfrak{C}) = (p \cup q')(\mathfrak{C})$; $q'(x)$ exists by part (1). Observe that every automorphism of $\mathfrak{C}$ fixing $\text{Dom}(p)$ maps $p(\mathfrak{C})$ onto itself and therefore maps $q(\mathfrak{C})$ onto itself.

It follows that if $c_1, c_2 \in p(\mathfrak{C})$ are such that $\text{tp}(c_1, \text{Dom}(p)) = \text{tp}(c_2, \text{Dom}(p))$, then $c_1 \in q(\mathfrak{C})$ if and only if $c_2 \in q(\mathfrak{C})$. Let $\mathbf{P} := \{\text{tp}(b, \text{Dom}(p)) : b \in q(\mathfrak{C})\}, \mathbf{P}(\mathfrak{C}) := \bigcup\{r(\mathfrak{C}) : r \in \mathbf{P}\}$. Then by the above explanation, $\mathbf{P}(\mathfrak{C}) \subseteq q(\mathfrak{C})$. By definition, $q(\mathfrak{C}) \subseteq p(\mathfrak{C})$, so they are equal. Let $q_{**} = \bigcap\{r : r \in \mathbf{P}\}$, so we have $q(\mathfrak{C}) \subseteq q_{**}(\mathfrak{C})$.

If they are equal, then we are done. Otherwise take $c_1 \in q_{**}(\mathfrak{C}) \backslash q(\mathfrak{C})$. Without loss of generality, let $\psi(x, \bar{d}) \in q$ be such that $\models \neg\psi(c_1, \bar{d})$.

By definition of $\mathbf{P}$ and c_1, for each $\theta(x, \bar{e}) \in \text{tp}(c_1, \text{Dom}(p))$ there exists some $p_{\theta(x,\bar{e})} \in \mathbf{P}$ such that $\theta(x, \bar{e}) \in p_{(x,\bar{e})}$ and therefore some $c_{\theta(x,\bar{e})} \in q(\mathfrak{C})$ satisfies $\theta(x, \bar{e})$. So $\text{tp}(c_1, \text{dom}(p)) \cup q'(x)$ is finitely satisfiable and is therefore realized by some c_2. Thus $\text{tp}(c_1, \text{Dom}(p)) = \text{tp}(c_2, \text{Dom}(p))$, but $c_1 \notin q'(\mathfrak{C}) = q(\mathfrak{C})$ and $c_2 \in (p \cup q')(\mathfrak{C})$, a contradiction.

3) By the proof of part (1). $\square_{1.1}$

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