Advances in Mathematics 290 (2016) 614–681

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Existence of optimal ultrafilters and the fundamental complexity of simple theories

M. Malliaris [a,*], S. Shelah [b,c]

[a] *Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA*
[b] *Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel*
[c] *Department of Mathematics, Hill Center – Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA*

ARTICLE INFO

Article history:
Received 15 June 2014
Received in revised form 8 December 2015
Accepted 9 December 2015
Available online 5 January 2016
Communicated by H. Jerome Keisler

MSC:
03C45
03C20
03E05
03G05
06E10

Keywords:
Model theory
Ultrafilters
Saturation of ultrapowers
Keisler's order
Classification of simple theories
Optimal ultrafilters

ABSTRACT

In the first edition of *Classification Theory*, the second author characterized the stable theories in terms of saturation of ultrapowers. Prior to this theorem, stability had already been defined in terms of counting types, and the unstable formula theorem was known. A contribution of the ultrapower characterization was that it involved sorting out the global theory, and introducing nonforking, seminal for the development of stability theory. Prior to the present paper, there had been no such ultrapower characterization of an unstable class. In the present paper, we first establish the existence of so-called optimal ultrafilters on (suitable) Boolean algebras, which are to simple theories as Keisler's good ultrafilters [15] are to all (first-order) theories. Then, assuming a supercompact cardinal, we characterize the simple theories in terms of saturation of ultrapowers. To do so, we lay the groundwork for analyzing the global structure of simple theories, in ZFC, via complexity of certain amalgamation patterns. This brings into focus

* Corresponding author.
E-mail address: mem@math.uchicago.edu (M. Malliaris).

http://dx.doi.org/10.1016/j.aim.2015.12.009

a fundamental complexity in simple unstable theories having no real analogue in stability.

1. Introduction

1.1. Background

We begin by giving some history and context of the power of ultraproducts as a tool in mathematics, and specifically in model theory. Ultrafilters on an infinite cardinal λ are maximal (under inclusion) subsets of the power set of λ which are closed under finite intersection, upward closed, and do not contain the empty set. These give a robust notion of largeness, allowing for infinite averaging arguments and the study of asymptotic or pseudofinite behavior in models. Early appearances were in the work of Tarski 1930 [35] on measures and Cartan 1937 [4,5] in general topology. The groundwork for their use in model theory was laid in the 1950s and early 1960s by Łoś [20], Tarski, Keisler [14], Frayne, Morel, and Scott [10], and Kochen [18] in terms of the *ultraproduct* construction. Given an ultrafilter $\mathcal{D}$ on λ, the ultraproduct N of a sequence of models $\langle M_\alpha : \alpha < \lambda \rangle$ in a fixed language $\mathcal{L}$ has as its domain the set of equivalence classes of elements of the Cartesian product $\prod_{\alpha<\lambda} M_\alpha$ under the equivalence relation of being equal on a set in $\mathcal{D}$. One then defines the relations, functions, and constants of $\mathcal{L}$ on each tuple of elements of the ultraproduct to reflect the average behavior across the index models. The fundamental theorem of ultraproducts, Łoś' theorem, says that the set of statements of first order logic true in the ultraproduct are precisely the statements true in a $\mathcal{D}$-large set of index models, i.e. the theory of N is the average theory of the models M_α. Model theorists concentrated further on so-called regular ultrafilters, as will be explained in due course.

This construction gave rise to some remarkable early transfer theorems. For example, Ax and Kochen [1–3] and independently Eršov [8] proved that for any nonprincipal (= containing all cofinite sets) ultrafilter $\mathcal{D}$ on the set of primes, the ultraproduct $\mathcal{Q}_p = \prod_p Q_p/\mathcal{D}$ of the p-adic fields Q_p and the ultraproduct $\mathcal{S}_p = \prod_p \mathbb{F}_p((t))/\mathcal{D}$ of the fields of formal power series over $\mathbb{F}_p$ are elementarily equivalent, i.e. satisfy the same first-order statements. Then from Lang's theorem that every homogeneous polynomial of degree $> d$ with more than d^2 variables has a nontrivial zero in $\mathbb{F}_p((t))$ for each p they deduce the corresponding theorem in Q_p for all but finitely many p.

Working with ultrapowers, meaning that all the index models are the same, a similar averaging process happens. The central "algebraic characterization of elementary equivalence" now appears: two models satisfy the same set of first order statements precisely when they have isomorphic ultrapowers, proved by Keisler 1961 under GCH [14] and by Shelah 1971 [32] in general.

The theorems of Ax–Kochen and Ersov just mentioned used only ultrafilters on ω, Łoś' transfer of the first order theory and $\aleph_1$-saturation. The first order theory is a relatively superficial description of models shared by many in the same class, e.g. all algebraically closed fields of characteristic 0. From a model-theoretic point of view, the deeper structure of ultrapowers has to do with the Stone space of types and the property of saturation.[1]

Keisler, one of the major architects of model theory beginning in the 1960s, in particular has done much on ultrapowers, see [16]. He proved that one could define and build a family of so-called good regular ultrafilters [15],[2] and he noticed that for any $\mathcal{D}$ in this family of good regular ultrafilters and any model M in a countable language, the ultrapower $M^\lambda/\mathcal{D}$ is sufficiently, i.e. λ^+-, saturated. Moreover, ultrapowers of certain theories are only saturated if the ultrafilter is good. In 1967 [17], Keisler proposed a means of comparing the complexity of theories according to the difficulty of saturating their ultrapowers (Definition 2.4). Progress on this far reaching program, known as Keisler's order, requires advances in model theory on one hand, and advances in ultrafilter construction on the other.

From the model theoretic point of view, a major motivation for understanding Keisler's order comes from the search for dividing lines, that is, properties whose presence gives complexity and whose absence gives a good structure theory. For example, the dividing line of stable versus unstable theories has been fundamental since the 1970s [34]. However, there are many unstable theories, and for some of them a 'positive theory' may be analyzed; so if one hopes to generalize stability theory, a natural approach is to find and develop other dividing lines one by one in response to suitable questions. By a 1978 theorem of Shelah, Keisler's order independently detects the dividing line at stability. This suggests that a fruitful and moreover uniform way of looking for meaningful divisions in the enormous class of unstable theories is to progress, if possible, in the unstable classification of Keisler's order. In short, Keisler's order provides a uniform point of view from which to approach the problem of looking for dividing lines for a large and central family of theories, and a model-theoretic incentive to characterize equivalence classes. A natural target is the family of simple theories, a central and popular family in model theory for more than two decades. For background on simple theories and some history, see the survey [11]. We mention here that simple unstable theories include examples such as pseudofinite fields [13] and have been a fertile ground of model-theoretic interaction with algebra, geometry, combinatorics and number theory.

[1] Saturation is a fullness condition. Informally, we may identify types over a given set A with orbits under automorphisms of some much larger universal domain which fix A pointwise, so call a model M κ-saturated if it contains representatives of all orbits under any automorphism of the larger universal domain which fix some subset of M of size $< \kappa$ pointwise. Syntatically, an A-type p is a maximal consistent set of formulas in a fixed number of free variables and parameters from A; it is realized in the model $M \supseteq A$ if for some $\bar{a} \in \mathrm{Dom}(M)$, $M \models \varphi(\bar{a})$ for all $\varphi \in p$; and a κ-saturated model is one in which all types over sets $|A| < \kappa$ are realized.

[2] Keisler's proof assumed GCH, an hypothesis removed by Kunen [19], in a proof which introduced some central techniques in ultrafilter construction.

For many years there was little progress on Keisler's order. Recently, a series of papers by the authors has significantly changed the landscape (see [26] for some history) and given us the leverage for the present work. In the current paper, we establish the existence of a new family of ultrafilters in parallel with developing the model theory of simple theories in a direction very different from prior work. Combining the two, we prove a characterization of the class of simple theories in terms of saturation of ultrapowers, assuming a supercompact cardinal. (For a discussion of our use of a large cardinal, see §2.2 below.) As would be expected, our work here has two complementary parts: on one hand we define and establish the existence of optimal ultrafilters, and on the other we extend the model theory of simple theories in order to show that such ultrafilters affect saturation. We hope to eventually be able to eliminate the large cardinal hypothesis in the main theorem of this paper, but for now it clarifies the model-theoretic content by allowing us to work with models as closed sets (as will be explained in due course).

Although there has been much interest in and work on simple theories, many basic structural questions about simple theories and the extent to which they may differ essentially from a few canonical examples remain wide open. One theme of this paper has to do with finding the right frame for seeing divisions in complexity classes among the simple unstable theories. The complexity we detect has primarily to do with amalgamation; it appears built on non-forking, and has no real analogue in the stable case (where we have amalgamation even of $\mathcal{P}^-(n)$-diagrams, see [34] Chapter XII). We give a model-theoretic formulation of this property, which we call explicit simplicity, in Section 3. The history of classification theory would suggest that future work may well reveal other formulations of this property; such a formulation would be likely to arise from progress on determining the identity of equivalence classes in Keisler's order among the simple theories (as opposed to the identification of dividing lines).

1.2. Results

We prove the following theorems. In each case, the results will hold for any four-tuple of infinite cardinals $(\lambda, \mu, \theta, \sigma)$ satisfying the following hypotheses, plus any additional requirements given in the theorem.

Definition 1.1. Call $\lambda, \mu, \theta, \sigma$ *suitable* when:

(a) $\sigma \leq \theta \leq \mu < \lambda$.
(b) θ is regular, $\mu = \mu^{<\theta}$ and $\lambda = \lambda^{<\theta}$.
(c) $(\forall \alpha < \theta)(2^{|\alpha|} < \mu)$.

Conditions 1.1(b) and (c) will essentially guarantee that certain equivalence relations defined in the course of our proofs are not unnecessarily large. We will be mainly interested in cases where $\mu < \lambda$. Note that the hypotheses of 1.1 hold when $\sigma = \theta = \aleph_0$ and $\mu < \lambda$ are any infinite cardinals, or when $\sigma = \theta$ is regular, $\mu = (2^\theta)^+$, and $\mu^+ = \lambda$

(hence the existence of a suitable tuple with uncountable σ is provable in ZFC), or when σ is uncountable and supercompact, $\sigma = \theta = \mu$, and $\mu^+ = \lambda$. It will follow from the main definitions that these cardinals $\lambda, \mu, \theta, \sigma$ each control specific aspects of both the model-theoretic and the set-theoretic picture, and varying their values, modulo the basic constraints of 1.1, will give useful information.

We first state the theorem which organizes our main results, before discussing "explicitly simple."

Theorem. *(Organizing theorem, Theorem 8.1 below.) Assume $(\lambda, \mu, \theta, \sigma)$ are suitable and that σ is an uncountable supercompact cardinal. There exists a regular ultrafilter $\mathcal{D}$ over λ such that for every model M in a countable signature, $M^\lambda/\mathcal{D}$ is λ^+-saturated if $Th(M)$ is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, and $M^\lambda/\mathcal{D}$ is not μ^{++}-saturated if $Th(M)$ is not simple.*

Regarding supercompact, usually "σ a compact cardinal" will suffice (keeping in mind that we will be working with σ-complete filters and ultrafilters), but the existence theorem for optimal ultrafilters given below assumes existence of an uncountable supercompact cardinal. As the statement of this theorem suggests, a model-theoretic contribution of the paper is the development of a notion we call $(\lambda, \mu, \theta, \sigma)$-explicitly simple, a measure of the complexity of amalgamation, discussed further in the introduction to Section 3. It will be clear from Section 3 that $(\lambda, \mu, \theta, \sigma)$-explicitly simple becomes weaker as μ increases, and that every $(\lambda, \mu, \theta, \sigma)$-explicitly simple theory is simple, even when $\mu = \lambda$. More remarkable is that it is possible to capture simplicity in this way.

Theorem. *(Simple theories are explicitly simple, Theorem 4.10 below.) Assume $(\lambda, \mu, \theta, \sigma)$ are suitable. If $\mu^+ = \lambda$, then every simple theory T with $|T| < \sigma$ is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, and moreover this characterizes simplicity of T.*

As discussed later in this paper, we believe this new characterization will re-open the research on simple theories, which has long been dominated by analogies to stability theory, by allowing for a classification of simple unstable theories according to the possible values of $\mu << \lambda$.

Complementing the development of explicit simplicity, we define and prove existence of a new family of ultrafilters, called $(\lambda, \mu, \theta, \sigma)$-optimal. These ultrafilters, defined in Section 5, may be thought of as an analogue of Keisler's *good* ultrafilters from [15] which handle patterns arising from explicitly simple theories. The Boolean algebra in the statement of this theorem will be defined in 2.19(3).

Theorem. *(Existence theorem for optimal ultrafilters, Theorem 5.9 below.) There exists a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter on the Boolean algebra $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$ whenever $(\lambda, \mu, \theta, \sigma)$ are suitable and $\sigma > \aleph_0$ is supercompact.*

On the connection of this ultrafilter on a Boolean algebra to a regular ultrafilter on λ, see Section 1.3 below. Finally, leveraging explicit simplicity and optimality, we prove the algebraic characterization of simple theories.

Theorem. *(Ultrapower characterization of simplicity, Theorem 8.2 below.) Suppose $(\lambda, \mu, \theta, \sigma)$ are suitable where σ is an uncountable supercompact cardinal and $\mu^+ = \lambda$. Then there is a regular ultrafilter $\mathcal{D}$ on λ such that for any model M in a countable signature, $M^\lambda/\mathcal{D}$ is λ^+-saturated if $Th(M)$ is simple and $M^\lambda/\mathcal{D}$ is not λ^+-saturated if $Th(M)$ is not simple.*

Theorem 8.2 has the following consequence for Keisler's order:

Conclusion. *(On Keisler's order $\trianglelefteq$, Conclusion 8.4 below.) Assume there exists an uncountable supercompact cardinal. If T, T' are complete countable theories, T is simple, and $T' \trianglelefteq T$, then T' is simple.*

Finally, we return to ultrafilters in the case $\sigma = \theta = \aleph_0$. We define and prove existence of so-called perfect ultrafilters for $\sigma = \theta = \aleph_0$, and prove that such ultrafilters are optimal. Theorem 9.4 is proved in ZFC.

Theorem. *(Existence of perfect ultrafilters, Theorem 9.4 below.) Let $(\lambda, \mu, \aleph_0, \aleph_0)$ be suitable. Let $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu}$. Then there exists a (λ, μ)-perfect ultrafilter on $\mathfrak{B}$.*

To conclude this catalogue of results, we record that the technology developed here has some surprising consequences, which will appear in subsequent work. Notably, overturning a longstanding conjecture that Keisler's order has finitely many classes, we prove in [29] that already within the simple theories there is substantial complexity:

Theorem A. *(See Malliaris and Shelah [29], in ZFC.) Keisler's order has infinitely many classes. In fact, there is an infinite strictly descending chain of simple low theories in Keisler's order, above the random graph.*

Thus Keisler's order is sensitive to the fine structure of amalgamation as measured by our criterion of explicit simplicity. This framework raises questions which we plan to address in work in progress, related to the natural interpretation of the coloring criterion developed here within particular classes of simple theories.

1.3. Introduction for set theorists

Here we briefly outline the innovations of the paper which may be of interest to set theorists, independent of the model-theoretic questions of saturation of simple theories. These are of two kinds (as we will explain). First is constructing ultrafilters on λ or

on related Boolean algebras. Second, the classes Mod_T, for T simple, include examples natural for set theorists such as the random graphs and certain classes of hypergraphs.

A major part of the paper has to do with the construction of regular ultrafilters *using* large cardinals. Historically, model theorists had typically focused on regular ultrafilters because of the connections to the compactness theorem, and in particular their nice saturation properties (see e.g. Theorem B, p. 622 below) whereas set theorists had typically focused on understanding quite complete ultrafilters under the relevant large cardinal hypotheses (or remnants of this like $\aleph_0^{\aleph_1}/\mathcal{D} = \aleph_1$). Our present approach, following our earlier paper [27], reunites the two. Let $\mathfrak{B}$ be a complete Boolean algebra of cardinality $\leq 2^\lambda$ with the $\leq \lambda^+$-c.c. Following [27], we build regular ultrafilters on λ by first building a regular, λ^+-good filter $\mathcal{D}_0$ so that $\mathcal{P}(\lambda)/\mathcal{D}_0$ is isomorphic to $\mathfrak{B}$, and then complete the construction by specifying an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$, which need not be regular. In the present paper, our main case is $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$, the completion of the free Boolean algebra generated by 2^λ independent partitions of size μ, where intersections of size $< \theta$ are nonempty. (Further work considering the case where $\mathfrak{B}$ is not necessarily the completion of a free Boolean algebra will be developed in [30].) In the present paper, we use $\mathfrak{B}$ exclusively to refer to one of these completions of a free Boolean algebra.

In this setup, our focus is on construction of appropriate ultrafilters $\mathcal{D}_*$ on $\mathfrak{B}$. The present paper introduces two new set-theoretic properties of ultrafilters on such Boolean algebras, "optimal" in 5.8 and "perfect" in 9.1, and proves that such ultrafilters exist. Both definitions capture in some sense being as good (c.f. Keisler's 'good' ultrafilters) as possible modulo some background cardinal constraints. We succeed to prove that if $\sigma \leq \theta \leq \lambda$ and σ is supercompact then on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$ an optimal ultrafilter exists; and we prove existence of a perfect ultrafilter on $\mathfrak{B}$ in ZFC assuming $\sigma = \theta = \aleph_0$, and show that any such ultrafilter is optimal. In the present paper, we require that σ, if uncountable, is supercompact rather than simply requiring existence of a σ-complete ultrafilter on λ, because this is what we use in the existence proof for optimal ultrafilters.

The model-theoretic usefulness of this "separation of variables" approach was established by [27], in particular Theorem F quoted in the next section, which says that the resulting saturation properties of the regular ultrafilter $\mathcal{D}$ induced on λ by $\mathcal{D}_0$ and $\mathcal{D}_*$ may be characterized in terms of related conditions on the ultrafilter $\mathcal{D}_*$. So we are free to address saturation problems by working with σ-complete ultrafilters $\mathcal{D}_*$ on Boolean algebras (in the present case, completions of free Boolean algebras), a much richer context. But it has also pure set-theoretic meaning: for instance, finding ultrafilters on λ which are flexible but not good, see 5.18 below, as asked by Dow 1985 [6].

Readers unfamiliar with simple theories may prefer to keep in mind one of the many natural combinatorial examples of such theories. For each $n > k \geq 2$, let $T_{n,k}$ be the theory of the unique countable generic hypergraph in a language with a $k+1$-ary graph hyperedge (a symmetric, irreflexive $k+1$-place relation) where the axioms say that the theory is infinite and any configuration of edges and non-edges is allowed provided that there are no complete hypergraphs on $n+1$ vertices (i.e. there do not exist $n+1$ vertices of which every distinct subset of size $k+1$ is a hyperedge). When $k = 1$ such theories

are not simple (e.g. the triangle-free random graph) but when $k \geq 2$ they are (e.g. the tetrahedron-free three-hypergraph), as proved by Hrushovski [13]. These examples will be central to a further analysis of simple theories via perfect ultrafilters in [29].

The reader interested primarily in these new ultrafilters may skip ahead to Sections 5 and 9. Such a reader may also find it useful to skim Section 2 for insight into the saturation claims we make about these ultrafilters, which are largely combinatorial in nature.

2. Overview and preparation

2.1. Overview

We begin by giving an overview of some main themes of the paper. For additional information on Keisler's order the reader may wish to consult Keisler 1967 [17], or the recent papers [26,23], and [31].

Convention 2.2. *(On types.) Given $N := M^\lambda/\mathcal{D}$ an ultrapower,*

(a) *Call a type or partial type p over A, $A \subseteq N$* small *if $|A| \leq \lambda$.*
(b) *Any small type may be enumerated (possibly with repetitions) as $\{\varphi_i(x, a_i) : i < \lambda\}$, where $\ell(a_i)$ need not be 1.*
(c) *For each parameter $a \in A \subseteq N$, fix in advance some lifting of a to M^λ. Then by the notation $a[t]$ we mean the t-th coordinate of this lifting of a. When a is a tuple $a_1, \ldots, a_n$, the notation $a[t]$ is understood to mean the tuple $a_1[t], \ldots, a_n[t]$.*[3] *We will use this notation throughout the paper.*
(d) *By Łoś' theorem, if p is a consistent partial type in N then we may define the* Łoś map $f : [p]^{<\aleph_0} \to \mathcal{D}$ *by*

$$u \mapsto \{t \in \lambda : M \models \exists x \bigwedge_{i \in u} \varphi_i(x, a_i[t])\}$$

(e) *$[A]^{<\kappa}$ denotes the set of all subsets of A of cardinality $< \kappa$.*
(f) *$\mathrm{Dom}(M)$ denotes the universe of a structure M, and $||M|| = |\mathrm{Dom}(M)|$.*

Definition 2.3. The ultrafilter $\mathcal{D}$ on λ is *regular* if it contains a regularizing family, that is, a set $\{X_i : i < \lambda\} \subseteq \mathcal{D}$ such that for any $u \subseteq \lambda$, $|u| \geq \aleph_0$,

$$\bigcap_{i \in u} X_i = \emptyset.$$

[3] Informally, $a[t]$ is the "projection of a to index t" or "to the index model M_t".

Equivalently, $\mathcal{D}$ is regular if every set of size $\leq \lambda$ in any $\mathcal{D}$-ultrapower is covered by a pseudofinite set.

The hypothesis *regular* entails that saturation of ultrapowers is a property of the (countable) theory, not the model chosen:

Theorem B. *(See Keisler [15] Cor. 2.1a.) When $\mathcal{D}$ is a regular ultrafilter on λ and $M \equiv N$ in a countable signature, then $M^\lambda/\mathcal{D}$ is λ^+-saturated iff $N^\lambda/\mathcal{D}$ is λ^+-saturated.*

Theorem B justifies the quantification over all models in the next, central definition.

Definition 2.4. (Keisler's order, Keisler 1967 [17].) Let T_1, T_2 be complete countable first order theories. Say $T_1 \trianglelefteq T_2$ if whenever $\lambda \geq \aleph_0$, $\mathcal{D}$ is a regular ultrafilter on λ, $M_1 \models T_1$, $M_2 \models T_2$ we have that

$$(M_2)^\lambda/\mathcal{D} \text{ is } \lambda^+\text{-saturated} \implies (M_1)^\lambda/\mathcal{D} \text{ is } \lambda^+\text{-saturated}$$

Keisler's order $\trianglelefteq$ is a preorder on theories, often thought of as a partial order on the $\trianglelefteq$-equivalence classes.

Question 2.5. *(See Keisler 1967.) Determine the structure of Keisler's order.*

The state of what was known about the structure Keisler's order through 2012 can be found in section 4 of the authors' paper [23]. Since that paper was written, and prior to the current paper, the following results have been obtained:

Theorem C. *(See Malliaris and Shelah [27].) Keisler's order has at least two classes among the simple unstable theories.*

Theorem D. *(See Malliaris and Shelah [25], announced in [26].) Any theory with the model-theoretic tree property SOP_2 belongs to the maximum class in Keisler's order.*

To explain what is, in general, at stake in questions of saturation of ultrapowers, we now discuss types in regular ultrapowers. For transparency, in this section, all languages and thus all theories are countable. (Beginning in §3, we will allow the language to be uncountable.)

Definition 2.6. Let $\mathcal{D}$ be a regular ultrafilter on λ, M a model in a countable signature, p a small partial type over $A \subseteq N := M^\lambda/\mathcal{D}$. A *distribution* of p is a map $d : [p]^{<\aleph_0} \to \mathcal{D}$ such that:

(a) d is monotonic, i.e. $u \subseteq v \implies d(v) \subseteq d(u)$, and $d(\emptyset) = \lambda$
(b) d refines the Łoś map f, meaning that $d(u) \subseteq f(u)$ for each $u \in [p]^{<\aleph_0}$
(c) the image of d is a regularizing family, 2.3 above.

In some sense, the problem of realizing types in ultrapowers is already visible in what Łoś' theorem does *not* guarantee. Although a type is "on average" (in the ultrapower) consistent, i.e. distributions exist, when we try to realize it by assigning finitely many formulas of the type to each index model via a distribution 2.6 it becomes apparent that there is no guarantee that the finite set of formulas $\{\varphi_i(x, a_i[t]) : t \in d(\{\varphi_i\})\}$ assigned to index t has a common realization.

Note that 2.6(a) is not necessary, as it may always be ensured (refining a given map by induction on the size of u).

Specifically, the following fact explains a basic mechanism controlling saturation of regular ultrapowers.

Fact 2.7. *(See [23] 1.8.) Let $\mathcal{D}$ be a regular ultrafilter on λ, M a model in a countable signature, p a small partial type over $A \subseteq N := M^\lambda/\mathcal{D}$. Then the following are equivalent:*

1. *p is realized in N.*
2. *Some distribution d of p has a multiplicative refinement, that is, a map $d' : [p]^{<\aleph_0} \to \mathcal{D}$ such that for any u, v, first, $d'(u) \subseteq d(u)$, and second, $d'(u) \cap d'(v) = d'(u \cup v)$.*

The property of (monotonic) maps from $[\lambda]^{<\aleph_0} \to \mathcal{D}$ admitting multiplicative refinements is a natural set-theoretic question:

Definition 2.8. (Good ultrafilters, Keisler [15].) The filter $\mathcal{D}$ on λ is said to be μ^+-*good* if every $f : [\mu]^{<\aleph_0} \to \mathcal{D}$ has a multiplicative refinement, where this means that for some $f' : [\mu]^{<\aleph_0} \to \mathcal{D}$, $u \in [\mu]^{<\aleph_0} \implies f'(u) \subseteq f(u)$, and $u, v \in [\mu]^{<\aleph_0} \implies f'(u) \cap f'(v) = f'(u \cup v)$.

Note that we may assume the functions f are monotonic.

$\mathcal{D}$ is said to be *good* if it is λ^+-good.

Keisler proved that good regular ultrafilters on λ always exist assuming GCH [15]; this was proved in ZFC by Kunen [19]. Thus, by Fact 2.7, for any λ there exists a regular ultrafilter on λ such that $M^\lambda/\mathcal{D}$ is λ^+-saturated for any M in a countable signature. In the other direction, there exist T able to code failures of goodness, e.g. $Th([\omega]^{<\aleph_0}, \subseteq)$, so that if $M \models T$ then $M^\lambda/\mathcal{D}$ is λ^+-saturated iff $\mathcal{D}$ is good (Keisler [17] Theorem 1.4c). This proves existence of a maximum class in Keisler's order.

Definition 2.9. Reflecting the saturation properties of good ultrafilters, when $\mathcal{D}$ is an ultrafilter on λ we will say that "$\mathcal{D}$ is good for T," or "$\mathcal{D}$ is (λ^+, T)-good," to mean that for any $M \models T$, $M^\lambda/\mathcal{D}$ is λ^+-saturated.

We now know that it is also possible for a theory to be Keisler-maximal without explicitly coding all failures of goodness:

Theorem E. *(See Shelah 1978 [34] VI.3.9.) Any theory with the strict order property is maximal in Keisler's order, e.g. Th*$(\mathbb{Q}, <)$.

In fact, SOP_2 suffices (Malliaris and Shelah, Theorem D above). The "basis" of functions whose multiplicative refinements ensure that of all others is not yet understood. We know the only essential complexity is local:

Fact 2.10. *(Local saturation suffices, Malliaris [21] Theorem 12.) Suppose* $\mathcal{D}$ *is a regular ultrafilter on* I *and* T *a countable complete first order theory. Then for any* $M^I/\mathcal{D}$*, the following are equivalent:*

1. $M^I/\mathcal{D}$ *is* λ^+*-saturated.*
2. $M^I/\mathcal{D}$ *realizes all* φ*-types over sets of size* $\leq \lambda$*, for all formulas* φ *in the language of* T.

To understand classes other than the Keisler-maximal class, as in the present paper, it is therefore necessary to realize some types while omitting others, that is, to understand how certain model-theoretically meaningful families of functions may have multiplicative refinements while others do not. A point of leverage on this problem was built in [27] and applied there to obtain the first ZFC dividing line among the unstable theories. It translates the problem just described into a problem about patterns in some quotient Boolean algebra, as we now explain. For 2.11, note that the notion of an λ^+-excellent filter is defined in [27]. It is proved in Theorem 12.3 of that paper that a filter is λ^+-excellent if and only if it is λ^+-good and in the present paper, λ^+-excellent and λ^+-good are used interchangeably.[4]

Definition 2.11. (Regular ultrafilters built from tuples, from [27].) Suppose $\mathcal{D}$ is a regular ultrafilter on I, $|I| = \lambda$. We say that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ when the following hold. Note that λ is given by $\mathcal{D}_0$, and if not mentioned otherwise, we will assume the index set of $\mathcal{D}$ is λ.

1. $\mathcal{D}_0$ is a regular, $|I|^+$-excellent filter on I
   (for the purposes of this paper, it is sufficient to use regular and good)
2. $\mathfrak{B}$ is a complete Boolean algebra of cardinality 2^λ and $\leq \lambda^+$-c.c.
3. $\mathcal{D}_*$ is an ultrafilter on $\mathfrak{B}$
4. there exists a surjective homomorphism $\mathbf{j} : \mathcal{P}(I) \to \mathfrak{B}$ such that:

[4] More precisely, one can define a notion of "excellent for a theory T" and likewise of "good for a theory T." What is proved in [27] is that "excellent" i.e. "excellent for all countable T" coincides with "good" i.e. "good for all countable T" i.e. every monotonic function from finite subsets of I into the filter has a multiplicative refinement. This is the property we need for Theorem F, so the reader may substitute good for excellent in that theorem. However, it is important to mention that for specific values of T, "excellent for T" and "good for T" need not coincide. If one wanted to work with more precise versions of Theorem F where $\mathcal{D}_0$ is excellent only for certain theories, the situation might be different.

(a) $\mathcal{D}_0 = \mathbf{j}^{-1}(\{1_{\mathfrak{B}}\})$
(b) $\mathcal{D} = \{A \subseteq I : \mathbf{j}(A) \in \mathcal{D}_*\}$.

We may make $\mathbf{j}$ explicit and write "built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*, \mathbf{j})$".

It was verified in [27] Theorem 8.1 that whenever $\mu \leq \lambda$ and $\mathfrak{B}$ is μ^+-c.c. there exists a regular good $\mathcal{D}_0$ on λ and a surjective homorphism $\mathbf{j} : \mathcal{P}(I) \to \mathfrak{B}$ such that $\mathcal{D}_0 = \mathbf{j}^{-1}(1)$. Thus, Definition 2.11 is meaningful, and this opens up many possibilities for ultrafilter construction.

We now state Theorem F, used throughout the present paper, and then define "morality" in 2.14.

Theorem F. *(Separation of variables, Malliaris and Shelah [27] Theorem 6.13; see Observation 2.15 below.) Let $\kappa \leq \lambda$. Suppose that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*, \mathbf{j})$, and $\mathcal{D}_0$ is excellent.*[5] *Then the following are equivalent:*

(A) *$\mathcal{D}_*$ is $(\kappa, \mathfrak{B}, T)$-moral, i.e. κ-moral for each formula φ of T.*
(B) *For any $M \models T$, $M^\lambda/\mathcal{D}_1$ is κ^+-saturated.*

The practical consequence of Theorem F is that one can construct regular ultrafilters in a two-step process. First, one constructs a λ^+-excellent *filter* $\mathcal{D}_0$ admitting the desired homomorphism $\mathbf{j}$ to a specified Boolean algebra $\mathfrak{B}$. One may ensure the non-saturation half of the argument at this stage by clever choice of $\mathfrak{B}$: for example, a key move of [27] is to show that if $\mathfrak{B}$ has the μ^+-c.c. for $\mu < \lambda$, then $\mathcal{D}$ cannot be good for non-low or non-simple theories, regardless of the choice of $\mathcal{D}_*$. See Section 2.2 below.[6] See §2.2 below for the present analogue. Second, one builds an appropriate ultrafilter $\mathcal{D}_*$ on the Boolean algebra $\mathfrak{B}$, usually focused on the positive (saturation) side of the argument. Theorem F ensures that $\mathcal{D}_*$ controls the resulting saturation properties of $\mathcal{D}$. This will be our strategy below.

We now explain the condition of "morality" on $\mathcal{D}_*$.

Definition 2.12. (Possibility patterns, c.f. [27].) Let $\mathfrak{B}$ be a Boolean algebra and $\overline{\varphi} = \langle \varphi_\alpha : \alpha < \lambda \rangle$ a sequence of formulas of $\mathcal{L}$. Say that $\overline{\mathbf{b}}$ is a $(\lambda, \mathfrak{B}, T, \overline{\varphi})$-possibility when:

1. $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$
2. $u \in [\lambda]^{<\aleph_0}$ implies $\mathbf{b}_u \in \mathfrak{B}^+$
3. if $v \subseteq u \in [\lambda]^{<\aleph_0}$ then $\mathbf{b}_u \subseteq \mathbf{b}_v$ (monotonicity) and $\mathbf{b}_u = 1_{\mathfrak{B}}$

[5] The requirement "$\mathcal{D}_0$ is λ^+-excellent" is assumed in the definition of "built from" but we repeat it here for emphasis.
[6] Informally, there is too little room in the Boolean algebra to account for the "wideness" of theories with significant forking: see [27] §9, specifically Conclusion 9.10. That proof uses in an essential way that (in our notation) $\theta = \aleph_0$, as will be explained in due course.

4. if $u_* \in [\lambda]^{<\aleph_0}$ and $\mathbf{c} \in \mathfrak{B}^+$ satisfies

$$(u \subseteq u_* \implies ((\mathbf{c} \leq \mathbf{b}_u) \ \vee \ (\mathbf{c} \leq 1 - \mathbf{b}_u)))$$

then we can find a model $M \models T$ and $a_\alpha \in M$ for $\alpha \in u_*$ such that for every $u \subseteq u_*$,

$$M \models (\exists x) \bigwedge_{\alpha \in u} \varphi_\alpha(x; a_\alpha) \ \text{ iff } \ \mathbf{c} \leq \mathbf{b}_u.$$

When the sequence $\overline{\varphi}$ is constant with each $\varphi_\alpha = \varphi$, say $\overline{\mathbf{b}}$ is a $(\lambda, \mathfrak{B}, T, \varphi)$-possibility.

Definition 2.12 ensures that $\overline{\mathbf{a}}$ could have plausibly arisen as the image under $\mathbf{j}$ of the distribution of a φ-type by asking that the Venn diagram of the elements $\mathbf{a}$ accurately reflects the complexity of φ: that is, whenever some nonzero element $\mathbf{b}$ of $\mathfrak{B}$ induces an ultrafilter on some $\{\mathbf{a}_v : v \subseteq u\}$, we can find a set of instances $\{\varphi_i : i \in u\}$ in a monster model of T whose pattern of intersection corresponds exactly to that dictated by $\mathbf{b}$.

Example 2.13. Let $\mathcal{D}$ be built from $(\mathcal{D}_1, \mathfrak{B}, \mathcal{D}_0, \mathbf{j})$. Let $p \in S(A), A \subseteq M^\lambda/\mathcal{D}_1$ be a small φ-type and, identifying p with λ, let $f : [\lambda]^{<\aleph_0} \to \mathcal{D}_1$ be the Łoś map of p. Let $\overline{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\lambda]^{<\aleph_0} \rangle$ be given by $\mathbf{a}_u = \mathbf{j}(f(u))$, so $\mathbf{a}_u \in \mathfrak{B}^+$. Then $\overline{\mathbf{a}}$ is a $(\lambda, \mathfrak{B}, T, \varphi)$-possibility.

Then morality, 2.14, is simply the Boolean algebra equivalent to a regular ultrafilter being good for a theory, see 2.7 and 2.9 above.

Definition 2.14. (Moral ultrafilters on Boolean algebras, [27].) We say that an ultrafilter $\mathcal{D}_*$ on the Boolean algebra $\mathfrak{B}$ is $(\lambda, \mathfrak{B}, T, \overline{\varphi})$-moral when for every $(\lambda, \mathfrak{B}, T, \overline{\varphi})$-possibility $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ such that $\mathbf{b}_u \in \mathcal{D}_*$ for each $u \in [\lambda]^{<\aleph_0}$, there is a multiplicative $\mathcal{D}_*$-refinement $\overline{\mathbf{b}'} = \langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0} \rangle$, i.e.

1. $u_1, u_2 \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_{u_1} \cap \mathbf{b}'_{u_2} = \mathbf{b}'_{u_1 \cup u_2}$
2. $u \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_u \subseteq \mathbf{b}_u$
3. $u \in [\lambda]^{<\aleph_0} \implies \mathbf{b}'_u \in \mathcal{D}_*$.

We write $(\lambda, \mathfrak{B}, T, \Delta)$-moral to mean $(\lambda, \mathfrak{B}, T, \varphi)$-moral for all $\varphi \in \Delta$. We write $(\lambda, \mathfrak{B}, T)$-moral to mean $(\lambda, \mathfrak{B}, T, \varphi)$-moral for all formulas φ.

In the last sentence of the definition of "moral", we could equivalently have said: we write $(\lambda, \mathfrak{B}, T)$-moral to mean $(\lambda, \mathfrak{B}, T, \overline{\varphi})$-moral for all sequences $\overline{\varphi}$ of formulas and all $(\lambda, \mathfrak{B}, T, \overline{\varphi})$-possibilities, because our theories T are countable. (On uncountable T, see [33].) The equivalence is by Fact 2.10 and Theorem F. Since in the present paper it is not usually necessary to restrict to φ-types, we will often use this second formulation.

The statement of [27] Theorem 6.13 was stated just for the case $\kappa = \lambda$. For completeness, we justify the use of $\kappa \leq \lambda$ in Theorem F above.

Observation 2.15. *Fix $|I| = \lambda$ and $\kappa \leq \lambda$. Then (1) iff (2).*

(1) *$\mathcal{D}$ is a regular ultrafilter on I, built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$, and $\mathcal{D}_*$ is $(\kappa, \mathfrak{B}, T)$-moral for all formulas φ of T.*
(2) *For any model $M \models T$, and any type $p \in \mathbf{S}(N)$ where $N \subseteq M$, $||N|| = \kappa$, p is realized in M.*

Proof. (1) implies (2): This is the direction we use in the present paper. Recall that "built from" implies $\mathcal{D}_0$ is λ^+-excellent. By regularity of $\mathcal{D}$, we may choose any model M of T, in particular we may choose M λ^+-saturated. By Fact 2.10, we may assume p is a φ-type. Let $\langle \varphi(x, \bar{a}^*_\alpha) : \alpha < \kappa \rangle$ be an enumeration of φ. Fix some lifting of the parameters so that we may write "$a[t]$" for $a \in N$ and $t \in I$. For each $u \in [\lambda]^{<\aleph_0}$, define

$$B_u = \{t \in I : (\exists x) \bigwedge_{\alpha \in u} \varphi_\alpha(x, \bar{a}_{v_\alpha}[t])\}.$$

Without loss of generality, $B_\emptyset = I$. Define $\langle A_u : u \in [\lambda]^{<\aleph_0} \rangle$ by: $A_u = B_{u \cap \kappa}$. Let $\mathbf{a}_u = \mathbf{j}(A_u) \in \mathcal{D}_*$, which gives us the sequence $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\lambda]^{<\aleph_0} \rangle$, which is a possibility pattern, by Łoś' theorem, c.f. 2.13. By hypothesis (1), there exists $\bar{\mathbf{a}}' = \langle \mathbf{a}'_u : u \in [\lambda]^{<\aleph_0} \rangle$ such that $\bar{\mathbf{a}}'$ is a sequence of elements of $\mathcal{D}_*$ which form a multiplicative refinement of $\bar{\mathbf{a}}$. For each $u \in [\lambda]^{<\aleph_0}$, choose A'_u such that $\mathbf{j}(A'_u) = \mathbf{a}'_u$. Let $A''_u = A'_u \cap A_u$. Then the sequence $\langle A''_u : u \in [\lambda]^{<\aleph_0} \rangle$ refines $\langle A_u : u \in [\lambda]^{<\aleph_0} \rangle$ and is multiplicative mod $\mathcal{D}_0$.

Now we use the definition of excellent, specifically Claim 4.9(1) of [27], using $\mathcal{D}_0$ and $\bar{A}''$ here for $\mathcal{D}$ and $\bar{A}$ there. By that Claim, there is a sequence $\bar{B}' = \langle B'_u : u \in [\lambda]^{<\aleph_0} \rangle$ such that first, $\bar{B}'$ refines $\bar{A}''$ so a fortiori $\bar{B}'$ refines $\bar{A}$, and second, $\bar{B}'$ is actually multiplicative, not just multiplicative mod $\mathcal{D}_0$. The map $f : [\kappa]^{<\aleph_0} \to \mathcal{D}$ given by $u \mapsto B'_u$ is therefore a multiplicative map, which means that for each $t \in I$, the set

$$\{\varphi(x, \bar{a}^*_\alpha[t]) : t \in f(\{\alpha\})\}$$

is a partial type in M. Since M is λ^+-saturated, we may choose some $b_*[t]$ realizing this type. Let $b_* = \prod_{t \in I} b_*[t]/\mathcal{D}$. Then b_* realizes p as desired.

(2) implies (1): This is immediate from Lemma 6.12 of [27] replacing λ by κ in conclusions (A) and (B) of that lemma and in the corresponding proof. □

2.2. Why a large cardinal?

The cardinal σ is supercompact iff on every set A of cardinality $\geq \sigma$ there exists a normal σ-complete ultrafilter on $[A]^{<\sigma}$ (see 5.2 below). This implies that σ is compact, i.e. that every σ-complete filter can be extended to a σ-complete ultrafilter.

Where do complete filters appear, given that all ultrafilters in Keisler's order are regular? The idea is that Theorem F allows us to build regular ultrafilters from complete

ones: $\mathcal{D}_*$ may be σ-complete for some uncountable σ, assuming the existence of $\sigma > \aleph_0$ compact.

Why is this useful? In the main theorem of [27], we proved existence of a ZFC dividing line in Keisler's order among the unstable theories, by separating the minimum unstable theory, the random graph, from all non-simple and simple non-low theories. The non-saturation half of that argument proved, in the context of Theorem F, that when the quotient Boolean algebra is $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\aleph_0}$ and $CC(\mathfrak{B}) = \mu^+ \leq \lambda$, i.e. the maximal size of an antichain in $\mathfrak{B}$ is μ, then the resulting $\mathcal{D}_1$ was not good for any non-low or non-simple theory, see also 9.6 below. It is crucial there that $\mathfrak{B}$ is the completion of a free Boolean algebra and that the last of the three cardinal subscripts for $\mathfrak{B}$ is $\aleph_0$, so in our notation, $\sigma = \theta = \aleph_0$. [The saturation part of the proof showed that the lack of global inconsistency in the random graph meant that its types could still be realized when μ was small.] To the extent that the random graph is typical of simple theories, one can ask whether higher octaves of those arguments would work to separate all simple theories from all non-simple ones. Our strategy here is, therefore, to continue working with completions of free Boolean algebras, and to continue to concentrate on the case where $CC(\mathfrak{B}) = \mu^+ \leq \lambda$. (We will consider other Boolean algebras in the paper [30] in preparation.) However, the ultrafilter $\mathcal{D}_*$ we construct is σ-complete for some uncountable σ, so in our present notation $\theta \geq \sigma > \aleph_0$, in order to have a chance at saturating simple theories which are non-low. The large cardinal assumption gives us enough room in the construction to deal with the extra amount of forking in simple non-low theories, while still allowing us to ensure non-saturation of any non-simple theory. The remarkable fact is that, after taking care of this one possible problem at lowness, we are able to leverage a new analysis of amalgamation in simple theories to build ultrafilters which precisely characterize the dividing line at simplicity.

We consider both $\sigma = \aleph_0$ and also σ uncountable and supercompact in our various ultrafilter existence proofs. We use the second case in this paper to characterize simplicity, and will use the first [which necessarily does not saturate non-low simple theories, but is in ZFC] in [29].

2.3. Structure of the paper

The structure of the paper is as follows. We assume throughout that our tuples $(\lambda, \mu, \theta, \sigma)$ of cardinals are suitable in the sense of 1.1 above. In §3, we develop the model-theoretic amalgamation condition called "$(\lambda, \mu, \theta, \sigma)$-explicitly simple." In §4, we characterize simple theories as explicitly simple using $\mu^+ = \lambda$. As discussed there and carried further in [29], varying the distance of μ and λ outlines a new approach to classifying the simple unstable theories. In §5, we define the new property of ultrafilters on certain Boolean algebras, called "$(\lambda, \mu, \theta, \sigma)$-optimal," and prove an existence theorem assuming σ is uncountable and supercompact. (Existence in the case $\sigma = \theta = \aleph_0$ will follow from Theorem 9.4 below.) If $\mathcal{D}$ is a regular ultrafilter on λ built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ where $\mathcal{D}_*$ is $(\lambda, \mu, \theta, \sigma)$-optimal, we will call $\mathcal{D}$ $(\lambda, \mu, \theta, \sigma)$-*optimized*. Assuming $\mu < \lambda$,

we then show how to ensure optimized ultrafilters do not saturate non-simple theories. §6 proves a technical lemma about arranging presentations to interact well with liftings in ultrapowers. In §7, assuming $\mu^+ = \lambda$ (as well as σ uncountable and supercompact to quote the ultrafilter existence theorem of §5), we prove that optimized ultrafilters saturate simple theories. §8 contains the paper's main theorems, characterizing simple theories via saturation of ultrapowers. §9 states and proves existence of so-called perfect ultrafilters on certain Boolean algebras, which will be useful for $\sigma = \aleph_0$ in future papers. §10 contains a list of open problems.

2.4. Basic definitions

For history on simple unstable theories, and for statements of theorems from the literature, we refer to the survey article [11]. We will use:

Definition 2.16 *(Simple theories).* Given a background theory T,

1. A formula $\varphi = \varphi(x, y)$ has the *k-tree property*, for $k < \omega$, when there exist parameters $\{a_\eta : \eta \in {}^{\omega>}\omega\}$, $\ell(a_\eta) = \ell(y)$, so that:
   (a) for each $\eta \in {}^{\omega>}\omega$, the set $\{\varphi(x, a_{\eta^\frown i}) : i < \omega\}$ is k-inconsistent
   (b) for each $\eta \in {}^{\omega}\omega$, the set $\{\varphi(x, a_{\eta|n}) : n < \omega\}$ is consistent.
2. A formula φ is *simple* if it does not have the tree property, i.e. it does not have the k-tree property for any k.

A theory is called simple if all of its formulas are.

Definition 2.17 *(D-rank, lowness).* Again fix T.

1. For each formula $\varphi(\bar{x}, \bar{y})$, an integer $k < \omega$, and a formula $\theta(\bar{x})$, all possibly with parameters, we define $D(\theta, \varphi, k)$ to be ≥ 0 if $\theta(\bar{x})$ is consistent, and $\geq \alpha + 1$ if there exists $\bar{a}_\alpha$ which forks over the parameters of θ such that $D(\theta(\bar{x}) \wedge \varphi(x, \bar{a}_\alpha), \varphi, k) \geq \alpha$. Equivalently, T is simple if and only if for all formulas φ and θ and all $k < \omega$, the rank $D(\theta, \varphi, k)$ is finite.
2. We say T is *low* if for each formula $\varphi(\bar{x}; \bar{y})$ there is $k < \omega$ such that for any indiscernible sequence $\langle \bar{a}_n : n < \omega \rangle$, with $\ell(\bar{a}_n) = \ell(y)$, we have that $\{\varphi(\bar{x}; \bar{a}_n) : n < \omega\}$ is consistent iff it is k-consistent.

Theorem G. *(Independence theorem, version of [11] Theorem 2.11.) Let T be simple and $M \models T$. Let A, B be sets such that $\mathrm{tp}(A/MB)$ does not fork over M. Let $p \in S(M)$. Let q be a nonforking extension of p over MA and r be a nonforking extension of p over MB. Then $q \cup r$ is consistent, moreover $q \cup r$ is a nonforking extension of p over MAB.*

Definition 2.18 *(Partitions).*

1. A partition of a Boolean algebra is a maximal set of pairwise disjoint nonzero elements. We may also apply this to sequences with no repetitions.
2. $CC(\mathfrak{B}) = \sup\{\mu^+ : \mathfrak{B} \text{ has a partition of size } \mu\}$.
3. When $\mathbf{a} \in \mathfrak{B}$ and $\bar{\mathbf{c}} = \langle \mathbf{c}_\epsilon : \epsilon < \mu \rangle$ is a partition of $\mathfrak{B}$, we say that $\bar{\mathbf{c}}$ *supports* $\mathbf{a}$ when $\epsilon < \mu \implies (\mathbf{c}_\epsilon \leq \mathbf{a}) \vee (\mathbf{c}_\epsilon \leq 1 - \mathbf{a})$ (in $\mathfrak{B}$).
4. When $\bar{\mathbf{c}} = \langle \mathbf{c}_\zeta : \zeta < \mu \rangle$, $\bar{\mathbf{d}} = \langle \mathbf{d}_\epsilon : \epsilon < \mu \rangle$ are partitions of $\mathfrak{B}$, say that $\bar{\mathbf{d}}$ *refines* $\bar{\mathbf{c}}$ if for each $\epsilon < \mu$, there is $\zeta < \mu$ such that $\mathbf{d}_\epsilon \leq \mathbf{c}_\zeta$.

We focus on completions of free Boolean algebras, mainly $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$, the completion of the Boolean algebra generated freely by 2^λ independent partitions of size μ, where intersections of fewer than θ nonzero elements are nonzero precisely when no two are from the same partition. It will be convenient to describe such objects as follows.

Definition 2.19 *(Boolean algebra notation[7]).* Let α be an ordinal, $\mu \geq \theta$ cardinals; the existence statement is 2.21.

1. Let
$$\mathrm{FI}_{\mu,\theta}(\alpha) = \{h : h \text{ is a function, } \mathrm{Dom}(h) \subseteq \alpha, \ |\mathrm{Dom}(h)| < \theta \text{ and } \mathrm{Range}(h) \subseteq \mu\}$$
2. $\mathfrak{B}^0 = \mathfrak{B}^0_{\alpha,\mu,\theta}$ is the Boolean algebra generated by:
   $\{\mathbf{x}_f : f \in \mathrm{FI}_{\mu,\theta}(\alpha)\}$ freely subject to the conditions that
   (a) $\mathbf{x}_{f_1} \leq \mathbf{x}_{f_2}$ when $f_1 \subseteq f_2 \in \mathrm{FI}_{\mu,\theta}(\alpha)$.
   (b) $\mathbf{x}_f \cap \mathbf{x}_{f'} = 0$ if f, f' are incompatible functions.[8]
3. $\mathfrak{B}^1_{\alpha,\mu,\theta}$ is the completion of $\mathfrak{B}^0_{\alpha,\mu,\theta}$.

In 1, 2, 3 when $\theta = \aleph_0$ we may omit it.

Convention 2.20. *We will assume that giving $\mathfrak{B}$ determines a set of generators $\langle \mathbf{x}_f : f \in \mathrm{FI}_{\mu,\theta}(\alpha_*) \rangle$, so also α_*, μ, θ.*

Fact 2.21. *Assuming $\lambda = \lambda^{<\theta}$, $\mathfrak{B}^0_{2^\lambda,\mu,\theta}$ and thus its completion exists.*

Proof. See Engelking and Karlowicz [7], Fichtenholz and Kantorovich [9], Hausdorff [12], or Shelah [34] Appendix, Theorem 1.5. □

Convention 2.22 *(Conventions on notation). Some effort has been made to standardize notation as follows (these objects will be defined below, and will be subject to further*

[7] Following [34] VI §3 or [24]. "FI" recalls the simplest case $\theta = \aleph_0$, i.e. "finite intersection."
[8] Note that 'iff' follows. It also follows that when $j < \theta$, $g = \bigcup_{i<j} f_i$ implies $\mathbf{x}_g = \bigcap_{i<j} \mathbf{x}_{f_i}$ in $\mathfrak{B}^0$ and in $\mathfrak{B}^1$.

hypotheses). The reader can quickly scan the following list at this point, and refer back to it later on as needed.

- *The letters $\mathcal{D}, \mathcal{E}$ always indicate a filter.*
- *When occurring together, the symbols $\mathcal{D}, \mathcal{D}_0, \mathcal{D}_*, \mathbf{j}$, I are used in compliance with Theorem F, p. 625.*
- *$\mathfrak{B}$ is a Boolean algebra; in the proofs, it is always a completion of a free Boolean algebra, so of the form $\mathfrak{B}^1_{2^\lambda,\mu,\theta}$, as defined in 2.19.*
- *$\mathfrak{B}^+$ is $\mathfrak{B} \setminus \{0\}$.*
- *When $\mathcal{D}$ is a filter on $\mathfrak{B}$, $\mathcal{D}^+ = \{\mathbf{a} \in \mathfrak{B} : \mathbf{a} \neq 0 \bmod \mathcal{D}\}$.*
- *$\lambda \geq \mu \geq \theta \geq \sigma$ are suitable infinite cardinals (1.1), where:*
  - *λ is the size of the index set for our background regular ultrafilter $\mathcal{D}_1$, thus, we are interested in realizing types in simple theories over sets of cardinality $\leq \lambda$.*
  - *$\mu \leq \lambda$, in the interesting case $\mu < \lambda$: this is the range of the coloring function we build on fragments of types in simple theories, and also the size of a maximal antichain in our Boolean algebra $\mathfrak{B}$.*
  - *θ (note $\sigma \leq \theta \leq \mu$) is the last parameter for the underlying Boolean algebra $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$, see 2.19.*
  - *if σ is an uncountable supercompact cardinal then we build $\mathcal{D}_*$ to be σ-complete.*

  We have kept θ and σ separate due to their different roles and requirements, but the casual reader will not lose much by assuming they are equal.
- *Boldface letters $\mathbf{c}, \mathbf{x}, \mathbf{b} \ldots$ are elements of $\mathfrak{B}$.*
- *Fraktur letters are generally used for objects of interest having multiple parts, e.g. $\mathfrak{m}$ for presentations, $\mathfrak{r}$ for elements of the set of type fragments $\mathcal{R}_\mathfrak{m}$ associated to a presentation $\mathfrak{m}$.*
- *$f, f_1, f_2 \ldots$ are elements of $\mathrm{FI}_{\mu,\theta}(\alpha_*)$, noting that $\mathbf{x}_f \in \mathfrak{B}$ is an element corresponding to the function f as in 2.19.*
- *$\Omega \subseteq [\lambda]^{<\sigma}$ is stationary, which means cofinal if $\sigma = \aleph_0$.*
- *u, v, w are subsets of λ; generally $u \in \Omega$, so $|u| < \sigma$, whereas w, v may be larger.*
- *ϵ, ζ, ξ are elements of μ, i.e. ordinals $< \mu$.*
- *$\delta_\mathfrak{m}$ is an ordinal $\leq |T|$, usually $\mathrm{cl}_\mathcal{M}(\emptyset)$ in the context of a presentation $\mathfrak{m}$.*

3. Definition of "explicitly simple"

In this section and the next we develop a new perspective on simplicity.

This section gives the first main definition of the paper: "the theory T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple." The definition makes sense for any suitable four-tuple of infinite cardinals $\lambda \geq \mu \geq \theta \geq \sigma > |T|$ in the sense of 1.1, and so varying these cardinals will give information about the theory. The parameter we are mainly interested in varying is μ. As mentioned, it will follow from the definitions in this section that $(\lambda, \mu, \theta, \sigma)$-explicitly simple becomes weaker as μ increases, and that every $(\lambda, \mu, \theta, \sigma)$-explicitly simple theory is simple, even when $\mu = \lambda$.

Recall from 2.6 that when analyzing saturation of ultrapowers, Łoś' theorem guarantees that while projections of finite pieces of a type to a given index model may each be consistent, their 'relative position' is a priori not preserved, so there is no guarantee that the union of these pieces is consistent.

An informal model-theoretic description of this problem is the following. Suppose, for clarity, that T is a theory whose only forking comes from equality, and p is a type over a set of size λ. Suppose that finitely many finite pieces of the type are moved by piecewise automorphisms of the monster model agreeing on common intersections and introducing no new forking. Is the union of these automorphic images consistent? Not necessarily: consider the effect of piecewise automorphisms f, g, h on three formulas $\{R(x,a,b)\}$, $\{R(x,b,c)\}$, $\{R(x,a,c)\}$ in the generic tetrahedron-free three-hypergraph where despite $f(a) = h(a), f(b) = g(b), g(c) = h(c)$ we may have $R(f(a), g(b), h(c))$. So instead we may try to gauge the complexity of the 'amalgamation problems' arising under such partial automorphisms by asking: can we color the pieces $[p]^{<\aleph_0}$ with no more than μ colors in such a way that within each color class, after piecewise automorphism, the union is always consistent? Note that when $\mu = \lambda$ there is trivially a coloring, as each piece gets its own color. To make the question precise, one will want to add some clarifying hypotheses, such as closure conditions on the finite pieces, and in the general case, some natural conditions on forking. After doing so, however, the question is whether a non-trivial coloring exists ($\mu < \lambda$).

Our picture is that all simple theories are in some sense close to what we see in these generic hypergraphs: the noise arising from forking may be muted so that the basic amalgamation problems controlling consistency rise to the surface. Enumerating each p in such a way that an algebra defined on its indices captures this additional noise, a precise general formulation of this partial-automorphism condition "T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple" may be given. The first main theorem of the paper, which we prepare for here and prove in the next section, will prove that we may essentially always find such a coloring when T is simple and $\mu^+ = \lambda$ (so using the first nontrivial number of colors), and moreover that this characterizes simplicity of T.

Context 3.1. *In this section we make the following assumptions. However, many of the definitions make sense under weaker hypothesis.*

1. $\aleph_0 < \sigma \leq \theta \leq \mu < \lambda$ *are suitable in the sense of 1.1. Note that the definitions in this section will also make sense in the case where* $\mu = \lambda$. *The reader may wish to assume* $\sigma = \theta$.
2. T *is a complete first order theory, with infinite models, and* $|T| < \sigma$. *The definition of 'explicitly simple' will entail that* T *is simple, i.e.* $\kappa(T)$ *exists, see Fact 3.8.*
3. $\mathfrak{C} = \mathfrak{C}_T$ *is the monster model for* T, *of cardinality* $> \lambda$.
4. *For transparency,* T *eliminates imaginaries, i.e.* $T = T_*^{eq}$ *for some complete theory* T_*. *In particular, we assume that whenever* $M \models T$, *every finite sequence of*

elements of M is coded by some $a \in \mathrm{Dom}(M)$. Otherwise, write T^{eq} and M^{eq} throughout.[9]

5. *"Independent" means nonforking and "dnf" means does not fork.*

We begin by stating the organizing definition. We will define the key items "$\mathfrak{m}$ is a presentation," "$\mathfrak{n}$ refines $\mathfrak{m}$," the set of type fragments "$\mathcal{R}_{\mathfrak{m}}$" associated to $\mathfrak{m}$, and "$G : \mathcal{R}_{\mathfrak{m}} \to \mu$ is an intrinsic coloring" over the course of the section, in 3.3, 3.6, 3.9, 3.11 respectively.

Definition 3.2 *(Explicitly simple).* Assume $(\lambda, \mu, \theta, \sigma)$ are suitable. We say T is $(\lambda, \mu, \theta, \sigma)$-*explicitly simple* if T is simple and for every $N \models T$, $||N|| = \lambda$, $p \in \mathbf{S}(N)$ nonalgebraic,

(a) there exists a presentation $\mathfrak{m}$ of p.
(b) for every presentation $\mathfrak{m}$ of p, there is a presentation $\mathfrak{n}$ of p refining $\mathfrak{m}$ and a function $G : \mathcal{R}_{\mathfrak{n}} \to \mu$ such that G is an intrinsic coloring of $\mathcal{R}_{\mathfrak{n}}$.

Note that Definition 3.2 makes sense because we will prove the existence of presentations $\mathfrak{m}$ for all simple theories in Section 4. Why is simplicity of T assumed in 3.2? See Discussion 3.12 below.

Next we define a presentation of a type. This will essentially be the data of a certain enumeration of that type along with an algebra to capture nonforking and amalgamation bases. By 'algebra' on λ we mean a first order structure with functions and no relations whose domain is λ. The closure of a set $u \subseteq \lambda$ in such an algebra $\mathcal{M}$, denoted $\mathrm{cl}_{\mathcal{M}}(u)$, is the substructure generated by u, so $u \subseteq \mathrm{cl}_{\mathcal{M}}(u) = \mathrm{cl}_{\mathcal{M}}(\mathrm{cl}_{\mathcal{M}}(u))$. We also give a value to $\mathrm{cl}_{\mathcal{M}}(\emptyset)$.

Definition 3.3. Suppose we are given $N \models T$, $||N|| = \lambda$, and $p \in \mathbf{S}(N)$. A $(\lambda, \theta, \sigma)$-*presentation* for p is the data of an enumeration and an algebra,

$$\mathfrak{m} = (\langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle, \mathcal{M})$$

where these objects satisfy:

1. $p = \langle \varphi_\alpha(x; a^*_\alpha) : \alpha < \lambda \rangle$ is an enumeration of p, which induces an enumeration $\langle a^*_\alpha : \alpha < \lambda \rangle$ of $\mathrm{Dom}(N)$, possibly with repetitions, and with the a^*_α possibly imaginary.
2. $\mathcal{M}$ is an algebra on λ with $< \theta$ functions.

[9] For general T this assumption indeed makes things clearer, although for certain specific T it may be more transparent to stick to elements. The use of imaginaries poses no problems in ultrapowers, since ultrapowers commute with reducts, so there is no issue in passing to a larger theory and proving realization of types there. We use imaginaries below in the definition of the algebra, as $\langle a^*_\alpha : \alpha < \lambda \rangle$ is allowed to be a sequence of imaginaries. However, by use of a more complex indexing scheme and algebra, this assumption could straightforwardly be avoided, as is done for the notationally simpler case of certain hypergraphs in [29].

3. For any finite $u \subseteq \lambda$, $|\operatorname{cl}_{\mathcal{M}}(u)| < \sigma$. Thus, for any $u \subseteq \lambda$, if $|u| < \sigma$ then $|\operatorname{cl}_{\mathcal{M}}(u)| < \sigma$, and if $|u| < \theta$ then $|\operatorname{cl}_{\mathcal{M}}(u)| < \theta$.
4. $\operatorname{cl}_{\mathcal{M}}(\emptyset)$ is an infinite cardinal $\leq |T|$, so an initial segment of λ.
   $M_* := N \restriction \{a^*_\alpha : \alpha < \operatorname{cl}_{\mathcal{M}}(\emptyset)\}$ is a distinguished elementary submodel of N, and we require that p does not fork over M_*.
5. Moreover, for each $u \in [\lambda]^{<\sigma}$, $N_u := N \restriction \{a^*_\alpha : \alpha \in \operatorname{cl}_{\mathcal{M}}(u)\}$ is an elementary submodel of N, and $\{\varphi_\alpha(x, a^*_\alpha) : \alpha \in \operatorname{cl}_{\mathcal{M}}(u)\}$ is a complete type over this submodel which dnf over M_*. (In particular, $\{\varphi_\alpha(x, a^*_\alpha) : \alpha \in \operatorname{cl}_{\mathcal{M}}(\emptyset)\}$ is a complete type over M_*.)
6. If $\alpha \in \operatorname{cl}_{\mathcal{M}}(u)$, $\beta \leq \alpha$, writing $A_\beta = \{a^*_\gamma : \gamma < \beta\}$, we have that $\operatorname{tp}(a^*_\alpha, A_\beta \cup M_*)$ does not fork over $\{a^*_\gamma : \gamma \in \operatorname{cl}_{\mathcal{M}}(u) \cap \beta\} \cup M_*$.

In a context where $(\lambda, \theta, \sigma)$ are given, "presentation" means "$(\lambda, \theta, \sigma)$-presentation."

Remark 3.4. From a presentation $\mathfrak{m}$, the following were unambiguously defined: M_* in item 4, $\operatorname{cl}_{\mathcal{M}}(\emptyset)$ in item 4, N_u in item 5 for any $u \subseteq \lambda$, A_α in item 6 for any $\alpha < \lambda$.[10]

Although we don't pursue this approach in the present paper, it is worth noting that in Definition 3.3, for certain less complicated theories (e.g. T with no function symbols and trivial forking, as is the case in [29]) we might prefer to allow $\operatorname{cl}_{\mathcal{M}}(u)$ to be a set, rather than requiring it to be a submodel, and in particular to only require of $\operatorname{cl}_{\mathcal{M}}(\emptyset)$ that the following observation holds.

Observation 3.5. *(Independence theorem over $\operatorname{cl}_{\mathcal{M}}(\emptyset)$, see [11] 2.13, p. 17.) By the definition of presentation and the simplicity of T, the following will be true for any presentation $\mathfrak{m}$. If $\ell = 1, 2$ are such that:*

1. $\operatorname{cl}_{\mathcal{M}}(\emptyset) \subseteq u_\ell \subseteq \lambda$
2. $u_\ell = \operatorname{cl}_{\mathcal{M}}(u_\ell)$, *thus* $u_1 \cap u_2 = \operatorname{cl}_{\mathcal{M}}(u_1 \cap u_2)$
3. $A_\ell = \{a^*_\alpha : \alpha \in u_\ell\}$ *and* A_1 *is independent from* A_2 *over* $A_1 \cap A_2$
4. $p_\ell \in \mathbf{S}(A_\ell)$ *dnf over* $\{a^*_\alpha : \alpha \in \operatorname{cl}_{\mathcal{M}}(\emptyset)\}$ *and* $p_\ell \supseteq p \restriction \{a^*_\alpha : \alpha \in A_\ell\}$

then $p_1 \cup p_2$ is a consistent type which does not fork over $\{a^*_\alpha : \alpha \in \operatorname{cl}_{\mathcal{M}}(\emptyset)\}$.

Definition 3.6 *(Refinements of presentations).* Suppose we are given $N \models T$, $||N|| = \lambda$, and $p \in \mathbf{S}(N)$. Let $\mathfrak{m} = (\bar{\varphi}_{\mathfrak{m}}, \mathcal{M}_{\mathfrak{m}})$, $\mathfrak{n} = (\bar{\varphi}_{\mathfrak{n}}, \mathcal{M}_{\mathfrak{n}})$ be presentations of p. We say that $\mathfrak{n}$ *refines* $\mathfrak{m}$ when:

[10] Although it is already a global assumption for the section, note that together items 3, 4, and 5 and the fact that σ is strongly inaccessible imply that $|T| < \sigma$ essentially, i.e. identifying two non-logical symbols under the relation of equivalence modulo T. That is, if T is a complete first order theory, T has a model $M = M_*$ of cardinality $\delta < \sigma$, and E is the equivalence relation on $\tau(T)$ which identifies functions, resp. predicates, iff they have the same interpretation in M, then we may conclude E has $\leq 2^\delta < \sigma$ classes.

(a) $\bar{\varphi}_{\mathfrak{m}} = \bar{\varphi}_{\mathfrak{n}}$.
(b) $\mathrm{cl}_{\mathcal{M}_{\mathfrak{m}}}(\emptyset) = \mathrm{cl}_{\mathcal{M}_{\mathfrak{n}}}(\emptyset)$.
(c) $\mathcal{M}_{\mathfrak{m}} \subseteq \mathcal{M}_{\mathfrak{n}}$, i.e. the algebra of $\mathfrak{n}$ extends that of $\mathfrak{m}$.

Since we allow the sequence $\langle a^*_\alpha : \alpha < \lambda \rangle$ to contain repetitions, some care was needed in the definition of the models N_u: the set $\{\alpha : a_\alpha \in |N_u|\}$ could have size λ, although $\mathrm{Dom}(N_u)$ has cardinality $< \theta$. Note also that the cardinal θ has two roles: first, the size of $\mathrm{cl}_{\mathcal{M}}(u)$ thus $||N_u||$ is $< \theta$, call this θ_1, and second, in $\mathrm{FI}_{\mu,\theta}(\alpha_*)$ in §7, call this θ_2. We don't separate them here, but what we use is $\theta_1 \leq \theta_2$, and we could have used $\sigma = \theta_1 < \theta_2$.

Let us emphasize that we have included simplicity of T in definition of explicitly simple, to avoid trivial satisfaction of the hypotheses (see also Discussion 3.12 below).

Observation 3.7. *Let T be a theory and let $(\lambda, \mu, \theta, \sigma)$ be suitable infinite cardinals. Suppose that for every $N \models T$ of size λ every nonalgebraic type $p \in \mathbf{S}(N)$ has a $(\lambda, \theta, \sigma)$-presentation. Then T is simple.*

A fortiori, if T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, then T is simple.

Proof. By the definition. □

Nonetheless, the assumption of simplicity is natural because we assume that p does not fork over a small set. Recall that:

Fact 3.8. *Let T be a complete theory. Then T is simple iff $\kappa(T)$ exists iff $\kappa(T) \leq |T|^+$, where*

$$\kappa(T) = \min\{\kappa : \text{ if } A \subseteq \mathfrak{C}, q \in \mathbf{S}(A) \text{ then } q \text{ dnf over some } B \subseteq A, |B| < \kappa\}.$$

Proof. See Theorems 3.4 and 3.6 of [11]. □

We now arrive to the right general analogue of a 'fragment of a type.' Its ingredients are a set of indices u, a closed set $w \supseteq u$ (containing e.g. forking of u), the type of a model in the variables $\bar{x}_w$, and a type over that model in the variables $x, \bar{x}_w$, satisfying some additional conditions suitable to automorphic images of pieces of p.

Definition 3.9 *(The set of quadruples $\mathcal{R}_{\mathfrak{m}}$).* Let $\mathfrak{m}$ be a presentation of a given type $p = p_{\mathfrak{m}}$. Then $\mathcal{R} = \mathcal{R}_{\mathfrak{m}}$ is the set of $\mathfrak{r} = (u, w, q, r)$ such that:

1. $u \in [\lambda]^{<\sigma}$, $w \in [\lambda]^{<\theta}$ and $w = \mathrm{cl}_{\mathcal{M}}(w)$.
2. $u \subseteq \mathrm{cl}_{\mathcal{M}}(u) \subseteq w$.
3. $q = q(\overline{x}_w)$ is a complete type in the variables $\overline{x}_w$ such that:
   (a) for any finite $v \subseteq \mathrm{cl}_{\mathcal{M}}(\emptyset)$, if $M_* \models \psi(\overline{a}^*_v)$ then $\psi(\overline{x}_v) \in q$.
   (b) for any finite $\{\alpha_0, \ldots, \alpha_n\} \subseteq u$, $\exists x \bigwedge_{i \leq n} \varphi_\alpha(x, a^*_\alpha) \in q$.

4. $r = r(x, \overline{x}_w)$ is a complete type in the variables $x, \overline{x}_w$, extending
$$q(\overline{x}_w) \cup \{\varphi_\alpha(x, x_\alpha) : \alpha \in u\}.$$
5. if $\overline{b}^*_w$ realizes $q(\overline{x}_w)$ in $\mathfrak{C}_T$ and $\alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset) \implies b^*_\alpha = a^*_\alpha$, then
   (a) $r(x, \overline{b}^*_w)$ is a type which does not fork over M_* and extends $p \restriction M_*$.
   (b) if $w' \subseteq w$ is $\mathcal{M}$-closed, $\mathfrak{C}_T \restriction \{b^*_\alpha : \alpha \in w'\} \preceq \mathfrak{C}_T$ and $r(x, \overline{b}^*_w) \restriction \overline{b}^*_{w'}$ is a complete type over this elementary submodel.
   (c) if $w' \subseteq w$ is $\mathcal{M}$-closed and $\alpha \in w'$ then $\mathrm{tp}(b^*_\alpha, \{b^*_\beta : \beta \in w \cap \alpha\})$ dnf over $\{b^*_\beta : \beta \in w' \cap \alpha\}$.

Note that for tuples in $\mathcal{R}$, the type r is like p in the sense of being a nonforking extension of $p \restriction M_*$ to a set including the domain of N_u, however this type is not guaranteed to be "correct" on all of w. Since the definition $\mathcal{R}_{\mathfrak{m}}$ is fairly unconstrained, in comparing elements of this set we will be most interested in cases which avoid trivial inconsistency.

Definition 3.10. Suppose we are given $\overline{\mathfrak{r}} = \langle \mathfrak{r}_t = (u_t, w_t, q_t, r_t) : t < t_* < \sigma \rangle$ from $\mathcal{R}_{\mathfrak{m}}$. Say that $\overline{b}^* = \langle b^*_\alpha : \alpha \in \bigcup_t w_t \rangle$, with each $b^*_\alpha \in \mathfrak{C}$ (possibly imaginary), is a *good instantiation* of $\overline{\mathfrak{r}}$ when the following conditions hold.

1. $\alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset) \implies b^*_\alpha = a^*_\alpha$.
2. For each $t < t_*$, $\overline{b}^* \restriction_{w_t}$ realizes $q_t(\overline{x}_{w_t})$.
3. For each $t < t' < t_*$, if $v \subseteq w_t \cap w_{t'}$ is finite, then:
   (a) for each formula $\psi(\overline{x}_v)$, $\psi(\overline{b}^*_v) \in q_t \iff \psi(\overline{b}^*_v) \in q_{t'}$.
   (b) for each formula $\psi(x, \overline{x}_v)$, $\psi(x, \overline{b}^*_v) \in r_t \iff \psi(x, \overline{b}^*_v) \in r_{t'}$.
4. If $\beta \in w_t$ for some $t < t_*$ then
$$\mathrm{tp}(b^*_\beta, \{b^*_\gamma : \gamma \in \bigcup_{s \le t} w_s \text{ and } \gamma < \beta\}) \text{ dnf over } \{b^*_\gamma : \gamma \in w_t \cap \beta\}.$$
5. For each $t < t_*$, if $w' \subseteq w$ and $\mathrm{cl}_{\mathcal{M}}(w') = w'$ then $\mathfrak{C}_T \restriction \{b^*_\alpha : \alpha \in w'\} \preceq \mathfrak{C}_T$ and $r_t(x, \overline{b}^*_{w'})$ is a complete type over this elementary submodel which does not fork over M_* (noting that the domain of M_* is $\{b^*_\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$ by the first item).

Now we arrive at the key point, coloring $\mathcal{R}_{\mathfrak{m}}$ with few ($\mu < \lambda$) colors to capture consistency.

Definition 3.11. Let $\mathfrak{m}$ be a $(\lambda, \theta, \sigma)$-presentation and $\mathcal{R} = \mathcal{R}_{\mathfrak{m}}$ be from 3.9. Call $G : \mathcal{R}_{\mathfrak{m}} \to \mu$ *an intrinsic coloring of* $\mathcal{R}_{\mathfrak{m}}$ if: whenever
$$\overline{\mathfrak{r}} = \langle \mathfrak{r}_t = (u_t, w_t, q_t, r_t) : t < t_* < \sigma \rangle$$
is a sequence of elements of $\mathcal{R}_{\mathfrak{m}}$ and $\overline{b}^* = \langle b^*_\alpha : \alpha \in \bigcup_{t<t_*} w_t \rangle$ is a good instantiation of $\overline{\mathfrak{r}}$,

if $G \restriction \{\mathfrak{r}_t : t < t_*\}$ is constant,
then the set of formulas

$$\{\varphi_\alpha(x, b^*_\alpha) :\ \alpha \in u_t,\ \varphi_\alpha \in r_t,\ t < t_*\}$$

is a consistent partial type which does not fork over M_*.

Note that in 3.11, we ask for $\langle b^*_\alpha : \alpha \in \bigcup_t w_t\rangle$ when we only aim for consistency of $\{\varphi_\alpha(x, b^*_\alpha) :\ \alpha \in u_t,\ \varphi_\alpha \in r_t,\ t < t_*\}$, however meeting the requirements of the larger type will affect the choice of $\bar{b}^*$ thus of $\bar{b}^* \restriction \bigcup_t u_t$.

We have now defined all terms necessary for 'explicitly simple,' so the reader may wish to re-read Definition 3.2. In the next section, we will use this definition to characterize simplicity.[11]

Discussion 3.12. Why is simplicity of T assumed in 3.2? Suppose we were to drop the assumption "$|T| < \sigma$" from 3.1 and also to drop the assumption "T is simple" from Definition 3.2; call this modified Definition 3.2*. It would continue to make sense to ask whether 3.2* holds for a given T and a given tuple of cardinals $(\lambda, \mu, \theta, \sigma)$. If e.g. $|T| > \lambda$, then it may be that there are no models of T of size λ and so the hypotheses of 3.2* are trivially satisfied, even when T is not simple. However, under our present hypotheses, existence of presentations for every nonalgebraic type over every model N of T of size λ necessarily implies T is simple:

Observation 3.13. *When $\lambda \geq \theta \geq \sigma > |T|$ and θ is regular (as holds by our Hypotheses 3.1 and 1.1), the statement that "for every $N \models T$, $||N|| = \lambda$, $p \in \mathbf{S}(N)$ nonalgebraic, there exists a presentation $\mathfrak{m}$ of p" implies "T is simple."*

[11] Our proof in the next section will also work for the following slightly different definition, by 4.2.2, which we include for interest. Note it entails that any reasonable enumeration may be extended to a presentation, making explicit what is proved in 4.2, but does not say that every presentation may be refined to one which works.

Definition 3.2A. Assume $(\lambda, \mu, \theta, \sigma)$ are suitable. We might alternatively have said that T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple if T is simple and whenever we are given:

(i) $N \models T$, $||N|| = \lambda$, $p \in \mathbf{S}(N)$ nonalgebraic,
(ii) an enumeration $\langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda\rangle$ of p, where each a^*_α is a singleton, possibly imaginary, $\{a^*_\alpha : \alpha < \lambda\} = \mathrm{Dom}(N)$, and
(iii) for some cardinal $\delta_{\mathfrak{m}} \leq |T|$, $\{a^*_\alpha : \alpha < \delta_{\mathfrak{m}}\}$ is the domain of an elementary submodel of N over which p does not fork, and $\{\varphi_\alpha(x, a^*_\alpha) : \alpha < \delta_{\mathfrak{m}}\}$ is a complete type over this submodel
(iv) $\mathcal{S}$ is an algebra on λ with $< \sigma$ functions, with $\{\alpha : \alpha < \delta_{\mathfrak{m}}\}$ closed under $\mathcal{S}$,

there exist

(a) an algebra $\mathcal{M} \supseteq \mathcal{S}$ of functions on λ such that $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \mathrm{cl}_{\mathcal{S}}(\emptyset)$ and $(\langle \varphi_\alpha : \alpha < \lambda\rangle, \mathcal{M})$ form a $(\lambda, \theta, \sigma)$-presentation $\mathfrak{m}$ of p (*thus* the set of type fragments $\mathcal{R}_{\mathfrak{m}}$ associated to $\mathfrak{m}$ is well defined)
(b) and a function $G : \mathcal{R}_{\mathfrak{m}} \to \mu$

such that G is an intrinsic coloring of $\mathcal{R}_{\mathfrak{m}}$.

Proof. This is because the definition of presentation, 3.3, requires that the type p not fork over some $N' \prec N$ of size $< \sigma \leq \lambda$. As any theory which is not simple has a formula with the 2-tree property, Definition 2.16, we may build a model $N \models T$ of size λ as the union of an increasing continuous elementary chain $\langle M_\alpha : \alpha < \theta \rangle$ and a nonalgebraic type $p \in \mathbf{S}(N)$ such that p forks over M_α for any $\alpha < \theta$. Since θ is regular and $\theta \geq \sigma > |T|$, if $A \subseteq N$ is any set of size $< \sigma$, and in particular if A is an elementary submodel of N of size $< \sigma$, then A must be contained in some M_α and so p will fork over A. □

Thus, to conclude Discussion 3.12, our current assumption "$|T| < \sigma$" implies that the hypothesis "T is simple" in Definition 3.2 is redundant. However, since the definition makes sense without the global hypothesis on σ, we prefer to leave this hypothesis to emphasize that Definition 3.2 is meant to be a strengthening of simplicity, even in contexts beyond that of this section.

Discussion 3.14 *(A classification program).* In the case of the random graph, any function G will work. In the case of an arbitrary simple theory, it will be shown in the next section that a suitable algebra and coloring can always be found assuming $\mu^+ = \lambda$. This outlines a program of stratifying the simple theories by determining which intermediate classes exist: that is, determine model-theoretic conditions which will characterize explicit simplicity for arbitrary μ, or just e.g. $\mu = \aleph_0$ or $\mu^+ < \lambda$. This work begins in [29].

4. Proof that simple theories are explicitly simple

Theorem 4.10 in this section will prove that *all* simple theories T with $|T| < \sigma$ are explicitly simple for suitable $(\lambda, \mu, \theta, \sigma)$, when $\lambda = \mu^+$, by judicious use of Skolem functions, $\kappa(T)$, and the independence theorem. Recall that the a^*_α may be imaginaries. In cases where we have more information about the theory T, it is to be expected that more direct arguments may be given. Is there a simple theory for which $\mu^+ = \lambda$ is necessary? See Section 10.

Context 4.1. *In this section we assume:*

1. T *is a simple theory with infinite models,* $T = T^{eq}$*, in the signature* τ*.*
2. $(\lambda, \mu, \theta, \sigma)$ *are suitable in the sense of 1.1.*
3. $|T| < \sigma \leq \theta$*, so in particular* $\aleph_0 < \sigma$*.*
4. $\mu^+ = \lambda$*.*
5. $N \models T$*,* $|N| = \lambda$*.*
6. $p \in \mathbf{S}(N)$ *is nonalgebraic, and* $p = p(x)$*.*

The main ingredient in the proof that every simple theory is explicitly simple is the next Lemma 4.2. By essentially the same proof, we will show that under our present hypotheses: presentations exist, moreover any reasonable enumeration of a type given with some basic algebra may be extended to a presentation, and moreover that any

presentation may be extended to one whose set of type fragments admits an intrinsic coloring.

Lemma 4.2. *Let $(\lambda, \mu, \theta, \sigma)$, T, N, p be as in 4.1, so $\mu^+ = \lambda$ and T is simple.*

1. *Whenever $\bar{\varphi} = \langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle$ is an enumeration of p satisfying*
   (a) *each a^*_α is a singleton, possibly imaginary;*
   (b) *$\{a^*_\alpha : \alpha < \lambda\} = \mathrm{Dom}(N)$;*
   (c) *for some cardinal $\delta \leq |T|$, $\{a^*_\alpha : \alpha < \delta\}$ is the domain of an elementary submodel M_* of N over which p does not fork, and $\{\varphi_\alpha(x, a^*_\alpha) : \alpha < \delta\}$ is a complete type over this submodel.*

   there exist an algebra $\mathcal{M}$ on λ and a function G such that $\mathfrak{m} = (\bar{\varphi}, \mathcal{M})$ is a presentation, and $G : \mathcal{R}_{\mathfrak{m}} \to \mu$ is an intrinsic coloring.
2. *Suppose that in addition to $\bar{\varphi}$ from (1) we are given an algebra $\mathcal{S}$ on λ with $< \theta$ functions, such that $\{\alpha : \alpha < \delta\}$ is closed under the functions of $\mathcal{S}$ and $u \in [\lambda]^{<\sigma}$ implies $\mathrm{cl}_{\mathcal{S}}(u) \in [\lambda]^{<\sigma}$. Then there exist an algebra $\mathcal{M} \supseteq \mathcal{S}$ on λ and a function G such that $\mathfrak{m} = (\bar{\varphi}, \mathcal{M})$ is a presentation, $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \delta$, and $G : \mathcal{R}_{\mathfrak{m}} \to \mu$ is an intrinsic coloring.*
3. *Suppose that in addition to $\bar{\varphi}$ from (1) we are given an algebra $\mathcal{S}$ on λ such that $(\bar{\varphi}, \mathcal{S})$ is a $(\lambda, \theta, \sigma)$-presentation, and $\delta = \mathrm{cl}_{\mathcal{S}}(\emptyset)$. Then there exist an algebra $\mathcal{M} \supseteq \mathcal{S}$ on λ and a function G such that $\mathfrak{m} = (\bar{\varphi}, \mathcal{M})$ is a presentation, $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \mathrm{cl}_{\mathcal{S}}(\emptyset) = \delta$, and $G : \mathcal{R}_{\mathfrak{m}} \to \mu$ is an intrinsic coloring.*

Proof. It will suffice to prove (2). Since T is simple, recall that $\kappa(T)$ exists and $\kappa(T) \leq |T|^+$, Fact 3.8. For clarity, rename δ as $\delta_{\mathfrak{m}}$ for the entirety of this proof. (Our construction will ensure that $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \delta = \delta_{\mathfrak{m}}$.)

The construction will take several steps and include some intermediate definitions and claims. First, we build a presentation by specifying an algebra $\mathcal{M}$. Let

$$X_0 = \{\delta < \lambda \ : \ \delta \geq \delta_{\mathfrak{m}} \text{ and } N \restriction_{\{a^*_\alpha : \alpha < \delta\}} \preceq N\}. \tag{4.1}$$

As λ is regular, X_0 is a club of λ. We will construct $\mathcal{M}$ to satisfy the following additional properties:

1. $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \{\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} = \{\alpha : \alpha < \delta_{\mathfrak{m}}\}$, so in particular $\mathrm{cl}_{\mathcal{M}}(\emptyset)$ is a cardinal $\leq |T|$.
2. if $u \in [\lambda]^{<\aleph_0}$ then $|\mathrm{cl}_{\mathcal{M}}(u)| < \sigma$, and if $u \in [\lambda]^{<\sigma}$ then also $|\mathrm{cl}_{\mathcal{M}}(u)| < \sigma$.
3. for each $v \in [\lambda]^{<\theta}$, $\mathrm{cl}_{\mathcal{M}}(v) \in Y_0$ where

$$\begin{aligned} Y_0 = \{ & w \subseteq \lambda \ : \ |w| < \theta, \{\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} \subseteq w, \\ & (\forall \delta \in X_0) \left(N \restriction_{\{a^*_\alpha \, : \, \alpha \in w \cap \delta\}} \preceq N \right), \\ & (\alpha \in w) \implies \mathrm{tp}(a^*_\alpha, \{a^*_\beta : \beta < \alpha\}, N) \text{ dnf over } \{a^*_\beta : \beta \in w \cap \alpha\}\}. \end{aligned}$$

4. if $w \in [\lambda]^{<\theta}$ and $w = \mathrm{cl}_{\mathcal{M}}(w)$ then, writing $Z_0 = \{\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} \cup X_0$:
   (i) $\alpha \in w \iff \alpha + 1 \in w$
   (ii) $\alpha \in w \implies \min(Z_0 \setminus \alpha) \in w \wedge \sup(Z_0 \cap (\alpha + 1)) \in w$
   (iii) $\alpha \in w \implies \alpha \cap w \subseteq \mathrm{cl}_{\mathcal{M}}(\{\alpha\} \cup (w \cap \mu))$
   (iv) if $v \in [w]^{<\aleph_0}$ then $(\exists \alpha \in w)(\{a^*_\beta : \beta \in v\} \subseteq a^*_\alpha)$.
5. $\mathcal{M}$ contains functions $F_i : \lambda \times \lambda \to \lambda$, $i = 1, 2$ such that:
   (i) if $\alpha \geq \mu$ then $\langle F_1(\alpha, \beta) : \beta < \alpha\rangle$ lists μ without repetition; otherwise, it lists $|\alpha|$.
   (ii) $\langle F_2(\alpha, \beta) : \beta < |\alpha| \leq \mu\rangle$ lists $\{\gamma : \gamma < \alpha\}$ without repetition.
   (iii) $\beta < \alpha \implies F_2(\alpha, F_1(\alpha, \beta)) = \beta$.
6. $\mathcal{M} \supseteq \mathcal{S}$.

In our construction of such an $\mathcal{M}$ we will do a little more than is necessary. The first collection of functions we add to the algebra are:

(A) Let $\delta_{\mathfrak{m}} = \delta$ be as in (1)(c) of the Lemma. For each $\epsilon < \delta_{\mathfrak{m}}$, let $F_\epsilon : \lambda \to \lambda$ be the constant function equal to ϵ. This will ensure that[12] $\{\alpha : \alpha < \delta_{\mathfrak{m}}\} \subseteq \mathrm{cl}_{\mathcal{M}}(\emptyset)$.
(B) Recalling $\kappa(T) \leq |T|^+$, for each $\epsilon \leq |T|$ add functions $F^0_\epsilon : \lambda \to \lambda$ such that $F^0_\epsilon(\alpha) \leq \alpha$ and $\mathrm{tp}(a^*_\alpha, A_\alpha)$ dnf over $\{a^*_{F^0_\epsilon(\alpha)} : \epsilon < |T|\}$.
(C) For each $\epsilon < |T|$ choose functions $F^1_\epsilon : \lambda \to \lambda$ such that $\langle F^1_\epsilon(\alpha) : \epsilon < |T|\rangle$ enumerates the elements of $\mathrm{acl}^{eq}(a^*_\alpha)$, possibly with repetition.
(D) We include all functions from $\mathcal{S}$, which, without loss of generality, are denoted by symbols distinct from all other functions we add to $\mathcal{M}$.
(E) It will be useful to add a family of Skolem functions which are guaranteed to choose the minimum witness with respect to our given enumeration $\langle a^*_\alpha : \alpha < \lambda\rangle$. Fix some enumeration $\langle \varphi_\epsilon : \epsilon < |T|\rangle$ of the $\mathcal{L}$-formulas. For each $\epsilon < |T|$ choose functions $F^2_\epsilon : \lambda \to \lambda$ such that $F^2_\epsilon(\alpha)$ is the minimal β such that a^*_β is a witness to $\varphi_\epsilon(x, a^*_\alpha)$, if this is well defined and nonempty, and 0 otherwise. (Alternately, we could consider F^2_ϵ as a k_ϵ-place function, but this is not necessary as we have coded finite sets in condition 4.(iv).)
(F) For each $\epsilon < |T|$, define $F^3_\epsilon : \lambda \to \lambda$ so that $F^3_\epsilon(\alpha) = \beta$ if $\varphi_\epsilon = \varphi(x, y, a^*_\alpha)$ is an equivalence relation with finitely many classes and $\varphi(x, a^*_\beta, a^*_\alpha) \in p$, and $F^3_\epsilon(\alpha) = 0$ otherwise.
(G) We'd like to ensure that the type restricted to closed sets is complete.[13] For each formula $\psi(x, \bar{y})$ in the signature τ, let G_ψ be a new function of arity $\ell(\bar{y})$ defined so that: $G_\psi(\alpha_{i_0}, \ldots, \alpha_{i_{k-1}}) = \beta$ if $\beta < \lambda$ is the least ordinal such that $\varphi_\beta(x, a^*_\beta)$ is equivalent mod T to either $\psi(x, a^*_{\alpha_{i_0}}, \ldots, a^*_{\alpha_{i_{k-1}}})$ or its negation. Since N is a model and p is complete, this is well defined.

[12] The reader may prefer to also add a single constant, 0.
[13] Up to this point, if $\alpha \in w$, our restricted type will include $\varphi_\alpha(x, a^*_\alpha)$ but won't necessarily decide the value of some other formula $\psi(x, a^*_\alpha)$ with the same parameter unless there is $\beta \in w$ such that $\varphi_\beta = \psi$ or $\neg\psi$ and $a^*_\beta = a^*_\alpha$.

Next we add some families of functions which will help in partitioning our eventual $\mathcal{R}_{\mathfrak{m}}$ into equivalence classes which are sufficiently tree-like to allow inductive amalgamation, since we have made no particular assumptions about the theory (beyond simplicity) which would otherwise guarantee such a coherence of patterns. The functions in family (I) use $\lambda = \mu^+$ in an essential way.

(H) $J_i : \lambda \to \lambda$, $i = 1, 2, 3$ where $J_1(\alpha) = \alpha + 1$ and $J_2(\alpha) = \min(Z_0 \setminus \alpha)$ and $J_3(\alpha) = \sup(Z_0 \cap (\alpha + 1))$.

(I) (Recalling $\lambda = \mu^+$) $F_i : \lambda \times \lambda$, $i = 1, 2$ such that:

1. if $\alpha \geq \mu$ then $\langle F_1(\alpha, \beta) : \beta < \alpha \rangle$ lists μ without repetition; otherwise, let it list $|\alpha|$.
2. $\langle F_2(\alpha, \beta) : \beta < |\alpha| \leq \mu \rangle$ lists $\{\gamma : \gamma < \alpha\}$ without repetition.
3. $\beta < \alpha \implies F_2(\alpha, F_1(\alpha, \beta)) = \beta$.

So for each $w \in [\lambda]^{<\theta}$, $\alpha \in w$ and $\beta \in \alpha \cap w$ implies $F_1(\alpha, \beta) \in w \cap \mu$ and $\beta = F_2(\alpha, F_1(\alpha, \beta))$.

Note that $\langle F_1(\alpha, \beta) : \beta < \alpha \rangle$ maps α into μ, whereas $\langle F_2(\alpha, \beta) : \beta < \alpha \rangle$ maps a subset of μ to α, so the third condition is natural. Note also that by choice of J_1, J_2 and unions of elementary chains we have that $w \in Y_0$ implies $\sup(w) \in X_0$.

Let $\mathcal{M}$ be the algebra including the functions from (A) through (I) above. Let us check that the numbered conditions of the claim are satisfied:

1. We ensured with family (A) that $\{\alpha : \alpha < \delta_{\mathfrak{m}}\} \subseteq \mathrm{cl}_{\mathcal{M}}(\emptyset)$. Let us check that equality holds by examining the functions of $\mathcal{M}$. The functions in family (B) are nonincreasing so will not change an initial segment. The functions in families (C) and (F) will map tuples from $\{\alpha : \alpha < \delta_{\mathfrak{m}}\}$ back to this set as $M_* \preceq N$ and $T = T^{eq}$. As for family (D), we required that $\{\alpha : \alpha < \delta_{\mathfrak{m}}\}$ be closed under the functions of $\mathcal{S}$. The new Skolem functions in (E) were chosen to give the least witness in the ordering inherited from the enumeration, and $M_* \preceq N$. As for the functions of (G), we assumed to begin with that $p \restriction M_*$ is complete. Recalling the definition of Z_0 in item (2) of the claim, the functions J_1 and J_2 are equal on $\mathrm{cl}_{\mathcal{M}}(\emptyset)$ and act by $\alpha \mapsto \alpha + 1$. Since $\mathrm{cl}_{\mathcal{M}}(\emptyset)$ is a limit ordinal, indeed a cardinal this poses no problem. Finally, as $|\mathrm{cl}_{\mathcal{M}}(\emptyset)| < \sigma \leq \mu$, the functions F_i on $\mathrm{cl}_{\mathcal{M}}(\emptyset) \times \mathrm{cl}_{\mathcal{M}}(\emptyset)$ will just list ordinals less than $\mathrm{cl}_{\mathcal{M}}(\emptyset)$. This proves that $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \delta_{\mathfrak{m}}$.
2. Immediate, as $u \in [\lambda]^{<\sigma}$ implies $\mathrm{cl}_{\mathcal{S}}(u) \in [\lambda]^{<\sigma}$ and $|\mathcal{M} \setminus \mathcal{S}| \leq |T| < \sigma$.
3. Given $v \in [\lambda]^{<\theta}$, we need to check that $w = \mathrm{cl}_{\mathcal{M}}(v) \in Y_0$. We know $|w| < \theta$, and $\{\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} \subseteq w$ by the addition of constant functions in item (A). The nonforking condition

$$(\alpha \in w) \implies \mathrm{tp}(a^*_\alpha, \{a^*_\beta : \beta < \alpha\}, N) \text{ dnf over } \{a^*_\beta : \beta \in w \cap \alpha\}$$

is guaranteed by the functions in item (B). Finally, why should

$$\delta \in X_0 \implies \left(N \restriction_{\{a^*_\alpha \,:\, \alpha \in w \cap \delta\}} \preceq N\right) \ ?$$

Suppose we don't get an elementary submodel, i.e. there is a formula with parameters in $\{a^*_\alpha \,:\, \alpha \in w \cap \delta\}$ which has a solution in N but not in this submodel. However, we chose the Skolem functions in (E) to select the minimum possible witness. Since $\delta \in X_0$, the minimal witness must be of index $\beta < \delta$, contradiction.

4. Conditions (i)–(ii) are ensured by the functions in family (H). For item (iii), recall from family (I) that for each $w \in [\lambda]^{<\theta}$, $\alpha \in w$ and $\beta \in \alpha \cap w$ implies $F_1(\alpha, \beta) \in w \cap \mu$ and $\beta = F_2(\alpha, F_1(\alpha, \beta))$. For (iv) remember that we assumed that T eliminates imaginaries; in fact, in item (F) we have coded all finite sets.
5. Ensured by the functions of family (I).
6. We assumed $\mathcal{M} \supseteq \mathcal{S}$.

Let us check that $(\bar{\varphi}, \mathcal{M})$ is a presentation, by checking the requirements of Definition 3.3. 3.3.1 was assumed in the Claim. 3.3.2–3 were just verified in item 2. For 3.3.4, the first line was verified in item 1. M_* follows by the hypothesis $1(c)$ of the Lemma and the fact that $\mathrm{cl}_{\mathcal{M}}(\emptyset) = \delta_{\mathfrak{m}}$. 3.3.5 was ensured by the functions of families (E), (G) respectively. Finally, 3.3.6 follows by the functions in (B). This shows that $\mathfrak{m} = (\bar{\varphi}, \mathcal{M})$ is indeed a presentation.

Fix this presentation $\mathfrak{m}$ for the remainder of the proof. (4.2)

Let $\mathcal{R}_{\mathfrak{m}}$ be the associated set of quadruples given by 3.9. We now work towards the definition of the coloring G. In particular, we look for underlying trees. This will require several definitions. Since we are assuming $|T| < \sigma \leq \theta$, we have $\theta \geq \aleph_1$ always. We could avoid referencing the order topology by assuming $\theta > \aleph_1$.

Definition 4.3. For α an ordinal and $w \subseteq \alpha$, write $\overline{w}$ for its topological closure, i.e. closure with respect to the order topology.

Definition 4.4. Recalling Y_0 from condition 1. on the algebra above[14] define $\Upsilon_1 \subseteq Y_0$ to be the set

$$\{w \in [\lambda]^{<\theta} \,:\, w = \mathrm{cl}_{\mathcal{M}}(w) \text{ and if } v \in [w]^{<\aleph_0} \text{ then } (\exists \alpha \in w)(\{a^*_\beta : \beta \in v\} \subseteq a^*_\alpha)\}.$$

By regularity of $\lambda > \aleph_1$, Υ_1 is cofinal in the natural partial order, and moreover is stationary. Since we constructed $\mathcal{M}$ so that the closure of a set $\mathrm{cl}_{\mathcal{M}}(u) \in Y_0$, and in addition we have coding of finite sets, $\mathrm{cl}_{\mathcal{M}}(u) \in \Upsilon_1$ for each $u \in \Omega$.

Definition 4.5. Define an equivalence relation E on Υ_1 by $w_1 \ E \ w_2$ when:

[14] and the fact that the a^*_α may be imaginary, so code finite sets.

1. $\{w_1, w_2\} \subseteq \Upsilon_1$.
2. $\mathrm{otp}(w_1) = \mathrm{otp}(w_2)$ and $\mathrm{otp}(\overline{w}_1) = \mathrm{otp}(\overline{w}_2)$.
3. $w_1 \cap \mu = w_2 \cap \mu$.
4. (a) if $h : w_1 \to w_2$ is order preserving onto then the map $a^*_\alpha \mapsto a^*_{h(\alpha)}$ is elementary.
   (b) if $h : \overline{w}_1 \to \overline{w}_2$ is order preserving onto then the map $a^*_\alpha \mapsto a^*_{h(\alpha)}$ is elementary.
5. If $\alpha \in \overline{w}_1 \setminus \mu$ and $\beta \in \overline{w}_1 \cap \mu$ then $F_1(\alpha, \beta) = F_1(h(\alpha), h(\beta))$.

Observation 4.6. *E is an equivalence relation with μ classes.*

Proof. There are $\leq \theta \leq \mu$ choices of order type in both clauses of (2). For (3), recall by 1.1(b) that $\mu^{<\theta} = \mu$. Then for (4), let $\rho = |w|$ or $= |\overline{w}|$, so $\rho < \theta$ since θ is regular. Let v be $\mathrm{otp}(w)$ and fix an order preserving bijection $\pi : v \to w$. As an upper bound, we count the v-indexed sequences of types $\langle p_i : i \in v \rangle$ where $p_i = p_i(x_{\pi(i)}, \{a^*_j : j \in w \cap \pi(i)\})$. For each $i \in v$ (without loss of generality $i \geq |T|$), suitability 1.1(c) implies that $i < \theta \implies 2^{|i|} < \mu$, so there are $< \mu$ choices for p_i thus no more than $\theta \cdot \mu \leq \mu$ such sequences. The case where $v = \mathrm{otp}(\overline{w})$ is analogous. Finally, for (5): if $\alpha \in \overline{w}_1 \setminus \mu$ then $\langle F_1(\alpha, \beta) : \beta < \alpha \rangle$ lists μ without repetition. So for each $\alpha \in \overline{w}_1$ (of which there are $< \theta$) and each $\beta \in \overline{w}_1 \cap \mu$ (of which there are $< \theta$) we need to know the value of $F_1(\alpha, \beta) \in \mu$. Since $\mu^{<\theta} = \mu$, (5) requires no more than μ classes. This completes the proof. □

Claim 4.7 *("Treeness"). Whenever wEv (so w, v are $\mathrm{cl}_{\mathcal{M}}$-closed), we have:*

1. *$w \cap v$ is $\mathrm{cl}_{\mathcal{M}}$-closed.*
2. *("treeness") $w \cap v \trianglelefteq w$ (and $w \cap v \trianglelefteq v$).*

Proof. By definition of E, $w \cap \mu = v \cap \mu$. Suppose we have $\mu \leq \beta < \alpha$ with $\alpha \in w \cap v$ and $\beta \in w$. Recalling the functions from family (I), $\beta' := F_1(\alpha, \beta) \in w \cap \mu$ and so $\beta' \in v$ since w and v agree on μ. As $v = \mathrm{cl}_{\mathcal{M}}(v)$, $F_2(\alpha, \beta') \in v$ and $F_2(\alpha, \beta') = F_2(\alpha, F_1(\alpha, \beta)) = \beta$ by definition of F_1, F_2. This shows (2.): $w \cap v \trianglelefteq w$, and $\trianglelefteq v$. Finally, closure (1.) is immediate because the algebra consists of functions and so the nonempty intersection of two $\mathrm{cl}_{\mathcal{M}}$-closed sets will be closed under these functions. □

We will refer to 4.7(2) as "by treeness" in the rest of the proof. We now define the coloring $G : \mathcal{R}_{\mathfrak{m}} \to \mu$.

Claim 4.8. *There is $G : \mathcal{R}_{\mathfrak{m}} \to \mu$ so that $G(u, w, q, r) = G(u', w', q', r')$ implies:*

(i) *$w/E = w'/E$, where E is the equivalence relation from 4.5.*
(ii) *$\beta \in w \cap w' \implies \mathrm{otp}(\beta \cap w) = \mathrm{otp}(\beta \cap w')$.*
(iii) *if $h : w \to w'$ is order preserving onto then h maps u to u'.*
(iv) *if $h : w \to w'$ is order preserving onto then h maps q to q' and r to r' in the obvious way.*
(v) $\mathrm{Range}(G) \subseteq \mu$.

Proof. For (i), recall that E has μ equivalence classes. Now (ii) will follow by the "treeness" condition 4.7. Recall $\mu = \mu^{<\theta}$ by 1.1(b). So for (iii), there are indeed no more than μ ways to choose a sequence of ordinal length $\alpha < \sigma \leq \theta$ from a sequence of length $< \theta$. For (iv), as in 4.6, there are likewise (relatively) few equivalence classes of types recalling $\alpha < \theta \implies 2^{|\alpha|} \leq \mu$ by 1.1(c). (Note that q need not be the type of $\{a^*_\alpha : \alpha \in w\}$.) So each condition requires no more than μ classes, and then (v) follows. □

For the rest of the proof, fix G satisfying 4.8. Fix $\bar{\mathfrak{r}} = \langle \mathfrak{r}_t = (u_t, w_t, p_t, r_t) : t < t_* < \sigma \rangle$ from $\mathcal{R}_{\mathfrak{m}}$. Let $w = \bigcup_t w_t$. Suppose that $G \restriction \bar{\mathfrak{r}}$ is constant and that $\bar{\mathbf{b}}^*_w$ is a good instantiation of $\bar{\mathfrak{r}}$ in the sense of 3.10. Recall that this definition ensures that $\{b^*_\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} = \mathrm{Dom}(M_*)$, and that if $\beta \in w_t$ then

$$\mathrm{tp}(b^*_\gamma, \{a_\beta : \beta \in \textstyle\bigcup_\xi w_\xi \cap \gamma\}) \text{ dnf over } \{b^*_\beta : \beta \in w_t \cap \gamma\}. \tag{4.3}$$

Prior to the main amalgamation, let us record a case of good behavior.

Claim 4.9. *Suppose $\gamma \in \bigcap_{t<t_*} u_t$. Then the set*

$$\{\varphi(x, \overline{b}^*_v) : \text{ there is } t < t_* \text{ s.t. } v \in [w_t \cap \beta]^{<\aleph_0} \text{ and } \varphi(x, \overline{x}_v) \in r_t(x, \overline{x}_{w_t})\}$$

is a partial type which does not fork over M_*.

Proof. This is simply because the treeness condition 4.7 along with conditions (i)–(iv) from the definition of G in 4.8 guarantee that for any $t < t' < t_*$, there is an order preserving map $h : w_t \to w_{t'}$, and this map is constant on the common initial segment of the w_t's. Thus for some, equivalently every, $t < t_*$ the type in the statement of the claim is simply the partial type

$$\{\varphi(x, \overline{b}^*_v) : v \in [w_t \cap \beta]^{<\aleph_0} \text{ and } \varphi(x, \overline{x}_v) \in r_t(x, \overline{x}_{w_t})\}$$

which is consistent and does not fork over M_* by definition of $\mathcal{R}$. □

It remains to show that the set of formulas

$$\{\varphi_\alpha(x, b^*_\alpha) : t < t_*, \alpha \in u_t\} \tag{4.4}$$

is consistent. Let $u = \bigcup_t u_t$ and recall $w = \bigcup_t w_t$, so $\{\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} \subseteq w$. For each $\gamma \leq \lambda$, define:

$$r^*_\gamma = \{\varphi(x, \overline{b}^*_v) : \text{ there is } t < t_* \text{ s.t. } v \in [w_t \cap \gamma]^{<\aleph_0} \text{ and } \varphi(x, \overline{x}_v) \in r_t(x, \overline{x}_{w_t})\} \tag{4.5}$$

We prove by induction on $\gamma \leq \lambda$ that r^*_γ is a consistent partial type which does not fork over $\{a^*_\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\} = \{b^*_\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$. Clearly this will imply that equation (4.4) is consistent.[15]

$\gamma = 0$: Trivial.

γ limit: Consistency is by compactness, and nonforking is by the finite character of nonforking in simple theories.

$\gamma = \beta + 1$: If $\beta \notin \bigcup_t w_t$, there is nothing to show.

Suppose then that $\beta \in w_t$ for at least one t. We can write t_* as the disjoint union of two sets

$$Z_0 := \{t < t_* : \beta \notin w_t\}, Z_1 := \{t < t_* : \beta \in w_t\}$$

where, by assumption, Z_1 is nonempty. For $i \in \{0, 1\}$ we define:

$$r^i_\gamma := \{\varphi(x, \overline{b}^*_v) : \text{ there is } t \in Z_i \text{ s.t. } v \in [w_t \cap \gamma]^{<\aleph_0} \text{ and } \varphi(x, \overline{x}_v) \in r_t(x, \overline{x}_{w_t})\}$$

Now both r^0_γ, r^1_γ are consistent partial types which moreover do not fork over M_*: the case of r^0_γ is by inductive hypothesis (it is contained in r^*_β by definition of Z_0), and the case of r^1_γ is 4.9. If Z_0 is empty, there is nothing to amalgamate, so we are finished. If not, define $W \subseteq \beta$ to be:

$$W := W_0 \cap W_1 \text{ where } W_0 := \bigcup\{w_t : t \in Z_0\} \cap \gamma, \ W_1 := \bigcup\{w_t : t \in Z_1\} \cap \gamma.$$

Let us show that

$$\mathfrak{C} \restriction \{b^*_\gamma : \gamma \in W\} \preceq \mathfrak{C}. \tag{4.6}$$

By 3.10.5, to prove equation (4.6) it will suffice to prove

$$W = \mathrm{cl}_{\mathcal{M}}(W). \tag{4.7}$$

Suppose equation (4.7) did not hold. Then for some finite tuple of elements $\beta_0, \ldots, \beta_n \in W$ (of an appropriate length) and one of the functions of the algebra, call it X, $X(\beta_0, \ldots, \beta_n) = \beta_* \notin W$. As $t < t_* \implies w_t = \mathrm{cl}_{\mathcal{M}}(w_t)$, $\beta_* \in w_t$ for each $t < t_*$. Now for any w_s, w_t with $s \in Z_0 \neq \emptyset$, $t \in Z_1 \neq \emptyset$, we have that $\beta_* \in w_s \cap w_t$ while by assumption, $\beta \in w_t$ and $\beta \notin w_s \cap w_t$. If $\beta_* > \beta$, this contradicts treeness, 4.7. However, if $\beta_* \leq \beta$ and $\beta_* \in \bigcup_{t<t_*} w_t$ then necessarily $\beta_* \in W$, also a contradiction. This completes the verification of equation (4.7) and so also of equation (4.6).

We now check nonforking. By 4.7, $W_1 = w_t \cap \gamma$ for some, equivalently every, $t \in Z_1 \neq \emptyset$. So by equation (4.3), $B_1 = \{b^*_\alpha : \alpha \in W_1\}$ is independent from $B_0 = \{b^*_\alpha : \alpha \in W_0\}$ over $B = \{b^*_\alpha : \alpha \in W\}$. B is the domain of an elementary submodel of $\mathfrak{C}$ by equation (4.6).

[15] We are actually proving something stronger than explicit simplicity as we will have consistency over all the w's, not just the u's.

Let $\delta = \sup(W)$. From equations (4.7) and (4.6) and Definition 3.10(5), $p_x := r^*_\delta \restriction B$ is a complete type over a model, which does not fork over M_*. By inductive hypothesis, r^0_γ and r^1_γ are consistent extensions of p_x to B_0, B_1 respectively, which do not fork over M_*. If necessary, we can complete r^0_γ, r^1_γ. Apply the independence theorem (Theorem G, p. 629) to complete the induction.

This proves that $\mathfrak{m}$, G satisfy 3.11 and completes the proof of Lemma 4.2. □

We arrive at the first main theorem of the paper:

Theorem 4.10 *(Simple is explicitly simple). Suppose that T is a complete first order theory. Suppose that $(\lambda, \mu, \theta, \sigma)$ are suitable cardinals in the sense of 1.1 and in addition:*

(a) $|T| < \sigma$
(b) $\mu^+ = \lambda$.

Then T is simple if and only if T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple.

Proof. If T is simple, apply Lemma 4.2. In the other direction, by Observation 3.13 and hypothesis (a), if T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple then it is simple. □

Note that by the remarks after Definition 1.1, this gives a characterization of simplicity in ZFC:

Corollary 4.11. *T is simple iff T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple for some (every) suitable tuple $(\lambda, \mu, \theta, \sigma)$ satisfying $|T| < \sigma$ and $\mu^+ = \lambda$.*

Proof. The existence of such suitable tuples is provable in ZFC, consider e.g. $\sigma = \theta$ regular and $> |T|$, $\mu = (2^\theta)^+$, and $\mu^+ = \lambda$. □

What did we use in the characterization of Theorem 4.10?

First, note that to apply the independence theorem in the inductive step in Lemma 4.2 we needed that $\mathfrak{C} \restriction \{b_\gamma : \gamma \in X\} \preceq \mathfrak{C}$; the addition of $\mathrm{cl}_{\mathcal{M}}$ ensured that this would happen. More precisely, we used that given $\{\mathfrak{r}_t : t < t_* < \sigma\}$ and $\beta + 1 = \gamma < \lambda$,

1. for each $\mathfrak{r}_t = (u_t, w_t, q_t, r_t)$, $w_t = \mathrm{cl}_{\mathcal{M}}(w_t)$.
2. for each $\beta < \lambda$, really $\beta \in \bigcup_t w_t$, $\mathfrak{C} \restriction \{b_\gamma : \gamma \in W\} \preceq \mathfrak{C}$ where W is the intersection of $W_0 := \bigcup\{w_t : t < t_*, \beta \notin w_t\} \cap \gamma$, $W_1 := \bigcup\{w_t : t < t_*, \beta \in w_t\} \cap \gamma$.

For some theories no problem would arise, e.g. when every type is stationary, so no algebra would be necessary.

Second, it is natural to ask about Theorem 4.10 when the hypothesis (a) $\mu^+ = \lambda$ is replaced by $\mu = \lambda$. If $|T| < \sigma$, one can satisfy the definition of presentation by simply choosing the algebra to contain Skolem functions and functions to cover nonforking (a

small part of what was done in the argument above). Provided the cardinals $\lambda, \mu, \theta, \sigma$ are such that the corresponding $\mathcal{R}_{\mathfrak{m}}$ has cardinality λ, the existence of the required G with range $\lambda = \mu$ may then be trivially satisfied by assigning each element of $\mathcal{R}$ its own color. Thus Theorem 4.10 remains true when "$\mu^+ = \lambda$" is replaced by $\mu = \lambda$ and $\mathcal{R}_{\mathfrak{m}}$ has cardinality λ.

The choices for how to present the parameters $\langle a^*_\alpha : \alpha < \lambda \rangle$ have different advantages. Listing imaginaries, as we currently do, makes the presentation of formulas much more compact but introduces a lot of redundancy: each singleton $c \in \mathrm{Dom}(N)$ appears in cofinally many tuples. This doesn't interfere with the nonforking condition in 3.11, as we assumed $\mu^+ = \lambda$. To show $(\lambda, \mu, \theta, \sigma)$-explicitly simple for $\mu^+ < \lambda$, however, would require addressing this, perhaps by listing the domain of N in terms of actual singletons.

Discussion 4.12. We have stated Theorem 4.10 as "T simple iff T explicitly simple," not "T simple iff T^{eq} explicitly simple," reflecting our claim that using imaginary elements is purely presentational. To see this, notice that throughout the entire proof of this theorem we could have considered each a^*_α (or b^*_α) as a finite set with respect to some fixed background enumeration of the singletons of $\mathrm{Dom}(N)$, and considered the corresponding variable x_α as the corresponding finite sequence of variables. In this setup the only essential addition would be an additional coding function, i.e. an enumeration, translating between the finite set coded by index α and vice versa, so as to be able to define and apply the functions of the algebra $\mathcal{M}$. Such a coding function could alternately be subsumed into the basic algebra $\mathcal{S}$, much as we subsume Skolem functions in 6.1.

Finally, as this discussion reflects, Theorem 4.10 raises the question of the nature of an intrinsic coloring in certain examples, since we have given only a proof of its existence. One central example, the generic k-ary hypergraphs omitting complete sub-hypergraphs on n vertices [13], will be developed in our paper [29].

5. Existence of optimal ultrafilters

Convention 5.1. *In this section we assume:*

- $\lambda, \mu, \theta, \sigma$ *are suitable cardinals.*
- $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$.
- σ *is* $\aleph_0$ *or is uncountable and supercompact, see 5.2.*
- *All ultrafilters* $\mathcal{D}_*$ *on* $\mathfrak{B}$ *mentioned in this section are* σ*-complete; we may repeat this for emphasis.*
- *When* $\mathcal{V} \subseteq 2^\lambda$*, we write* $\mathfrak{B} \restriction \mathcal{V}$ *to mean the subalgebra generated by* $\{\mathbf{x}_f : f \in \mathrm{FI}_{\mu,\theta}(\mathcal{V})\}$.

Readers familiar with normal ultrafilters may wish to skip 5.2, 5.3 and 5.4, starting again with 5.5.

Definition 5.2. (See e.g. Jech, p. 137.)

1. Call the uncountable cardinal σ *supercompact* if for any A, $|A| \geq \sigma$ there exists an ultrafilter $\mathcal{E}$ on $I = [A]^{<\sigma}$ which is:
   (a) σ-complete.
   (b) fine, meaning that in addition for any $a \in A$, $\{X \in I : a \in X\} \in \mathcal{E}$.
   (c) normal, meaning that in addition $\mathcal{E}$ is closed under diagonal intersections: if $\{X_a : a \in A\} \subseteq \mathcal{E}$ then $\{X \in I : X \in \bigcap_{a \in X} X_a\} \in \mathcal{E}$.
2. If in 1. $A = \sigma$ and $\mathcal{E}$ is normal on $I = [\sigma]^{<\sigma}$, the set $\{X \in I : X$ is an ordinal $< \sigma\} \in \mathcal{E}$, so we may say "$\mathcal{E}$ is an ultrafilter on σ."

Fact 5.3. *If $\mathcal{E}$ is a normal ultrafilter on κ, then $\mathcal{E}$ contains every closed unbounded subset of κ. Moreover any $f : \kappa \to \kappa$ which is regressive on a set in $\mathcal{E}$ must be constant on a set in $\mathcal{E}$.*

Observation 5.4. *Let χ be large enough. Fix $\mathfrak{A} = (\mathcal{H}(\chi), \epsilon)$ and let $A = \mathcal{H}(\chi)$. Let J be the set $[A]^{<\sigma}$ and let $\mathcal{E}$ be a normal ultrafilter on J. Then the set $I := \{X \in J :$ X is an elementary submodel of $\mathfrak{A}\} \in \mathcal{E}$.*

Proof. Let $\langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of A and expand $\mathfrak{A}$ to $\mathfrak{A}^*$ by Skolem functions which choose the least witness according to this enumeration. $Th(\mathfrak{A}^*)$ eliminates imaginaries and has Skolem functions. For each of the countably many Skolem functions g, define $f_g : A \to A$ by: $f_g(X)$ is the least β such that $b_\beta \in X$ but $g(b_\beta) \notin X$, if this exists, and otherwise $f_g(X) = X$. If $f_g(X) \neq X$ on an $\mathcal{E}$-large set, then it is regressive on an $\mathcal{E}$-large set, and so by normality constant and equal to some $b \in A$ on an $\mathcal{E}$-large set, contradicting the fact that $\mathcal{E}$ is fine. So it must be that $f_g(X) = X$ on an $\mathcal{E}$-large set Y_g for each of the countably many Skolem functions g, and since $\mathcal{E}$ is σ-complete, $\bigcap_g Y_g \in \mathcal{E}$ is the desired set of elementary submodels. □

Definition 5.5 *(Continuous sequence).* Let $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\sigma} \rangle$ be a sequence of elements of $\mathfrak{B}^+$. We call $\bar{\mathbf{b}}$ *continuous* when it is monotonic, meaning $u \subseteq v$ implies $\mathbf{b}_u \geq \mathbf{b}_v$, and in addition for all infinite $u \in [\lambda]^{<\sigma}$,

$$\mathbf{b}_u = \bigcap \{\mathbf{b}_v : v \subseteq u, |v| < \aleph_0\}.$$

So if $\sigma = \aleph_0$ then "continuous" is just "monotonic."

Definition 5.6 *(Support of a sequence).* Let $\overline{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\sigma} \rangle$ be a sequence of elements of $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$.

1. We say X is a support of $\overline{\mathbf{b}}$ in $\mathfrak{B}$ when $X \subseteq \{\mathbf{x}_f : f \in \mathrm{FI}_{\mu,\theta}(\alpha)\}$ and for each $u \in [\lambda]^{<\aleph_0}$ there is a maximal antichain of $\mathfrak{B}$ consisting of elements of X all of which are either $\leq \mathbf{b}_u$ or $\leq 1 - \mathbf{b}_u$. Though there is no canonical choice of support we will write $\mathrm{supp}(\bar{\mathbf{b}})$ to mean *some* support.

2. When a support $\mathrm{supp}(\bar{\mathbf{b}})$ is given, write

$$\mathfrak{B}^+_{\mathrm{supp}(\bar{\mathbf{b}}),\mu,\theta} \text{ to mean } \mathfrak{B}^+_{\alpha_*,\mu,\theta}$$

where $\alpha_* < 2^\lambda$ is minimal such that $\bigcup\{\mathrm{Dom}(f) : \mathbf{x}_f \in \mathrm{supp}(\bar{\mathbf{b}})\} \subseteq \alpha_*$.
3. When $\mathcal{V} \subseteq 2^\lambda$, we say "$\bar{\mathbf{b}}$ is supported by $\mathfrak{B} \restriction \mathcal{V}$" to mean that there is a support for $\bar{\mathbf{b}}$ contained in $\mathfrak{B} \restriction \mathcal{V}$, recalling the notation from 5.1.

We emphasize that the support need not be unique. In the next definition, λ, μ, θ come from the Boolean algebra $\mathfrak{B}$ and σ comes from the sequence $\bar{\mathbf{b}}$.

Definition 5.7 *(Key Property).* Let $(\lambda, \mu, \theta, \sigma)$ be suitable, and $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$. Let $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\sigma} \rangle$ be a continuous sequence of elements of $\mathfrak{B}^+$. We say $\bar{\mathbf{b}}$ has the $(\lambda, \mu, \theta, \sigma)$-Key Property when there exist

(a) $\mathcal{V} \subseteq 2^\lambda$, $|\mathcal{V}| \leq \lambda$, such that a support of $\bar{\mathbf{b}}$ is contained in $\mathfrak{B} \restriction \mathcal{V}$
(b) a closed unbounded $\Omega_* \subseteq [\lambda]^{<\sigma}$

such that for every $\alpha < 2^\lambda$ with $\mathcal{V} \subseteq \alpha$, there is a sequence

$$\bar{\mathbf{b}}' = \bar{\mathbf{b}}'(\alpha) = \langle \mathbf{b}'_{\alpha,\{i\}} : i < \lambda \rangle$$

of elements of $\mathfrak{B}^+$ which generates a multiplicative refinement $\langle \mathbf{b}'_{\alpha,u} : u \in [\lambda]^{<\sigma} \rangle$ of $\bar{\mathbf{b}}$ such that for each $f \in \mathrm{FI}_{\mu,\theta}(\alpha)$, and each $u \in \Omega_*$, if $\mathbf{x}_f \leq \mathbf{b}_u$ then we may extend $f \subseteq f' \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$ so that $\mathbf{x}_{f'} \leq \mathbf{b}'_u$.

Definition 5.8. Assume $\lambda, \mu, \theta, \sigma$ are suitable. $\mathcal{D}_*$ is $(\lambda, \mu, \theta, \sigma)$-optimal if:

- $\mathcal{D}_*$ is a σ-complete ultrafilter on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$, and
- whenever $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\sigma} \rangle$ is a continuous sequence of elements of $\mathcal{D}_*$ with the $(\lambda, \mu, \theta, \sigma)$-Key Property there is a multiplicative sequence $\bar{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in [\lambda]^{<\sigma} \rangle$ of elements of $\mathcal{D}_*$ which refines $\bar{\mathbf{b}}$.

Theorem 5.9. *Suppose $(\lambda, \mu, \theta, \sigma)$ are suitable, $\sigma > \aleph_0$ is supercompact, and $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$.*

1. *There exists a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$.*
2. *Let $\mathcal{D}^*_0$ be a σ-complete filter on $\mathfrak{B}$ generated by $< 2^\lambda$ sets, or[16] just generated by a set supported by $\mathfrak{B} \restriction \mathcal{V}$, $\mathcal{V} \subseteq 2^\lambda$, $|\mathcal{V}| < 2^\lambda$. Then there exists a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$ which extends $\mathcal{D}^*_0$.*

[16] Note that if $\mu < \lambda$ then $\mu^+ < 2^\lambda$, a case we will use.

Proof. Clearly it suffices to prove the second. Let $\mathcal{D}^*_0$ be given. Fix a set $X_* \subseteq \mathcal{D}^*_0$ of generators for this filter, with $|X_*| < 2^\lambda$. Choose $\mathcal{V}_* \subseteq 2^\lambda$, $|\mathcal{V}_*| < 2^\lambda$ which contains a support for X_*. Without loss of generality, $\mathcal{V}_* = \alpha_* < 2^\lambda$ (i.e. if necessary, use a permutation of 2^λ mapping the relevant support into an ordinal $< 2^\lambda$). We will use X_*, α_* in our inductive construction. If no $\mathcal{D}^*_0$ is given, let $X_* = \emptyset$ and let $\alpha_* = 0$.

There are at most $2^\lambda = |{}^\lambda\mathfrak{B}|$ sequences $\bar{\mathbf{b}}$ with the Key Property. For each such $\bar{\mathbf{b}}$, fix $\mathcal{V} \in [2^\lambda]^{\leq\lambda}$ and $\Omega_* \subseteq [\lambda]^{<\sigma}$ satisfying 5.7(a)–(b). Choose an enumeration of these tuples

$$\bar{\mathfrak{s}} = \langle (\bar{\mathbf{b}}_\delta, \mathcal{V}_\delta, \Omega_\delta) : \delta \in S \rangle$$

which satisfies

- $S \subseteq \{\delta < 2^\lambda : \delta > \alpha_* \text{ and } \delta \text{ is divisible by } \lambda\}$.
- $\mathcal{V}_\delta \subseteq \delta$ for $\delta \in S$.

For each $\delta \in S$, let $\bar{\mathbf{b}}'_\delta$ be a multiplicative refinement of $\bar{\mathbf{b}}_\delta$ as guaranteed by Definition 5.7 in the case where α of 5.7 is replaced by δ. Without loss of generality[17] $\bar{\mathbf{b}}'_\delta$ is supported by $\mathfrak{B} \restriction (\delta + \lambda)$.

Let $\mathfrak{A}, I_0, \mathcal{E}$ be given by 5.4, so each $N \in I_0$ is an elementary submodel of $\mathfrak{A}$ of size $< \sigma$. Since $\mathcal{E}$ is fine, we may assume that on an $\mathcal{E}$-large set $I \subseteq I_0$ the elements $\lambda, \mu, \theta, \sigma, \mathfrak{B}, \bar{\mathfrak{s}}$ belong to N. When $N \in I$ and $\lambda, \sigma, \delta, \Omega_\delta \in N$, $N \models$ "Ω_δ is closed and unbounded in $[\lambda]^{<\sigma}$", so from our external point of view, $N \in I$ and $\delta \in S \cap N$ implies $\lambda \cap N \in \Omega_\delta$.

Since each N is small it will (from an external point of view) only contain a small part of each of these objects. Both $\lambda \cap N$ and $S \cap N$ are of size $< \sigma$. Given N and $\delta \in S \cap N$, $\mathbf{b}_{\delta, \lambda \cap N}$ is (from an external point of view) an element of $\langle \mathbf{b}_{\delta,u} : u \in [\lambda]^{<\sigma} \rangle$, which we may call the "canonical element" for the sequence $\bar{\mathbf{b}}_\delta$ as seen by N.

Fix for awhile $N \in I$. Let $\mathbf{d}_N = \cap\{\mathbf{d} : \mathbf{d} \in N \cap X_*\}$, so $\mathbf{d}_N \in \mathcal{D}^*_0$ is supported by $\mathfrak{B} \restriction \alpha_*$.

Enumerate $S \cap N = \langle \delta_\epsilon : \epsilon < \epsilon_N \rangle$ in increasing order, for some limit ordinal $\epsilon_N < \sigma$. Working in the large background model, by induction on $\epsilon \leq \epsilon_N$ we will build an increasing continuous sequence $\langle f_\epsilon = f_{N,\delta_\epsilon} : \epsilon \leq \epsilon_N \rangle$ such that:

(a) each $f_\epsilon \in \mathrm{FI}_{\mu,\theta}(\delta_\epsilon)$, so necessarily $\mathbf{x}_{f_\epsilon} \in \mathfrak{B}^+$.
(b) if $\gamma < \epsilon \leq \epsilon_N$ then $f_\gamma \subseteq f_\epsilon$, and if $\epsilon \leq \epsilon_N$ is a nonzero limit ordinal, then $f_\epsilon = \bigcup\{f_\gamma : \gamma < \epsilon\}$, i.e. the sequence is continuous and increasing.
(c) if $\epsilon = \gamma + 1 < \epsilon_N$ then either

$$\mathbf{x}_{f_\epsilon} \leq \mathbf{b}_{\delta_\gamma, \lambda \cap N}$$

or else $\mathbf{x}_{f_\epsilon}$ is disjoint to some $\mathbf{b}_{\delta_\gamma, v}$ where $v \in [\lambda \cap N]^{<\aleph_0}$.

[17] We may always appeal to an automorphism of $\mathfrak{B}$ which is the identity on $\mathfrak{B} \restriction \delta$ to find a sequence of appropriate support, as e.g. in Observation 9.2 below.

(d) if $\epsilon = \gamma + 1 < \epsilon_N$ and $\mathbf{x}_{f_\epsilon} \leq \mathbf{b}_{\delta_\gamma, \lambda \cap N}$, then in addition $\mathbf{x}_{f_\epsilon} \leq \mathbf{b}'_{\delta_\gamma, \lambda \cap N}$.
(e) $\mathbf{x}_{f_0} \leq \mathbf{d}_N$.

If $\epsilon = 0$, choose f_0 to satisfy (a) and (e), recalling that all elements of S are $\geq \alpha_*$. Otherwise, arriving to ϵ, let $g = \bigcup_{\gamma < \epsilon} f_\epsilon$. If $\epsilon = \epsilon_N$ or a nonzero limit ordinal, this suffices, so let $f_\epsilon = g$. Otherwise, suppose $\epsilon = \gamma + 1$, so $g = f_\gamma \in \mathrm{FI}_{\mu,\theta}(\delta_\gamma)$ by hypothesis (b). As observed above, working in $\mathfrak{B}$ we see that the canonical object for the sequence $\mathbf{b}_{\delta_\gamma}$ in N, $\mathbf{b}_{\delta_\gamma, \lambda \cap N}$, is an element $\mathbf{c}$ of $\mathfrak{B}$ which is, by choice of S, supported by $\mathfrak{B} \restriction \delta_\gamma$. So we may extend g to $g' \in \mathrm{FI}_{\mu,\theta}(\delta_\gamma)$ such that either $\mathbf{x}_{g'} \leq \mathbf{c}$ or $\mathbf{x}_{g'} \cap \mathbf{c} = 0$. (Either $\mathbf{x}_g \cap \mathbf{c} = 0$ already, or not, and if not we can find g' so that $\mathbf{x}_{g'} \leq \mathbf{c}$ using the fact that $\bar{\mathbf{b}}_{\delta_\gamma}$ is continuous; so clause (c) holds.) Since we had chosen $\bar{\mathbf{b}}'_{\delta_\gamma}$ to satisfy the Key Property and to be supported by $\mathfrak{B} \restriction (\delta_\gamma + \lambda)$, and $\delta_\gamma + \lambda \leq \delta_\epsilon$ by choice of S, the Key Property ensures that if $\mathbf{x}_{g'} \leq \mathbf{c} = \mathbf{b}_{\delta_\epsilon, \lambda \cap N}$, then we may extend g' to $g'' \in \mathrm{FI}_{\mu,\theta}(\delta_\epsilon)$ so that

$$\mathbf{x}_{g''} \leq \mathbf{b}'_{\delta_\epsilon, \lambda \cap N}.$$

Let $f_\epsilon = g''$. This completes the inductive successor step and therefore the construction. Let $f_N = f_{\epsilon_N}$.

Having constructed f_N for each $N \in I$, we now consider the Boolean algebra $\mathfrak{B}$. Each f_N corresponds to the nonzero element $\mathbf{x}_{f_N} \in \mathfrak{B}^+$. Observe that for each $N \in I$ we may choose a σ-complete ultrafilter D_N on $\mathfrak{B}$ which contains $\mathbf{x}_{f_N}$. Moreover for each N, the choice of $\mathbf{x}_{f_0}$ in condition (e) above ensures that D_N contains $\{\mathbf{d} : \mathbf{d} \in N \cap X_*\}$.

In the remainder of the proof, we will build the optimal ultrafilter $\mathcal{D}$ on $\mathfrak{B}$ by averaging these ultrafilters. Recalling Fact 5.2, let $\mathcal{E}$ be an ultrafilter on I which is σ-complete and normal. Define

$$\mathcal{D}_* = \mathrm{Av}_{\mathcal{E}}(\langle D_N : N \in I \rangle) = \{\mathbf{a} \in \mathfrak{B} : \{N \in I : \mathbf{a} \in D_N\} \in \mathcal{E}\}. \tag{5.1}$$

Let us verify that $\mathcal{D}_*$ is σ-complete. Given a sequence $\langle \mathbf{a}_i : i < i_* < \sigma \rangle$ of elements of $\mathcal{D}_*$, by definition, for each i the set $X_i := \{N \in I : \mathbf{a}_i \in \mathcal{D}_N\} \in \mathcal{E}$. Since $\mathcal{E}$ is σ-complete, $X := \bigcap\{X_i : i < i_*\} \in \mathcal{E}$. $N \in X$ means that $\{\mathbf{a}_i : i < i_*\} \subseteq D_N$, so by the σ-completeness of D_N we have that $\mathbf{a} := \bigcap\{\mathbf{a}_i : i < i_*\} \in D_N$. Then $\{N \in I : \mathbf{a} \in D_N\} \supseteq X \in \mathcal{E}$, so $\mathbf{a} \in \mathcal{D}_*$ by definition of $\mathcal{D}_*$. Clearly $\mathcal{D}_* \supseteq \mathcal{D}^*_0$ because $\mathbf{d} \in X_*$ implies that $\{N \in I : \mathbf{d} \in N\} \in \mathcal{E}$ which implies that $\{N \in I : \mathbf{d}_N \leq \mathbf{d}\} \in \mathcal{E}$ which implies that $\{N : \mathbf{x}_{f_N} \leq \mathbf{d}\} \in X_*$ which implies that $\mathbf{d} \in \mathcal{D}_*$. This suffices recalling that X_* generates $\mathcal{D}^*_0$.

Now suppose $\bar{\mathbf{b}}$ has the $(\lambda, \mu, \theta, \sigma)$-Key Property. Let $\delta \in S$ be such that $\bar{\mathbf{b}} = \bar{\mathbf{b}}_\delta$ in the enumeration above. Let $\bar{\mathbf{b}}'$ be its canonical multiplicative refinement $\bar{\mathbf{b}}'_\delta$, chosen at the beginning of the proof. It will suffice to show that if $\mathbf{b}_{\delta,u} \in \mathcal{D}_*$ for each $u \in [\lambda]^{<\aleph_0}$, then for each $i < \lambda$,

$$\mathbf{b}'_{\{i\}} \in \mathcal{D}_*.$$

Fix for awhile such an i. Consider any $u \in [\lambda]^{<\sigma}$. By our present assumptions, $\mathbf{b}_u \in \mathcal{D}_*$. By definition of $\mathcal{D}_*$, this means that

$$\{N \in I : \mathbf{b}_u \in D_N\} \in \mathcal{E}$$

and moreover that $\mathbf{b}_u$ occurs $\mathcal{E}$-a.a. in models N such that $\{\delta, i\} \subseteq N$, i.e. models which consider $\mathbf{b}_u$ to be part of the correct problem and contain the index i:

$$\{N \in I : \{\delta, i\} \in N \text{ and if } u \in ([\lambda]^{<\sigma}) \cap N \text{ then } \mathbf{b}_u \in D_N\} \in \mathcal{E}.$$

For any N in this set, $\delta \in S \cap N$ and so by construction (see clause (d))

$$\mathbf{x}_{f_{N,\delta}} \leq \bigcap \{\mathbf{b}_u : u \in [\lambda]^{<\sigma} \cap N\}.$$

So by clause (d) above, $\mathbf{x}_{f_N} \leq \mathbf{b}'_{\{i\}}$. This shows that $\mathbf{b}'_{\{i\}} \in \mathcal{D}_*$, and as $i < \lambda$ was arbitrary, this completes the proof. □

Remark 5.10. Regarding the case $\sigma = \aleph_0$, we will prove existence of a $(\lambda, \mu, \aleph_0, \aleph_0)$-optimal ultrafilter in Corollary 9.7 below.

Our next task is to build a useful choice for $\mathcal{D}^*_0$ mentioned in the proof of Theorem 5.9, with the aim that any $\mathcal{D}$ built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ where $\mathcal{D}_0$ is regular and excellent and $\mathcal{D}_*$ is from Theorem 5.9 and this given $\mathcal{D}^*_0$ will not saturate any non-simple theory. The following fact is well known; we include a proof for completeness.

Fact 5.11. *Assume $\mu = \mu^{<\sigma}$. Then there is $\langle u^\alpha_\epsilon : \epsilon < \mu, \alpha < \mu^+ \rangle$ such that:*

1. $u^\alpha_\epsilon \in [\alpha]^{<\sigma}$
2. $\beta \in u^\alpha_\epsilon \implies u^\beta_\epsilon = u^\alpha_\epsilon \cap \beta$
3. *if $u \in [\mu^+]^{<\sigma}$ then for some $\epsilon < \mu$ we have that $(\forall \beta \in u)(u \cap \beta \subseteq u^\beta_\epsilon)$.*

Proof. To begin, fix a sequence $\langle f_\alpha : \alpha < \mu^+ \rangle$ where each $f_\alpha : \alpha \to \mu$ is injective and in addition is surjective whenever $|\alpha| \geq \mu$. Define a symmetric binary relation E on $u_1, u_2 \in [\mu^+]^{<\sigma}$ by: $E(u_1, u_2)$ if (a) $\mathrm{otp}(u_1) = \mathrm{otp}(u_2)$, and (b) if $h : u_1 \to u_2$ is order-preserving and onto, and $\beta < \alpha$ are from u_1, then $f_\alpha(\beta) = f_{h(\alpha)}(h(\beta))$. Then E is an equivalence relation with $\leq \mu$ classes. [To see this, note the number of its classes is bounded by the following count: for each set u, we first choose an order type ($\leq \sigma$ options), and then for each $\alpha \in u$ (of which there are $< \sigma$), there are $\leq \mu^{<\sigma}$ possible choices for the values of f_α on $u \cap \alpha$. As we assumed $\mu^{<\sigma} = \mu$, E is therefore an equivalence relation with $\sigma \cdot (\mu^{<\sigma})^{<\sigma} = \mu$ equivalence classes.] Let $\langle E_\epsilon : \epsilon < \mu \rangle$ list these classes.

Now for each $u \in [\alpha]^{<\sigma}$, let $u^\alpha_\epsilon = u$ iff for some $w \in E_\epsilon$ and $\gamma \in w$

$$(w \cap (\gamma+1))E(u \cup \{\alpha\})$$

Otherwise, u^α_ϵ is empty. This sequence will satisfy (1) by construction. To see that (2) is satisfied, suppose we are given $u = u^\alpha_\epsilon$, w, γ satisfying the previous equation, and $\beta \in u^\alpha_\epsilon$. Then letting $w' = u^\alpha_\epsilon \cap \beta$ and $\gamma' = h^{-1}(\beta)$ suffices. Finally, condition (3) follows from condition (2) and the fact that for each α and at least one ϵ, u^α_ϵ is well defined (non-empty). □

Claim 5.12. *Let $(\lambda, \mu, \theta, \sigma)$ be suitable, $\mu = \mu^{<\theta}$, σ uncountable and compact.*[18] *There exists a σ-complete filter $\mathcal{D}^*_0$ on $\mathfrak{B}^1_{2^\lambda,\mu,\theta}$ generated by μ^+ sets such that:*

if $\mathcal{D}$ is a regular ultrafilter on λ built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*, \mathbf{j})$ where $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$ and $\mathcal{D}_*$ is a σ-complete ultrafilter on $\mathfrak{B}$ extending $\mathcal{D}^*_0$, then $\mathcal{D}$ is not μ^{++}-good for any non-simple theory.

Proof. Let $\langle u^\alpha_\epsilon : \epsilon < \mu, \alpha < \mu^+\rangle$ be given by Fact 5.11. Let $\langle \mathbf{c}_\gamma : \gamma < \mu\rangle$ be a maximal antichain of $\mathfrak{B}$. Define

$$\mathcal{D}^*_0 = \{\bigcup_{\gamma \in A} \mathbf{c}_\gamma : \text{ for some } u \in [\mu^+]^{<\sigma},\ A \supseteq \{\epsilon : (\forall \alpha \in u)(u \cap \alpha \subseteq u^\alpha_\epsilon)\}\}.$$

$\mathcal{D}^*_0$ is a σ-complete filter on μ by Fact 5.11. It is supported by $\mathfrak{B} \restriction \mathcal{V}$ when it supports all the $\mathbf{c}_\gamma$'s, and so satisfies the requirement in 5.9.2.

Let $\mathcal{C}$ be the monster model of a non-simple theory. We now look for an omitted type in $\mathcal{C}^I/\mathcal{D}$. Let $\varphi(x;\overline{y})$ have the tree property in $\mathfrak{C}$ (so without loss of generality it will have TP_1 or TP_2; we observe this distinction, but ultimately don't really use it). We choose $\overline{a}_\eta \in {}^{\ell(\overline{y})}\mathfrak{C}$ for $\eta \in {}^{\sigma>}\mu^+$ so that:

1. if $\eta \in {}^\sigma(\mu^+)$ then $\{\varphi(x;\overline{a}_{\eta|_{i+1}}) : i < \sigma\}$ is a 1-type.
2. if φ has TP_1 then for η, ν incomparable elements of ${}^{\sigma>}(\mu^+)$, the set

$$\{\varphi(x;\overline{a}_\eta), \varphi(x;\overline{a}_\nu)\}$$

   is inconsistent,
3. if φ has TP_2 then for $i \in \sigma$ and η, ν incomparable (or equivalently, not equal) elements of ${}^i(\mu^+)$, the set $\{\varphi(x;\overline{a}_\eta), \varphi(x;\overline{a}_\nu)\}$ is inconsistent.

Now we use 5.11 to pick a proposed path through the tree. For $\epsilon < \mu$, $\alpha < \mu^+$ let $\eta_{\epsilon,\alpha}$ list $u^\alpha_\epsilon \cup \{\alpha\}$ in increasing order, so $\eta_{\epsilon,\alpha} \in {}^{\sigma>}\mu^+$. Fix a partition $\langle C_\gamma : \gamma < \mu\rangle$ of I such

[18] The assumption that "σ is compact" is just used to ensure that there exists some σ-complete ultrafilter $\mathcal{D}_*$ extending $\mathcal{D}^*_0$ is nonempty, so that a $\mathcal{D}$ as described by the Claim actually exists.

that $\mathbf{j}(C_\gamma) = \mathbf{c}_\gamma$ for each $\gamma < \mu$. For each $\alpha < \lambda$, define the function f_α from I to ${}^{\ell(\overline{y})}\mathfrak{C}$ by: if $t \in C_\epsilon$ then $f_\alpha(t) = \overline{a}_{\eta_{\epsilon,\alpha}}$.

Let $p = \{\varphi(x, f_\alpha/\mathcal{D}) : \alpha < \mu^+\}$. Then p is a set of formulas of the language of T with parameters from $\mathfrak{C}^I/\mathcal{D}$. Let us check that p is a consistent partial 1-type. Since consistency depends on comparability in the partial ordering, it suffices to check all pairs. If $\beta < \alpha < \mu^+$, let $A_1 = \{\epsilon < \mu : \beta \in u^\alpha_\epsilon\}$. By construction, $\bigcup_{\gamma \in A_1} \mathbf{c}_\gamma \in \mathcal{D}^*_0$, so we may choose $X_1 \in \mathcal{D}$ in the $\mathbf{j}$-preimage of this set. Unraveling the construction, this shows that p is indeed a consistent partial φ-type.

Assume for a contradiction that we have $f \in {}^I\mathfrak{C}$ such that $f/\mathcal{D}$ realizes p in $\mathfrak{C}^I/\mathcal{D}$. For each $\alpha < \mu^+$, let

$$A_\alpha = \{t \in I : \mathfrak{C} \models \varphi(f(t), f_\alpha(t))\} \in \mathcal{D}$$

Let $\mathbf{a}_\alpha = A_\alpha/\mathcal{D}$ be the corresponding member of $\mathfrak{B}$.

As $\langle \mathbf{c}_\gamma : \gamma < \mu\rangle$ is an antichain of $\mathfrak{B}$, for each $\alpha < \mu^+$ there is $\gamma(\alpha) < \mu$ such that $\mathbf{b}_\alpha = \mathbf{a}_\alpha \cap \mathbf{c}_{\gamma(\alpha)} \in \mathfrak{B}^+$. By the pigeonhole principle, there is $\zeta < \mu$ such that $|\mathcal{U}_1| = \mu^+$, where

$$\mathcal{U}_1 = \{\alpha < \mu^+ : \gamma(\alpha) = \zeta\}.$$

As $\{\mathrm{otp}(u_\alpha) : \alpha \in \mathcal{U}_1\}$ has cardinality $\leq \sigma < \mu^+$, there is $\rho < \sigma$ such that $|\mathcal{U}_2| = \mu^+$, where

$$\mathcal{U}_2 = \{\alpha \in \mathcal{U}_1 : \mathrm{otp}(u_\alpha) = \rho\}$$

However, $\mathfrak{B}$ has the μ^+-c.c. So for some $\alpha \neq \beta \in \mathcal{U}_2$, we have that $\mathbf{b} := \mathbf{b}_\alpha \cap \mathbf{b}_\beta$ is positive. Let $B \subseteq I$ be such that $\mathbf{j}(B) = \mathbf{b}$, so $B \neq \emptyset \bmod \mathcal{D}_0$. Since B is contained in each of A_α, A_β, and $C_\zeta \bmod \mathcal{D}_0$, we may choose $t \in B \cap (A_\alpha \cap A_\beta \cap C_\zeta)$.

Recall that $\eta_{\epsilon,\alpha}$ lists $u^\alpha_\epsilon \cup \{\alpha\}$ in increasing order, and $A_\alpha = \{t \in I : \mathfrak{C} \models \varphi(f(t), f_\alpha(t))\}$. Thus by our choice of t and ζ, $f(t)$ realizes

$$\{\varphi(x, f_\alpha(t)), \varphi(x, f_\beta(t))\} = \{\varphi(x, \overline{a}_{\eta_{\zeta,\alpha}}), \varphi(x, \overline{a}_{\eta_{\zeta,\beta}})\}$$

But $\eta_{\zeta,\alpha}, \eta_{\zeta,\beta}$ are distinct members of ${}^{\sigma>}(\mu^+)$ of the same length. Thus by our choice of parameters in the tree, the set $\{\varphi(x, \overline{a}_{\eta_{\zeta,\alpha}}), \varphi(x, \overline{a}_{\eta_{\zeta,\beta}})\}$ is inconsistent. [Note that this does not depend on whether we are in the case of TP_1 or of TP_2.] This contradiction shows that p is not realized, and so completes the proof. □

Conclusion 5.13. *Let $(\lambda, \mu, \theta, \sigma)$ be suitable, $\mu < \lambda$, and suppose σ is uncountable and supercompact. Let $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu,\theta}$. Then there is an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}$ such that:*

(a) *$\mathcal{D}_*$ is $(\lambda, \mu, \theta, \sigma)$-optimal.*
(b) *whenever $\mathcal{D}$ is a regular ultrafilter built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ then $\mathcal{D}$ is not μ^{++}-good for any non-simple theory,* thus *not μ^{++}-good.*

Proof. Use the filter $\mathcal{D}_0^*$ from Claim 5.12 in Theorem 5.9(2). □

Definition 5.14. Let $(\lambda, \mu, \theta, \sigma)$ be suitable and suppose that σ is uncountable and that a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter exists. We say the ultrafilter $\mathcal{D}$ on I, $|I| = \lambda$ is $(\lambda, \mu, \theta, \sigma)$-*optimized* when there exists a regular excellent filter $\mathcal{D}_0$ on I and a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter $\mathcal{D}_*$ on $\mathfrak{B}^1_{2^\lambda, \mu, \theta}$ such that $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$. Note that any such $\mathcal{D}$ will be regular.

We now record the following connection. On the relevance of this property, see section 2.2, "Why a large cardinal?"

Definition 5.15. (Flexible filters, Malliaris [22].) We say that the filter $\mathcal{D}$ on a set I is λ-flexible if for any $f \in {}^I\mathbb{N}$ with $n \in \mathbb{N} \implies n <_{\mathcal{D}} f$, we can find $X_\alpha \in \mathcal{D}$ for $\alpha < \lambda$ such that for all $t \in I$

$$f(t) \geq |\{\alpha : t \in X_\alpha\}|$$

Informally, we can find a λ-regularizing family below any given nonstandard integer.

Observation 5.16. *Let $(\lambda, \mu, \theta, \sigma)$ be suitable and $\sigma > \aleph_0$ supercompact. Let $\mathcal{D}$ be a $(\lambda, \mu, \theta, \sigma)$-optimized ultrafilter. Then $\mathcal{D}$ is flexible.*

Proof. It was proved in [22] Section 8 that any regular ultrafilter which is good for some non-low simple theory must be flexible. So this Observation will be an immediate corollary of the theorem, proved in the next section, that any such optimized ultrafilter is good for any countable simple theory.

One can also give a direct proof, which we only sketch as it is not central for our arguments. We know $\mathcal{D}$ is built from some $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ where $\mathcal{D}_0$ is excellent (therefore good, therefore flexible) and $\mathcal{D}_*$ is σ-complete for $\sigma > \aleph_0$. Then the argument is exactly that worked out in Malliaris and Shelah [24] Claim 7.8. In particular, nothing about optimality of $\mathcal{D}_*$ is used, only its σ-completeness on a completion of a free Boolean algebra. □

This gives a new solution to an old question of Dow [6], which we had also answered in an earlier paper [23] assuming a measurable cardinal. In our present terminology, using 'flexible' for 'ok,' the question is:

Question 5.17. *(Dow 1985, in our terminology) Does there exist a regular ultrafilter which is κ^+-flexible and not κ^+-good?*

Conclusion 5.18. *Let $(\lambda, \mu, \theta, \sigma)$ be suitable and $\sigma > \aleph_0$ supercompact. Then there is a regular ultrafilter $\mathcal{D}$ on λ which is λ-flexible and not μ^{++}-good.*

Proof. An optimized ultrafilter will fit the bill by 5.13 and 5.16. □

Our final claim shows that "not μ^{++}-good" in 5.13(b) is best possible.

Claim 5.19. *Let $(\lambda, \mu, \theta, \sigma)$ be suitable. If $\mathcal{D}_*$ is a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$ then:*

1. *$\mathcal{D}_*$ is μ^+-good.*
2. *if $\mathcal{D}$ is a $(\lambda, \mu, \theta, \sigma)$-optimized ultrafilter built from some regular excellent $\mathcal{D}_0$ along with $\mathcal{D}_*$ and $\mathfrak{B}$, then $\mathcal{D}$ is μ^+-good.*

Proof. For (1) it will suffice to show that if $\langle \mathbf{b}^1_u : u \in \Omega = [\mu]^{<\sigma} \rangle$ is a continuous sequence of elements of $\mathcal{D}$ then it has a multiplicative refinement. Define $\bar{\mathbf{b}}^2 = \langle \mathbf{b}^2_u : u \in [\lambda]^{<\sigma} \rangle$ by: $\mathbf{b}^2_u = \mathbf{b}^1_{u \cap \mu}$. It suffices to show that $\bar{\mathbf{b}}^2$ has the Key Property 5.7, and therefore has a multiplicative refinement by optimality. Choose $\mathcal{V} \subseteq 2^\lambda$, $|\mathcal{V}| \leq \mu$ as in 5.7(a). Let $\alpha < 2^\lambda$ be any ordinal such that $\alpha \supseteq \mathcal{V}$. Define a new maximal antichain of $\mathfrak{B}$ by $\langle \mathbf{c}_\epsilon : \epsilon < \mu \rangle$ where $\mathbf{c}_\epsilon = \mathbf{x}_{f_\epsilon}$ and f_ϵ is the function with domain $\{\alpha\}$ and range ϵ. Let $\langle u_\epsilon : \epsilon < \mu \rangle$ list $\Omega = [\mu]^{<\sigma}$. Define $\mathbf{b}'_{\{i\}}$ for $i < \lambda$ by: if $i \geq \mu$, then $\mathbf{b}'_{\{i\}} = 1_{\mathfrak{B}}$, and if $i < \mu$ then $\mathbf{b}'_{\{i\}} = \bigcap \{\mathbf{b}^2_{u_\epsilon} \cap \mathbf{c}_\epsilon : i \in u_\epsilon\}$. Let $\langle \mathbf{b}'_u : u \in [\lambda]^{<\sigma} \rangle$ be the multiplicative sequence generated by $\langle \mathbf{b}'_{\{i\}} : i < \lambda \rangle$. It remains to check that 5.7(b) holds. Fix $u \in [\lambda]^{<\sigma}$ and let ϵ_* be such that $u \cap \mu = u_{\epsilon_*}$. If $\mathbf{x}_f \leq \mathbf{b}^2_u$, then $\mathbf{x}_f \cap \mathbf{c}_{\epsilon_*} \cap \mathbf{b}^2_{u_{\epsilon_*}} > 0$, and $\mathbf{c}_{\epsilon_*} \cap \mathbf{b}^2_{u_{\epsilon_*}} \leq \mathbf{b}'_{\{i\}}$ for each $i \in u = u_{\epsilon_*}$. So as $\bar{\mathbf{b}}'$ is multiplicative, $\mathbf{x}_f \cap \mathbf{c}_{\epsilon_*} \cap \mathbf{b}^2_{u_{\epsilon_*}} \leq \mathbf{b}'_u$ as desired. This completes the proof that $\bar{\mathbf{b}}^2$ has the Key Property, and so the proof of (1).

(2) follows from (1) by Theorem F, p. 625. □

In Claim 5.19, it would be natural to consider adding a monotonicity clause of the form "$\mathcal{D}_*$ is $(\lambda, \mu, \theta, \sigma)$-optimal and $\lambda' \in [\mu, \lambda)$ implies $\mathcal{D}_*$ is $(\lambda', \mu, \theta, \sigma)$-optimal." This would require a slight change in the definition, since we have tied λ to the size of the underlying Boolean algebra 2^λ, so we omit it. If we were to add an additional parameter so as to separate these two uses of λ, the size of the Boolean algebra and the length of the sequence $\bar{\mathbf{b}}$, then we have monotonicity in the second.

6. Presentations in ultrapowers

In this section we prove a lemma saying that presentations for types in ultrapowers can be arranged to interact with a choice of lifting of the parameters in a nice way. This lemma will proceed by building an algebra $\mathcal{S}$ which is the optional input to Lemma 4.2.2 above.

Lemma 6.1. *Suppose $(\lambda, \mu, \theta, \sigma)$ are suitable, T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, T eliminates imaginaries, T is complete and simple with infinite models, and $|T| < \sigma$. Suppose we are given:*

1. $\mathcal{D}$ *a regular ultrafilter on* I, $|I| = \lambda$.
2. $M \models T$ *is* λ^+*-saturated, and admits an expansion* M^+ *by new Skolem functions for formulas of* T. *Let* $T^+ = Th(M^+)$.
3. $N \preceq M^I/\mathcal{D}$, $||N|| = \lambda$, N *admits an expansion to* $N^+ \models T^+$ *such that* $N^+ \subseteq (M^+)^I/\mathcal{D}$, *and* $p \in \mathbf{S}(N)$.

Then there exist

- *a* $(\lambda, \theta, \sigma)$*-presentation* $\mathfrak{m} = (\langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle, \mathcal{M})$ *of* p
- *an intrinsic coloring* $G : \mathcal{R}_{\mathfrak{m}} \to \mu$, *which we may assume has range* $= \mu$
- *and a choice of lifting* $\{a^*_\alpha : \alpha < \lambda\} \to M^I$

such that identifying each a^*_α *with its image under this lifting, we have that for any* $w = \mathrm{cl}_{\mathcal{M}}(w) \subseteq \lambda$,

(a) *for every* $t \in I$,

$$\mathfrak{C}_T \restriction \{a^*_\alpha[t] : \alpha \in w\} \preceq \mathfrak{C}_T,$$

(b) *for each finite sequence* $\langle \alpha_{i_0}, \dots, \alpha_{i_{k-1}} \rangle$ *of elements of* w, *and each formula* $\varphi(x, \bar{y})$ *of* T *with* $\ell(\bar{y}) = k$, *there is* $\beta \in w$ *such that for* all $t \in I$,

$$M \models (\exists x)\varphi(x, a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{k-1}}[t]) \implies \varphi(a^*_\beta[t], a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{k-1}}[t]).$$

Before proving Lemma 6.1, we record that such types are enough.

Observation 6.2. *Let* T, $\mathcal{D}$ *be as in Lemma 6.1. To prove that* $\mathcal{D}$ *is good for* T *it would suffice to show that every* p *arising in the form 6.1.3 is realized.*

Proof. Since the ultrafilter $\mathcal{D}$ is regular, we are free to choose any infinite model $M \models T$ as the index model, in particular we may choose it to be sufficiently saturated. Fix any $A \subseteq M^I/\mathcal{D}$, $|A| \leq \lambda$, and p_0 a type over A. Let M^+ be any expansion of M by Skolem functions and let $T^+ = Th(M)$. We may assume $|T^+| = |T| < \sigma$. Since ultrapowers commute with reducts, there is an expansion of the ultrapower $M^I/\mathcal{D}$ to a model of T^+. In particular there is an elementary submodel N^+ of the ultrapower (in the expanded language) such that $A \subseteq \mathrm{Dom}(N^+)$ and $||N^+|| = \lambda$. Let N be the reduct of N^+ to $\tau(T)$ and let $p \in \mathbf{S}(N)$ be any type extending p_0. Clearly to realize p_0 it suffices to realize p. Finally, regarding the range of the function G: if G is an intrinsic coloring of some presentation $\mathfrak{m}$, let E_G be the equivalence relation on elements of $\mathcal{R}_{\mathfrak{m}}$ given by $E_G(\mathfrak{r}, \mathfrak{r}')$ iff $G(\mathfrak{r}) = G(\mathfrak{r}')$. Then any function $G' : \mathcal{R}_{\mathfrak{m}} \to \mu$ such that $E_{G'}$ refines E_G will also be an intrinsic coloring of $\mathcal{R}_{\mathfrak{m}}$, so we may assume the range of G is exactly μ.

This completes the proof. □

Proof of Lemma 6.1. Let τ^+ denote the signature of N^+ and τ that of N, and let $\mathfrak{C}$ denote $\mathfrak{C}_T$, the monster model for T. Let $E \subseteq \lambda$ denote the set of even ordinals less than λ. To begin, let

$$\langle \varphi_\alpha(x, a^*_\alpha) : \alpha \in E \rangle \tag{6.1}$$

be an enumeration of p which satisfies:

(a) each a^*_β is a singleton, possibly imaginary;
(b) $\{a^*_\beta : \alpha \in E\} = \mathrm{Dom}(N)$;
(c) $\{a^*_\beta : \alpha \in |T| \cap E\}$ is the domain of an elementary submodel M_* of N over which p does not fork, and $\{\varphi_\alpha(x, a^*_\beta) : \alpha \in |T| \cap E\}$ is a complete type over this submodel.

For each $\alpha \in E$, choose $g_\alpha \in {}^I M$ such that first, $a^*_\alpha = g_\alpha/\mathcal{D}$, and second, if $\alpha, \alpha' \in E$ and $a^*_\alpha = a^*_{\alpha'}$, then $g_\alpha = g_{\alpha'}$.

Let Terms be the set of all terms built up inductively from function symbols of τ^+ and the free variables $\{x_\alpha : \alpha \in E\}$. Choose a map $\rho : \mathrm{Terms} \to \lambda$ such that:

(i) ρ is one-to-one and onto.
(ii) for each $\alpha \in E$, $x_\alpha \mapsto \alpha \in E$.
(iii) $\rho^{-1}(\{\alpha : \alpha < |T|\}$ consists precisely of the elements of Terms whose free variables are among $\{x_\alpha : \alpha < |T|\}$.

Now we define functions $\{g_\alpha : \alpha \in \lambda \setminus E\} \subseteq {}^I M$, i.e. we need to define the value of g_α when α is odd. Fix for awhile $\alpha \in \lambda \setminus E$. Then $\rho^{-1}(\alpha)$ is a term, say $\mathfrak{t} = \mathfrak{t}(x_{i_0}, \ldots, x_{i_{k-1}}) \in$ Terms, where k is finite and depends on $\mathfrak{t}$ and this notation means that the free variables of $\mathfrak{t}$ are precisely $x_{i_0}, \ldots, x_{i_{k-1}}$ (in particular $\mathfrak{t}$ is *not* necessarily a term arising as a single function applied to a series of variables). Fix $t \in I$. Since $M^+ \models T^+$, there is a unique $a \in \mathrm{Dom}(M^+)$ such that

$$M^+ \models \text{“}a = \mathfrak{t}[g_{i_0}(t), \cdots, g_{i_{k-1}(t)}]\text{”}$$

where the expression in quotations means that the term $\mathfrak{t}$ evaluates in M^+, on the given sequence of values, to a. Assign $g_\alpha(t) = a$. As $\alpha \in \lambda \setminus E$ and $t \in I$ were arbitrary, this completes the definition of $\{g_\alpha : \alpha < \lambda \setminus E\}$. Note that this definition applied to g_α for α even would just return g_α.

Before continuing, let us prove that if N and $\{a^*_\alpha : \alpha < |T|\}$ are closed under the functions of $(M^+)^I/\mathcal{D}$ then $\{g_\alpha/\mathcal{D} : \alpha \in X\} = \{g_\alpha/\mathcal{D} : \alpha \in E \cap X\} = X$ where $X \in \{|T|, \lambda\}$. In other words, we have not actually added new elements to either $\mathrm{Dom}(M_*)$ or $\mathrm{Dom}(N)$, but have simply repeated existing elements in a larger enumeration. Note that the second equality was ensured by (b)–(c) above and the right to left inclusions are obvious. We prove the remaining inclusion, $\{g_\alpha/\mathcal{D} : \alpha \in X\} \subseteq \{g_\alpha/\mathcal{D} : \alpha \in E \cap X\}$,

by induction on the complexity of the term $\rho^{-1}(\alpha)$. If $\rho^{-1}(\alpha)$ is a single variable x_β with $\beta \in X$, then by our construction we know that $2\beta = \alpha$ and so $\alpha \in E \cap X$. Suppose then that $\rho^{-1}(\alpha)$ is a term of the form $f_\varphi(\mathfrak{t}_{i_0}, \dots, \mathfrak{t}_{i_{\ell-1}})$, where in slight abuse of notation, we write this to mean that f_φ is an ℓ-place function symbol from $\tau^+ \setminus \tau$ applied to the terms $\mathfrak{t}_{i_0}, \dots, \mathfrak{t}_{i_{\ell-1}}$. By inductive hypothesis, for each $j < \ell$ there is $\alpha_j \in E \cap X$ such that

$$g_{\rho(\mathfrak{t}_{i_j})}/\mathcal{D} = g_{\alpha_{i_j}}/\mathcal{D}.$$

Then as both M_* and N were expanded to models of T^+, writing "$\mathfrak{C}^+ \restriction X$" for the appropriate expansion, there is some $\beta \in E \cap X$ such that

$$\mathfrak{C}^+ \restriction X \models f(g_{\alpha_{i_0}}/\mathcal{D}, \dots, g_{\alpha_{i_{\ell-1}}}/\mathcal{D}) = g_\beta/\mathcal{D}$$

and then unraveling the definition of g_α in the previous paragraph, clearly $g_\alpha/\mathcal{D} = g_\beta/\mathcal{D}$.

For $\alpha \in \lambda \setminus E$, let φ_α be the formula "$x \neq y$", recalling that p is nonalgebraic (of course even simpler formulas would work e.g. "$x = x$"). For each $\alpha < \lambda$, let $a^*_\alpha = g_\alpha/\mathcal{D}$. Then the enumeration

$$\bar{\varphi} = \langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle$$

will satisfy the hypotheses of Lemma 4.2. So we have our enumeration and our lifting $a^*_\alpha \mapsto g_\alpha$ (for $\alpha < \lambda$), and it remains to translate these Skolem functions in the natural way into an algebra $\mathcal{S}$ on λ and to prove this algebra has the desired properties.

For each function symbol $f_\varphi \in \tau^+ \setminus \tau$, of arity $k = k_\varphi$, add a function F_φ of the same arity to the algebra defined as follows. Although similar to the argument just given, this definition will have an important additional uniqueness property. For each $\langle \alpha_{i_0}, \dots, \alpha_{i_{k-1}} \rangle \in {}^k\lambda$, in slight abuse of notation, define

$$F_\varphi(\alpha_{i_0}, \dots, \alpha_{i_{k-1}}) = \rho(\text{ "}f_\varphi(\rho^{-1}(\alpha_{i_0}), \cdots, \rho^{-1}(\alpha_{i_{k-1}}))\text{" })$$

where the expression in quotation denotes the element of Terms formed by applying the k-place function symbol f_φ to the sequence of terms $\rho^{-1}(\alpha_{i_0})$, $\cdots$, $\rho^{-1}(\alpha_{i_{k-1}})$. As ρ was a bijection, this value is unique and well defined. Let $\mathcal{S}$ be the algebra given by the functions $\{F_\varphi : f_\varphi \in \tau^+ \setminus \tau\}$, so clearly $|\mathcal{S}| = |\tau^+ \setminus \tau| \leq |T| < \sigma$. We now make several observations about how the algebra $\mathcal{S}$ interacts with the enumeration $\bar{\varphi}$ and the functions of $\tau^+ \setminus \tau$. First, our construction has guaranteed that:

(1) For each Skolem function $f_\varphi \in \tau^+ \setminus \tau$, say of arity k, and every distinct $\alpha_{i_0}, \dots, \alpha_{i_{k-1}}$ from λ, there exists $\beta < \lambda$ such that for *all* $t \in I$, $M^+ \models$ "$f_\varphi(a^*[t]_{\alpha_{i_0}}, \dots, a^*_{\alpha_{i_{k-1}}}[t]) = a^*_\beta[t]$."

This is more than would be guaranteed a priori by Łoś' theorem: Łoś would say that if we fix an enumeration and a lifting, then for any such sequence of α's we may find a β which

works almost everywhere. Here we have a single β, namely the value of $F_\varphi(\alpha_{i_0}, \dots, \alpha_{i_{k-1}})$, which works everywhere.[19]

(2) For each $k < \omega$ and each $f_\varphi \in \tau^+ \setminus \tau$ of arity k, there is a function $F_\varphi \in \mathcal{S}$ of arity k such that whenever $\alpha_{i_0}, \dots, \alpha_{i_{k-1}}, \beta$ satisfy condition (a),

$$F_\varphi^{\mathcal{S}}(\alpha_{i_0}, \dots, \alpha_{i_{k-1}}) = \beta.$$

This says that the F_φ translate the action of the Skolem functions in the natural way. Finally, let us prove that:

(3) For any nonempty $w = \mathrm{cl}_{\mathcal{S}}(w) \subseteq \lambda$ and $t \in I$, $\mathfrak{C} \restriction \{a_\alpha^*[t] : \alpha \in w\} \preceq \mathfrak{C}$.

Since we have fixed a lifting, $a_\alpha^*[t] = g_\alpha(t)$, so we will use these interchangeably. Fix some such $w = \mathrm{cl}_{\mathcal{S}}(w) \subseteq \lambda$ and some $t \in I$. Since this set is a subset of the index model M, it will suffice to prove that $M \restriction \{g_\alpha(t) : \alpha \in w\} \preceq M$. Suppose for a contradiction that $\{g_\alpha(t) : \alpha \in w\} = \{a_\alpha^*[t] : \alpha \in w\}$ is not the domain of an elementary submodel of M. Then there are a formula $\varphi = \varphi(x, \bar{y})$ and $\alpha_{i_0}, \dots, \alpha_{i_{\ell(\bar{y})-1}} \in w$ such that

$$M \models (\exists x)\varphi(x, a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{\ell(\bar{y})-1}}[t])$$

but there does not exist $\gamma \in w$ such that

$$M \models \varphi[a^*_\gamma[t], a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{\ell(\bar{y})-1}}[t]].$$

Let $f_\varphi \in \tau^+$ be the function symbol whose interpretation in T^+ corresponds to the Skolem function for φ. Then

$$M^+ \models (\exists x)\varphi(x, a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{\ell(\bar{y})-1}}[t]) \implies$$
$$\varphi(f_\varphi(a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{\ell(\bar{y})-1}}[t]), a^*_{\alpha_{i_0}}[t], \dots, a^*_{i_{\ell(\bar{y})-1}}[t]).$$

Moreover, by observation (1), there is $\beta < \lambda$ such that for all elements of I, and in particular for the t we have chosen,

$$M^+ \models f_\varphi(a^*[t]_{\alpha_{i_0}}, \dots, a^*_{\alpha_{i_{k-1}}}[t]) = a^*_\beta[t].$$

By observation (2), $w = \mathrm{cl}_{\mathcal{S}}(w)$ means that necessarily $\beta \in w$. This contradiction completes the proof of (3).

Notice that if $\mathcal{M} \supseteq \mathcal{S}$ is any larger algebra and $w = \mathrm{cl}_{\mathcal{M}}(w) \subseteq \lambda$, then a fortiori $w = \mathrm{cl}_{\mathcal{S}}(w)$ so (3) remains true.

[19] Note that we accomplished this by "padding" our original enumeration so that $\langle a^*_\alpha : \alpha < \lambda \rangle$ may contain many repetitions.

Then the enumeration $\bar{\varphi}$ and the algebra $\mathcal{S}$ satisfy the hypotheses of Definition 3.2. As we have assumed that T is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, we may apply that Definition to obtain a presentation $\mathfrak{m} = (\bar{\varphi}, \mathcal{M})$ and an intrinsic coloring $G : \mathcal{R}_{\mathfrak{m}} \to \mu$, with $\mathcal{M} \supseteq \mathcal{S}$. This presentation $\mathfrak{m}$, the coloring G, and the lifting $\langle g_\alpha : \alpha < \lambda \rangle$ are as desired. Note that by definition of presentation, any $\mathrm{cl}_{\mathcal{M}}$-closed set is nonempty, so we no longer need the proviso "nonempty" when applying (3) in the context of a presentation, Definition 3.3.

This completes the proof of the claim. □

7. Ultrapower types in simple theories

In this section we assume the following:

- $(\lambda, \mu, \theta, \sigma)$ are suitable.
- $\sigma > \aleph_0$ is strongly inaccessible.[20]
- $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$.
- T is complete, countable, first-order, and $(\lambda, \mu, \theta, \sigma)$-explicitly simple.

In the next Theorem, we have assumed existence of an optimal ultrafilter rather than "σ is uncountable and supercompact". This is because supercompactness was used for expediency to *construct* an optimal ultrafilter but nothing about the definition of optimal or optimized (Definition 5.14) seems to suggest its necessity, and supercompactness is not otherwise used in the proof.

Theorem 7.3. *Suppose $(\lambda, \mu, \theta, \sigma)$ are suitable. Suppose a $(\lambda, \mu, \theta, \sigma)$-optimal ultrafilter exists. Let T be a complete, countable theory which is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, let $M \models T$, and let $\mathcal{D}$ be a $(\lambda, \mu, \theta, \sigma)$-optimized ultrafilter on I, $|I| = \lambda$. Then $M^I/\mathcal{D}$ is λ^+-saturated.*

Proof. As ultraproducts commute with reducts, we may assume T eliminates imaginaries (if not, work in T^{eq} throughout). To begin, quoting Lemma 6.1 and Observation 6.2, let us fix:

1. $\mathcal{D}_0$, $\mathcal{D}_*$, $\mathbf{j}$ witnessing that $\mathcal{D}$ is optimized.
2. $N \preceq M^I/\mathcal{D}$, $||N|| = \lambda$, and $p \in \mathbf{S}(N)$ nonalgebraic, for which there exist:
   (a) a $(\lambda, \theta, \sigma)$-presentation $\mathfrak{m} = (\langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle, \mathcal{M})$ of p
   (b) an intrinsic coloring $G : \mathcal{R}_{\mathfrak{m}} \to \mu$, with range exactly μ
   (c) and a choice of lifting $\{a^*_\alpha : \alpha < \lambda\} \to M^I$

[20] What we use in this section is first, that there is an optimal, thus σ-complete, ultrafilter on $\mathfrak{B}$, and second, we use that σ is strongly inaccessible in Claim 7.6. As σ will be strongly inaccessible in our cases of interest (e.g. σ compact or strongly compact) there is no present cost to using this in Claim 7.6; however, that Claim could likely be circumvented.

[i.e. let $F : M^I/\mathcal{D} \to M^I$ be a choice function, meaning that $F(a/\mathcal{D}) \in a/\mathcal{D}$, and in the rest of the proof we allow ourselves to write a^*_α instead of $F(a^*_\alpha)$ when this is clear from the context, so for $\alpha < \lambda$ and $t \in I$ the value "$a^*_\alpha[t]$" is well defined] we have that for any $w = \mathrm{cl}_{\mathcal{M}}(w) \subseteq \lambda$,

(d) for any $t \in I$,

$$\mathfrak{C}_T \restriction \{a^*_\alpha[t] : \alpha \in w\} \preceq \mathfrak{C}_T,$$

(e) for each finite sequence $\langle \alpha_{i_0}, \dots, \alpha_{i_{k-1}} \rangle$ of elements of w and formula $\varphi(x, \bar{y})$ of T with $\ell(\bar{y}) = k$, there is $\beta \in w$ s.t. for *all* $t \in I$,

$$M \models (\exists x)\varphi(x, a^*_{\alpha_{i_0}}[t], \dots, a^*_{\alpha_{i_{k-1}}}[t]) \implies \varphi(a^*_\beta[t], a^*_{\alpha_{i_0}}[t], \dots, a^*_{\alpha_{i_{k-1}}}[t]).$$

3. Recall that the properties ensured by the presentation $\mathfrak{m}$ include:
   (a) $\langle a^*_\alpha : \alpha < \lambda \rangle$ is an enumeration of $N = N^{eq}$ (note that the next few conditions put some restrictions on this enumeration). We write A_α for the set $\{a_\beta : \beta < \alpha\}$.
   (b) $\bar{\varphi} = \langle \varphi_\alpha(x, a^*_\alpha) : \alpha < \lambda \rangle$, corresponding enumeration of the type p.
   (c) $\mathcal{M}$ is an algebra on λ with $\leq \theta$ functions such that for each $u \in \lambda$, if $|u| < \theta$ then $|\mathrm{cl}_{\mathcal{M}}(u)| < \theta$, and if $|u| < \sigma$ then $|\mathrm{cl}_{\mathcal{M}}(u)| < \sigma$.
   (d) $\{\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$ is a cardinal $\leq |T|$.
   (e) $N \restriction \{a^*_\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)\} = M_* \preceq N$, where $||M_*|| \leq |T| < \sigma$ and p does not fork over M_*.
   (f) for each $u \in [\lambda]^{<\sigma}$, $N \restriction \{a^*_\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(u)\}$ is the domain of an elementary submodel $N_u \preceq N$, and if in addition $u = \mathrm{cl}_{\mathcal{M}}(u)$ then the partial type $\{\varphi_\alpha(x, a^*_\alpha) : \alpha \in u\}$ implies the elementary diagram of N_u (and is an element of $\mathbf{S}(N_u)$ which does not fork over M_*).
   (g) for each $u \in [\lambda]^{<\sigma}$ and $\beta \leq \alpha \in \mathrm{cl}_{\mathcal{M}}(u)$, we have that $\mathrm{tp}(a^*_\alpha, A_\beta \cup M_*)$ does not fork over $\{a^*_\gamma : \gamma \in \mathrm{cl}_{\mathcal{M}}(u) \cap \beta\} \cup M_*$.
4. $\Omega = [\lambda]^{<\sigma}$.

With the stage set, our first task is to build a continuous sequence $\langle \mathbf{b}_u : u \in \Omega \rangle$ of elements of $\mathfrak{B}$ which form a possibility pattern corresponding to the type p. Towards this, we shall describe N by a type in λ variables in the natural way. Let $\langle x_\alpha : \alpha < \lambda \rangle$ be a sequence of singleton variables (possibly they will be filled by imaginaries) and as before, for v a sequence of elements of λ, let $\bar{x}_v$ denote $\langle x_\alpha : \alpha \in v \rangle$. [If v was defined to be a set, interpret $\bar{x}_v$ by considering v as a sequence listing its elements in increasing order. The main set Γ_1 of formulas, in the language $\mathcal{L}$ of T, is closed under permuting the variables.] Define

$$\Gamma_1 = \{\psi(\bar{x}_v) : v \in {}^{\omega >}\lambda,\ \psi \text{ an } \mathcal{L}\text{-formula in } |v| \text{ free variables } \}. \tag{7.1}$$

For each finite $u \subseteq \lambda$, define

$$\varphi_u = \varphi_u(x, \bar{x}_u) = \bigwedge_{\alpha \in u} \varphi_\alpha(x, x_\alpha). \tag{7.2}$$

These are collected in the set

$$\Gamma_2 = \{\varphi_u : u \in [\lambda]^{<\aleph_0}\}. \tag{7.3}$$

Note that since $p \in \mathbf{S}(N)$, $(\exists x)\varphi_u(x, \bar{x}_u) \in \Gamma_1$ for each $\varphi_u \in \Gamma_2$. Now we invoke Łoś' map. For each $\psi(\bar{x}_v) \in \Gamma_1$, define

$$\mathbf{a}_{\psi(\bar{x}_v)} = \mathbf{j}(A_{\psi(\bar{x}_v)}) \text{ where } A_{\psi(\bar{x}_v)} = \{t \in I : M \models \psi[\bar{a}_v[t]]\}. \tag{7.4}$$

It will be useful to name the element of $\mathfrak{B}$ recording that "M_* appears correctly":

$$\mathbf{a}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)} := \bigcap\{\mathbf{a}_{\psi(\bar{x}_v)} : v \in [\mathrm{cl}_{\mathcal{M}}(\emptyset)]^{<\omega} \ \wedge \ \psi(\bar{x}_v) \in \Gamma_1 \ \wedge \ \models \psi[\bar{a}_v]\} \in \mathcal{D}_*. \tag{7.5}$$

Note that in equation (7.5) we make essential use of $\sigma > |T|$. Likewise, for each $\varphi_u \in \Gamma_2$, define

$$\mathbf{b}_{\varphi_u} = \mathbf{j}(B_{\varphi_u}) \text{ where } B_{\varphi_u} = \{t \in I : M \models (\exists x)\varphi_u(x, \bar{a}_u[t])\}. \tag{7.6}$$

By Łoś' theorem, each $\mathbf{b}_{\varphi_u(x,\bar{x}_u)}$ belongs to $\mathcal{D}_*$ and a fortiori to $\mathfrak{B}^+$. As $\mathcal{D}_*$ is σ-complete, we may define $\mathbf{b}_u$ for all $u \in \Omega$ by setting $\mathbf{b}_u = \mathbf{b}_{\varphi_u(x,\bar{x}_u)}$ when u is finite and $\mathbf{b}_u = \bigcap\{\mathbf{b}_v : v \in [u]^{<\aleph_0}\}$ when u is infinite. Then

$$\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in \Omega \rangle \tag{7.7}$$

is a continuous sequence of elements of $\mathcal{D}_*$ in the sense of 5.5 above, and is a possibility pattern 2.12 (for the formulas in the sequence $\bar{\varphi}$) by Łoś' theorem. So by Theorem F, showing $\bar{\mathbf{b}}$ has a multiplicative refinement $\langle \mathbf{b}'_u : u \in [\lambda]^{<\sigma} \rangle$ in $\mathcal{D}_*$ will suffice to realize the type p.

Notice at this point that for any $u \subseteq \lambda$,

$$\mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} \leq \mathbf{a}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)}. \tag{7.8}$$

This follows from condition 3.(f) from the beginning of the proof: "for each $u \in [\lambda]^{<\sigma}$, $N \restriction \{a^*_\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(u)\}$ is the domain of an elementary submodel $N_u \preceq N$, and if in addition $u = \mathrm{cl}_{\mathcal{M}}(u)$ then the partial type $\{\varphi_\alpha(x, a^*_\alpha) : \alpha \in u\}$ implies the elementary diagram of N_u."

Our next task is to define a family of supporting sequences for $\bar{\mathbf{b}}$, recalling Definition 5.6. We will set the stage by defining several progressively more refined families

$$\mathcal{F} \subseteq \mathrm{FI}_{\mu,\theta}(2^\lambda) \times [\lambda]^{<\theta}.$$

In each case, $\leq_{\mathcal{F}}$ is the natural partial order on elements of $\mathcal{F}$ given by

$$(f, w) \leq_{\mathcal{F}} (f', w') \text{ when } f \subseteq f', w \subseteq w'$$

where recall $f \subseteq f'$ implies $\mathbf{x}_{f'} \leq \mathbf{x}_f$, as befits having more information. In each case, the family $\mathcal{F}$ will be defined as $\bigcup\{\mathcal{F}_u : u \in \Omega\}$.

For each $u \in \Omega$, we define $\mathcal{F}^0_u$ to be the set of pairs (f, w) such that

(i) $f \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$, $w \in [\lambda]^{<\theta}$.
(ii) $u \subseteq w = \mathrm{cl}_{\mathcal{M}}(w)$, so w is closed and contains the closure of u.

For each $u \in \Omega$ we define $\mathcal{F}^1_u \subseteq \mathcal{F}^0_u$ to be the set of pairs (f, w) which are, in addition, decisive "on w":

(iii) $\mathbf{x}_f \leq \mathbf{b}_{\varphi_v(x,\overline{x}_v)}$ or $\mathbf{x}_f \leq 1 - \mathbf{b}_{\varphi_v(x,\overline{x}_v)}$ when $v \in [w]^{<\aleph_0}$ and $\varphi_v \in \Gamma_2$.
(iv) $\mathbf{x}_f \leq \mathbf{a}_{\psi(\overline{x}_v)}$ or $\mathbf{x}_f \leq 1 - \mathbf{a}_{\psi(\overline{x}_v)}$ when $v \in [w]^{<\aleph_0}$ and $\psi(\overline{x}_v) \in \Gamma_1$.

The family $\mathcal{F}^1_u$ is dense in $\mathcal{F}^0_u$, that is, for any $(f, w) \in \mathcal{F}^0_u$ there is $(f', w') \in \mathcal{F}^1_u$ with $(f, w) \leq_{\mathcal{F}^0_u} (f', w')$. This is because the generators $\langle \mathbf{x}_f : f \in \mathrm{FI}_{\mu,\theta}(2^\lambda)\rangle$ are dense in $\mathfrak{B}$. Notice also that both families are closed under limits which are not too large, i.e. if $\alpha < \theta$ is a limit ordinal and $\langle (f_\beta, w_\beta) : \beta < \alpha\rangle$ is a strictly increasing sequence of elements of $\mathcal{F}^1_u$, then $(\bigcup_{\beta<\alpha} f_\beta, \bigcup_{\beta<\alpha} w_\beta) \in \mathcal{F}^1_u$. To prove this we need to check that the limit $\bigcup_{\beta<\alpha} w_\beta$ remains closed, which is true because each w_β is closed.

Following an idea from [28], we now settle collisions. For $\alpha, \beta \in \lambda$, write

$$A_{a_\alpha = a_\beta} = \{t \in I : M \models a_\alpha[t] = a_\beta[t]\} \text{ and } \mathbf{a}_{a_\alpha = a_\beta} = \mathbf{j}(A_{a_\alpha = a_\beta}). \tag{7.9}$$

Clearly for any $f \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$ and $\alpha < \lambda$, $\mathbf{x}_f \leq \mathbf{a}_{a_\alpha = a_\alpha}$. For each $u \in \Omega$, let $\mathcal{F}^2_u \subseteq \mathcal{F}^1_u$ be the set of pairs (f, w) such that, in addition,

(v) for each $\alpha \in w$, f decides equality for α, meaning that there is $\beta \leq \alpha$, $\beta \in w$ such that: $\mathbf{x}_f \leq \mathbf{a}_{a_\alpha = a_\beta}$ and for no $f' \supseteq f$ and $\gamma < \beta$ (not necessarily from w) do we have

$$\mathbf{x}_{f'} \leq \mathbf{a}_{a_\alpha = a_\gamma}.$$

To prove that $\mathcal{F}^2_u$ is dense in $\mathcal{F}^1_u$, it suffices to show that for each $(f, w) \in \mathcal{F}^1_u$ and $\alpha \in w$ there is (f_α, w_α) with $(f, w) \leq_{\mathcal{F}^1_u} (f_\alpha, w_\alpha)$ in which this condition is met for α. Suppose we are given $(f, w) \in \mathcal{F}^1_u$ and $\alpha \in w$. Let $\beta_0 = \alpha$, $f_0 = f$. Arriving to i, $\mathbf{x}_f \leq \mathbf{a}_{a_\alpha = a_{\beta_i}}$ and if in addition $\mathbf{x}_f \cap \mathbf{a}_{a_\alpha = a_\beta} = 0$ for all $\beta < \beta_i$, then the condition is satisfied for α. Otherwise, let $\beta_{i+1} < \beta_i$ be a counterexample and as the generators are dense, we may choose $f_{i+1} \supseteq f_i$ so that $\mathbf{x}_{f_{i+1}} \leq \mathbf{a}_{a_\alpha = a_{\beta_{i+1}}}$. Since the ordinals are well ordered, this process stops at some finite stage $n = n(\alpha)$. Let w_n be the closure of $w \cup \{\beta_n\}$. Fix $(f_\alpha, w_\alpha) \in \mathcal{F}^1_u$ such that $(f_n, w_n) \leq_{\mathcal{F}^1_u} (f_\alpha, w_\alpha)$. Then (f_α, w_α) is as required. As $\mathcal{F}^1_u$ is

closed under limits of cofinality less than θ, we may build the desired $(f_*, w_*) \in \mathcal{F}^2_u$ as a limit of the elements (f_α, w_α), indexing suitably to handle the new βs added along the way.

Note that whenever $(f, w) \in \mathcal{F}^2_u$ and $\alpha \in w$ we may define

$$\rho_\alpha(f) = \min\{\beta \leq \alpha : \mathbf{x}_f \leq \mathbf{a}_{a_\alpha = a_\beta} \text{ and for all } \gamma < \beta \ \mathbf{x}_f \cap \mathbf{a}_{a_\alpha = a_\gamma} = 0\}. \tag{7.10}$$

Moreover, $\rho_\alpha(f) \in w$ and also $\rho_\alpha(f') = \rho_\alpha(f)$ for any $f \subseteq f' \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$, so this value is robustly defined.[21]

We now record how elements of $\mathcal{F}^2_u$ naturally induce types. The key families $\mathcal{F}^3_u$, $\mathcal{F}^4_u \subseteq \mathcal{F}^2_u$ will be defined in terms of conditions on these types. In what follows $q(\dots)$ is a type of parameters, and $r(x, \dots)$ may or may not correspond to a fragment of p; the notation is meant to invoke 3.9. For each $u \in \Omega$ and each $(f, w) \in \mathcal{F}^2_u$,

define $q_{f,w}$ to be the type in the variables $\bar{x}_w$ given by:

$$q_{f,w}(\bar{x}_w) = \{\psi(\overline{x}_v) : \mathbf{x}_f \leq \mathbf{a}_{\psi(\overline{x}_v)},\ \psi(\overline{x}_v) \in \Gamma_1\}$$

$$\cup \{\neg\psi(\overline{x}_v) : \mathbf{x}_f \leq 1 - \mathbf{a}_{\psi(\overline{x}_v)},\ \psi(\overline{x}_v) \in \Gamma_1\}. \tag{7.11}$$

Notice that if $\mathbf{x}_f \leq \mathbf{a}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)}$, then the restriction of $q_{f,w}$ to the variables $\bar{x}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)} = \bar{x}_{\delta_{\mathfrak{m}}}$ is realized by $\langle a^*_\alpha : \alpha < \delta_{\mathfrak{m}}\rangle$. Next, for each $u \in \Omega$ and each $(f, w) \in \mathcal{F}^2_u$,

$$\text{choose } r_{f,w} = r_{f,w}(x, \bar{x}_w) \supseteq q_{f,w} \text{ to be a complete type such that:} \tag{7.12}$$

(a) if $\overline{b}_w$ realizes q in $\mathfrak{C}$ then $r(x, \overline{b}_w)$ dnf over $\{b_\alpha : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$.
(b) if consistent, choose r so that in addition
$r \supseteq \{\varphi_\alpha(x, x_\alpha) : \alpha \in u\} \cup \{\varphi_\alpha(x, x_\alpha) : \alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$.

The choice of each $r_{f,w}$ will be fixed for the rest of the proof. Note in particular that if $\bar{b}_w$ satisfies $\alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset) \implies b_\alpha = a^*_\alpha$ then $r(x, \overline{b}_w)$ dnf over M_*.

Let $\mathcal{F}^3_u \subseteq \mathcal{F}^2_u$ be the set of (f, w) such that in addition:

(vi) whenever $\bar{b}_w$ is a sequence of elements of the monster model realizing $q_{f,w}$, then for any $v \subseteq w$, $v = \mathrm{cl}_{\mathcal{M}}(v)$ we have that $\{b_\alpha : \alpha \in v\}$ is the domain of an elementary submodel.

Let us prove that $\mathcal{F}^3_u = \mathcal{F}^2_u$ for all $u \in \Omega$. Suppose for a contradiction that (vi) fails for some $q_{f,w}$ and $\bar{b}_w$. Then there is a finite $v \subseteq w$ and a formula $\varphi(x, \bar{y})$ of T with $\ell(\bar{y}) = |v|$ witnessing the failure, i.e. $\mathfrak{C} \models \exists x \varphi(x, \bar{b}_v)$ but there does not exist $\gamma \in w$ such that

[21] In some sense we have found a small region where the elements $a_\alpha, a_{\rho_\alpha(f)}$ move 'in lockstep'. Note that we are simply tying them to each other, not to any particular values in the monster model.

$\mathfrak{C} \models \varphi(b_\gamma, \bar{b}_v)$. Since $q_{f,w}$ is a complete type, $\psi(\bar{x}_v) = \exists x \varphi(x, \bar{b}_v) \in q_{f,w}$. By definition of $q_{f,w}$, it must be that

$$\mathbf{x}_f \leq \mathbf{a}_{\psi(\bar{x}_v)} \text{ i.e. } \mathbf{x}_f \leq \mathbf{j}(A_{\psi(\bar{x}_v)} = \{t \in I : M \models \exists x \varphi(x, \bar{a}^*_v[t])\}).$$

By assumption (2)(e) at the beginning of the proof, there exists $\beta \in w$ such that for all $t \in I$,

$$M \models (\exists x)\varphi(x, \bar{a}^*_v[t]) \implies \varphi(a^*_\beta[t], \bar{a}^*_v[t]). \tag{7.13}$$

We will show $\mathbf{x}_f \leq \mathbf{a}_{\varphi(x_\beta, \bar{x}_v)}$. If not, $\mathbf{a}_{\psi(\bar{x}_v)} \cap (1 - \mathbf{a}_{\varphi(x_\beta, \bar{x}_v)}) > \mathbf{x}_f > 0$, thus

$$A_{\psi(\bar{x}_v)} \setminus A_{\varphi(x_\beta, \bar{x}_v)} \neq \emptyset.$$

Let t be any element of this supposedly nonempty set. Then $M \models (\exists x)\varphi(x, \bar{a}^*_v[t])$ but it is not the case that $M \models \varphi(a^*_\beta[t], \bar{a}^*_v[t])$, contradicting (7.13). This contradiction proves that $\mathbf{x}_f \leq \mathbf{a}_{\varphi(x_\beta, \bar{x}_v)}$. So by the definition of $q_{f,w}$, it must be that $\varphi(x_\beta, \bar{x}_v) \in q_{f,w}$. This contradiction proves that (vi) will always hold, i.e. Lemma 6.1 has arranged that $\mathcal{F}^2_u = \mathcal{F}^3_u$ for each $u \in \Omega$.

There is one more family to define, $\mathcal{F}^4_u$. First, let us record that the "if consistent" clause in the definition of $r_{f,w}$, (7.12)(b), is often activated.

Observation 7.4. *Suppose $u \in \Omega$, $(f, w) \in \mathcal{F}^3_u$, and $\mathbf{x}_f \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$. Then $r_{f,w}$ will always contain $\{\varphi_\alpha(x, x_\alpha) : \alpha \in u\} \cup \{\varphi_\alpha(x, x_\alpha) : \alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$.*

Proof. In other words, we will show the "if consistent" from the definition of $r_{f,w}$ is consistent. Denote by N_u the elementary submodel $\mathfrak{C} \restriction \{a^*_\alpha : \alpha \in \mathrm{cl}_{\mathcal{M}}(u)\}$. (By our conditions on the algebra this is an elementary submodel, which includes M_*.) The hypothesis on $\mathbf{x}_f$ means that $q \restriction \bar{x}_{\mathrm{cl}_{\mathcal{M}}(u)}$ is realized by N_u, under an appropriate enumeration. Since nonforking is invariant under automorphism and all the types in question are complete, suppose without loss of generality that we are given $\bar{b}_w$ realizing $q_{f,w}$ such that $\bar{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$ enumerates $\mathrm{Dom}(M)$, and $\alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)$ implies $b_\alpha = a^*_\alpha$. By the definition of presentation, $p \restriction N_u$ is a type which includes $\{\varphi_\alpha(x, a^*_\alpha) : \alpha \in u\} \cup \{\varphi_\alpha(x, a^*_\alpha) : \alpha < \mathrm{cl}_{\mathcal{M}}(\emptyset)\}$ and does not fork over M_*. We may choose $r_*(x, \bar{b}_w)$ to be any nonforking extension of $p \restriction N_u$ to the elementary submodel $\mathfrak{C} \restriction \bar{b}_w$. Let $r(x, \bar{x}_w)$ be the translation of r_* to a type over the empty set in the variables $x, \bar{x}_w$. This shows that "if consistent" indeed is, so we may assume $r_{f,w}$ has the stated properties, though it need not have arisen in this way. *Observation 7.4* □

To motivate our remaining step in the construction of a support, consider for a moment a sequence of four-tuples induced by the same f, say

$$\langle (v_t, w_t, q_{f_t,w_t}, r_{f_t,w_t}) : t < t_* < \sigma \rangle$$

where $(v_t, w_t, q_{f_t,w_t}, r_{f_t,w_t})$ arises from $(f_t, w_t) \in \mathcal{F}^3_{u_t}$ and $v_t \subseteq u_t$, and there is a single f with $f_t \subseteq f$ for all t. A priori, this four-tuple need not be from $\mathcal{R}_{\mathfrak{m}}$ (e.g. if $\mathbf{x}_f \cap \mathbf{a}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)} = 0$) nonetheless we may begin to analyze its properties. By definition of $q_{f,w}$, existence of such an f means the union $\bigcup_{t<t_*} q_{f_t,w_t}$ will always be a partial type in the variables $\bar{x}_w$, where $w = \bigcup\{w_t : t < t_*\}$. Let $\bar{b}^*_w$ be any sequence realizing this type. Then the types $r_{f_t,w_t}(x, \bar{b}^*_{w_t})$ may be explicitly contradictory as their definition allowed for arbitrary choices. Going forwards, our strategy will be to handle the issue of explicit inconsistency in the r's with the construction of $\bar{\mathbf{b}}'$; before that, we ensure the necessary nonforking with the following definition.

Recall D-rank from 2.17 above. In the following, we do not require that D-rank is definable, only that the value is constant in the sense described.

Let $\mathcal{F}^4_u$ be the set of pairs $(f, w) \in \mathcal{F}^3_u$ such that in addition:

(vii) for every $(f', w') \in \mathcal{F}^3_u$ with $(f, w) \leq_{\mathcal{F}^0_u} (f', w')$, and every sequence $\langle b_\alpha : \alpha \in w' \rangle$ of elements of the monster model which realizes $q_{f',w'}(\overline{x}_{w'})$, and every $\alpha \in w$,
- $\mathrm{tp}(b_\alpha, \{b_\gamma : \gamma \in w' \cap \alpha\})$ dnf over $\{b_\gamma : \gamma \in w \cap \alpha\}$.
- for any formula φ and $k < \omega$,

$$D(\mathrm{tp}(b_\alpha, \{b_\gamma : \gamma \in w' \cap \alpha\}), \varphi, k) = D(\mathrm{tp}(b_\alpha, \{b_\gamma : \gamma \in w \cap \alpha\}), \varphi, k).$$

Claim 7.5. *As T is simple and θ is regular, $\mathcal{F}^4_u$ is $\leq_{\mathcal{F}^0_u}$-dense in $(\mathcal{F}^1_u, \leq_{\mathcal{F}^1_u})$.*

Proof. It suffices to prove it is dense in $\mathcal{F}^3_u$. Suppose for a contradiction that $\mathcal{F}^4_u$ is not dense. By induction on $\alpha < \theta$ we choose b_α, q_α, f_α, w_α such that:

- $(f_\alpha, w_\alpha) \in \mathcal{F}^3_u$
- $\beta < \alpha$ implies $(f, w) \leq_{\mathcal{F}^1_u} (f_\beta, w_\beta) \leq_{\mathcal{F}^1_u} (f_\alpha, w_\alpha)$
- if α is a limit then $(f_\alpha, w_\alpha) = (\bigcup_{\beta<\alpha} f_\beta, \bigcup_{\beta<\alpha} w_\beta)$
- $q_\alpha = q_{f_\alpha,w_\alpha}$, so $\beta < \alpha \implies q_\beta \subseteq q_\alpha$
- $\langle b_\beta : \beta < \alpha \rangle$ realizes $q_\alpha \restriction \{x_\gamma : \gamma \in w_\alpha \cap \alpha\}$
- if $\alpha = \beta + 1$ then for some $\mathbf{i} \in w_\beta$
  (α) $\mathrm{tp}(b_{\mathbf{i}}, \{b_\gamma : \gamma \in w_\alpha \cap \mathbf{i}\})$ forks over $\{b_\gamma : \gamma \in w_\beta \cap \mathbf{i}\}$
  (β) for some $\varphi \in \mathcal{L}(\tau_T)$ and finite k,

$$D(\mathrm{tp}(b_{\mathbf{i}}, \{b_\gamma : \gamma \in w_\alpha \cap \mathbf{i}\}), \varphi, k) < D(\mathrm{tp}(b_{\mathbf{i}}, \{b_\gamma : \gamma \in w_\beta \cap \mathbf{i}\}), \varphi, k).$$

As $\theta = \mathrm{cof}(\theta) > |T|$, by Fodor's lemma this contradicts the assumption that T is a simple theory. *Claim 7.5* □

We are ready to choose partitions supporting each $\mathbf{b}_u$, $u \in \Omega$. The next claim states our requirements, and adds a coherence condition.

Claim 7.6. *There exists*

$$\overline{f} = \langle \overline{f}_u : u \in \Omega \rangle = \langle \ \langle (f_{u,\zeta}, w_{u,\zeta}) : \zeta < \mu \rangle \ : \ u \in \Omega \rangle$$

which is a good choice of partitions *for* $\bar{\mathbf{b}}$*, where this means*

1. *for each* $u \in \Omega$, $\bar{f}_u = \langle (f_{u,\zeta}, w_{u,\zeta}) : \zeta < \mu \rangle$ *is a sequence of elements of* $\mathcal{F}^4_u$ *such that* $\langle \mathbf{x}_{f_{u,\zeta}} : \zeta < \mu \rangle$ *is a maximal antichain of* $\mathfrak{B}$ *and for each* $\zeta < \mu$*, either* $\mathbf{x}_{f_{u,\zeta}} \leq \mathbf{b}_u$ *or* $\mathbf{x}_{f_{u,\zeta}} \leq 1 - \mathbf{b}_u$.
2. (Coherence) *writing* $\mathcal{V} = \bigcup \{ \mathrm{Dom}(f_{u,\zeta}) : u \in \Omega, \zeta < \mu \}$, *if* $v_0 \in [\mathcal{V}]^{<\theta}$ *and* $u_1 \in \Omega$*, then for some* u_* *we have:* $u_1 \subseteq u_* \in \Omega$ *and* $\zeta < \mu \implies v_0 \subseteq \mathrm{Dom}(f_{u_*,\zeta})$ *and* $\bar{f}_{u_*}$ *refines* $\bar{f}_{u_1}$.

Proof. $\mathfrak{B}$ satisfies the μ^+-c.c. and has maximal antichains of cardinality μ, so for any $u \in \Omega$ we may choose $\overline{f}^x_u = \{ (f_{u,\epsilon}, w_{u,\epsilon}) : \epsilon < \mu \} \subseteq \mathcal{F}^4_u$, such that:

$$\langle \mathbf{x}_{f_{u,\zeta}} : \zeta < \mu \rangle \text{ is a maximal antichain of } \mathfrak{B}, \text{ so } \epsilon \neq \zeta < \mu \implies f_{u,\epsilon} \neq f_{u,\zeta}.$$

[We build such a partition by induction on ζ, using the density of $\mathcal{F}^4_u$. In doing so we may assume, without loss of generality, that $0 \in \mathrm{Dom}(f)$ for each f used in this partition; then the partition will have size at least μ, and since $\mathfrak{B}$ has the μ^+-c.c., the construction will stop at some ordinal $< \mu^+$. Renumbering, we may assume the sequence is indexed by $\zeta < \mu$.]

Say that $\overline{f}_{u_i}$ *refines* $\overline{f}_{u_j}$ if for each $\epsilon < \mu$, $f_{u_i,\epsilon}$ extends $f_{u_k,\zeta}$ for some $\zeta < \mu$. To ensure coherence across the family, 1.1 ensures that we may enumerate $\Omega \times [\lambda]^{<\theta}$ as $\langle (u_i, v_i) : i < \lambda \rangle$. Build $\overline{f}_{u_i}$ by induction on $i < \lambda$ as follows. Arriving to i,

(a) if $(\exists j < i) \big((u_i \subseteq u_j) \wedge (\zeta < \mu \implies v_i \subseteq \mathrm{Dom}(f_{u_j,\zeta})) \big)$ then let $j(i)$ be the least such j and let $\overline{f}_{u_i}$ be a common refinement of $\overline{f}^x_u$ and $\overline{f}_{u_j}$.
(b) If there is no such j, choose $\overline{f}_{u_i}$ such that it refines $\overline{f}^x_{u_i}$ and $\overline{f}_{u_k}$ whenever $k < i$ and $u_k \subseteq u_i$ and $\zeta < \mu \implies \mathrm{Dom}(f_{u_i,\zeta}) \supseteq v_i$. We can do this because there are $\leq 2^{|u_i|} < \theta$ such k, recalling that $\sigma \leq \theta$ and σ is (by our hypotheses in this section) compact, thus strongly inaccessible.

For each $u \in \Omega$, let $\overline{f}_u = \langle (f_{u,\zeta}, w'_{u,\zeta}) : \zeta < \mu \rangle$ be the resulting family. This completes the construction. *Claim 7.6* □

For the remainder of the proof, fix the support just built as well as

$$\mathcal{V} \subseteq 2^\lambda, |V| = \lambda \text{ such that } \bigcup \{ \mathrm{Dom}(f_{u,\zeta}) : u \in \Omega, \zeta < \mu \} \subseteq \mathcal{V}. \tag{7.14}$$

Having built a support for the sequence $\bar{\mathbf{b}}$, our next task is to define the proposed multiplicative refinement $\bar{\mathbf{b}}'$. Towards this, let us take stock. For each $u \in \Omega$ and $\zeta < \mu$, we may henceforth unambiguously write

$$q_{u,\zeta} \text{ for } q_{f_{u,\zeta},w_{u,\zeta}},\ r_{u,\zeta} \text{ for } r_{f_{u,\zeta},w_{u,\zeta}}. \tag{7.15}$$

Claim 7.7. *For each $u \in \Omega$ and $\zeta < \mu$,*

$$\text{if } \mathbf{x}_{f_{u,\zeta}} \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} \text{ then } (u, w_{u,\zeta}, q_{u,\zeta}, r_{u,\zeta}) \in \mathcal{R}_{\mathfrak{m}}.$$

Proof. We check Definition 3.9. For 3.9.1–2, $u \in [\lambda]^{<\sigma}$, $w \in [\lambda]^{<\theta}$, and $u \subseteq \mathrm{cl}_{\mathcal{M}}(u) \subseteq w = \mathrm{cl}_{\mathcal{M}}(w)$ by the definition of $\mathcal{F}^0$. Towards 3.9.3, $q_{f,w}$ is always a complete type in the variables $\bar{x}_w$, and if $\mathbf{x}_f \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$ then $\mathbf{x}_f \leq \mathbf{a}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)}$ by equation (7.8). So 3.9.3(a), holds by definition of $q_{f,w}$ (really, of Γ_1) and 3.9.3(b) holds by the remark after equation (7.3). 3.9.4+5(a) follow from Observation 7.4. 3.9.5(b) is because $r_{f,w}$ is a complete type (notice that the condition "if $w' \subseteq w$ is $\mathcal{M}$-closed then $\mathfrak{C}_T \restriction \{b^*_\alpha : \alpha \in w'\} \preceq \mathfrak{C}_T$ and $r_{f,w}(x, \overline{b}^*_{w'})$ is a complete type over this elementary submodel" does not ask for a given enumeration to have any closure properties, but simply that all formulas are decided). Finally 3.9.5(c) is by the fact that $(f, w) \in \mathcal{F}^4_u$. *Claim 7.7* □

The four-tuple from the statement of Claim 7.7 is therefore in the domain of the function G giving the intrinsic coloring for $\mathfrak{m}$, which was fixed as part of the presentation at the beginning of the proof. (We may trivially extend G to all four-tuples arising from some $(f_{u,\zeta}, w_{u,\zeta})$ by setting G to be ∞ if it is not otherwise defined.) We will want to amalgamate certain such tuples later in the proof, but first we take advantage of the ultrapower setup to eliminate some extraneous noise. Define E to be the equivalence relation on pairs $(u, \zeta) \in \Omega \times \mu$ given by:

$$(u_1, \zeta_1) E (u_2, \zeta_2) \text{ iff} \tag{7.16}$$

1. $(u_1, w_{u_1,\zeta})$ and $(u_2, w_{u_2,\zeta})$ satisfy:
   (i) $\mathrm{otp}(u_1) = \mathrm{otp}(u_2)$, $\mathrm{otp}(w_1) = \mathrm{otp}(w_2)$
   (ii) if $\gamma \in w_1 \cap w_2$ then $\mathrm{otp}(\gamma \cap w_1) = \mathrm{otp}(\gamma \cap w_2)$
   (iii) if $\gamma \in w_1 \cap w_2$ then $\gamma \in u_1$ iff $\gamma \in u_2$
   (iv) the order preserving map from w_1 to w_2 carries u_1 to u_2.
2. There is an order preserving function h from $\mathrm{Dom}(f_1)$ onto $\mathrm{Dom}(f_2)$, s.t.:
   (i) (implied) $\mathrm{otp}(\mathrm{Dom}(f_1)) = \mathrm{otp}(\mathrm{Dom}(f_2))$
   (ii) $\gamma \in \mathrm{Dom}(f_1) \implies f_2(h(\gamma)) = f_1(\gamma)$
   (iii) $\gamma \in \mathrm{Dom}(f_1) \cap \mathrm{Dom}(f_2) \implies h(\gamma) = \gamma$.
3. $\zeta_1 = \zeta_2$.
4. $G(u_1, w_{u_1,\zeta_1}, q_{u_1,\zeta_1}, r_{u_1,\zeta_1}) = G(u_2, w_{u_2,\zeta_2}, q_{u_2,\zeta_2}, r_{u_2,\zeta_2})$, i.e the values are both defined and equal or both ∞.

Claim 7.8. *E from (7.16) is an equivalence relation with μ classes, which we will list as*

$$\langle E_\epsilon : \epsilon < \mu \rangle.$$

Proof. For counting purposes we may assume $\zeta < \mu$ is fixed.

For the first condition, the equivalence class of $(u, w_{u,\zeta})$ is fixed if we determine the ordinals $\gamma = \mathrm{otp}(w_{u,\zeta})$, $\delta = \mathrm{otp}(u)$, and determine which function from δ into γ gives u. There are $\leq \theta \leq \mu$ choices of γ, $\leq \sigma \leq \mu$ choices of δ, and for any given $\gamma < \theta$, $\delta < \sigma$, we have by 1.1*(c)* that $|{}^\delta\gamma| < \theta \leq \mu$. So by 1.1*(b)*, the total count is bounded by $\mu \cdot \mu \cdot \mu = \mu$.

For the second condition, the equivalence class of $f_{u,\zeta}$ is fixed if we first determine the ordinal $\mathrm{otp}(\mathrm{Dom}(f_{u,\zeta}))$, call it β, and then determine which function from β into μ determines the values of the function. Since $\mathrm{otp}(\mathrm{Dom}(f_{u,\zeta}))$ is an ordinal $< \theta$, the number of possible results is bounded by $\theta \cdot \mu^{<\theta} = \theta \cdot \mu = \mu$.

This shows the number of equivalence classes is $\leq \mu \cdot \mu = \mu$, and by our choice of G it is exactly μ. *Claim 7.8* □

Given $\langle E_\epsilon : \epsilon < \mu \rangle$ from Claim 7.8, fix also a choice of representatives by specifying some function

$$h : \mu \to \Omega \times [\lambda]^{<\theta} \times \mu \times \mu. \tag{7.17}$$

We ask that $h(\epsilon) = (u_{h(\epsilon)}, w_{h(\epsilon)}, \zeta_{h(\epsilon)}, \xi_{h(\epsilon)})$ satisfies: for some $(u, \zeta) \in E_\epsilon$, $u = u_{h(\epsilon)}$, $\zeta = \zeta_{h(\epsilon)}$, $w_{h(\epsilon)} = w_{u,\zeta}$, and $\xi_{h(\epsilon)} = G(u, w, q_{u,\zeta}, r_{u,\zeta})$ or ∞.

The crucial set for each $\alpha < \lambda$, $\epsilon < \mu$ will be

$$\mathcal{U}_{\alpha,\epsilon} = \{u : u \in \Omega \text{ and } (u, \zeta_{h(\epsilon)}) \in E_\epsilon \text{ and } \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}\}. \tag{7.18}$$

Recall $\mathcal{V}$ from (7.14) above. Let

$$\alpha_* < 2^\lambda \text{ be such that } \mathcal{U} \subseteq \alpha_*. \tag{7.19}$$

Let Code_λ denote some fixed coding function from ${}^{\omega>}\lambda$ to λ, and let Code_μ denote some fixed coding function from ${}^{\omega>}\mu$ to μ. Let $\langle X_0, \ldots, X_5 \rangle$ be a partition of λ into six sets of cardinality λ. For $\gamma < \lambda$ and $n < 6$, let $\rho(\gamma, X_n)$ denote the image of γ under a fixed one-to-one map of λ into X_n. Let tv denote the truth value of an expression (either 0 or 1).

$$\text{For each } u \in \Omega, \zeta < \mu \text{ define } f^* = f^*_{u,\zeta} \text{ as follows.} \tag{7.20}$$

1. $\mathrm{Dom}(f^*) \subseteq \alpha + \lambda$ is of cardinality $< \theta$, $\mathrm{Range}(f^*) \subseteq \mu$, and f^* is determined by the remaining conditions.

2. If $\gamma \in \mathrm{Dom}(f_{u,\zeta})$ then

$$f^*(\alpha + \rho(\gamma, X_0)) = \mathrm{Code}_\mu(\ \langle f_{u,\zeta}(\gamma), \mathrm{otp}(\gamma \cap \mathrm{Dom}(f_{u,\zeta}))\rangle\ ).$$

3. If $\gamma \in w_{u,\zeta}$, then

$$f^*(\alpha + \rho(\gamma, X_1)) = \mathrm{Code}_\mu(\mathrm{tv}(\gamma \in u), \mathrm{otp}(\gamma \cap u), \mathrm{otp}(\gamma \cap w_{u,\zeta})).$$

4. If $\beta \neq \alpha$ are from $w_{u,\zeta}$ then

$$f^*(\alpha + \rho(\gamma, X_2)) = \mathrm{tv}(\ \rho_\alpha(f_{u,\zeta}) = \rho_\beta(f_{u,\zeta})\ ).$$

5. If $v \subseteq u$, $\bar{v}$ is this set listed in increasing order, and $\mathrm{Code}_\lambda(\ \bar{v}\ ) = \gamma$, then

$$f^*(\alpha + \rho(\gamma, X_3)) = \mathrm{tv}(\ \mathbf{x}_{f_{u,\zeta}} \leq \mathbf{b}_v\ ).$$

6. If $v \in {}^{\omega>}(w_{u,\zeta})$, φ is the k-th $\mathcal{L}$-formula in $|v|$ free variables under some enumeration fixed in advance, and $\gamma = \mathrm{Code}_\lambda(\ \langle v, k\rangle\ )$, then

$$f^*(\alpha + \rho(\gamma, X_4)) = \mathrm{tv}(\mathbf{x}_{f_{u,\zeta}} \leq \mathbf{a}_{\varphi(\bar{x}_v)} = \mathrm{tv}(\ \varphi(\bar{x}_v) \in q_{u,\zeta}\ ).$$

7. If $v \in {}^{\omega>}(w_{u,\zeta})$, φ is the k-th $\mathcal{L}$-formula in $1 + |v|$ free variables under some enumeration fixed in advance, and $\gamma = \mathrm{Code}_\lambda(\ \langle v, k\rangle\ )$, then

$$f^*(\alpha + \rho(\gamma, X_5)) = \mathrm{tv}(\ \varphi(x, \bar{x}_v) \in r_{u,\zeta}\ ).$$

This completes the definition (7.20). Fix also a new maximal antichain:

$$\text{let } \bar{\mathbf{c}} = \langle \mathbf{c}_\epsilon : \epsilon < \mu\rangle \text{ be given by } \mathbf{c}_\epsilon = \mathbf{x}_{\{(\alpha+\lambda,\epsilon)\}}\}. \tag{7.21}$$

We arrive at the definition of $\bar{\mathbf{b}}'$. For each $\alpha < \lambda$, let

$$\mathbf{b}'_{\{\alpha\}} = \Big(\bigcup\{\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \ :\ \epsilon < \mu,\ u \in \mathcal{U}_{\alpha,\epsilon}\}\Big) \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(\{\alpha\})}. \tag{7.22}$$

Why is (7.22) nonzero? For each $\epsilon < \mu$ such that $\mathcal{U}_{\alpha,\epsilon} \neq \emptyset$, and each $u \in \mathcal{U}_{\alpha,\epsilon}$,

$$\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}\ >\ 0.$$

since the domains of the functions corresponding to $\mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}}$, $\mathbf{c}_\epsilon$ and $\mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}}$ are mutually disjoint, and adding $\mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$ is allowed by the definition of $\mathcal{U}_{\alpha,\epsilon}$ in (7.18). By monotonicity, $\mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(\{\alpha\})}$ for any $u \in \mathcal{U}_{\alpha,\epsilon}$. This verifies that (7.22) is nonzero.

For each $u \in \Omega \setminus \emptyset$, define

$$\mathbf{b}'_u = \bigcap\{\mathbf{b}'_{\{\alpha\}} : \alpha \in u\}. \tag{7.23}$$

Let $\mathbf{b}'_{\emptyset} = 1_{\mathfrak{B}}$. This completes the definition of the sequence $\bar{\mathbf{b}}'$:

$$\bar{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in \Omega \rangle. \tag{7.24}$$

By construction, $\bar{\mathbf{b}}'$ is multiplicative, and if $|u| = 1$ then $\mathbf{b}'_u \leq \mathbf{b}_u$.

An immediate consequence (as $\langle \mathbf{c}_\epsilon : \epsilon < \mu \rangle$ is a maximal antichain) of this definition is that whenever $\mathbf{c} \in \mathfrak{B}^+_{\alpha_*}$ and $0 < \mathbf{c} \leq \mathbf{c}_\epsilon \cap \mathbf{b}'_{\{\alpha\}}$,

$$\bigcup \{ \mathbf{c} \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} : u \in \mathcal{U}_{\alpha,\epsilon} \} \neq 0 \tag{7.25}$$

so in particular there is $u \in \mathcal{U}_{\alpha,\epsilon}$ such that

$$\mathbf{c} \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} > 0 \qquad \text{thus} \qquad \mathbf{c} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} > 0 \tag{7.26}$$

where again the conjunct "$\cap\, \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$" is by the definition of $\mathcal{U}_{\alpha,\epsilon}$ in equation (7.18). We now work towards proving that:

Claim 7.9. $\bar{\mathbf{b}}'$ *is a multiplicative refinement of* $\bar{\mathbf{b}}$.

Proof. Suppose for a contradiction that there is some $u_* \in \Omega$ such that

$$0 < \mathbf{c} \leq \bigcap_{\alpha \in u_*} \mathbf{b}'_{\{\alpha\}} \setminus \mathbf{b}_{u_*}. \tag{7.27}$$

By continuity of $\bar{\mathbf{b}}$, we may assume that u_* is finite. Let $f \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$ be such that $\mathbf{x}_f \leq \mathbf{c}$, and as $\langle \mathbf{c}_\epsilon : \epsilon < \mu \rangle$ is a maximal antichain, without loss of generality there is some $\epsilon < \mu$ such that $\mathbf{x}_f \leq \mathbf{c}_\epsilon$. Since $\mathbf{x}_f \leq \mathbf{b}'_{\{\alpha\}}$ by construction, necessarily $\mathcal{U}_{\alpha,\epsilon} \neq \emptyset$ for each $\alpha \in u_*$. Enumerate u_* as $\langle \alpha_t : t < t_* \rangle$. By induction on $t \leq t_*$, we will choose functions $f_t \in \mathrm{FI}_{\mu,\theta}(2^\lambda)$ and sets $u_t \in \mathcal{U}_{\alpha_t,\epsilon}$ such that for each t:

(i) $f_t \supseteq f$, *thus* for each t, $\mathbf{x}_{f_t} \leq \bigcap \{ \mathbf{b}'_{\{\alpha\}} : \alpha \in u_* \} \cap \mathbf{c}_\epsilon$
(ii) $t' < t \implies f_{t'} \subseteq f_t$
(iii) $f_t \supseteq f_{u_t,\zeta_{h(\epsilon)}} \cup f^*_{u_t,\zeta_{h(\epsilon)}}$
(iv) $\mathbf{x}_{f_t} \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u_t)}$.

Let $f_{-1} = f$ so that f_0 is the case "$-1+1$." Arriving to $t+1$, condition (i) implies that $\mathbf{x}_{f_t} \leq \mathbf{b}'_{\{\alpha_t\}} \cap \mathbf{c}_\epsilon$. So adding the latter two onto a conjunction will not affect whether or not we get 0. Apply (7.25)–(7.26) to choose $u_t \in \mathcal{U}_{\alpha_t,\epsilon}$ such that

$$\mathbf{x}_{f_\ell} \cap \mathbf{x}_{f_{u_t,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f^*_{u_t,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} > 0.$$

Let $f_{t+1} = f_t \cup f_{u_t,\zeta_{h(\epsilon)}} \cup f^*_{u_t,\zeta_{h(\epsilon)}}$. This completes the induction. Note that (iv) will be satisfied by the definition (7.18).

Let $f_* = f_{t_*}$ be the function so constructed.

Recalling the definition of E_ϵ in 7.16, since ϵ is fixed all the $\zeta_{h(\epsilon)}$'s are the same, so going forward we will write ζ for $\zeta_{h(\epsilon)}$. Then by Claim 7.7 and the fact that $\mathbf{x}_{f_*} \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u_t)}$ for each $\alpha_t \in u_*$, we have that for each $t < t_*$,

$$\mathfrak{r}_t := (u_t, w_{u_t,\zeta}, q_t = q_{u_t,\zeta}, r_t = r_{u_t,\zeta}) \in \mathcal{R}_{\mathfrak{m}}. \tag{7.28}$$

Let $\bar{\mathfrak{r}} = \langle \mathfrak{r}_t : t < t_* \rangle$. Moreover, the pairs (u_t, ζ) all belong to the E_ϵ class of our equivalence relation from (7.16). So equation (7.28) ensures that each $\mathfrak{r}_t$ is in the domain of G and item 4 of the definition of E ensures that the value of G is fixed. That is,

$$G \restriction \langle \mathfrak{r}_t : t < t_* \rangle \text{ is constant.} \tag{7.29}$$

Thanks to the $f^*_{u_t,\zeta}$ from (7.20), we may now find a good instantiation of $\bar{\mathfrak{r}}$:

Subclaim 7.10. *Let $w = \bigcup_t w_t$. We can find $\langle b^*_\alpha : \alpha \in w \rangle$ realizing $\bigcup_t q_t$ such that $\bar{b}^*_w$ is a good instantiation for $\bar{\mathfrak{r}}$ and*

$$\mathfrak{C} \models \neg(\exists x) \bigwedge_{\alpha \in u_*} \varphi_\alpha(x, b^*_\alpha). \tag{7.30}$$

Proof. As the q_t are all induced by the same $f_* = f_{t_*}$ the set $q = \bigcup_{t<t_*} q_t$ is consistent. First we will show it is consistent with $\neg\varphi_{u_*}$, recalling (7.2). Suppose not, so let $\Sigma_0 = \{\varphi_0(\bar{x}_{v_0}), \cdots, \varphi_k(\bar{x}_{v_k})\} \subseteq q$ be finite such that $\Sigma = \Sigma_0 \cup \{\neg\varphi_{u_*}\}$ is inconsistent. Let $v = v_0 \cup \cdots \cup v_k$. Let θ be an arbitrary finite conjunction of formulas from $q \restriction \bar{x}_{\mathrm{cl}_{\mathcal{M}}(\emptyset)}$. Recalling notation (7.4), $\mathbf{x}_{f_*}$ witnesses that the set

$$A_\theta \cap A_{\varphi_0(\bar{x}_{v_0})} \cap \cdots \cap A_{\varphi_k(\bar{x}_{v_k})} \setminus B_{\varphi_u} > 0.$$

Let t be any element of this set. Then $\{a^*_\alpha[t] : \alpha \in v\}$ realizes Σ, contradiction. This proves we may find some sequence $\bar{b}^*_w$ of elements of $\mathfrak{C}$, possibly imaginary, realizing q such that $b^*_\alpha = a^*_\alpha$ for $\alpha \in \mathrm{cl}_{\mathcal{M}}(\emptyset)$ and $\models \neg(\exists x) \bigwedge_{\alpha \in u_*} \varphi_\alpha(x, b^*_\alpha)$. Finally, let us check that this sequence is a good instantiation, 3.10. Conditions (1)–(2) of that definition we've just checked. (3) is ensured by the fact that

$$\mathbf{x}_{f_{t_*}} \leq \mathbf{x}_{f^*_{u_t,\zeta}}$$

for each u_t, recalling (7.20)6+7. (4) was ensured by $\mathcal{F}^4$. (5) was ensured by $\mathcal{F}^3$, and already checked by membership in $\mathcal{R}_{\mathfrak{m}}$, 3.9(5). *Subclaim 7.10* □

Let $\bar{b}^*_w$ be given by Subclaim 7.10. By equation (7.29), we may apply Definition 3.11, "G is an intrinsic coloring of $\mathcal{R}_{\mathfrak{m}}$", to conclude that $\{\varphi_{\alpha_t}(x, b^*_{\alpha_t}) : t <$

$t_*\} = \{\varphi_\alpha(x, b^*_\alpha) : \alpha \in u_*\}$ is consistent. This contradicts the choice of $\bar{b}^*$, specifically equation (7.30). This contradiction shows that $\mathbf{c}$ from equation (7.27) cannot exist, i.e. $\bar{\mathbf{b}}'$ must be a multiplicative refinement of $\bar{\mathbf{b}}$. This proves the Claim. *Claim 7.9* □

The last part of the argument is to verify that $\bar{\mathbf{b}}$ has the Key Property 5.7 using $\mathcal{V}$ from (7.14), α_* from (7.19), $\Omega_* = \{u \in [\lambda]^{<\sigma} : u = \mathrm{cl}_{\mathcal{M}}(u)\}$, and $\bar{\mathbf{b}}'$. Suppose $u \in \Omega_*$ and $f \in \mathrm{FI}_{\mu,\theta}(\alpha_*)$ are such that $\mathbf{x}_f \leq \mathbf{b}_u$. We hope to show

$$\mathbf{x}_f \cap \mathbf{b}'_u > 0. \tag{7.31}$$

As $\mathbf{x}_f \leq \mathbf{b}_u$, by choice of $u \in \Omega_*$,

$$\mathbf{x}_f \leq \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}. \tag{7.32}$$

Since $\overline{f}_u$ is a partition, after possibly extending f we have

$$\mathbf{x}_f \leq \mathbf{x}_{f_{u,\zeta_*}} \text{ for some } \zeta_* < \mu. \tag{7.33}$$

Choose $\epsilon < \mu$ such that $(u, \zeta_*) \in E_\epsilon$. Then $h(\epsilon) = (u, w_{h(\epsilon)}, \zeta_{h(\epsilon)} = \zeta_*, \xi_{h(\epsilon)})$. For each $\alpha \in u$, checking the definition (7.18) we have that

$$u \in \mathcal{U}_{\alpha,\epsilon}. \tag{7.34}$$

As $\bar{\mathbf{b}}'$ is a multiplicative sequence, $\mathbf{b}'_u = \bigcap_{\alpha\in u} \mathbf{b}'_{\{\alpha\}}$. So to show (7.31) it would suffice to show that

$$\mathbf{x}_f \cap \bigcap_{\alpha\in u} \mathbf{b}'_{\{\alpha\}}. \tag{7.35}$$

Recalling the definitions (7.22), and equation (7.34), the expression

$$\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \tag{7.36}$$

appears as a disjunct for each $\mathbf{b}'_{\{\alpha\}}$, and by (7.18) has nonzero intersection with $\mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$, so to show (7.35) it would suffice to show that

$$\mathbf{c}_\epsilon \cap \mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)} \cap \mathbf{x}_f > 0.$$

This is true because on one hand, $\mathbf{x}_f \leq \mathbf{x}_{f_{u,\zeta_{h(\epsilon)}}} \cap \mathbf{b}_{\mathrm{cl}_{\mathcal{M}}(u)}$ by our above argument, and on the other hand, the support of $\mathbf{x}_f$, c_ϵ, and $\mathbf{x}_{f^*_{u,\zeta_{h(\epsilon)}}}$ are pairwise disjoint and thus cannot cause inconsistency. This completes the proof that $\mathbf{x}_f \cap \mathbf{b}'_u > 0$.

As $\bar{\mathbf{b}}$ has the $(\lambda, \mu, \theta, \sigma)$-Key Property, in our optimal ultrafilter $\mathcal{D}_*$ it has a multiplicative refinement. Thus by Theorem F, the original type p is realized in the ultrapower $M^I/\mathcal{D}$. This completes the proof of Theorem 7.3. □

8. The ultrapower characterization of simple theories

We state our main result in two different ways, the first to underline the structure of the proof.

Theorem 8.1. *Assume $(\lambda, \mu, \theta, \sigma)$ are suitable and σ is an uncountable supercompact cardinal. There exists a regular ultrafilter $\mathcal{D}$ over λ such that for every model M in a countable signature, $M^\lambda/\mathcal{D}$ is λ^+-saturated if $Th(M)$ is $(\lambda, \mu, \theta, \sigma)$-explicitly simple, and $M^\lambda/\mathcal{D}$ is not μ^{++}-saturated if $Th(M)$ is not simple.*

Proof. As σ is uncountable and supercompact, Theorem 5.9 and Conclusion 5.13 prove existence of a regular ultrafilter $\mathcal{D}$ on I such that: $\mathcal{D}$ is built from $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ where $\mathcal{D}_0$ is regular and excellent on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda, \mu, \theta}$ and $\mathcal{D}_*$ is $(\lambda, \mu, \theta, \sigma)$-optimal, and moreover $\mathcal{D}$ is not μ^{++}-good for any non-simple theory. By Theorem 7.3, any such $\mathcal{D}$ is good for any $(\lambda, \mu, \theta, \sigma)$-explicitly simple theory. □

Theorem 8.2 *(The ultrapower characterization of simple theories). Assume $(\lambda, \mu, \theta, \sigma)$ are suitable, $\mu^+ = \lambda$ and σ is an uncountable supercompact cardinal. Then there is a regular ultrafilter $\mathcal{D}$ on λ such that for any model M in a countable signature, $M^\lambda/\mathcal{D}$ is λ^+-saturated whenever $Th(M)$ is simple, and $M^\lambda/\mathcal{D}$ is not λ^+-saturated whenever $Th(M)$ is not simple.*

Proof. Apply Theorem 8.1 assuming in addition that $\mu^+ = \lambda$. Then every countable simple theory is $(\lambda, \mu, \theta, \sigma)$-explicitly simple by Theorem 4.10. □

Remark 8.3. In Theorem 8.2, 'countable signature' may clearly be weakened to 'signature of size $< \sigma$.'

Assuming existence of an uncountable supercompact cardinal, Theorem 8.2 has the following immediate consequence for the structure of Keisler's order. Thanks to the referee for suggesting the formulation.

Conclusion 8.4. *Assume there exists an uncountable supercompact cardinal. If T, T' are countable theories, T is simple, and $T' \trianglelefteq T$, then T' is simple.*

9. Perfect ultrafilters

In this section we shall give a natural set-theoretic condition on ultrafilters, called 'perfect', which essentially says that they solve as many problems as possible modulo the cardinal constraints. We will use perfect ultrafilters in [29] in the case $\sigma = \aleph_0$ in applying the analysis of this paper to theories with trivial forking. Recall the definition of support, 5.6 above.

Definition 9.1 *(Perfect ultrafilters for the case $\sigma = \theta = \aleph_0$).* Let $(\lambda, \mu, \aleph_0, \aleph_0)$ be suitable. We say that an ultrafilter $\mathcal{D}_*$ on $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$ is (λ,μ)-*perfect* when (A) implies (B):

(A) $\langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0}\rangle$ is a monotonic sequence of elements of $\mathcal{D}_*$
and $\mathrm{supp}(\bar{\mathbf{b}})$ is a support for $\bar{\mathbf{b}}$ of cardinality $\leq \lambda$, see 5.6, such that for every $\alpha < 2^\lambda$ with $\bigcup\{\mathrm{Dom}(f) : \mathbf{x}_f \in \mathrm{supp}(\bar{\mathbf{b}})\} \subseteq \alpha$, there exists a multiplicative sequence

$$\langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0}\rangle$$

of elements of $\mathfrak{B}^+$ such that
(a) $\mathbf{b}'_u \leq \mathbf{b}_u$ for all $u \in [\lambda]^{<\aleph_0}$,
(b) for every $\mathbf{c} \in \mathfrak{B}^+_{\alpha,\mu} \cap \mathcal{D}_*$, no intersection of finitely many members of $\{\mathbf{b}'_{\{i\}} \cup (1 - \mathbf{b}_{\{i\}}) : i < \lambda\}$ is disjoint to $\mathbf{c}$.

(B) there is a multiplicative sequence $\bar{\mathbf{b}}' = \langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0}\rangle$ of elements of $\mathcal{D}_*$ which refines $\bar{\mathbf{b}}$.

Observation 9.2. *Suppose $\alpha < 2^\lambda$ is fixed, D_α is an ultrafilter on $\mathfrak{B}^1_{\alpha,\mu} \subseteq \mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$, and $\langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0}\rangle$ is a sequence of elements of D_α. Suppose that there exists a multiplicative sequence $\langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0}\rangle$ of elements of $\mathfrak{B}^+$ such that*

(a) $\mathbf{b}'_u \leq \mathbf{b}_u$ *for all* $u \in [\lambda]^{<\aleph_0}$,
(b) *for every* $\mathbf{c} \in \mathfrak{B}^+_{\alpha,\mu} \cap \mathcal{D}_\alpha$, *no intersection of finitely many members of* $\{\mathbf{b}'_{\{i\}} \cup (1 - \mathbf{b}_{\{i\}}) : i < \lambda\}$ *is disjoint to* $\mathbf{c}$.

Then there is a multiplicative sequence $\langle \mathbf{b}''_u : u \in [\lambda]^{<\aleph_0}\rangle$ such that (a), (b) hold with $\mathbf{b}'_u$, $\mathbf{b}'_{\{i\}}$ replaced by $\mathbf{b}''_u$, $\mathbf{b}''_{\{i\}}$ respectively, and such that some support of $\bar{\mathbf{b}}''$ is contained in $\mathfrak{B}^1_{\alpha+\lambda,\mu}$.

Remark 9.3. Note that in 9.2(b), omitting "$1 - \mathbf{b}_{\{i\}}$" gives an equivalent condition.

Proof of Observation 9.2. Without loss of generality there is $\mathcal{U}$ of cardinality λ such that some support of $\bar{\mathbf{b}}'$ is contained in $\{\mathbf{x}_f : f \in \mathrm{FI}_\mu(\mathcal{U})\}$. Let π be a permutation of 2^λ which is the identity on α and takes $\mathcal{U}$ into $\alpha + \lambda$. This induces an automorphism ρ of $\mathfrak{B}$ which is the identity on $\mathfrak{B}^1_{\alpha,\mu}$, so in particular is the identity on $\mathcal{D}_\alpha$ and thus on $\bar{\mathbf{b}}$. For each $u \in [\lambda]^{<\aleph_0}$, let $\mathbf{b}''_u = \rho(\mathbf{b}'_u)$. Then clearly $\bar{\mathbf{b}}''$ fits the bill. □

Theorem 9.4 *(Existence of perfect ultrafilters). Let $(\lambda, \mu, \aleph_0, \aleph_0)$ be suitable. Let $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$. Then there exists a (λ,μ)-perfect ultrafilter on $\mathfrak{B}$.*

Proof. Begin by letting $\langle \bar{\mathbf{b}}_\delta = \langle \mathbf{b}_{\delta,u} : u \in [\lambda]^{<\aleph_0}\rangle : \delta < 2^\lambda\rangle$ be an enumeration of the monotonic sequences of elements of $\mathfrak{B}^+$, each occurring cofinally often. Let $z : 2^\lambda \to 2^\lambda$ be an increasing continuous function which satisfies: $z(0) \geq 0$ and for all $\beta < 2^\lambda$, $z(\beta) + \lambda =$

$z(\beta+1)$. By induction on $\delta < 2^\lambda$ we will construct $\langle D_\delta : \delta < 2^\lambda\rangle$, an increasing continuous sequence of filters with each D_δ an ultrafilter on $\mathfrak{B}_{z(\delta),\mu}$, to satisfy:

(*) if $\delta = \beta + 1$, if it is the case that

> $\langle \mathbf{b}_{\beta,u} : u \in [\lambda]^{<\aleph_0}\rangle$ is a monotonic sequence of elements of $\mathcal{D}_\beta$ and there exists a choice of $\mathrm{supp}(\bar{b})$ with $\bigcup\{\mathrm{Dom}(f) : \mathbf{x}_f \in \mathrm{supp}(\overline{\mathbf{b}})\} \subseteq \beta$ and there exists a multiplicative sequence
>
> $$\langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0}\rangle$$
>
> of elements of $\mathfrak{B}^+$ such that
> (a) $\mathbf{b}'_u \leq \mathbf{b}_{\beta,u}$ for all $u \in [\lambda]^{<\aleph_0}$,
> (b) for every $\mathbf{c} \in \mathfrak{B}^+_{z(\beta),\mu} \cap \mathcal{D}_\beta$, no intersection of finitely many members of $\{\mathbf{b}'_{\{i\}} \cup (1 - \mathbf{b}_{\beta,\{i\}}) : i < \lambda\}$ is disjoint to $\mathbf{c}$.
>
> *then* there is a sequence $\bar{\mathbf{b}}'' = \langle \mathbf{b}''_u : u \in [\lambda]^{<\aleph_0}\rangle$ of elements of $\mathfrak{B}^+$ such that:
> (i) $\mathbf{b}''_u \leq \mathbf{b}_{\beta,u}$ for all $u \in [\lambda]^{<\aleph_0}$,
> (ii) for every $\mathbf{c} \in \mathfrak{B}^+_{z(\beta),\mu} \cap \mathcal{D}_\beta$, no intersection of finitely many members of $\{\mathbf{b}''_{\{i\}} \cup (1 - \mathbf{b}_{\beta,\{i\}}) : i < \lambda\}$ is disjoint to $\mathbf{c}$.
> (iii) some support of $\bar{\mathbf{b}}''$ is contained in $\mathfrak{B}_{z(\delta),\mu}$, and
> (iv) D_δ is an ultrafilter on $\mathfrak{B}_{z(\delta),\mu}$ which extends $D_\beta \cup \{\mathbf{b}'_u : u \in [\lambda]^{<\aleph_0}\}$.

The induction may be carried out at limit stages because all of the D_δ are ultrafilters. Suppose $\delta = \beta + 1$. If $\bar{\mathbf{b}}$ satisfies the quoted condition, then let $\bar{\mathbf{b}}''$ be given by Observation 9.2, using $z(\beta)$ here for α there. Then (i), (ii), (iii) are satisfied, so we need to prove that

$$D_\beta \cup \{\mathbf{b}''_u : u \in [\lambda]^{<\aleph_0}\}$$

has the finite intersection property. As D_β is an ultrafilter on $\mathfrak{B}_{z(\beta),\mu}$, and $\bar{\mathbf{b}}'$ is a multiplicative sequence, it suffices to prove that for any $\mathbf{c} \in \mathcal{D}_\beta$ and any finite $u \subseteq \lambda$,

$$\mathbf{c} \cap \bigcap\{\mathbf{b}''_{\{i\}} : i \in u\} > 0.$$

As $\mathbf{b}_{\{i\}} \in \mathcal{D}_\beta$ for each $i \in u$, we may assume that $\mathbf{c} \cap (1 - \mathbf{b}_{\{i\}}) = 0$ for each $i \in u$. Then we are finished by assumption (ii). This completes the induction. Let $\mathcal{D}_* = \bigcup_{\delta<2^\lambda} D_\delta$.

Let us check that $\mathcal{D}_*$ is indeed a perfect ultrafilter. If $\bar{\mathbf{b}}$ satisfies condition 9.1(A), let $\mathcal{U}$ be as there, and let $\delta = \beta+1$ be an ordinal $< 2^\lambda$ such that $\bar{\mathbf{b}}_\beta = \bar{\mathbf{b}}$ and $\mathcal{U} \subseteq \mathfrak{B}_{\beta,\mu}$, which is possible as we listed each sequence cofinally often. Then since D_β was an ultrafilter, $\mathcal{D}_* \restriction \mathfrak{B}_{\beta,\mu} = \mathcal{D}_\beta$ so at stage δ condition (*) of the inductive hypothesis will be activated and we will have ensured that $\bar{\mathbf{b}}$ has a multiplicative refinement in $\mathcal{D}_*$. □

Corollary 9.5. *Suppose* $(\lambda, \mu, \aleph_0, \aleph_0)$ *are suitable. and let* $\mathcal{D}_*$ *be an ultrafilter on* $\mathfrak{B}^1_{2^\lambda,\mu}$. *If* $\mathcal{D}_*$ *is* (λ,μ)*-perfect, then it is* $(\lambda, \mu, \aleph_0, \aleph_0)$*-optimal.*

Proof. We need to show that any sequence with the so-called Key Property 5.7 has a multiplicative refinement. Suppose then that $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\lambda]^{<\aleph_0} \rangle$ is a monotonic sequence of elements of $\mathcal{D}_*$ with the Key Property, and fix a support $\mathrm{supp}(\bar{\mathbf{b}})$ as given by that property. Let α be an ordinal $< 2^\lambda$ such that $\{\mathbf{b}_u : u \in [\lambda]^{<\aleph_0}\} \subseteq D_\alpha := \mathcal{D}_* \restriction \mathfrak{B}^1_{\alpha,\mu}$ and $\bigcup\{\mathrm{Dom}(f) : \mathbf{x}_f \in \mathrm{supp}(\bar{\mathbf{b}})\} \subseteq \alpha$. Write $\mathfrak{B}_\alpha$ for $\mathfrak{B}^1_{\alpha,\mu}$ for the remainder of this proof. The Key Property guarantees the existence of a cofinal $\Omega \subseteq [\lambda]^{<\aleph_0}$ and a sequence $\bar{\mathbf{b}}' = \langle \mathbf{b}'_{\{i\}} : i < \lambda \rangle$ of elements of $\mathfrak{B}^+$ which generates a multiplicative refinement $\langle \mathbf{b}'_u : u \in [\lambda]^{<\aleph_0} \rangle$ of $\bar{\mathbf{b}}$ such that for each $f \in \mathrm{FI}_{\mu,\aleph_0}(\alpha)$, and each $u \in \Omega$, if $\mathbf{x}_f \leq \mathbf{b}_u$ then we may extend $f \subseteq f' \in \mathrm{FI}_{\mu,\aleph_0}(2^\lambda)$ so that $\mathbf{x}_{f'} \leq \mathbf{b}'_u$.

In order to guarantee that our perfect ultrafilter will have given $\bar{\mathbf{b}}$ a multiplicative refinement, it will suffice to show that for every $\mathbf{c} \in \mathfrak{B}^+_\alpha \cap \mathcal{D}_\alpha$, no intersection of finitely many members of $\{\mathbf{b}'_{\{i\}} \cup (1 - \mathbf{b}_{\{i\}}) : i < \lambda\}$ is disjoint to $\mathbf{c}$. Let such an $\mathbf{c}$ be given, let $v \in [\lambda]^{<\aleph_0}$ and choose any u with $v \subseteq u \in \Omega$. As $\mathbf{c} \in \mathcal{D}_\alpha$ and $\mathbf{b}_u \in \mathcal{D}_\alpha$, without loss of generality $\mathbf{c} \leq \mathbf{b}_u$. As the generators are dense in the completion, we may choose $\mathbf{x}_f$ with $0 < \mathbf{x}_f \leq \mathbf{c}$ and $f \in \mathrm{FI}_{\mu,\aleph_0}(\alpha)$. Then $\mathbf{x}_f \leq \mathbf{b}_u$ so by the Key Property, we may extend $f \subseteq f'$ so that $\mathbf{x}_{f'} \leq \mathbf{b}'_u$. This proves that $\mathbf{x}_f \cap \bigcap\{\mathbf{b}'_{\{i\}} : i \in u\} > 0$, as desired.

Then $\bar{\mathbf{b}}$ contains a multiplicative refinement by the definition of 'perfect,' which proves that the ultrafilter is indeed optimal. □

Conclusion 9.6. *Let* $(\lambda, \mu, \aleph_0, \aleph_0)$ *be suitable. Let* $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$. *Then there is an ultrafilter* $\mathcal{D}_*$ *on* $\mathfrak{B}$ *such that:*

(a) $\mathcal{D}_*$ *is* (λ,μ)*-perfect, and indeed* $(\lambda, \mu, \aleph_0, \aleph_0)$*-optimal.*
(b) *if* $\mathcal{D}$ *is any regular ultrafilter built from* $(\mathcal{D}_0, \mathfrak{B}, \mathcal{D}_*)$ *where* $\mathcal{D}_0$ *is a regular* λ^+*-excellent filter on* λ, *we have that* $\mathcal{D}$ *is not good for any non-low or non-simple theory.*

Proof. It remains to justify clause (b) by quoting known results. Ref. [34] VI.3.23, p. 364 proves that any ultrafilter constructed by means of such an independent family of functions where $\mu < \lambda$ will not be μ^+-good, however the proof shows more: that it will not be μ^+-flexible. An alternate discussion is given in [27] Section 9.

The fact that an ultrafilter which is not flexible is not good for any non-low or non-simple theory was proved by [22] and [25]. More precisely, in [22] Section 8 it was proved that any regular ultrafilter which is good for a theory which has TP_2 or is simple and non-low, must be flexible. In [25] it was proved that any regular ultrafilter which is good for a theory with SOP_2 is good, therefore a fortiori flexible. Since any non-simple theory has either TP_2 or SOP_2, this completes the proof. □

Corollary 9.7. *Let* $(\lambda, \mu, \aleph_0, \aleph_0)$ *be suitable and let* $\mathfrak{B} = \mathfrak{B}^1_{2^\lambda,\mu}$. *Then there exists a* $(\lambda, \mu, \aleph_0, \aleph_0)$*-optimal ultrafilter on* $\mathfrak{B}$, *and moreover we may arrange that the non-saturation condition 9.6 holds.*

10. Some further questions

The theorems in this paper suggest a broad classification program for simple theories according to their "explicit simplicity". We believe the most urgent questions have to do with determining the identity of the equivalence classes of simple theories in Keisler's order. The following specific natural questions also arise. Assume $(\lambda, \mu, \theta, \sigma)$ are suitable.

For the first question, recall that from the point of view of explicit simplicity, μ bounds the range of the coloring function G on $\mathcal{R}_{\mathfrak{m}}$. However, because we have asked in 1.1 that $\mu \geq \theta \geq \sigma$, the value of μ may be larger than the actual number of colors needed. The first question essentially asks if we can build a simple theory where the range of this coloring function is truly uncountable but does not depend on λ. (Without the negative condition, the random graph would be a trivial example.)

Question 10.1. *Let* $\sigma = \theta = \aleph_1$, $\mu = (2^\theta)^{++}$. *Then whenever* $\lambda = \lambda^{<\theta} \geq \mu$, $(\lambda, \mu, \theta, \sigma)$ *is suitable and also* $(\lambda, \kappa, \theta, \sigma)$ *is suitable, where* $\kappa^+ = \mu$. *Does there exist a countable simple theory* T *such that for all* $\lambda = \lambda^{<\theta} \geq \mu$, T *is* $(\lambda, \mu, \theta, \sigma)$*-explicitly simple but not* $(\lambda, \kappa, \theta, \sigma)$*-explicitly simple? What about other uncountable constant values of* μ*?*

Recall that by our arguments above the random graph requires only one color, and it is not difficult to produce examples of essentially the same complexity requiring finitely or countably many colors. We have just asked for a simple theory requiring an uncountable (i.e. $> |T|$) but constant number of colors. The case where $\mu^{+n} = \lambda$, for n finite and greater than 1, is studied in the forthcoming paper [29]. The next question asks about $n = 1$:

Question 10.2 *(A maximal simple theory). Suppose* σ *is uncountable. Is there a countable simple theory which is* $(\lambda, \mu, \theta, \sigma)$*-explicitly simple if and only if* $\mu^+ = \lambda$*?*

Recall that in his paper [17] Keisler had developed the notion of a "versatile" formula to describe when theories T were saturated precisely by good regular ultrafilters. The next question asks whether something analogous can be done inside simplicity.

Question 10.3. *For which, if any, values of* $(\lambda, \mu, \theta, \sigma)$ *does there exist a simple theory* T *which is saturated by a regular ultrafilter* $\mathcal{D}$ *on* λ *iff* $\mathcal{D}$ *is* $(\lambda, \mu, \theta, \sigma)$*-optimized, or iff* $\mathcal{D}$ *is* (λ, μ)*-perfected in the sense of 9.1 above?*

The ultrapower characterization of stable theories from [34] Chapter VI proceeded by proving that a model of a stable theory is λ^+-saturated if and only if it is $\kappa(T)$ saturated

and every maximal indiscernible sequence has cardinality at least λ^+. It would be interesting to develop, perhaps from the arguments above, an analogous characterization of saturation in simple theories.

Question 10.4. *Give an analogous characterization of the saturated models of simple theories.*

Finally, we record the fundamental question of the minimum unstable Keisler class. The regular ultrafilters which saturate this class are known; see, for example, [23] Section 4.

Question 10.5. *Give an internal model-theoretic characterization of the equivalence class of the random graph in Keisler's order.*

Acknowledgments

Malliaris was partially supported by NSF grant DMS-1300634, by a Sloan fellowship, by a Gödel research prize fellowship, and by a research membership at MSRI funded by NSF grant 0932078 000 (Spring 2014). Malliaris' visits to Shelah at Rutgers were partially supported via NSF grant 1362974. Shelah was partially supported by European Research Council grant 338821. This is paper 1030 in Shelah's list.

We would like to thank the anonymous referee for precise and thoughtful reports which have significantly improved the presentation of the paper.

References

[1] J. Ax, S. Kochen, Diophantine problems over local fields. I, Amer. J. Math. 87 (1965) 605–630.
[2] J. Ax, S. Kochen, Diophantine problems over local fields. II, Amer. J. Math. 87 (1965) 631–648.
[3] J. Ax, S. Kochen, Diophantine problems over local fields. III, Ann. of Math. (2) 83 (1966) 437–456.
[4] H. Cartan, Théorie des filtres, C. R. Math. Acad. Sci. Paris 205 (1937) 595–598.
[5] H. Cartan, Filtres et ultrafiltres, C. R. Math. Acad. Sci. Paris 205 (1937) 777–779.
[6] A. Dow, Good and OK ultrafilters, Trans. Amer. Math. Soc. 290 (1) (1985) 145–160.
[7] R. Engelking, M. Karłowicz, Some theorems of set theory and their topological consequences, Fund. Math. 57 (1965) 275–285.
[8] Yu.L. Eršov, On the elementary theory of maximal normed fields, Dokl. Akad. Nauk SSSR 165 (1965);
E. translation, J. Sov. Math. 165 (1965) 1390–1393.
[9] G. Fichtenholz, L. Kantorovitch, Sur les opérations linéaires dans l'espace des fonctions bornées, Studia Math. 5 (1934) 69–98.
[10] T. Frayne, A. Morel, D. Scott, Reduced direct products, Fund. Math. 51 (1962) 195–248.
[11] R. Grossberg, J. Iovino, O. Lessmann, A primer of simple theories, Arch. Math. Logic 41 (6) (2002) 541–580.
[12] F. Hausdorff, Über zwei Sätze von G. Fichtenholz und L. Kantorovitch, Studia Math. 6 (1936) 18–19.
[13] E. Hrushovski, Pseudofinite fields and related structures, in: Model Theory and Applications, in: Quad. Mat., vol. 11, Aracne, Rome, 2002, pp. 151–212.
[14] H.J. Keisler, Ultraproducts and elementary classes, Indag. Math. 23 (1961) 477–495.
[15] H.J. Keisler, Good ideals in fields of sets, Ann. of Math. (2) 79 (1964) 338–359.

[16] H.J. Keisler, A survey of ultraproducts, in: Y. Bar-Hillel (Ed.), Logic, Methodology and Philosophy of Science, in: Proc. of the 1964 Int. Congress in Jerusalem, North-Holland, Amsterdam, 1965, pp. 112–126.
[17] H.J. Keisler, Ultraproducts which are not saturated, J. Symbolic Logic 32 (1967) 23–46.
[18] S. Kochen, Ultraproducts in the theory of models, Ann. of Math. (2) 74 (2) (1961) 221–261.
[19] K. Kunen, Ultrafilters and independent sets, Trans. Amer. Math. Soc. 172 (1972) 299–306.
[20] J. Łoś, Quelques remarques, théorèmes et problemes sur les classes definissables d'algebres, in: Mathematical Interpretation of Formal Systems, Studies in Logic, 1955, pp. 98–113.
[21] M. Malliaris, Realization of φ-types and Keisler's order, Ann. Pure Appl. Logic 157 (2–3) (2009) 220–224.
[22] M. Malliaris, Hypergraph sequences as a tool for saturation of ultrapowers, J. Symbolic Logic 77 (1) (2012) 195–223.
[23] M. Malliaris, S. Shelah, Constructing regular ultrafilters from a model-theoretic point of view, Trans. Amer. Math. Soc. 367 (2015) 8139–8173, http://dx.doi.org/10.1090/S0002-9947-2015-06303-X.
[24] M. Malliaris, S. Shelah, Model-theoretic properties of ultrafilters built by independent families of functions, J. Symbolic Logic 79 (1) (2014) 103–134.
[25] M. Malliaris, S. Shelah, Cofinality spectrum problems in model theory, set theory and general topology, J. Amer. Math. Soc. 29 (1) (2016) 237–297, http://dx.doi.org/10.1090/jams830.
[26] M. Malliaris, S. Shelah, General topology meets model theory, on $\mathfrak{p}$ and $\mathfrak{t}$, Proc. Natl. Acad. Sci. USA 110 (33) (2013) 13300–13305.
[27] M. Malliaris, S. Shelah, A dividing line within simple unstable theories, Adv. Math. 249 (2013) 250–288.
[28] M. Malliaris, S. Shelah, Saturating the random graph with an independent family of small range, in: A. Hirvonen, J. Kontinen, R. Kossak, A. Villaveces (Eds.), Logic Without Borders, in Honor of Jouko Väanänen, DeGruyter, 2015.
[29] M. Malliaris, S. Shelah, Keisler's order has infinitely many classes, http://arxiv.org/abs/1503.08341.
[30] M. Malliaris, S. Shelah, Boolean algebras underlying simple theories, manuscript F1484, in preparation.
[31] J. Moore, Model theory and the cardinal numbers $\mathfrak{p}$ and $\mathfrak{t}$, Proc. Natl. Acad. Sci. USA 110 (33) (2013) 13238–13239.
[32] S. Shelah, Every two elementarily equivalent models have isomorphic ultrapowers, Israel J. Math. 10 (1971) 224–233.
[33] S. Shelah, Saturation of ultrapowers and Keisler's order, Ann. Math. Logic 4 (1972) 75–114.
[34] S. Shelah, Classification Theory and the Number of Non-Isomorphic Models, rev. ed., North-Holland, 1990 (first edition 1978).
[35] A. Tarski, Une contribution à la théorie de la mesure, Fund. Math. 15 (1930) 42–50.