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COLORING FINITE SUBSETS OF UNCOUNTABLE SETS

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ABSTRACT. It is consistent for every $1 \le n < \omega$ that $2^\omega = \omega_n$ and there is a function $F : [\omega_n]^{<\omega} \to \omega$ such that every finite set can be written in at most $2^n - 1$ ways as the union of two distinct monocolored sets. If GCH holds, for every such coloring there is a finite set that can be written at least $\frac{1}{2}\sum_{i=1}^{n}\binom{n+i}{n}\binom{n}{i}$ ways as the union of two sets with the same color.

1. INTRODUCTION

In [6] we proved that for every coloring $F : [\omega_n]^{<\omega} \to \omega$ there exists a set $A \in [\omega_n]^{<\omega}$ which can be written at least $2^n - 1$ ways as $A = H_0 \cup H_1$ for some $H_0 \ne H_1$, $F(H_0) = F(H_1)$, and that for $n = 1$ there is in fact a function F for which this is sharp. Here we show that for every $n < \omega$ it is consistent that $2^\omega = \omega_n$ and for some function F as above for every finite set A there are at most $2^n - 1$ solutions of the above equation. We use historic forcing, which was first used in [1] and [7], then in [5] and [4]. Under GCH, we improve the positive result of [6] by showing that for every F as above some finite set can be written at least $T_n = \frac{1}{2}\sum_{i=1}^{n}\binom{n+i}{n}\binom{n}{i}$ ways as the union of two sets with the same F value.

With the methods of [6] it is easy to show the following corollary of our independence result. It is consistent that $2^\omega = \omega_n$ and there is a function $f : \mathbb{R} \to \omega$ such that if x is a real number then x cannot be written more than $2^n - 1$ ways as the arithmetic mean of some $y \ne z$ with $f(y) = f(z)$. ((y, z) and (z, y) are not regarded as distinct.)

Notation. We use the standard set theory notation. If S is a set, κ a cardinal, then $[S]^\kappa = \{A \subseteq S : |A| = \kappa\}$, $[S]^{<\kappa} = \{A \subseteq S : |A| < \kappa\}$, $[S]^{\le\kappa} = \{A \subseteq S : |A| \le \kappa\}$. $P(S)$ is the power set of S. If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$.

2. THE INDEPENDENCE RESULT

Theorem 1. *For $1 \le n < \omega$ it is consistent that $2^\omega = \omega_n$ and there is a function $F : [\omega_n]^{<\omega} \to \omega$ such that for every $A \in [\omega_n]^{<\omega}$ there are at most $2^n - 1$ solutions of $A = H_0 \cup H_1$ with $H_0 \ne H_1$, $F(H_0) = F(H_1)$.*

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For $\alpha < \omega_n$ fix a bijection $\varphi_\alpha : \alpha \to |\alpha|$. For $x \in [\omega_n]^{<\omega}$ define $\gamma_i(x)$ for $i < k = \min(n, |x|)$ as follows: $\gamma_0(x) = \max(x)$,

$$\gamma_{i+1}(x) = \varphi^{-1}_{\gamma_0(x)}\Big(\gamma_i\big(\varphi_{\gamma_0(x)}[x \cap \gamma_0(x)]\big)\Big),$$

and $\gamma(x) = \{\gamma_0(x), \dots, \gamma_{k-1}(x)\}$.

So, for example, if $n = 0$ then $\gamma(x) = \emptyset$, if $n = 1$, $x \neq \emptyset$, then $\gamma(x) = \{\gamma_0(x)\} = \{\max(x)\}$.

Lemma 1. *Given $s \in [\omega_n]^{\leq n}$, there are at most countably many $x \in [\omega_n]^{<\omega}$ such that $\gamma(x) = s$.*

Proof. By induction on n. □

Let $\Phi(s) = \bigcup\{x : \gamma(x) \subseteq s\}$, a countable set for $s \in [\omega_n]^{<\omega}$.

Definition. The two sets x, $y \in [\omega_n]^{<\omega}$ are *isomorphic* if the structures $(x; <, \gamma_0(x), \dots, \gamma_{k-1}(x))$, $(y; <, \gamma_0(y), \dots, \gamma_{k-1}(y))$ are isomorphic, i.e., $|x| = |y|$ and the positions of the elements $\gamma_i(x)$, $\gamma_i(y)$ are the same.

Notice that for every finite j there are just finitely many isomorphism types of j-element sets.

The elements of P, the applied notion of forcing, will be *some* structures of the form $p = (s, f)$ where $s \in [\omega_n]^{<\omega}$ and $f : P(s) \to \omega$.

The only element of P_0 is $\mathbf{1}_P = (\emptyset, \langle\emptyset, 0\rangle)$; it will be the largest element of P. The elements of P_1 are of the form $p = (\{\xi\}, f)$ where $f(\emptyset) = 0 \neq f(\{\xi\})$ for $\xi < \omega_n$.

Given P_t, $p = (s, f)$ is in P_{t+1} if the following is true. $s = \Delta \cup a \cup b$ is a disjoint decomposition. $p' = (\Delta \cup a, f')$ and $p'' = (\Delta \cup b, f'')$ are in P_t, where $f' = f|P(\Delta \cup a)$, $f'' = f|P(\Delta \cup b)$. There is $\pi : \Delta \cup a \to \Delta \cup b$, an isomorphism between $(\Delta \cup a, <, P(\Delta \cup a), f')$ and $(\Delta \cup b, <, P(\Delta \cup b), f'')$. $\pi|\Delta$ is the identity. For $H \subseteq \Delta \cup a$ the sets H and $\pi[H]$ are isomorphic. $a \cap \Phi(\Delta) = b \cap \Phi(\Delta) = \emptyset$. $f - f' - f''$ is one-to-one and takes only values outside $\mathrm{Ran}(f')$ (which is the same as $\mathrm{Ran}(f'')$). $P = \bigcup\{P_t : t < \omega\}$. We make $p \leq p', p''$ and the ordering on P is the one generated by this.

Lemma 2. *$(P, \leq)$ is ccc.*

Proof. Assume that $p_\alpha \in P$ $(\alpha < \omega_1)$. We can assume by thinning and using the Δ-system lemma and the pigeonhole principle that the following hold. $p_\alpha \in P_t$ for the same $t < \omega$. $p_\alpha = (\Delta \cup a_\alpha, f_\alpha)$ where the structures $(\Delta \cup a_\alpha, <, P(\Delta \cup a_\alpha), f_\alpha)$ and $(\Delta \cup a_\beta, <, P(\Delta \cup a_\beta), f_\beta)$ are isomorphic for $\alpha, \beta < \omega_1$, $\{\Delta, a_\alpha : \alpha < \omega_1\}$ pairwise disjoint. We can also assume that if π is the isomorphism between $(\Delta \cup a_\alpha, <, f_\alpha)$ and $(\Delta \cup a_\beta, <, f_\beta)$ then H and $\pi[H]$ are isomorphic for $H \subseteq \Delta \cup a_\alpha$. Moreover, if we assume that Δ occupies the same positions in the ordered sets $\Delta \cup a_\alpha$ $(\alpha < \omega_1)$ then π will be the identity on Δ. As $\Phi(\Delta)$ is countable, by removing countably many indices we can also assume that $\Phi(\Delta) \cap a_\alpha = \emptyset$ for $\alpha < \omega_1$. Now any p_α and p_β are compatible, as we can take $p = (\Delta \cup a_\alpha \cup a_\beta, f) \leq p_\alpha$, p_β where $f \supseteq f_\alpha$, f_β is an appropriate extension, i.e., $f - f_\alpha - f_\beta$ is one-to-one and takes values outside $\mathrm{Ran}(f_\alpha)$. □

Lemma 3. *If $(s, f) \in P$ and $H_0, H_1 \subseteq s$ have $f(H_0) = f(H_1)$, then H_0, H_1 are isomorphic.*

Proof. Set $(s,f)\in P_t$. We prove the statement by induction on t. There is nothing to prove for $t<2$. Assume now that $(s,f)\in P_{t+1}$, $s=\Delta\cup a\cup b$, $\pi:\Delta\cup a\to\Delta\cup b$ as in the definition of $(P,\le)$. As $f(H_0)$ is a value taken twice by f, both H_0 and H_1 must be subsets of either $\Delta\cup a$ or $\Delta\cup b$. We are done by induction unless $H_0\subseteq\Delta\cup a$ and $H_1\subseteq\Delta\cup b$ (or vice versa). Now H_0 and $\pi[H_0]$ are isomorphic and $f(H_0)=f(\pi[H_0])=f(H_1)$, so by the inductive hypothesis $\pi[H_0]$ and H_1 are ismorphic, and then so are H_0, H_1. □

Lemma 4. *If $(s,f)\in P$, $H_0,H_1\subseteq s$, $f(H_0)=f(H_1)$, $x\in H_0\cap H_1$, then x occupies the same position in the ordered sets H_0, H_1.*

Proof. Similarly to the proof of the previous lemma, by induction on t, for $(s,f)\in P_t$. With similar steps, we can assume that $(s,f)=(\Delta\cup a\cup b,f)\le(\Delta\cup a,f'),(\Delta\cup b,f'')$, $H_0\subseteq\Delta\cup a$, $H_1\subseteq\Delta\cup b$. Notice that $x\in\Delta$. Now, as $\pi(x)=x$, x is a common element of $\pi[H_0]$ and H_1, and also $f''(\pi[H_0])=f''(H_1)$. By induction we get that x occupies the same position in $\pi[H_0]$ and H_1, so by pulling back we get that this is true for H_0 and H_1. □

Lemma 5. *If $(s,f)\in P$, $A\subseteq s$, $0\le j\le n$, then A can be written at most 2^j-1 ways as $A=H_0\cup H_1$ with H_0, H_1 distinct, $f(H_0)=f(H_1)$, and $|\gamma(H_0)\cap\gamma(H_1)|\ge n-j$.*

Proof. By induction on j and, inside that induction, by induction on t, for $(s,f)\in P_t$. The case $t<2$ will always be trivial.

Assume first that $j=0$. In this case our lemma reduces to the following statement. There are no $H_0\ne H_1$ such that $f(H_0)=f(H_1)$ and $\gamma(H_0)=\gamma(H_1)$. In the inductive argument we assume as usual that $s=\Delta\cup a\cup b$ and so $(s,f)\in P_{t+1}$ was created from $(\Delta\cup a,f')$ and $(\Delta\cup b,f'')$, $H_0\subseteq\Delta\cup a$, $H_1\subseteq\Delta\cup b$. As $\gamma(H_0)=\gamma(H_1)$, $\gamma(H_0)\subseteq\Delta$, but then, as $\Phi(\Delta)\cap a=\emptyset$, H_0 can have no points outside Δ and similarly for H_1, so we can go back, say to $(\Delta\cup a,f')\in P_t$, which concludes the argument.

Assume now that the statement is proved for j and we have $p=(s,f)\in P_{t+1}$, $s=\Delta\cup a\cup b$ and p was created from $p'=(\Delta\cup a,f')$ and $p''=(\Delta\cup b,f'')$. In $A\subseteq\Delta\cup a\cup b$ we can assume that $y=A\cap a\ne\emptyset$, $z=A\cap b\ne\emptyset$, as otherwise we can pull back to p' or p''. But then, if $A=H_0\cup H_1$, then, if, say, $H_0\subseteq\Delta\cup a$, $H_1\subseteq\Delta\cup b$ hold, then necessarily $H_0\cap a=y$, $H_1\cap b=z$, so $H_0=x_0\cup y$, $H_1=x_1\cup z$ where $x_0\cup x_1=x=A\cap\Delta$. We can create decompositions of $B=x\cup\pi[y]\cup z$ by taking $B=\pi[H_0]\cup H_1$. But some of these decompositions will not be different and it may happen that we get a non-proper (i.e., one-piece) decomposition. This can only happen if $\pi[y]=z$, and then the two decompositions $A=(x_0\cup y)\cup(x_1\cup z)$ and $A=(x_1\cup y)\cup(x_0\cup y)$ produce the same decomposition of B, namely, $B=(x_0\cup z)\cup(x_1\cup z)$, and there is but one decomposition, $A=(x\cup y)\cup(x\cup z)$, which cannot be mapped to a decomposition of B. If this (i.e., $\pi[y]=z$) does not happen, we are done by induction. If this does happen, we know that $\gamma(H_0)=\gamma(x_0\cup y)$ has an element in y (by the argument at the beginning of the proof). As $f(x_0\cup y)=f(x_1\cup z)$, by Lemmas 3 and 4, both $H_0=x_0\cup y$ and $H_1=x_1\cup z$ have an element in the γ-subset, at the same positions which are mapped onto each other by π. We get that $\gamma(x_0\cup z)\cap\gamma(x_1\cup z)$ has at least $n-j$ elements, so by our inductive assumption we have at most 2^j-1 decompositions, which gives at most $2\cdot(2^j-1)+1=2^{j+1}-1$ decompositions of A. □

Let $G \subseteq P$ be a generic subset. Set

$$S = \bigcup \{s : (s, f) \in G\}, \qquad F = \bigcup \{f : (s, f) \in G\}.$$

Lemma 6. *There is a $p \in P$ such that $p \Vdash |S| = \aleph_n$.*

Proof. Otherwise $\mathbf{1} \Vdash \sup(S) < \omega_n$. By ccc, there is an ordinal $\xi < \omega_n$ for which $\mathbf{1} \Vdash \sup(S) < \xi$, but this is impossible as there are conditions in P_1 forcing that $\xi \in S$. $\square$

Now we can conclude the proof of the theorem. If G is generic, and $p \in G$ with the condition p of Lemma 6, then in $V[G]$ F witnesses the theorem by Lemma 5 (for $j = n$) on the ground set S. As $|S| = \omega_n$ we can replace it by ω_n. $\square$

3. The GCH result

Set

$$T_n = \frac{1}{2} \sum_{i=1}^{n} \binom{n+i}{n} \binom{n}{i}.$$

So $T_1 = 1$, $T_2 = 6$, $T_3 = 31$. In general, T_n is asymptotically $c(3 + 2\sqrt{2})^n / \sqrt{n}$ for some c.

Theorem 2 (GCH)**.** *If $F : [\omega_n]^{<\omega} \to \omega$, then some $A \in [\omega_n]^{<\omega}$ has at least T_n decompositions as $A = H_0 \cup H_1$, $H_0 \neq H_1$, $F(H_0) = F(H_1)$.*

Proof. By the Erdős-Rado theorem (see [2], [3]) there is a set $\{x_\alpha : \alpha < \omega_1\}$ which is $(n-1)$-end-homogeneous, i.e., for some $g : [\omega_1]^{<\omega} \to \omega$, if $\alpha_1 < \cdots < \alpha_k < \beta_1 < \cdots < \beta_{n-1} < \omega_1$ then

$$f(\{x_{\alpha_1}, \ldots, x_{\alpha_k}, x_{\beta_1}, \ldots, x_{\beta_{n-1}}\}) = g(\alpha_1, \ldots, \alpha_k).$$

Select $S_1 \in [\omega_1]^{\omega_1}$ in such a way that $g(\alpha) = c_0$ for $\alpha \in S_1$. Set $\gamma_1 = \min(S_1)$. In general, if γ_i, S_i are given ($1 \leq i < n$), pick $S_{i+1} \in [S_i - (\gamma_i + 1)]^{\omega_1}$ so that $g(\gamma_1, \ldots, \gamma_i, \alpha) = c_i$ for $\alpha \in S_{i+1}$, and set $\gamma_{i+1} = \min(S_{i+1})$. Given $\gamma_1, \ldots, \gamma_n$ and S_n, let $\gamma_{n+1}, \ldots, \gamma_{2n}$ be the n least elements of $S_n - (\gamma_n + 1)$.

Our set will be $A = \{x_{\gamma_1}, \ldots, x_{\gamma_{2n}}\}$. For $0 \leq i < n$ the color of any $(n+i)$-element subset of A containing $x_{\gamma_1}, \ldots, x_{\gamma_i}$ will be c_i. We can select $\frac{1}{2}\binom{2n-i}{n}\binom{n}{i}$ different pairs of those sets which cover A. In toto, we get T_n decompositions of A. $\square$

References

1. J. E. Baumgartner and S. Shelah, *Remarks on superatomic Boolean algebras*, Annals of Pure and Applied Logic **33** (1987), 109–129. MR **88d:**03100
2. P. Erdős, A. Hajnal, A. Máté, and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Studies in Logic, **106**, North-Holland, 1984. MR **87j:**04002
3. P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489. MR **18:**458a
4. M. Gilchrist and S. Shelah, *On identities of colorings of pairs for $\aleph_n$* (to appear).
5. P. Komjáth, *A set mapping with no infinite free subsets*, Journal of Symbolic Logic **56** (1991), 1400–1402. MR **92i:**03054

6. P. Komjáth and S. Shelah, *On uniformly antisymmetric functions*, Real Analysis Exchange **19** (1993–1994), 218–225. MR **95b:**26001
7. S. Shelah and L. Stanley, *A theorem and some consistency results in partition calculus*, Annals of Pure and Applied Logic **36** (1987), 119–152. MR **89d:**03045

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