Math. Log. Quart. **52**, No. 2, 151 – 164 (2006) / **DOI** 10.1002/malq.200410058

Recursive logic frames

Saharon Shelah[*1] and **Jouko Väänänen**[**2]

[1] Institute of Mathematics, Hebrew University, Jerusalem, Israel
[2] Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland

Received 21 October 2004, revised 5 December 2005, accepted 20 December 2005
Published online 1 March 2006

Key words Generalized quantifiers, identities, compact logics.
MSC (2000) 03C95, 03C80, 03C55

We define the concept of a *logic frame*, which extends the concept of an abstract logic by adding the concept of a syntax and an axiom system. In a *recursive* logic frame the syntax and the set of axioms are recursively coded. A recursive logic frame is called *complete* (*recursively compact*, $\aleph_0$*-compact*), if every finite (respectively: recursive, countable) consistent theory has a model. We show that for logic frames built from the cardinality quantifiers "there exists at least λ" completeness always implies $\aleph_0$-compactness. On the other hand we show that a recursively compact logic frame need not be $\aleph_0$-compact.

1 Introduction

For the definition of an abstract logic and a generalized quantifier the reader is referred to [4, 13, 14]. Undoubtedly the most important among abstract logics are the ones that have a complete axiomatization of validity. In many cases, most notably when we combine even the simplest generalized quantifiers, completeness of an axiomatization cannot be proved in ZFC alone but depends of principles like CH or $\diamond$. Examples of logics that have a complete axiomatization are:

1. the infinitary language $L_{\omega_1\omega}$ (see [10]);
2. logic with the generalized quantifier[1)] (see [27])
$$\exists^{\geq\aleph_1} x\, \varphi(x,\vec{y}\,) \Leftrightarrow |\{x\, :\, \varphi(x,\vec{y}\,)\}| \geq \aleph_1;$$
3. logic with the *cofinality quantifier* (see [24])
$$Q^{\mathrm{cof}}_{\aleph_0} xy\, \varphi(x,y,\vec{z}\,) \Leftrightarrow \{\langle x,y\rangle\, :\, \varphi(x,y,\vec{z}\,)\} \text{ is a linear order of cofinality } \aleph_0;$$
4. logic with the *cub-quantifier* (see [24])
$$Q^{\mathrm{cub}}_{\aleph_1} xy\, \varphi(x,y,\vec{z}\,) \Leftrightarrow \{\langle x,y\rangle\, :\, \varphi(x,y,\vec{z}\,)\} \text{ is an } \aleph_1\text{-like linear order in which a cub of initial segments have a sup;}$$
5. logic with the *Magidor-Malitz quantifier*, assuming $\diamond$ (see [15]),
$$Q^{\mathrm{MM}}_{\aleph_1} xy\, \varphi(x,y,\vec{z}\,) \Leftrightarrow \exists X\, (|X| \geq \aleph_1 \wedge (\forall x,y \in X)\, \varphi(x,y,\vec{z}\,)).$$

The extension $L(\exists^{\geq\kappa})$ of first order logic was introduced by Andrzej Mostowski in 1957 [18]. Here $\exists^{\geq\kappa}$ is the generalized quantifier
$$\mathfrak{M} \models \exists^{\geq\kappa} x\, \varphi(x,\vec{a}\,) \Leftrightarrow |\{b \in M\, :\, \mathfrak{M} \models \varphi(b,\vec{a}\,)\}| \geq \kappa.$$

* e-mail: shelah@math.huji.ac.il
** Corresponding author: e-mail: jouko.vaananen@helsinki.fi

1) This quantifier is usually denoted by Q_1.

Mostowski asked whether $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact (i. e. every countable set of sentences, every finite subset of which has a model, has itself a model) and observed that $L(\exists^{\geq\aleph_0})$ is not. In 1963 Gerhard Fuhrken [7] proved that $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact if $\aleph_0$ is small for κ (i. e. if $\lambda_n < \kappa$ for $n < \omega$, then $\prod_{n<\omega} \lambda_n < \kappa$). His proof was based on the observation that the usual Łoś Lemma

$$\prod_{n<\omega} \mathfrak{M}_n/F \models \varphi \Leftrightarrow \{n < \omega : \mathfrak{M}_n \models \varphi\} \in F$$

for ultrafilters F on ω and first order sentences φ can be proved for $\varphi \in L(\exists^{\geq\kappa})$ if $\aleph_0$ is small for κ. The $\aleph_0$-compactness follows from the Łoś Lemma immediately.

Vaught [27] proved $\aleph_0$-compactness of $L(\exists^{\geq\aleph_1})$ by proving what is now known as Vaught's Two-Cardinal Theorem and Chang [5] extended this to $L(\exists^{\geq\kappa^+})$ by proving $(\omega_1, \omega) \to (\kappa^+, \kappa)$, when $\kappa^{<\kappa} = \kappa$. Jensen [9] extended this to all κ under the assumption GCH$+\Box_\kappa$, which he showed to follow from $V = L$. Keisler [11] proved with a different method $\aleph_0$-compactness of $L(\exists^{\geq\kappa})$ for κ a singular strong limit cardinal. This led to the important observation that if $V = L$ holds and every regular cardinal is a successor cardinal (i. e. there are no weakly inaccessible cardinals), then $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact for all $\kappa > \omega$. We still do not know if this is provable in ZFC:

Open Problem 1.1 *Is it provable in* ZFC *that $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact for all $\kappa > \omega$? In particular, is it provable in* ZFC *that $L(\exists^{\geq\aleph_2})$ is $\aleph_0$-compact?*

The best result today towards solving this problem is:

Theorem 1.2 [26] *It is consistent, relative to the consistency of* ZF *that $L(\exists^{\geq\aleph_1}, \exists^{\geq\aleph_2})$ is not $\aleph_0$-compact.*

Our approach is to look for ZFC-provable relationships between completeness, recursive compactness and $\aleph_0$-compactness in the context of a particular logic in the hope that such relationships would reveal important features of the logic even if we cannot settle any one of these properties per se. For example, the $\aleph_0$-compactness of the logic $L_{\omega\omega}(\exists^{\geq\aleph_1}, \exists^{\geq\aleph_2}, \exists^{\geq\aleph_3}, \ldots)$ cannot be decided in ZFC, but we prove in ZFC that if this logic is recursively compact, it is $\aleph_0$-compact. We show by example that recursive compactness does not in general imply $\aleph_0$-compactness.

2 Logic frames

Our concept of a logic frame captures the combination of syntax, semantics and proof theory of an extension of first order logic. This is a very general concept and is not defined here with mathematical exactness, as we do not prove any general results about logic frames. All our results are about concrete examples.

Definition 2.1

1. A *logic frame* is a triple $L^* = \langle \mathcal{L}, \models_{\mathcal{L}}, \mathcal{A}\rangle$, where $\langle \mathcal{L}, \models_{\mathcal{L}}\rangle$ is a logic in the sense of [4, Definition 1.1.1], $\mathcal{A}$ is a class of L^*-axioms and L^*-inference rules. We write $\vdash_{\mathcal{A}} \varphi$ if φ is derivable using the axioms and rules in $\mathcal{A}$, and call a set T of L^*-sentences *$\mathcal{A}$-consistent* if no sentence together with its negation is derivable from T.

2. A logic frame $L^* = (\mathcal{L}, \models_{\mathcal{L}}, \mathcal{A})$ is *recursive* if

   (a) there is an effective algorithm which gives for each finite vocabulary τ the set $\mathcal{L}[\tau]$ and for each $\varphi \in \mathcal{L}[\tau]$ a second order[2)] formula which defines the semantics of φ;

   (b) there is an effective algorithm which gives the axioms and rules of $\mathcal{A}$.

3. A logic frame $L^* = \langle \mathcal{L}, \models_{\mathcal{L}}, \mathcal{A}\rangle$ is a *$\langle\kappa, \lambda\rangle$-logic frame*, if each sentence contains less than λ predicate, function and constant symbols, and $|\mathcal{L}[\tau]| \leq \kappa$ whenever the vocabulary τ has less that λ symbols altogether.

4. A logic frame $L^* = \langle \mathcal{L}, \models_{\mathcal{L}}, \mathcal{A}\rangle$ is

   (a) *complete* if every finite $\mathcal{A}$-consistent L^*-theory has a model;

   (b) *recursively compact* if every $\mathcal{A}$-consistent L^*-theory, which is recursive in the set of axioms and rules, has a model;

   (c) *(κ, λ)-compact* if every L^*-theory of cardinality $\leq \kappa$, every subset of cardinality $< \lambda$ of which is $\mathcal{A}$-consistent, has a model;

   (d) *κ-compact*, if it is (κ, ω)-compact.

2) Second order logic represents a strong logic with an effectively defined syntax. It is not essential, which logic is used here as long as it is powerful enough.

Note that

$$\aleph_0\text{-compactness} \Rightarrow \text{recursive compactness} \Rightarrow \text{completeness}.$$

The weakest condition is thus completeness. We work in this paper almost exclusively with complete logic frames investigating their compactness properties. In the following definition we use the concept of "possible universe". By this we mean an inner model or a forcing extension. The exact meaning of this concept is not at all critical for our results. We do not want to use "provable in ZFC" instead because we have ordinal parameters. For example, the logic $L(\exists^{\geq\kappa})$ has κ as a parameter.

Definition 2.2 A logic frame $L^* = \langle \mathcal{L}, \models_{\mathcal{L}}, \mathcal{A}\rangle$ has

1. *finite recursive character* if for every possible universe V',

$$V' \models (L^* \text{ is complete} \Rightarrow L^* \text{ is recursively compact});$$

2. *finite character* if for every possible universe V',

$$V' \models (L^* \text{ is complete} \Rightarrow L^* \text{ is } \aleph_0\text{-compact});$$

3. *recursive character* if for every possible universe V',

$$V' \models (L^* \text{ is recursively compact} \Rightarrow L^* \text{ is } \aleph_0\text{-compact}).$$

Finite (recursive) "(κ, λ)-character" means finite (respectively recursive) character with "$\aleph_0$-compact" replaced by "(κ, λ)-compact". "Strong character", means (κ, ω)-character for all κ.

An extension of first order logic by finitely many generalized quantifiers has finite recursive character (see [4]).

Example 2.3 Let

$$L(\exists^{\geq\kappa^+}) = \langle \mathcal{L}(\exists^{\geq\kappa^+}), \models_{L(\exists^{\geq\kappa^+})}, \mathcal{A}(\exists^{\geq\kappa^+})\rangle,$$

where

$$\mathfrak{M} \models \exists^{\geq\kappa} x\, \varphi(x, \vec{y}\,) \Leftrightarrow |\{x : \mathfrak{M} \models \varphi(x, \vec{y}\,)| \geq \kappa$$

and $\mathcal{A}(\exists^{\geq\kappa^+})$ has as axioms the basic axioms of first order logic and

1. $\neg\exists^{\geq\kappa^+} x\,(x = y \vee x = z)$;
2. $\forall x\,(\varphi \to \psi) \to (\exists^{\geq\kappa^+} x\,\varphi \to \exists^{\geq\kappa^+} x\,\psi)$;
3. $\exists^{\geq\kappa^+} x\,\varphi(x,\dots) \leftrightarrow \exists^{\geq\kappa^+} y\,\varphi(y,\dots)$, where $\varphi(x,\dots)$ is a formula of $L(\exists^{\geq\kappa^+})$ in which y does not occur;
4. $\exists^{\geq\kappa^+} y \exists x\,\varphi \to \exists x \exists^{\geq\kappa^+} y\,\varphi \vee \exists^{\geq\kappa^+} x \exists y\,\varphi$;

and Modus Ponens as the only rule. The logic $L(\exists^{\geq\kappa^+})$ was introduced by Mostowski [18] and the above frame for $\kappa = \aleph_0$ by Keisler [12]. The logic frame $L(\exists^{\geq\kappa^+})$ is an effective $\langle\omega,\omega\rangle$-logic frame. The logic frame $L(\exists^{\geq\aleph_1})$ is $\aleph_0$-compact, hence has finite character for a trivial reason. If $\kappa = \kappa^{<\kappa}$, then by Chang's Two-Cardinal Theorem, $L(\exists^{\geq\kappa^+})$ is $\aleph_0$-compact, in fact (κ,ω)-compact (see [22]). If $V = L$, then $L(\exists^{\geq\kappa^+})$ is (κ,ω)-compact for all κ (Jensen [9]).

Example 2.4 Suppose κ is a singular strong limit cardinal. Let

$$L(\exists^{\geq\kappa}) = \langle \mathcal{L}(\exists^{\geq\kappa}), \models_{L(\exists^{\geq\kappa})}, \mathcal{A}(\exists^{\geq\kappa})\rangle,$$

where $\mathcal{A}(\exists^{\geq\kappa})$ has as axioms the basic axioms of first order logic, a rather complicated set of special axioms from [11] (no simple set of axioms is known at present), and Modus Ponens as the only rule. The logic frame $L(\exists^{\geq\kappa})$ is (λ,ω)-compact for each $\lambda < \kappa$ (see [11, 22]).

Example 2.5 Suppose κ is strong limit ω-Mahlo[3)] cardinal. Let

$$L(\exists^{\geq\kappa}) = \langle \mathcal{L}(\exists^{\geq\kappa}), \models_{L(\exists^{\geq\kappa})}, \mathcal{A}(\exists^{\geq\kappa})\rangle,$$

where $\mathcal{A}(\exists^{\geq\kappa})$ has as axioms the basic axioms of first order logic, axioms given in [20], and Modus Ponens as the only rule. The logic frame $L(\exists^{\geq\kappa})$ is (λ,ω)-compact for each $\lambda < \kappa$ (see [21, 22]).

3) κ is 0-Mahlo if it is regular, $(n+1)$-Mahlo, if there is a stationary set of n-Mahlo cardinals below κ, and ω-Mahlo if it is n-Mahlo for all $n < \omega$.

Example 2.6 Suppose κ is a regular cardinal. Let

$$L(Q_\kappa^{\mathrm{cof}}) = \langle \mathcal{L}(Q_\kappa^{\mathrm{cof}}), \models_{L(Q_\kappa^{\mathrm{cof}})}, \mathcal{A}(Q_\kappa^{\mathrm{cof}})\rangle,$$

where $\mathfrak{M} \models Q_\kappa^{\mathrm{cof}} xy\, \varphi(x, y, \vec{z})$ if and only if $\{\langle x, y\rangle : \mathfrak{M} \models \varphi(x, y, \vec{z})\}$ is a linear order of cofinality κ, and $\mathcal{A}(Q_\kappa^{\mathrm{cof}})$ has as axioms the basic axioms of first order logic, the axioms from [24], and Modus Ponens as the only rule. The logic frame $L(Q_\kappa^{\mathrm{cof}})$ is *fully compact*, i. e. (κ, ω)-compact for all κ, hence has finite character for a trivial reason (see [24]).

Example 2.7 Let

$$L(Q_{\aleph_1}^{\mathrm{cub}}) = \langle \mathcal{L}(Q_{\aleph_1}^{\mathrm{cub}}), \models_{L(Q_{\aleph_1}^{\mathrm{cub}})}, \mathcal{A}(Q_{\aleph_1}^{\mathrm{cub}})\rangle,$$

where $\mathfrak{M} \models Q_{\aleph_1}^{\mathrm{cub}} xy\, \varphi(x, y, \vec{z})$ if and only if $\{\langle x, y\rangle : \mathfrak{M} \models \varphi(x, y, \vec{z})\}$ is an $\aleph_1$-like linear order in which a cub of initial segments have a sup, and $\mathcal{A}(Q_{\aleph_1}^{\mathrm{cub}})$ has as axioms the basic axioms of first order logic, and axioms from [3]. The logic frame $L(Q_{\aleph_1}^{\mathrm{cub}})$ is $\aleph_0$-compact (see [24]), hence has finite character for a trivial reason. We shall give explicit axioms for this logic frame later.

Example 2.8 Magidor-Malitz quantifier logic frame is

$$L(Q_\kappa^{\mathrm{MM}}) = \langle \mathcal{L}(Q_\kappa^{\mathrm{MM}}), \models, \mathcal{A}_\kappa^{\mathrm{MM}}\rangle,$$

where

$$Q_\kappa^{\mathrm{MM}} xy\, \varphi(x, y, \vec{z}) \Leftrightarrow \exists X\, (|X| \geq \kappa \wedge X \times X \subseteq \{\langle x, y\rangle : \varphi(x, y, \vec{z})\})$$

and $\mathcal{A}_\kappa^{\mathrm{MM}}$ is the set of axioms and rules introduced by Magidor and Malitz in [15]. The logic frame $L(Q_{\kappa^+}^{\mathrm{MM}})$ is an effective $\langle \omega, \omega\rangle$-logic frame. The logic frame $L(Q_{\kappa^+}^{\mathrm{MM}})$ is complete, if we assume $\diamondsuit$, $\diamondsuit_\kappa$ and $\diamondsuit_{\kappa^+}$, but there is a forcing extension in which $L(Q_{\aleph_1}^{\mathrm{MM}})$ is not $\aleph_0$-compact [1].

Example 2.9 Let

$$L_{\kappa\lambda} = \langle \mathcal{L}_{\kappa\lambda}, \models_{L_{\kappa\lambda}}, \mathcal{A}_{\kappa\lambda}\rangle,$$

where $\mathcal{A}_{\kappa\lambda}$ has as axioms the obvious axioms and Chang's Distributive Laws, and as rules Modus Ponens, Conjunction Rule, Generalization Rule and the Rule of Dependent Choices from [10]. This an old example of a logic frame introduced by Tarski in the late 50's and studied intensively, e. g. by Karp [10]. The logic frame $L_{\kappa\lambda}$ is a $\langle \kappa^\kappa, \kappa\rangle$-logic frame. It is effective and (μ, ω)-compact for all μ, if $\kappa = \lambda = \omega$. It is complete, if $\kappa = \omega_1$, $\lambda = \omega$. The logic frame $L_{\kappa\lambda}$ is complete also if

1. $\kappa = \mu^+$ and $\mu^{<\lambda} = \mu$, or
2. κ is strongly inaccessible, or
3. κ is weakly inaccessible, λ is regular and $(\forall \alpha < \kappa)(\forall \beta < \lambda)(\alpha^\beta < \kappa)$,

(see [10]) although in these cases the completeness is not as useful as in the case of $L_{\omega\omega}$ and $L_{\omega_1\omega}$. $L_{\kappa\lambda}$ is not complete if $\kappa = \lambda$ is a successor cardinal (D. Scott, see [10]). $L_{\kappa\lambda}$ is not (κ, κ)-compact unless κ is weakly compact, and then also $L_{\kappa\kappa}$ is (κ, κ)-compact. $L_{\kappa\lambda}$ is not (μ, κ)-compact for all $\mu \geq \kappa$ unless κ is strongly compact and then also $L_{\kappa\kappa}$ is. The logic frame $L_{\kappa\lambda}$ is not of finite (κ, κ)-character, unless $\kappa = \omega$, since it is in some possible universes complete, but not (κ, κ)-compact.

Example 2.10 Let $\aleph_0 < \kappa < \lambda$ be strongly compact cardinals. A sublogic $\mathcal{L}^1$ of $L_{\lambda\lambda}$, extending $L(\exists^{\geq\kappa})$, is defined in [8]. This logic is like $L_{\lambda\lambda}$ in that it allows quantification over sequences of variables of length $< \lambda$, but instead of conjunctions and disjunctions of length $< \kappa$, the logic $\mathcal{L}^1$ allows conjunctions and disjunctions over sets of formulas indexed by a set in a κ-complete ultrafilter on a cardinal $< \lambda$. The logic $\mathcal{L}^1$ is (μ, ω)-compact for $\mu < \kappa$, (μ, λ)-compact for all μ, has the interpolation property and other nice properties.

The definition of logic frames leaves many details vague, e. g. the exact form of axioms and rules. Also the conditions of a recursive logic frame would have to be formulated more exactly for any general results. Going into such details would take us too much astray from the main purpose of this paper.

Sh:790

3 Logics with recursive character

We now investigate the following quite general question involving an infinite sequence $(\kappa_n)_{n<\omega}$ of uncountable cardinals:

Question 3.1 *For which sequences $(\kappa_n)_{n<\omega}$ of uncountable cardinals is the logic $L(\exists^{\geq\kappa_n})_{n<\omega}$ $\aleph_0$-compact?*

As the preceding discussion indicates we cannot expect a general solution in ZFC. Extreme cases are

1. $\kappa_n = \aleph_1$ for all $n < \omega$,
2. $\aleph_0$ is small for each κ_n,
3. some κ_n is the supremum of a subset of the others,

where we have a trivial solution (in case 2. we have Łoś Lemma and therefore $\aleph_0$-compactness, and in case 3. we have an easy counter-example to $\aleph_0$-compactness).

Let us call a logic *recursively compact* if every recursive set of sentences, every finite subset of which has a model, itself has a model. Naturally this concept is meaningful only for logics which possess a canonical Gödel numbering of its sentences. Let us call a logic *recursively axiomatizable* if the set of (Gödel numbers of) valid sentences of the logic is recursively enumerable. By a result of Per Lindström [14] (see also [4]) any recursively axiomatizable logic of the form $L(\exists^{\geq\kappa_n})_{n\leq m}$ is actually recursively compact. This raises the question:

Question 3.2 *For which sequences $(\kappa_n)_{n<\omega}$ of uncountable cardinals is the logic $L(\exists^{\geq\kappa_n})_{n<\omega}$ recursively axiomatizable?*

We give an axiomatization $\mathcal{A}$ of $L(\exists^{\geq\kappa_n})_{n<\omega}$. We do not know in general whether this $\mathcal{A}$ is recursive (or recursively enumerable). We give a combinatorial characterization of sequences $(\kappa_n)_{n<\omega}$ for which the logic frame $(L(\exists^{\geq\kappa_n})_{n<\omega}, \models, \mathcal{A})$ is complete.

In the presence of an axiomatization $\mathcal{A}$ we can redefine our compactness properties. Rather than requiring that every finite subtheory has a model we can require that every finite subtheory is *$\mathcal{A}$-consistent* in the sense that no contradiction can be derived from it by means of the axioms and rules of $\mathcal{A}$. It turns out that this change is not significant in the sense that in our main result we could use either. However, this modified concept of compactness reveals an interesting connection between completeness and compactness: we can think of completeness (every consistent sentence has a model) as a compactness property of one-element theories. In this sense recursive compactness is a strengthening of completeness.

For example, if A^* is the Keisler axiomatization (from [12]) for $L(\exists^{\geq\aleph_1})$, it is consistent that $\langle L(\exists^{\geq\aleph_2}), A^*\rangle$ is complete (this follows from GCH), and it is also consistent that $\langle L(\exists^{\geq\aleph_2}), A^*\rangle$ is incomplete (this follows from $(\aleph_1, \aleph_0) \nrightarrow (\aleph_2, \aleph_1)$ which is consistent by [16]). However, we know it has provably finite character (see Proposition 3.19).

The main result of this paper (proved in Corollary 3.24) is the following:

Theorem 3.3 *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. There is a canonical axiomatization $\mathcal{A}$ of $L(\exists^{\geq\kappa_n})_{n<\omega}$ such that the logic frame $\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \models, \mathcal{A}\rangle$ has recursive character.*

It is noteworthy that the above theorem is a result in ZFC. The proof is based on formulating a partition theoretic equivalent condition for the $\aleph_0$-compactness (equivalently recursive compactness) of $L(\exists^{\geq\kappa_n})_{n<\omega}$.

There is a basic reduction of generalized quantifiers of the form $\exists^{\geq\kappa}$ to first order logic. This was established by Fuhrken [6]. A model $\langle M, \dots, A, <, \dots\rangle$ is called *λ-like* if $\langle A, <\rangle$ is a λ-like linear order (i. e. of cardinality λ with all initial segments of cardinality $< \lambda$). Fuhrken established a canonical translation $\varphi \longmapsto \varphi^+$ of $L(\exists^{\geq\kappa})$ to first order logic so that

$$\varphi \text{ has a model} \Leftrightarrow \varphi^+ \text{ has a } \kappa\text{-like model.}$$

Thus the questions of axiomatization and $\aleph_0$-compactness of $L(\exists^{\geq\kappa})$ were reduced to questions of axiomatization and $\aleph_0$-compactness of first order logic restricted to κ-like models.

If $\kappa = \lambda^+$, the reduction is slightly simpler. Then we can use *(κ, λ)-models*, i. e. models $\langle M, \dots, A, \dots\rangle$, where $|M| = \kappa$ and $|A| = \lambda$. The study of model theory of (κ, λ)-models makes, of course, sense also if $\kappa \neq \lambda^+$ even if this more general case does not arise from a reduction of $L(\exists^{\geq\kappa})$.

There is an immediate translation of the logic $L(\exists^{\geq\kappa_n})_{n<\omega}$ to first order logic on models that have for each $n < \omega$ a unary predicate P_n and a κ_n-like linear order $<_n$ on P_n. Let us call such models *$(\kappa_n)_{n<\omega}$-like models*. Mutatis mutandis, our approach applies also to logics of the form $L(\exists^{\geq\kappa_n})_{n<m}$.

For easier notation we fix $\langle A_n, <_n\rangle$ such that the sets A_n are disjoint and for each n the structure $\langle A_n, <_n\rangle$ is a well-order of order type κ_n. We say that $\langle a_0, \dots, a_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ is *increasing* if its restriction to any $\langle A_m, <_m\rangle$ is increasing in $\langle A_m, <_m\rangle$.

Definition 3.4 A triple

$$\mathcal{F} = \langle\langle E_a : a \in \textstyle\bigcup_{n<\omega} A_n\rangle, \langle\langle A_n, <_n\rangle : n < \omega\rangle, \langle h_n : n < \omega\rangle\rangle,$$

where

(E1) each E_a is an equivalence relation on $[\bigcup_{n<\omega} A_n]^{<\omega}$ such that equivalent sets have the same cardinality;

(E2) if $a \in A_n$, the number of equivalence classes of E_a is $< \kappa_n$;

(E3) $h_n : [\bigcup_{n<\omega} A_n]^{<\omega} \longrightarrow A_n$;

is called a $(\kappa_n)_{n<\omega}$-*pattern*.

Let us now try to use the pattern to construct a $(\kappa_n)_{n<\omega}$-like model. Let us assume that our starting theory T has the property that every finite subset has a $(\kappa_n)_{n<\omega}$-like model. We assume the vocabulary L of T has cardinality $< \min\{\kappa_n : n < \omega\}$. Let L^* be the Skolem-expansion of L and T^* the Skolem-closure of T. Let c_a, $a \in \bigcup_{n<\omega} A_n$, be new constant symbols. Let $<_n$ be the predicate symbol the interpretation of which we want to be κ_n-like. Consider the axioms

(T1) T^* (Skolem-closure of T);

(T2) $c_\alpha <_n c_\beta$ for $\alpha <_n \beta$ in A_n;

(T3) $P_n(c_a)$ for $a \in A_n$;

(T4) $P_m(t(c_{a_0}, \dots, c_{a_n})) \rightarrow t(c_{a_0}, \dots, c_{a_n}) <_m c_{h_m(\{a_0,\dots,a_n\})}$, where $\langle a_0, \dots, a_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ is increasing and t is a Skolem-term;

(T5) $t(c_{a_0}, \dots, c_{a_n}) = t(c_{b_0}, \dots, c_{b_n}) \vee (\neg(t(c_{a_0}, \dots, c_{a_n}) <_m c_a) \wedge \neg(t(c_{b_0}, \dots, c_{b_n}) <_m c_a))$ for all Skolem-terms t and all increasing $\langle a_0, \dots, a_n\rangle, \langle b_0, \dots, b_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ such that $\{a_0, \dots, a_n\} E_a \{b_0, \dots, b_n\}$, whenever $a \in A_m$.

Let Σ be an arbitrary finite subset of (T1) – (T5). Let $\mathfrak{M}$ be a $(\kappa_n)_{n<\omega}$-like model of $\Sigma \cap T^*$. Let D_m be the set of $a \in A_m$ such that c_a occurs in Σ. Let us expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ by adding interpretations to all the constants c_a, $a \in \bigcup_{n<\omega} A_n$, in such a way that they increase in $\langle P_m^{\mathfrak{M}}, <_m^{\mathfrak{M}}\rangle$ with $a \in P_m^{\mathfrak{M}}$ and are cofinal in $<_m^{\mathfrak{M}}$. The model $\mathfrak{M}'$ and Σ *induce* in a canonical way a $(\kappa_n)_{n<\omega}$-pattern

(1) $$\mathcal{F}' = \langle\langle E'_a : a \in \textstyle\bigcup_{n<\omega} A_n\rangle, \langle\langle A_n, <_n\rangle : n < \omega\rangle, \langle h'_n : n < \omega\rangle\rangle$$

as follows: If $a \in A_m$, then define for increasing $\langle a_0, \dots, a_n\rangle, \langle b_0, \dots, b_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$

$$\begin{aligned}\{a_0, \dots, a_n\} E'_a \{b_0, \dots, b_n\} \Leftrightarrow \mathfrak{M}' \models\ & t(c_{a_0}, \dots, c_{a_n}) = t(c_{b_0}, \dots, c_{b_n}) \\ & \vee (\neg(t(c_{a_0}, \dots, c_{a_n}) <_m c_a) \wedge \neg(t(c_{b_0}, \dots, c_{b_n}) <_m c_a)), \\ & \text{for all Skolem-terms } t \text{ occurring in } \Sigma,\end{aligned}$$

and

$$h'_m(\{a_0, \dots, a_n\}) = \min\{b \in A_m : t(c_{a_0}, \dots, c_{a_n})^{\mathfrak{M}'} <_m c_b^{\mathfrak{M}'} \text{ for all Skolem-terms } t \text{ occurring in } \Sigma\}.$$

We now stop for a moment to contemplate on the concept of identity. An ω-*cardinal identity* is a triple

(2) $$\mathfrak{I} = \langle\langle E_a : a \in \textstyle\bigcup_{n<\omega} D_n\rangle, \langle\langle D_n, <_n\rangle : n < \omega\rangle, \langle h_n : n < \omega\rangle\rangle,$$

where:

(I1) The $\langle D_m, <_m\rangle$ are disjoint finite linear orders, $D_m = \emptyset$ for all but finitely many m. The cardinality of $\bigcup_{n<\omega} D_n$ is called the *size of* $\mathfrak{I}$. The smallest l such that $D_m = \emptyset$ for $m > l$ is called the *length of* $\mathfrak{I}$.

(I2) Each E_a, $a \in D_m$, is an equivalence relation on $\mathcal{P}(D_m)$ such that equivalent sets have the same cardinality.

(I3) $h_m : [\bigcup_{n<\omega} D_n]^{<\omega} \longrightarrow D_m$ is a partial function.

An example of an ω-cardinal identity is the restriction

$$\mathcal{F}\restriction D = \langle\langle E_a\restriction D : a \in D \cap \bigcup_{n<\omega} D_n\rangle, \langle\langle D_n, <_n\rangle\restriction D : n < \omega\rangle, \langle h_n\restriction D : n < \omega\rangle\rangle$$

of $(\kappa_n)_{n<\omega}$-pattern to a finite D. An ω-cardinal identity

$$\mathfrak{I} = \langle\langle E_a : a \in \bigcup_{n<\omega} D_n\rangle, \langle\langle D_n, <_n\rangle : n < \omega\rangle, \langle h_n : n < \omega\rangle\rangle$$

is a *subidentity* of another ω-cardinal identity

$$\mathfrak{I}' = \langle\langle E'_a : a \in \bigcup_{n<\omega} D'_n\rangle, \langle\langle D'_n, <_n\rangle : n < \omega\rangle, \langle h'_n : n < \omega\rangle\rangle,$$

in symbols $\mathfrak{I} \leq \mathfrak{I}'$, if there is an order-preserving mapping $\pi : \bigcup_{n<\omega} D_n \longrightarrow \bigcup_{n<\omega} D'_n$ such that

(S1) $\pi\restriction D_m : \langle D_m, <_m\rangle \longrightarrow \langle D'_m, <'_m\rangle$ is order-preserving;

(S2) for $\{d_0, \ldots, d_n\}, \{d'_0, \ldots, d'_n\} \in [\bigcup_{n<\omega} D_n]^n$,

$$\text{if } \{d_0, \ldots, d_n\}E_a\{d'_0, \ldots, d'_n\}, \text{ then } \{\pi d_0, \ldots, \pi d_n\}E_{\pi a}\{\pi d'_0, \ldots, \pi d'_n\};$$

(S3) $\pi h_m(\{d_0, \ldots, d_n\}) \leq'_m h'_m(\{\pi d_0, \ldots, \pi d_n\})$ if $\{d_0, \ldots, d_n\} \in [\bigcup_{n<\omega} D_n]^n$.

Let $\mathfrak{I}(\mathcal{F})$ be the set of all subidentities of $\mathcal{F}\restriction D$ for finite D. We write $(\kappa_n)_{n<\omega} \to (\mathfrak{I})$ if $\mathfrak{I}$ belongs to $\mathfrak{I}(\mathcal{F})$ for every $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Let $\mathfrak{I}((\kappa_n)_{n<\omega})$ be the set of all $\mathfrak{I}$ such that $(\kappa_n)_{n<\omega} \to (\mathfrak{I})$, i. e.

$$\mathfrak{I}((\kappa_n)_{n<\omega}) = \bigcap\{\mathfrak{I}(\mathcal{F}) : \mathcal{F} \text{ is a } (\kappa_n)_{n<\omega}\text{-pattern}\}.$$

Definition 3.5 A $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$ is *fundamental* if $\mathfrak{I}(\mathcal{F}) = \mathfrak{I}((\kappa_n)_{n<\omega})$.

Suppose now that there is a fundamental $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Let us see how we can finish the construction of a κ-like model for T. We built up a $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}'$ from the model $\mathfrak{M}'$. Since $\mathcal{F}$ is fundamental, there is a finite set D' such that $\mathcal{F}\restriction D \leq \mathcal{F}'\restriction D'$. Thus $\mathfrak{M}$ can be expanded to a model of Σ.

To sum up, we have proved the following result:

Theorem 3.6 *If there is a fundamental $(\kappa_n)_{n<\omega}$-pattern, then first order logic on $(\kappa_n)_{n<\omega}$-models is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$. In particular, $L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$.*

The question of existence of fundamental $(\kappa_n)_{n<\omega}$-patterns is, of course, quite difficult. Let us recall some earlier results obtained by means of a construction of a fundamental pattern:

Theorem 3.7

1. *If $\aleph_0$ is small for κ, then $L(\exists^{\geq\kappa})$ is λ-compact for all $\lambda < \kappa$* (see [22]).

2. *If $\lambda^\omega = \lambda$ and $\kappa \geq \lambda$, then first order logic on (κ, λ)-models is λ-compact. In particular, then $L(\exists^{\geq\lambda^+})$ is λ-compact* (see [22]).

3. *If $\beth_\omega(\lambda) \leq \kappa$, then first order logic on (κ, λ)-models is λ-compact* (see [28]).

4. *If $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, λ singular, κ singular strong limit, then first order logic on (κ, λ)-models is recursively axiomatizable and λ-compact* (see [23]).

5. *If κ is singular strong limit, $L(\exists^{\geq\kappa})$ is λ-compact and recursively axiomatizable for each $\lambda < \kappa$* (see [23]; see [19] for details).

6. *If κ is ω-Mahlo, then $L(\exists^{\geq\kappa})$ is λ-compact and recursively axiomatizable for each $\lambda < \kappa$* (see [21]).

If $\aleph_0$ is small for each κ_n, then a simple enumeration argument gives a fundamental $(\kappa_n)_{n<\omega}$-pattern.

Corollary 3.8 [22] *If $\aleph_0$ is small for each κ_n, then first order logic on $(\kappa_n)_{n<\omega}$-like models is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$. In particular, then $L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$.*

If each κ_n is singular strong limit and no κ_n is a supremum of some of the others, then there is a fundamental $(\kappa_n)_{n<\omega}$-pattern $\mathcal{E}$, and $\mathfrak{I}((\kappa_n)_{n<\omega})$ is recursive and independent of the cardinals κ_n [23] (see [19] for details). Thus we have:

Corollary 3.9 [23] *If each κ_n is singular strong limit and no κ_n is a supremum of some of the others, then $L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact and recursively axiomatizable for each $\lambda < \min\{\kappa_n : n < \omega\}$.*

Example 3.10 $L(\exists^{\geq \beth_{\omega\cdot n}})_{0<n<\omega}$ is λ-compact and recursively axiomatizable for all $\lambda < \beth_\omega$.

Example 3.11 The logic $L(\exists^{\geq \beth_{\omega\cdot n}})_{0<n\leq\omega}$ fails to be $\aleph_0$-compact for trivial reasons. Still every fragment containing only finitely many generalized quantifiers is $\aleph_0$-compact.

If each κ_n is ω-Mahlo, then any κ-pattern is fundamental.

Corollary 3.12 [21] *If each κ_n is ω-Mahlo, then $L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact and recursively axiomatizable for each $\lambda < \min\{\kappa_n : n < \omega\}$.*

The results of this section could have been proved also for a finite sequence $(\kappa_n)_{n<m}$ of uncountable cardinals, with obvious modifications.

3.1 The character of $L(\exists^{\geq\kappa_n})_{n<\omega}$

Our goal in this section is to give the axioms $\mathcal{A}$ of $L(\exists^{\geq\kappa_n})_{n<\omega}$ and prove that $\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ has recursive character. Since $L(\exists^{\geq\kappa_n})_{n<\omega}$ is the union of its fragments $L(\exists^{\geq\kappa_n})_{n<m}$, where $n < \omega$, we first introduce an axiomatization of $L(\exists^{\geq\kappa_n})_{n<m}$ and discuss its completeness.

3.1.1 Logic with finitely many quantifiers

Keisler gave a simple and elegant complete axiomatization for $L(\exists^{\geq\aleph_1})$ based on a formalization of the principle that if an uncountable set is divided into non-empty parts, then either there are uncountably many parts or one part is uncountable. If $\kappa = \kappa^{<\kappa}$, this works also for $L(\exists^{\geq\kappa^+})$, but it certainly does not work for $L(\exists^{\geq\kappa})$ if κ is singular. Keisler gave a different axiomatization for $L(\exists^{\geq\kappa})$ when κ is a singular strong limit cardinal. We give a general axiomatization $\mathcal{A}_m$ for $L(\exists^{\geq\kappa_n})_{n<m}$, whatever $(\kappa_n)_{n<m}$ is, plus a criterion when this is complete. The question whether $\mathcal{A}_m$ is a *recursive* axiomatization remains open. In certain cases we can assert its recursiveness. We use this axiomatization to prove the finite character of the logic frame $\langle L(\exists^{\geq\kappa_n})_{n<m}, \mathcal{A}_m\rangle$.

In fact, we do not give the axioms of $\mathcal{A}_m$ explicitly but only give a criterion for their choice. Because of the nature of this criterion the set of Gödel numbers of the axioms is recursively enumerable. The method of "straightening Henkin-formulas" introduced by Barwise [2], could be used to turn our criterion into an explicit, albeit probably very complicated, set of axioms.

We defined above what it means for a $(\kappa_n)_{n<m}$-like model to induce a $(\kappa_n)_{n<m}$-pattern. If we have a model that is not necessarily $(\kappa_n)_{n<m}$-like, it may fail to induce a $(\kappa_n)_{n<\omega}$-pattern but it still induces some ω-cardinal identities. The concept of inducing an identity is defined as follows: The model $\mathfrak{M}$ of a vocabulary $L^* \cup \{c_a : a \in \bigcup_{n\in\mathbb{N}} A_n\}$, L containing unary predicates P_n, $n \in \mathbb{N}$, and the finite set Σ of first order sentences in the vocabulary of $\mathfrak{M}$ *induce* the ω-cardinal identity

$$\mathfrak{I} = \langle\langle E_a : a \in \textstyle\bigcup_{n<\omega} D_n\rangle, \langle\langle D_n, <_n\rangle : n < \omega\rangle, \langle h_n : n < \omega\rangle\rangle$$

defined as follows: Let D_n be the set of $a \in A_n$ for which c_a occurs in Σ. If $a \in D_m$, then define for increasing $\langle a_0, \ldots, a_n\rangle, \langle b_0, \ldots, b_n\rangle \in [\bigcup_{n<\omega} D_n]^{<\omega}$

$$\begin{aligned}\{a_0, \ldots, a_n\} E'_a \{b_0, \ldots, b_n\} \Leftrightarrow \mathfrak{M}' \models\ & t(c_{a_0}, \ldots, c_{a_n}) = t(c_{b_0}, \ldots, c_{b_n}) \\ & \vee (\neg(t(c_{a_0}, \ldots, c_{a_n}) <_m c_a) \wedge \neg(t(c_{b_0}, \ldots, c_{b_n}) <_m c_a)), \\ & \text{for all Skolem-terms } t \text{ occurring in } \Sigma,\end{aligned}$$

and

$$\begin{aligned}h'_m(\{a_0, \ldots, a_n\}) = \min\{b \in D_m :\ & t(c_{a_0}, \ldots, c_{a_n})^{\mathfrak{M}'} <_m c_b^{\mathfrak{M}'} \text{ for all} \\ & \text{Skolem-terms } t \text{ occurring in } \Sigma\} \text{ (or undefined).}\end{aligned}$$

This concept is the heart of our axiom system $\mathcal{A}_m$. Suppose φ is a sentence in $L(\exists^{\geq\kappa_n})_{n<m}$. Fuhrken introduced a reduction method by means of which there is a first order sentence φ^+ in a larger vocabulary such that φ has a model if and only if φ^+ has a $(\kappa_n)_{n<m}$-like model.

Definition 3.13 A sentence φ of $L(\exists^{\geq\kappa_n})_{n<m}$ in the vocabulary L is said to be $\mathcal{A}_m$*-consistent*, if for all $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$ and all finite $\Sigma \subseteq \{\varphi^+\}^*$ there is a model $\mathfrak{M}$ of Σ such that $\mathfrak{M}$ and Σ induce $\mathfrak{I}$. The set $\mathcal{A}_m$ of axioms of $L(\exists^{\geq\kappa_n})_{n<m}$ consists of all sentences φ of $L(\exists^{\geq\kappa_n})_{n<m}$ for which $\neg\varphi$ is not $\mathcal{A}_m$-consistent.

The definition of the axioms $\mathcal{A}_m$ may seem trivial as we seem to take all "valid" sentences as axioms. However, whether all "valid" sentences are actually axioms depends on whether we can prove the completeness of our axioms. Also, while there is no obvious reason why the set of valid sentences should be recursively enumerable in $\mathfrak{I}((\kappa_n)_{n<\omega})$, the set $\mathcal{A}_m$ of axioms certainly is.

Lemma 3.14 *Suppose φ is a sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ and φ has a model. Then φ is $\mathcal{A}_m$-consistent.*

Proof. Suppose $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$ and $\Sigma \subseteq \{\varphi^+\}^*$ is finite. Suppose $\mathfrak{M}$ is a $(\kappa_n)_{n<\omega}$-like model of Σ. Then $\mathfrak{M}$ and Σ induce a $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Since $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<\omega})$, there is a finite D such that $\mathfrak{I} \leq \mathcal{F}\restriction D$. Thus $\mathfrak{M}$ and Σ induce $\mathfrak{I}$. □

Lemma 3.15 *If there is a fundamental $(\kappa_n)_{n<m}$-pattern, then every $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ has a model.*

Proof. Suppose φ is an $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<m}$. Let $\mathcal{F}$ be a fundamental $(\kappa_n)_{n<m}$-pattern. Let $T = \{\varphi^+\}$. It suffices to show that the theory (T1) – (T5) constructed from $\mathcal{F}$ and T is finitely consistent. Let Σ be a finite part of (T1) – (T5) and let D be the set of $a \in \bigcup_{n<\omega} A_n$ for which c_a occurs in Σ. Note, that if we let $\mathfrak{I} = \mathcal{F}\restriction D$, then $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$. By assumption, $\Sigma \cap T^*$ has a model $\mathfrak{M}$ such that $\mathfrak{M}$ and Σ induce $\mathcal{F}\restriction D$. Thus $\mathfrak{M}$ can be expanded to a model of Σ. □

Proposition 3.16 *If every $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ has a model, then there is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Proof. Let $\mathfrak{I}$ be an arbitrary ω-cardinal identity, as in (2). Let the size of $\mathfrak{I}$ be k and length of $\mathfrak{I}$ be l. Let

$$\bigcup_{i\leq l} D_i = \{d_1, \dots, d_k\},$$

and let $\vec{s}$ denote a sequence $\langle s_i : i \leq l\rangle$ of natural numbers $\leq k$. We will say that $\{a_0, \dots, a_l\} \in \{d_1, \dots, d_k\}$ *is of type* $\vec{s}$ if the intersection of $\{a_0, \dots, a_l\}$ with D_i has size s_i for each $i \leq l$. Consider the following sentences of $L(\exists^{\geq\kappa_n})_{n<m}$ in a vocabulary consisting of a unary predicate P_i, a binary predicate $<_i$ and function symbols $F_i^{\vec{s}}$ and $H_i^{\vec{s}}$ for each $i < m$ and $n \leq k$. Let $\sigma_{\mathfrak{I}}$ be the conjunction of:

1. $\langle P_n, <_n\rangle$ is a κ_n-like linear order for $n < m$.
2. $F_i^{\vec{s}}$ is a function mapping sets $\{a_0, \dots, a_l\}$ of type $\vec{s}$ to P_i for $n < k$ and $i < m$.
3. The range of $F_i^{\vec{s}}$ is bounded in P_i.
4. $H_i^{\vec{s}}$ is a function mapping sets $\{a_0, \dots, a_l\}$ of type $\vec{s}$ to P_i for $n < k$ and $i < m$.
5. There are no $x_0, \dots, x_l$ of type $\vec{s}$ which would satisfy
   (a) $F_i^{\vec{s}}(x_{r_0}, \dots, x_{r_n}) = F_i^{\vec{s}}(x_{r'_0}, \dots, x_{r'_n}) \vee (F_i^{\vec{s}}(x_{r_0}, \dots, x_{r_n}) \geq_i d_a \wedge F_i^{\vec{s}}(x_{r'_0}, \dots, x_{r'_n}) \geq_i d_a)$ whenever

$$\langle d_{r_0}, \dots, d_{r_n}\rangle, \langle d_{r'_0}, \dots, d_{r'_n}\rangle \in [\bigcup_{n<\omega} D_n]^{<\omega}$$

   are increasing of type $\vec{s}$, $\{d_{r_0}, \dots, d_{r_n}\} E_a \{d_{r'_0}, \dots, d_{r'_n}\}$, and $a \in D_i$, and
   (b) $x_{h(\{d_{r_0}, \dots, d_{r_n}\})} \leq_i H_i^n(x_{r_0}, \dots, x_{r_n})$ whenever $h(\{d_{r_0}, \dots, d_{r_n}\}) \in D_i$.

Any model $\mathfrak{M}$ of $\sigma_{\mathfrak{I}}$ and any choice of a cofinal suborder $\langle A'_n, <_n\rangle$ of $\langle P_n, <_n\rangle^{\mathfrak{M}}$ of type κ_n (for $n < \omega$) gives rise to a $(\kappa_n)_{n<m}$-pattern $\mathcal{F}'$ as in (1), where for $a \in A'_i$

$$\begin{aligned}\{a_0, \dots, a_n\} E'_a \{b_0, \dots, b_n\} \Leftrightarrow {} & \text{if } (F_i^n)^{\mathfrak{M}}(a_0, \dots, a_n) <_i a \text{ or } (F_i^n)^{\mathfrak{M}}(b_0, \dots, b_n) <_i a, \\ & \text{then } (F_i^n)^{\mathfrak{M}}(a_0, \dots, a_n) = (F_i^n)^{\mathfrak{M}}(b_0, \dots, b_n),\end{aligned}$$

and

$$h'_i(\{a_0, \dots, a_n\}) = (H_i^n)^{\mathfrak{M}}(a_0, \dots, a_n).$$

We have written into the sentence $\sigma_{\mathfrak{I}}$ the condition that $\mathfrak{I}$ is not in $\mathfrak{I}(\mathcal{F}')$. On the other hand, if $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$, it is easy to construct a model of $\sigma_{\mathfrak{I}}$. Moreover, if $\mathfrak{I}_0, \dots, \mathfrak{I}_n \notin \mathfrak{I}((\kappa_n)_{n<m})$, it is not hard to construct a model of $\sigma_{\mathfrak{I}_0} \wedge \dots \wedge \sigma_{\mathfrak{I}_n}$.

Let $\mathfrak{I}_n$, $n < \omega$, be a list of all $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$. Without loss of generality, this list is recursive in $\mathcal{A}_m$. Suppose the set of valid $L(\exists^{\geq\kappa_n})_{n<m}$-sentences is recursively enumerable in $\mathcal{A}_m$. Now we use an argument (due to Per Lindström [14]) from abstract model theory. Let A be a set of natural numbers which is co-recursively enumerable in $\mathcal{A}_m$ but not recursively enumerable in $\mathcal{A}_m$. Say,

$$n \in A \Leftrightarrow \forall k\,((n,k) \in B),$$

where B is recursive in $\mathcal{A}_m$. Let P be a new unary predicate symbol and θ_n the first order sentence saying that P has exactly n elements. Let T be the theory

$$\{\theta_n \to \sigma_{\mathfrak{I}_i} : (\forall k \leq i)((n,k) \in B)\},$$

and let $C = \{n : T \models \neg\theta_n\}$. We show that $C \subseteq A$. Suppose that $T \models \neg\theta_n$. If $n \notin A$, then there is k such that $(n,k) \notin B$. Let $\mathfrak{M}$ be a model of $\{\sigma_{\mathfrak{I}_j} : i < k\} \cup \{\theta_m\}$. If $\theta_n \to \sigma_{\mathfrak{I}_i} \in T$, then $i < k$, whence $\mathfrak{M} \models \sigma_{\mathfrak{I}_i}$. So $\mathfrak{M} \models T$, a contradiction. Since C is recursively enumerable in $\mathcal{A}$, there is $n \in A \setminus C$. Thus there is $\mathfrak{M} \models T$ such that $\mathfrak{M} \models \theta_n$. Since $\forall k\,((n,k) \in B)$, the sentence $\theta_n \to \sigma_{\mathfrak{I}_i}$ is in T, and thereby true in $\mathfrak{M}$ for every i. Since $\mathfrak{M} \models \theta_n$, $\mathfrak{M} \models \sigma_{\mathfrak{I}_i}$ for all i. Let $\mathcal{F}$ be the $(\kappa_n)_{n<m}$-pattern that $\mathfrak{M}$ gives rise to. $\mathcal{F}$ is necessarily a fundamental $(\kappa_n)_{n<m}$-pattern. □

Summing up:

Theorem 3.17 *Suppose $(\kappa_n)_{n<m}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\mathcal{A}_m$ is a complete axiomatization of $L(\exists^{\geq\kappa_n})_{n<m}$.*
2. *$\langle L(\exists^{\geq\kappa_n})_{n<m}, \mathcal{A}_m\rangle$ is recursively compact.*
3. *$L(\exists^{\geq\kappa_n})_{n<m}$ is λ-compact for all $\lambda < \min\{\kappa_0, \ldots, \kappa_{m-1}\}$.*
4. *There is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Corollary 3.18 *$\langle L(\exists^{\geq\kappa_n})_{n<m}, \mathcal{A}_m\rangle$ has finite character.*

We do not know if $\mathcal{A}_m$ is recursive, except in such special cases as in Corollaries 3.9 and 3.12.

Recall the definition of $\mathfrak{I}(\kappa^+, \kappa)$ in [26] and [19].

Proposition 3.19

1. *Suppose that $\mathfrak{I}(\kappa^+, \kappa)$ is recursive, and that either A^* is recursive or there exists a universe $V' \supseteq V$ in which $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is recursively compact, then $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ has finite character.*

2. *Suppose that $\langle L(\exists^{\geq\aleph_1}), A^*\rangle$ is coherent (i. e. if a sentence has a model, then it is consistent with A^*). Then $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ has finite character.*

Proof.

1. Suppose $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is complete. Let $\Phi \in L(\exists^{\geq\kappa^+})$ say in the language of set theory that $\sigma_{\mathfrak{I}}$ holds for all $\mathfrak{I} \notin \mathfrak{I}(\kappa)$. Since $\mathfrak{I}(\kappa^+, \kappa)$ is recursive, this can be written in $L(\exists^{\geq\kappa^+})$. We show that Φ is consistent with the axioms A^*: If A^* is recursive, $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is recursively compact and there is a fundamental (κ^+, κ)-pattern, whence Φ is consistent with A^*. On the other hand, if there is a universe V' in which $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is recursively compact, then in V' there is a fundamental (κ^+, κ)-pattern, and hence in V' the sentence Φ is consistent with A^*. Thus Φ is consistent with A^* also in V. By completeness Φ has a model. Thus there is a fundamental (κ^+, κ)-pattern and $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is $\aleph_0$-compact.

2. Completeness implies $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$. We know that $\mathfrak{I}(\aleph_1, \aleph_0)$ is recursive (see [25]). $\mathfrak{I}(\kappa^+, \kappa)$ is also recursive, since $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ implies $\mathfrak{I}(\aleph_1, \aleph_0) = \mathfrak{I}(\kappa^+, \kappa)$. Now we use part 1. □

Corollary 3.20 *The logic frame $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$, where A^* is the Keisler axiomatization for $L(\exists^{\geq\aleph_1})$ and κ is an arbitrary cardinal, has finite character.*

3.1.2 Logic with infinitely many quantifiers

The axioms $\mathcal{A}$ are simply all the axioms $\mathcal{A}_m$, $m < \omega$, put together.

Proposition 3.21 *If $\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ is recursively compact, then there is a fundamental $(\kappa_n)_{n<\omega}$-pattern.*

P r o o f. Let $\mathfrak{I}_n$, $n < \omega$, be a list of all $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<\omega})$. Without loss of generality, this list is recursive in $\mathcal{A}$. Note that if $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<\omega})$, then there is m such that $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$, so we can use the sentences $\sigma_{\mathfrak{I}_n}$. Let T be the set of all $\sigma_{\mathfrak{I}_n}$, $n < \omega$. This theory is recursive in $\mathcal{A}$ and it is finitely consistent. Hence it has a model. The $(\kappa_n)_{n<\omega}$-pattern the model $\mathfrak{M}$ gives rise to is clearly fundamental. □

Theorem 3.22 *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\mathcal{A}$ is a complete axiomatization of $L(\exists^{\geq\kappa_n})_{n<\omega}$.*
2. *For every $m < \omega$ there is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Theorem 3.23 *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ is recursively compact.*
2. *$L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$.*
3. *There is a fundamental $(\kappa_n)_{n<\omega}$-pattern.*

Corollary 3.24 *$\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ has recursive character.*

Example 3.25 If $\kappa_n = \beth_{\omega\cdot n}$ for $0 < n \leq \omega$, then $\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ is complete but not $\aleph_0$-compact and thereby does not have finite character.

Above we investigated $\vec{\kappa}$-like models and related them to logic frames arising from generalized quantifiers. Similar results can be proved for models with predicates of given cardinality and also for models with a linear order in which given predicates have given cofinalities, but these results do not have natural formulations in terms of generalized quantifiers.

4 A logic which does not have recursive character

We show that there is a logic frame L^* which is recursively compact but not $\aleph_0$-compact. We make use of the quantifier Q^{St} from [24]. To recall the definition of Q^{St} we adopt the following notation:

Definition 4.1 Let $\mathfrak{A} = (A, R)$ be an arbitrary $\aleph_1$-like linearly ordered structure. We use $H(\mathfrak{A})$ to denote the set of all initial segments of $\mathfrak{A}$. A *filtration of* $\mathfrak{A}$ is a subset X of $H(\mathfrak{A})$ such that $A = \bigcup_{I\in X} I$ and X is closed under unions of increasing sequences. Let $D(\mathfrak{A})$ be the filter on $H(\mathfrak{A})$ generated by all filtrations of $\mathfrak{A}$.

Definition 4.2 The generalized quantifier Q^{St} is defined by $\mathfrak{A} \models Q^{\mathrm{St}}xy\,\varphi(x, y, \vec{a}\,)$ if and only if (A, R_φ), where $R_\varphi = \{(b, c) : \mathfrak{A} \models \varphi(b, c, \vec{a}\,)\}$, is an $\aleph_1$-like linearly ordered structure such that

$$\{I \in H(\mathfrak{A}) : I \text{ does not have a sup in } R_\varphi\} \notin D(\mathfrak{A}).$$

The generalized quantifier Q^{Cub}, definable in terms of Q^{St} and $\exists^{\geq\aleph_1}$, is defined by $\mathfrak{A} \models Q^{\mathrm{Cub}}xy\,\varphi(x, y, \vec{a}\,)$ if and only if (A, R_φ), where $R_\varphi = \{(b, c) : \mathfrak{A} \models \varphi(b, c, \vec{a}\,)\}$, is an $\aleph_1$-like linearly ordered structure such that

$$\{I \in H(\mathfrak{A}) : I \text{ does not have a sup in } R_\varphi\} \in D(\mathfrak{A}).$$

It follows from [24] and [3] that $L_{\omega\omega}(Q^{\mathrm{St}})$ equipped with some natural axioms and rules is a complete $\aleph_0$-compact logic frame.

Definition 4.3 If $S \subseteq \omega_1$, then the generalized quantifier Q^{St}_S is defined by $\mathfrak{A} \models Q^{\mathrm{St}}_S xy\,\varphi(x, y, \vec{a}\,)$ if and only if R_φ is an $\aleph_1$-like linear order of A with a filtration $\{I_\alpha : \alpha < \omega_1\}$ such that

$$(\forall\alpha < \omega_1)(I_\alpha \text{ has a sup in } R_\varphi \Leftrightarrow \alpha \in S).$$

The syntax of the logic L^{St} is defined as follows: L^{St} extends first order logic by the quantifiers $\exists^{\geq\aleph_1}$, Q^{St} and the infinite number of new formal quantifiers $Q^{\mathrm{St}}_{X_n}$ (we leave X_n unspecified).

If we fix a sequence $\langle S_0, S_1, \ldots\rangle$ and let $Q^{\mathrm{St}}_{X_n}$ be interpreted as $Q^{\mathrm{St}}_{S_n}$, we get a definition of semantics of L^{St}. We call this semantics the $\langle S_0, S_1, \ldots\rangle$*-interpretation of* L^{St}. Thus L^{St} has a fixed syntax and fixed axioms, given below, but many different semantics, depending on our interpretation of $\langle X_0, X_1, \ldots\rangle$ by various $\langle S_0, S_1, \ldots\rangle$.

Definition 4.4 We call a finite sequence $\sigma = \langle S_0, S_1, \ldots, S_n\rangle$ (or an infinite sequence $\langle S_0, S_1, \ldots\rangle$) of subsets of ω_1 *stationary independent*, if all finite Boolean combinations of the sets S_i are stationary.

If φ is a formula and $d \in 2$, let $(\varphi)^d$ be φ, if $d = 0$, and $\neg\varphi$, if $d = 1$. If $S \subseteq \omega_1$, then $(S)^d$ is defined similarly.

Definition 4.5 The axioms of $\mathcal{A}$ are:

(Ax1) The usual axioms and rules of $L(Q_1)$.

(Ax2) $Q^{\mathrm{St}}xy\,\varphi(x,y,\vec{z}\,) \to$ "R_φ is an $\aleph_1$-like linear order".

(Ax3) $Q^{\mathrm{St}}_{X_n}xy\,\varphi(x,y,\vec{z}\,) \to Q^{\mathrm{St}}xy\,\varphi(x,y,\vec{z}\,)$.

(Ax4) Independence Axiom Schema: Any non-trivial Boolean combination of the set S_n interpreting the X_n is stationary, i. e. $\Phi \to \Psi$, where Φ is the conjunction of the formulas

(a) "R_φ is an $\aleph_1$-like linear order";

(b) $Q^{\mathrm{St}}_{X_i}xy\,(\varphi(x,y,\vec{z}\,) \wedge \neg\theta_i(x,\vec{z}\,) \wedge \neg\theta_i(y,\vec{z}\,))$, $i = 0,\dots,l$;

and Ψ is the conjunction of the formulas $Q^{\mathrm{St}}xy\,(\varphi(x,y,\vec{z}\,) \wedge \bigwedge_{i<l}(\theta(y,\vec{z}\,))^{\eta(i)})$, for all $\eta : l \longrightarrow 2$.

(Ax4) Pressing Down Axiom Schema:

$$[Q^{\mathrm{St}}xy\,\varphi(x,y,\vec{u}\,) \wedge \forall x \exists z\,(\varphi(z,x,\vec{u}\,) \wedge \psi(x,z,\vec{u}\,))]$$
$$\to \exists z\, Q^{\mathrm{St}}xy\,(\varphi(x,y,\vec{u}\,) \wedge \psi(x,z,\vec{u}\,) \wedge \psi(y,z,\vec{u}\,)).$$

The axioms of $L^{\mathrm{St}}_{\omega_1\omega}$ are the above added with the usual axioms and rules of $L_{\omega_1\omega}$.

Definition 4.6 Suppose $\langle S_0, S_1, \dots\rangle$ is stationary independent. We define a new recursive logic frame

$$L^{\mathrm{St}}(S_0, S_1, \dots) = \langle L^{\mathrm{St}}(S_0, S_1, \dots), \models, \mathcal{A}\rangle,$$

where $\mathcal{A}$ is as in Definition 4.5. Let $L^{\mathrm{St}}_{\omega_1\omega}(S_0, S_1, \dots)$ be the extension of $L^{\mathrm{St}}(S_0, S_1, \dots)$ obtained by allowing countable conjunctions and disjunctions.

The standard proof (see e. g. [3]) shows:

Lemma 4.7

1. *The logic frames $L^{\mathrm{St}}(S_0, S_1, \dots)$ and $L^{\mathrm{St}}_{\omega_1\omega}(S_0, S_1, \dots)$ are complete for all stationary independent $\langle S_0, S_1, \dots\rangle$.*

2. *$\varphi \in L^{\mathrm{St}}(S_0, S_1, \dots)$ has a model in an $\langle S_0, S_1, \dots\rangle$-interpretation for some stationary independent $\langle S_0, S_1, \dots\rangle$ if and only if φ has a model in an $\langle S_0, S_1, \dots\rangle$-interpretation for all stationary independent $\langle S_0, S_1, \dots\rangle$.*

An immediate consequence of Lemma 4.7 is that the set $\mathrm{Val}(L^{\mathrm{St}})$ of sentences of L^{St} which are valid under $\langle S_0, S_1, \dots\rangle$-interpretation for some (equivalently, all) stationary independent $\langle S_0, S_1, \dots\rangle$ is recursively enumerable, provably in ZFC, and the predicate "φ has a model", where $\varphi \in L^{\mathrm{St}}_{\omega_1\omega}$, is a Σ_1^{ZFC}-definable property of φ.

By making different choices for the stationary independent $\langle S_0, S_1, \dots\rangle$, we can get logics with different properties. Clearly there is a trivial choice of $\langle S_0, S_1, \dots\rangle$ for which L^{St} fails to have $\aleph_0$-compactness. On the other hand, CH fails if and only if there is a choice of $\langle S_0, S_1, \dots\rangle$ which will make L^{St} $\aleph_0$-compact. We make now a choice of $\langle S_0, S_1, \dots\rangle$ which will render $L^{\mathrm{St}}(S_0, S_1, \dots)$ recursively compact but not $\aleph_0$-compact.

Let us fix a countable vocabulary τ which contains infinitely many symbols of all arities. Let T_n, $n < \omega$, list all $\mathcal{A}$-consistent recursive L^{St}-theories in the vocabulary τ. Let τ^n be a new disjoint copy of τ for each $n < \omega$. Let τ^* consist of the union of all the τ_n, the new binary predicate symbol $<^*$, and new unary predicate symbols P_n for $n < \omega$. For any $\eta : \omega \longrightarrow 2$ let $\psi_\eta \in L^{\mathrm{St}}_{\omega_1\omega}$ be the conjunction of the following sentences of the vocabulary τ^*:

(a) T_n translated into the vocabulary τ^n.

(b) "$<^*$ is an $\aleph_1$-like linear order of the universe".

(c) $Q^{\mathrm{St}}_{X_n}xy\,(x <^* y \wedge P_n(x) \wedge P_n(y))$.

(d) $\neg\exists x\,(\bigwedge_n (P_n(x))^{\eta(n)})$.

Lemma 4.8 *There is $\eta : \omega \longrightarrow 2$ such that ψ_η has a model.*

P r o o f. Let Γ consist of the sentences (a) – (c). By Lemma 4.7, Γ has a model $\mathfrak{M}$ of cardinality $\aleph_1$ in the $\langle S_0, S_1, \dots\rangle$-interpretation for some stationary independent $\langle S_0, S_1, \dots\rangle$. Get a new $\eta : \omega \longrightarrow 2$ by Cohen-forcing. Then in the extension $V[\eta]$

$$\textstyle\bigcap_n (S_n)^{\eta(n)} = \emptyset.$$

Thus $V[\eta]$ satisfies the Σ_1-sentence

(3) $\exists\eta\,(\psi_\eta \text{ has a model}).$

By the Levy-Shoenfield Absoluteness Lemma and Proposition 4.7 there is η in V such that (3) holds in V. □

Now let $\langle S_0^*, S_1^*, \ldots\rangle$ be stationary independent such that ψ_η has a model $\mathfrak{M}^*$ in the $\langle S_0^*, S_1^*, \ldots\rangle$-interpretation.

Theorem 4.9 *The recursive logic frame $L^{\mathrm{St}}(S_0^*, S_1^*, \ldots)$ is recursively compact but not $\aleph_0$-compact.*

P r o o f. Suppose T is a consistent recursive theory in L^{St}. W. l. o. g. $T = T_m$ for some $m < \omega$. Thus $\mathfrak{M}^* \restriction \tau^n$ gives immediately a model of T. To prove that L^{St} is not $\aleph_0$-compact, let T be a theory consisting of the following sentences:

(i) "$<^*$ is an $\aleph_1$-like linear order".

(ii) $Q^{\mathrm{St}}_{S_n^*}xy\,(x <^* y \wedge P_n(x) \wedge P_n(y))$ for $n < \omega$.

(iii) $Q^{\mathrm{St}}xy\,(x <^* y \wedge P(x) \wedge P(y))$.

(iv) $\forall x\,(P(x) \rightarrow (P_n(x))^{\eta(n)})$ for $n < \omega$.

Any finite subtheory of T contains only predicates $P_0, \ldots, P_m$ for some m, and has therefore a model: we let $P_i = S_i^*$ for $i = 0, \ldots, m$ and

$$P = (P_0)^{\eta(0)} \cap \cdots \cap (P_m)^{\eta(m)}.$$

On the other hand, suppose $\langle A, <^*, P, P_0, P_1, \ldots\rangle \models T$. By (ii) there are filtrations $\langle D_\alpha^n : \alpha < \omega_1\rangle$ of $<^*$ and clubs E^n such that for all n and for all $\alpha \in E^n$

$$\{\alpha < \omega_1 : D_\alpha^n \text{ has a sup in } \langle A, <^*\rangle\} = S_n^*.$$

By (iii) there is a filtration $\langle F_\alpha : \alpha < \omega_1\rangle$ of $<^*$ such that

$$B = \{\alpha < \omega_1 : F_\alpha \text{ has a sup in } P\}$$

is stationary. Let $E^* \subseteq \bigcap_n E_n$ be a club such that $C_\alpha = D_\alpha^n = F_\alpha$ for $\alpha \in E^*$ and $n < \omega$. Let $\delta \in E^* \cap B$ and $a = \sup F_\delta$. Then $a \in P$, hence $a \in \bigcap_n (P_n)^{\eta(n)}$ by (iv). As $a = \sup D_\delta^n$ for all n, we have $a \in \bigcap_n (S_n^*)^{\eta(n)}$, contrary to the choice of η. We have proved that theory T has no models. □

Thus $L^{\mathrm{St}}(S_0^*, S_1^*, \ldots)$ does not have finite character. We end with an example of a logic which, without being provably complete, has anyhow finite character: Recall that $\diamondsuit_S$ for $S \subseteq \omega_1$ is the statement that there are sets $A_\alpha \subseteq \alpha$, $\alpha \in S$, such that for any $X \subseteq \omega_1$, the set $\{\alpha \in S : X \cap \alpha = A_\alpha\}$ is stationary.

Definition 4.10 Let $\mathcal{L}^\diamondsuit$ be the extension of $L_{\omega\omega}$ by $\exists^{\geq\aleph_1}$, Q^{St} and $Q^{\mathrm{St}}_{\mathcal{S}}$, where

$$\mathcal{S} = \begin{cases} \emptyset & \text{if there is no bistationary } S \text{ with } \diamondsuit_S,\\ \omega_1 & \text{if there is a bistationary } S \text{ with } \diamondsuit_S \text{ but no maximal one},\\ S & \text{if } S \text{ is a maximal bistationary } S \text{ with } \diamondsuit_S. \end{cases}$$

We get a recursive logic frame $L^\diamondsuit = \langle \mathcal{L}^\diamondsuit, \models, \mathcal{A}\rangle$ by adapting the set $\mathcal{A}$ to the case of just one bistationary set.

Theorem 4.11 *$L^\diamondsuit$ has finite character.*

P r o o f. Let us first suppose there is no bistationary S with $\diamondsuit_S$. Then the consistent sentence

$$\text{“} < \text{is an } \aleph_1\text{-like linear order”} \wedge Q^{\mathrm{St}}xy\,(x < y) \wedge Q^{\mathrm{St}}_{\mathcal{S}}(x < y)$$

has no model, so $\mathcal{L}$ is incomplete. Suppose then there is a bistationary S with $\diamondsuit_S$ but no maximal one. Then the consistent sentence

$$\text{“} < \text{is an } \aleph_1\text{-like linear order”} \wedge Q^{\mathrm{St}}xy\,(x < y \wedge P(x)) \wedge Q^{\mathrm{St}}_{\mathcal{S}}(x < y \wedge \neg P(x))$$

has no model, so $\mathcal{L}$ is again incomplete. Finally, suppose there is a maximal bistationary S with $\diamondsuit_S$. Now $\mathcal{L}$ is $\aleph_0$-compact by an analog of Lemma 4.7. □

Our results obviously do not aim to be optimal. We merely want to indicate that the concept of a logic frame offers a way out of the plethora of independence results about generalized quantifiers. The logic $L_{\omega\omega}(\exists^{\geq\aleph_{n+1}})_{n<\omega}$ is a good example. The results about its $\aleph_0$-compactness under GCH and $\aleph_0$-incompactness in other models of set theory leave us perplexed about the nature of the logic. Having recursive character reveals something conclusive and positive, and raises the question, do other problematic logics also have recursive character. Our logic L^{St} is the other extreme: it is always completely axiomatizable, but a judicious choice of $\langle S_0, S_1, \ldots\rangle$ renders it recursively compact without being $\aleph_0$-compact.

Open Question 4.12 *Does the Magidor-Malitz logic $L(Q_1^{\mathrm{MM}})$ have recursive character?*

$L(Q_1^{\mathrm{MM}})$ is $\aleph_0$-compact whenever $\diamond$ holds (see [15]). But $L(Q_1^{\mathrm{MM}})$ may fail to be $\aleph_0$-compact (see [1]). The question is whether $L(Q_1^{\mathrm{MM}})$ is $\aleph_0$-compact in every model in which it is recursively compact.

Acknowledgements This is an extended version of Preprint # 593 of the Centre de Recerca Matemàtica, Barcelona. The second author is grateful for the hospitality of the CRM in April 2004.

The first author's research was supported by the Israel Science Foundation. Publication number [ShVa:790]. The second author's research was partially supported by grant 40734 of the Academy of Finland.

References

[1] U. Abraham and S. Shelah, A Δ^2_2 well-order of the reals and incompactness of $L(Q^{\mathrm{MM}})$. Ann. Pure Applied Logic **59**, 1 – 32 (1993).
[2] J. Barwise, Some applications of Henkin quantifiers. Israel J. Math. **25** (1/2), 47 – 63 (1976).
[3] J. Barwise, M. Kaufmann, and M. Makkai, Stationary logic. Ann. Math. Logic **13**, 171 – 224 (1978).
[4] J. Barwise and S. Feferman (eds.), Model-theoretic logics (Springer-Verlag, 1985).
[5] C. C. Chang, A note on the two cardinal problem. Proc. Amer. Math. Soc. **16**, 1148 – 1155 (1965).
[6] G. Fuhrken, Skolem-type normal forms for first-order languages with a generalized quantifier. Fund. Math. **54**, 291 – 302 (1964).
[7] G. Fuhrken, Languages with added quantifier "there exist at least $\aleph_\alpha$". In: Theory of Models, Proceedings 1963 International Symposium Berkeley, pp. 121 – 131 (North-Holland, 1965).
[8] W. Hodges and S. Shelah, There are reasonably nice logics. J. Symbolic Logic **56**, 300 – 322 (1991).
[9] R. B. Jensen, The fine structure of the constructible hierarchy. Ann. Math. Logic **4**, 229 – 308 (1972).
[10] C. R. Karp, Languages with expressions of infinite length (North-Holland, 1964).
[11] H. J. Keisler, Models with orderings. In: Logic, Methodology and Philosophy of Science III, Proceedings Third International Congress, Amsterdam, 1967, pp. 35 – 62 (North-Holland, 1968).
[12] H. J. Keisler, Logic with the quantifier "there exist uncountably many". Ann. Math. Logic **1**, 1 – 93 (1970).
[13] P. Lindström, First order predicate logic with generalized quantifiers. Theoria **32**, 186 – 195 (1966).
[14] P. Lindström, On extensions of elementary logic. Theoria **35**, 1 – 11 (1969).
[15] M. Magidor and J. Malitz, Compact extensions of $L(Q)$. Ann. Math. Logic **11**, 217 – 261 (1977).
[16] W. Mitchell, Aronszajn trees and the independence of the transfer property. Ann. Math. Logic **5**, 21 – 46 (1972/73).
[17] M. Morley, Partitions and models. In: Proceedings of the Summer School in Logic, Leeds, 1967, pp. 109 – 158 (Springer-Verlag, 1968).
[18] A. Mostowski, On a generalization of quantifiers. Fund. Math. **44**, 12 – 36 (1957).
[19] J. H. Schmerl, Transfer theorems and their applications to logics. In: Model-theoretic logics, Perspectives in Mathematical Logic, pp. 177 – 209 (Springer-Verlag, 1985).
[20] J. H. Schmerl, An axiomatization for a class of two-cardinal models. J. Symbolic Logic **42**, 174 – 178 (1977).
[21] J. H. Schmerl and S. Shelah, On power-like models for hyperinaccessible cardinals. J. Symbolic Logic **37**, 531 – 537 (1972).
[22] S. Shelah, Two cardinal compactness. Israel J. Math. **9**, 193 – 198 (1971).
[23] S. Shelah, On models with power-like orderings. J. Symbolic Logic **37**, 247 – 267 (1972).
[24] S. Shelah, Generalized quantifiers and compact logic. Trans. Amer. Math. Soc. **204**, 342 – 364 (1975).
[25] S. Shelah, Appendix to: "Models with second-order properties II. Trees with no undefined branches". Ann. Math. Logic **14**, 73 – 87 and 223 – 226 (1978).
[26] S. Shelah, The pair $(\aleph_n, \aleph_0)$ may fail $\aleph_0$-compactness. In: Logic Colloquium 2001, Lecture Notes in Logic 20, pp. 402 – 433 (ASL, 2005).
[27] R. L. Vaught, The completeness of logic with the added quantifier "there are uncountably many". Fund. Math. **54**, 303 – 304 (1964).
[28] R. L. Vaught, A Löwenheim-Skolem theorem for cardinals far apart. In: Theory of Models, Proceedings 1963 International Symposium Berkeley, pp. 390 – 401 (North-Holland, 1965).

