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Algebra Universalis

On Depth and Depth$^+$ of Boolean Algebras

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Abstract. We show that the Depth$^+$ of an ultraproduct of Boolean Algebras cannot jump over the Depth$^+$ of every component by more than one cardinal. Consequently we have similar results for the Depth invariant.

1. introduction

Monk [2] has dealt systematically with cardinal invariants of Boolean algebras. In particular he dealt with the question how an invariant of an ultraproduct of a sequence of Boolean algebras relates to the ultraproduct of the sequence of the invariants of each of the Boolean algebras. That is, the relationship of $\operatorname{inv}(\prod_{\epsilon<\kappa} \mathbf{B}_\epsilon/D)$ with $\prod_{\epsilon<\kappa} \operatorname{inv}(\mathbf{B}_\epsilon)/D$. One of the invariants he dealt with is the depth of a Boolean algebra, Depth($\mathbf{B}$). We continue [7] here, obtaining weaker results without "large cardinal axioms". On related results see [1], [6], [3]. Further results on Depth and Depth$^+$ by the authors are contained in [4].

Definition 1.1. Let $\mathbf{B}$ be a Boolean Algebra.

$$\operatorname{Depth}(\mathbf{B}) := \sup\{\theta : \exists \bar{b} = (b_\gamma : \gamma < \theta),\ \text{increasing sequence in } \mathbf{B}\}.$$

Dealing with questions of Depth, Saharon Shelah noticed that investigating a slight modification of Depth, namely - Depth$^+$, might be helpful (see [7] for the behavior of Depth and Depth$^+$ above a compact cardinal).

Definition 1.2. Let $\mathbf{B}$ be a Boolean Algebra.

$$\operatorname{Depth}^+(\mathbf{B}) := \sup\{\theta^+ : \exists \bar{b} = (b_\gamma : \gamma < \theta),\ \text{increasing sequence in } \mathbf{B}\}.$$

This article deals mainly with Depth$^+$, in the aim to get results for the Depth. It follows [7], both in the general ideas and in the method of the proof.

Let us take a look at the main claim of [7]:

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Claim 1.3. *Assume*

(a) $\kappa < \mu \leq \lambda$.
(b) μ *is a compact cardinal.*
(c) $\lambda = \mathrm{cf}(\lambda)$.
(d) $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$.
(e) $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$.
(f) $\mathbf{B} = \prod_{i<\kappa} \mathbf{B}_i / D$.

Then $\mathrm{Depth}^+(\mathbf{B}) \leq \lambda$.

So, λ bounds the $\mathrm{Depth}^+(\mathbf{B})$, where $\mathbf{B}$ is an ultraproduct of the Boolean Algebras $\mathbf{B}_i$, if it bounds the Depth^+ of every $\mathbf{B}_i$. That requires some reasonable assumptions on λ, and also a pretty high price for that result — you should raise your view to a very large λ, above a compact cardinal. Now, the existence of large cardinals is an interesting philosophical question. You might think that adding a compact cardinal to your world is a natural extension of ZFC. But, mathematically, it is important to check what happens without a compact cardinal (or below the compact, even if the compact cardinal exists).

In this article we drop the assumption of a compact cardinal. Consequently, we phrase a weaker conclusion. We prove that if λ bounds the Depth^+ of every $\mathbf{B}_i$, then the Depth^+ of $\mathbf{B}$ cannot jump beyond λ^+.

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2. Bounding Depth^+

Notation 2.1. (a) κ, λ are infinite cardinals.
(b) D is an ultrafilter on κ.
(c) $\mathbf{B}_i$ is a Boolean Algebra, for any $i < \kappa$.
(d) $\mathbf{B} = \prod_{i<\kappa} \mathbf{B}_i / D$.

We now state our main result:

Theorem 2.2. *Assume*

(a) $\lambda = \mathrm{cf}(\lambda)$,
(b) $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$,
(c) $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda$ *for every* $i < \kappa$.

Then $\mathrm{Depth}^+(\mathbf{B}) \leq \lambda^+$.

Remark 2.3. We can improve 2.2 (b), demanding only $\lambda^\kappa = \lambda$. We intend to give a detailed proof in a subsequent paper.

Corollary 2.4. *Assume*

(a) $\lambda^\kappa = \lambda$;
(b) $\mathrm{Depth}(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$.

Then $\mathrm{Depth}(\mathbf{B}) \leq \lambda^+$.

Proof. By (b), $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda^+$ for every $i < \kappa$. By (a), $\alpha < \lambda^+ \Rightarrow |\alpha|^\kappa < \lambda^+$. Now, λ^+ is a regular cardinal, so the pair (κ, λ^+) satisfies the requirements of Theorem 2.2. So, $\mathrm{Depth}^+(\mathbf{B}) \leq \lambda^{+2}$, and that means that $\mathrm{Depth}(\mathbf{B}) \leq \lambda^+$. □

Remark 2.5. If λ is inaccessible (or even strong limit, with cofinality above κ), and $\mathrm{Depth}(\mathbf{B}_i) < \lambda$ for every $i < \kappa$, you can easily verify that $\mathrm{Depth}(\mathbf{B}) < \lambda$, using Theorem 2.2 and simple cardinal arithmetic.

Proof of Theorem 2.2. Let $\langle M_\alpha : \alpha < \lambda^+ \rangle$ be a continuous and increasing sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ for sufficiently large χ with the following properties:

(a) $(\forall \alpha < \lambda^+)(\|M_\alpha\| = \lambda)$,
(b) $(\forall \alpha < \lambda^+)(\lambda + 1 \subseteq M_\alpha)$,
(c) $(\forall \beta < \lambda^+)(\langle M_\alpha : \alpha \leq \beta \rangle \in M_{\beta+1})$.

Choose $\delta^* \in S^{\lambda^+}_\lambda (:= \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\})$, such that $\delta^* = M_{\delta^*} \cap \lambda^+$. Assume toward a contradiction that $(a_\alpha : \alpha < \lambda^+)$ is an increasing sequence in $\mathbf{B}$. Let us write a_α as $\langle a^\alpha_i : i < \kappa \rangle / D$ for every $\alpha < \lambda^+$. We may assume that $\langle a^\alpha_i : \alpha < \lambda^+, i < \kappa \rangle \in M_0$.

We will try to create a set Z, in the Lemma below, with the following properties:

(a) $Z \subseteq \lambda^+, |Z| = \lambda$,
(b) $\exists i_* \in \kappa$ such that for every $\alpha < \beta, \alpha, \beta \in Z$, we have $\mathbf{B}_{i_*} \models a^\alpha_{i_*} < a^\beta_{i_*}$.

Since $|Z| = \lambda$, we have an increasing sequence of length λ in $\mathbf{B}_{i_*}$, so $\mathrm{Depth}^+(\mathbf{B}_{i_*}) \geq \lambda^+$, contradicting the assumptions of the claim. □

Lemma 2.6. *There exists* Z *as above.*

Proof. For every $\alpha < \beta < \lambda^+$, define:

$$A_{\alpha,\beta} = \{i < \kappa : \mathbf{B}_i \models a^\alpha_i < a^\beta_i\}$$

By the assumption, $A_{\alpha,\beta} \in D$ for all $\alpha < \beta < \lambda^+$. For all $\alpha < \delta^*$, let A_α denote the set A_{α,δ^*}.

Let $\langle v_\alpha : \alpha < \lambda \rangle$ be increasing and continuous, such that for every $\alpha < \lambda$,

(i) $v_\alpha \in [\delta^*]^{<\lambda}$ for every $\alpha < \lambda$,
(ii) v_α has no last element, for every $\alpha < \lambda$,
(iii) $\delta^* = \bigcup_{\alpha < \lambda} v_\alpha$.

Let $u \subseteq \delta^*$, $|u| \leq \kappa$. Define

$$S_u = \{\beta < \delta^* : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

Now define

$$C = \{\delta < \lambda : \delta \text{ is a limit ordinal and } (\forall \alpha < \delta)[(u \subseteq v_\alpha) \wedge (|u| \leq \kappa) \Rightarrow \sup(v_\delta) = \sup(S_u \cap \sup(v_\delta))]\}.$$

Since $\lambda = \mathrm{cf}(\lambda)$ and $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$, and since $|v_\delta| < \lambda$ for all $\delta < \lambda$, C is a club set of λ.

The fact that $|D| = 2^\kappa < \mathrm{cf}(\lambda) = \lambda$ implies that there exists $A_* \in D$ such that $S = \{\alpha < \lambda : \mathrm{cf}(\alpha) > \kappa \text{ and } A_{\sup(v_\alpha)} = A_*\}$ is a stationary subset of λ.

C is a club and S is stationary, so $C \cap S$ is also stationary. Choose $\delta^1_0 = \min(\mathrm{C}\cap\mathrm{S})$. Choose $\delta^1_{\epsilon+1} \in C \cap S$ for every $\epsilon < \lambda$ such that $\epsilon < \zeta \Rightarrow \sup\{\delta^1_{\epsilon+1} : \epsilon < \zeta\} < \delta^1_{\zeta+1}$. Define δ^1_ϵ to be the limit of $\delta^1_{\gamma+1}$, when $\gamma < \epsilon$, for every limit $\epsilon < \lambda$. Since C is closed, we have

(a) $\{\delta^1_\epsilon : \epsilon < \lambda\} \subseteq C$;
(b) $\langle \delta^1_\epsilon : \epsilon < \lambda \rangle$ is increasing and continuous;
(c) $\delta^1_{\epsilon+1} \in S$, for every $\epsilon < \lambda$.

Lastly, define $\delta^2_\epsilon = \sup(v_{\delta^1_\epsilon})$, for every $\epsilon < \lambda$. Define, for every $\epsilon < \lambda$, the family

$$\mathfrak{A}_\epsilon = \{S_u \cap \delta^2_{\epsilon+1} \setminus \delta^2_\epsilon : u \in [v_{\delta^2_{\epsilon+1}}]^{\leq \kappa}\}.$$

We get a family of non-empty sets, which is downward κ^+-directed. So, there is a κ^+-complete filter E_ϵ on $[\delta^2_\epsilon, \delta^2_{\epsilon+1})$, with $\mathfrak{A}_\epsilon \subseteq E_\epsilon$, for every $\epsilon < \lambda$.

Define, for any $i < \kappa$ and $\epsilon < \lambda$, the sets $W_{\epsilon,i} \subseteq [\delta^2_\epsilon, \delta^2_{\epsilon+1})$ and $B_\epsilon \subseteq \kappa$, by:

$$W_{\epsilon,i} := \{\beta : \delta^2_\epsilon \leq \beta < \delta^2_{\epsilon+1} \text{ and } i \in A_{\beta,\delta^2_{\epsilon+1}}\},$$
$$B_\epsilon := \{i < \kappa : W_{\epsilon,i} \in E^+_\epsilon\}.$$

Finally, take a look at $W_\epsilon := \cap\{[\delta^2_\epsilon, \delta^2_{\epsilon+1}) \setminus W_{\epsilon,i} : i \in \kappa \setminus B_\epsilon\}$. For every $\epsilon < \lambda, W_\epsilon \in E_\epsilon$, since E_ϵ is κ^+-complete, so clearly $W_\epsilon \neq \emptyset$.

Choose $\beta = \beta_\epsilon \in W_\epsilon$. If $i \in A_{\beta,\delta^2_{\epsilon+1}}$, then $W_{\epsilon,i} \in E^+_\epsilon$, so $A_{\beta,\delta^2_{\epsilon+1}} \subseteq B_\epsilon$ (by the definition of B_ϵ). But, $A_{\beta,\delta^2_{\epsilon+1}} \in D$, so $B_\epsilon \in D$, and consequently $A_* \cap B_\epsilon \in D$, for any $\epsilon < \lambda$.

Choose $i_\epsilon \in A_* \cap B_\epsilon$, for every $\epsilon < \lambda$. You choose λ i_ϵ-s from A_*, and $|A_*| = \kappa$, so we can arrange a fixed $i_* \in A_*$ such that the set $Y = \{\epsilon < \lambda : \epsilon$ is an even ordinal, and $i_\epsilon = i_*\}$ has cardinality λ.

The last step will be as follows: define $Z = \{\delta^2_{\epsilon+1} : \epsilon \in Y\}$. Clearly, $Z \in [\delta^*]^\lambda \subseteq [\lambda^+]^\lambda$. We will show that for $\alpha < \beta$ from Z we get $\mathbf{B}_{i_*} \models a^\alpha_{i_*} < a^\beta_{i_*}$. The idea is that if $\alpha < \beta$ and $\alpha, \beta \in Z$, then $i_* \in A_{\alpha,\beta}$.

Why? Recall that $\alpha = \delta^2_{\epsilon+1}$ and $\beta = \delta^2_{\zeta+1}$, for some $\epsilon < \zeta < \lambda$ (that's the form of the members of Z). Define

$$U_1 = S_{\{\delta^2_{\epsilon+1}\}} \cap [\delta^2_\zeta, \delta^2_{\zeta+1}) \in \mathfrak{A}_\zeta \subseteq E_\zeta.$$
$$U_2 = \{\gamma : \delta^2_\zeta \leq \gamma < \delta^2_{\zeta+1} \text{ and } i_* \in A_{\gamma,\delta^2_{\zeta+1}}\} \in E^+_\zeta.$$

So, $U_1 \cap U_2 \neq \emptyset$.

Choose $\iota \in U_1 \cap U_2$. Now the following statements hold:

(a) $\mathbf{B}_{i_*} \models a^\alpha_{i_*} < a^\iota_{i_*}$. [Why? Well, $\iota \in U_1$, so $A_{\delta^2_{\epsilon+1,\iota}} = A_{\delta^2_{\epsilon+1}} = A_*$. But, $i_* \in A_*$, so $i_* \in A_{\delta^2_{\epsilon+1,\iota}}$, which means that $\mathbf{B}_{i_*} \models a^{\delta^2_{\epsilon+1}}_{i_*} (= a^\alpha_{i_*}) < a^\iota_{i_*}$].

(b) $\mathbf{B}_{i_*} \models a^\iota_{i_*} < a^\beta_{i_*}$. [Why? Well, $\iota \in U_2$, so $i_* \in A_{\iota,\delta^2_{\zeta+1}}$, which means that $\mathbf{B}_{i_*} \models a^\iota_{i_*} < a^{\delta^2_{\zeta+1}}_{i_*} (= a^\beta_{i_*})$].

(c) $\mathbf{B}_{i_*} \models a^\alpha_{i_*} < a^\beta_{i_*}$. [Why? By (a)+(b)].

So, we are done. □

Without a compact cardinal, we may have a 'jump' of the Depth$^+$ in the ultraproduct of the Boolean Algebras (see [5, §5]). So, we can have $\kappa < \lambda$, $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$ for every $i < \kappa$, and $\text{Depth}^+(\mathbf{B}) = \lambda^+$. We can show that if there exists such an example for κ and λ, then you can create an example for every regular θ between κ and λ.

Claim 2.7. *Assume*

(a) $\kappa < \lambda, D$ *is an ultrafilter on* κ
(b) $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$, *for every* $i < \kappa$
(c) $\text{Depth}^+(\mathbf{B}) = \lambda^+$
(d) $\theta \in \text{Reg} \cap [\kappa, \lambda)$.

Then there exist Boolean algebras $\mathbf{C}_j$, $j < \theta$, *and a uniform ultrafilter* E *on* θ *such that* $\text{Depth}^+(\mathbf{C}_j) \leq \lambda$ *for every* $j < \theta$ *and* $\text{Depth}^+(\mathbf{C}) := \text{Depth}^+(\prod_{j<\theta} \mathbf{C}_j / E) = \lambda^+$.

Proof. Break θ into θ sets $(u_\alpha : \alpha < \theta)$ such that for every $\alpha < \theta$,

(a) $|u_\alpha| = \kappa$,
(b) $\bigcup_{\alpha<\theta} u_\alpha = \theta$,
(c) $\alpha \neq \beta \Rightarrow u_\alpha \cap u_\beta = \emptyset$.

For every $\alpha < \theta$, let $f_\alpha : \kappa \to u_\alpha$ be one to one, onto and order preserving. Define D_α on u_α in the following way: if $A \subseteq u_\alpha$, then $A \in D_\alpha$ iff $f^{-1}_\alpha(A) \in D$. For θ itself, define a filter E_* on θ in the following way: if $A \subseteq \theta$, then $A \in E_*$ iff $A \cap u_\alpha \in D_\alpha$ for every (except, maybe $< \theta$ ordinals) $\alpha < \theta$. Now, choose any ultrafilter E on θ, such that $E_* \subseteq E$.

Define $\mathbf{C}_{f_\alpha(i)} = \mathbf{B}_i$, for every $\alpha < \theta$ and $i < \kappa$. You will get $(\mathbf{C}_j : j < \theta)$ such that $\text{Depth}^+(\mathbf{C}_j) \leq \lambda$ for every $j < \theta$. But, we will show that $\text{Depth}^+(\mathbf{C}) \geq \lambda^+$ (remember that $\mathbf{C} = \prod_{j<\theta} \mathbf{C}_j/E$).

Well, let $(a_\xi : \xi < \lambda)$ testify $\text{Depth}^+(\mathbf{B}) = \lambda^+$. Recall, a_ξ is $\langle a_i^\xi : i < \kappa\rangle/D$. We may write $f_\alpha(a_\xi)$ for $\langle f_\alpha(a_i^\xi) : i < \kappa\rangle/D_\alpha$, where $\alpha < \theta$. Clearly, $(f_\alpha(a_\xi) : \xi < \lambda)$ testifies $\text{Depth}^+(\mathbf{C}^\alpha) = \lambda^+$ where $\mathbf{C}^\alpha := \prod_{i<\kappa} \mathbf{C}_{f_\alpha(i)}/D_\alpha$.

Now, $\langle (f_\alpha(a_\xi) : \alpha < \theta) : \xi < \lambda\rangle/E$ is an increasing sequence in $\mathbf{C}$. □

Remark 2.8. (1) Claim 2.7 applies, in a similar fashion, to the Depth invariant.
(2) Claim 2.7 is useful for comparing $\text{Depth}(\mathbf{C})$ to $\prod_{j<\theta} \text{Depth}(\mathbf{C}_j)/E$, when $\lambda^\theta = \lambda$.

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