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# Embeddings of Cohen Algebras

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## INTRODUCTION

Complete Boolean algebras proved to be an important tool in topology and set theory. Two of the most prominent examples are $\mathbb{B}(\kappa)$, the algebra of Borel sets modulo measure zero ideal in the generalized Cantor space $\{0, 1\}^\kappa$ equipped with product measure, and $\mathbb{C}(\kappa)$, the algebra of regular open sets in the space $\{0, 1\}^\kappa$, for $\kappa$ an infinite cardinal. $\mathbb{C}(\kappa)$ is much easier to analyse than $\mathbb{B}(\kappa)$: $\mathbb{C}(\kappa)$ has a dense subset of size $\kappa$, while the density of $\mathbb{B}(\kappa)$ depends on the cardinal characteristics of the real line, and the definition of $\mathbb{C}(\kappa)$ is simpler. Indeed, $\mathbb{C}(\kappa)$ seems to have the simplest definition among all algebras of its size. In the Main Theorem of this paper we show that in a certain precise sense, $\mathbb{C}(\aleph_1)$ has the simplest *structure* among all algebras of its size, too.

MAIN THEOREM. *If ZFC is consistent then so is ZFC* $+ 2^{\aleph_0} = \aleph_2 +$ "*every complete Boolean algebra* $\mathfrak{B}$ *of uniform density* $\aleph_1$ *has a complete subalgebra isomorphic to* $\mathbb{C}(\aleph_1)$."

There is another interpretation of the result. Let $\langle \mathrm{BA}(\kappa), \lessdot \rangle$ denote the class of complete Boolean algebras of uniform density $\kappa$ quasi-ordered by complete embeddability. This quasi-order can be understood as a rough measure of complexity of the algebras concerned. Now $\mathrm{BA}(\aleph_0)$ has just one element up to isomorphism; it is $\mathbb{C}(\aleph_0)$. The class $\mathrm{BA}(\aleph_1)$ can already be

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immensely rich, permitting of no simple classification; this is the case say under the continuum hypothesis. The Main Theorem shows that the class $\mathrm{BA}(\aleph_1)$ can have a smallest element. Note that this smallest element must then be $\mathbb{C}(\aleph_1)$, since by [5, Proposition 7] $\mathbb{C}(\aleph_1)$ is minimal in $\mathrm{BA}(\aleph_1)$.

The techniques introduced in this paper provide us with much more information. Most notably we get

COROLLARY 14. *Under* $\mathrm{MA}_{\aleph_1}$ $\mathbb{C}(\aleph_1)$ *embeds into every complete c.c.c. Boolean algebra of uniform density* $\aleph_1$.

COROLLARY 37. *Under* PFA $\mathbb{C}(\aleph_1)$ *embeds into every complete Boolean algebra of uniform density* $\aleph_1$.

The search for complex objects which have to be embedded into complete Boolean algebras of small size has been going on for some time. It has been proved that every algebra in the class $\mathrm{BA}(\aleph_1)$ may have to add a real [8], indeed a Cohen real [13]. Every uncountable Boolean algebra may have to have an uncountable independent subset [10].

The proof of the Main Theorem is an iteration argument. The heart of the matter lies in introducing a regular embedding of $\mathbb{C}(\aleph_1)$ to a given algebra $\mathfrak{B}$ of uniform density $\aleph_1$ by a sufficiently mild forcing. This problem is solved in the first three sections. Section 1 introduces the crucial auxiliary notion of an avoidable subset of the algebra $\mathfrak{B}$, Section 2 deal with productively c.c.c. $\mathfrak{B}$ as an easier special case, proving Corollary 14 and setting the stage for the attack at the general case in Section 3. At the end of Section 3 we are able to demonstrate the Main Theorem. Section 4 is devoted to a couple of relevant ZFC examples of algebras of bigger density. Finally, Section 5 suggests several open problems.

The arguments in the paper are given a nested structure, in the priority order Theorem, Lemma, Claim. It is advisable, for example, upon the first reading of the proof of Theorem X to leave out the arguments for the lemmas. Our notation follows the set-theoretic standard as set forth in [4]. Throughout the paper we work with separative partially ordered sets representing dense subsets of the Boolean algebras in question rather than with the algebras themselves. "Algebra" stands for "complete Boolean algebra" and "embedding", "embeds" stand for "complete embedding", "completely embeds" respectively. In a forcing notion we write $p \geqslant q$ to mean that $q$ is more informative than $p$ (i.e., the Western way); and $p \perp q$ to mean that $p$ and $q$ are incompatible, that is, no $r$ is less than both $p$ and $q$. All partial orders in this paper will have a maximal element by default, denoted by 1. A poset $P$ is *separative* if for $p \nleqslant q$ there is $r \leqslant p$, $r \perp q$. We say that $P$ has *uniform density* $\kappa$ if $|P| = \kappa$ and for no $p \in P$, $R \in [P]^{<\kappa}$ $R$ is dense below $p$. An algebra has uniform density $\kappa$ if it has a dense subset of uniform density $\kappa$. If $p \in P$ then $P \restriction p$ stands for $\{r \in P: r \leqslant p\}$. We write $P \lessdot Q$ (*P embeds into Q*) if there is $\dot{H}$, a $Q$-name such that $Q \Vdash$ "$\dot{H} \subset \check{P}$ is generic over $V$". Thus $P \lessdot Q$ iff RO($P$) embeds into RO($Q$) and we can reasonably use $\lessdot$ for embedding of algebras. $\mathbb{C}(\kappa)$ is construed as RO($C_\kappa$), where $C_\kappa = \{h: h$ is a function and $\mathrm{dom}(h) \in [\kappa]^{<\aleph_0}$, $\mathrm{rng}(h) \subset 2\}$ ordered by reverse inclusion. For an ordinal $\alpha$ and a set $X$ of ordinals we write $\alpha^{*X}$ for $\min(X \setminus \alpha + 1)$. $H_\kappa$ is the collection of all sets hereditarily of size $< \kappa$. For two models $M$, $N$, $M \prec N$ means that $M$ is an elementary submodel of $N$ and the special predicates will be often understood from the context. ▮C105 marks the end of the proof of Claim 105, ▮T61 marks the end of the proof of Theorem 61, etc.

The results in this paper were obtained during the meeting of the two authors at Rutgers University in September 1994 and the week following it. The second author thanks Rutgers University for its hospitality during this time. Theorem 8, Definition 20, and Lemma 21 are due to the first author, Lemma 42 is due to both authors independently, and the other results are due to the second author. The results of this paper appeared in Chap. 2 of the second author's Ph.D. thesis [14].

## 1. THE OVERALL STRATEGY

Of course, the proof of the Main Theorem is by a forcing iteration argument. The basic challenge is, given a poset $P$ of uniform density $\aleph_1$, to find a sufficiently mild forcing $Q$ such that $Q \Vdash$ "$C_{\aleph_1} \lessdot P$". Then we can hope to iterate the procedure to obtain a model for the desired statement.

The following notion plays a very important role in our argument.

Definition 1. Let $P$ be an arbitrary poset. A set $D \subset P$ is called almost avoidable if for every $p \in P$ there is a finite set $\mathrm{tr}(p) \subset D$, called a trace of $p$ in $D$, such that for any $b \in [D]^{<\aleph_0}$ with $b \cap \mathrm{tr}(p) = 0$ there is $p' \leqslant p$ which is incompatible with every element of $b$.

For example, any finite set $D \subset P$ is almost avoidable (set $\mathrm{tr}(p) = D$ for every $p \in P$) and any antichain $D \subset P$ is almost avoidable (set $\mathrm{tr}(p) = \{r\}$, where $r \in D$ is some element of $D$ compatible with $p$, for every $p \in P$). However, we shall be interested in finding a *dense* almost avoidable set $D \subset P$. Here is a canonical example of such a situation. Let $P$ be the Cohen poset ${}^{<\omega}\omega$ ordered by reverse extension. Then $P$, as a subset of itself, is almost avoidable; just set $\mathrm{tr}(s) = \{t \in P: t \subset s\}$. If $b$ is a finite set in ${}^{<\omega}\omega$ with $b \cap \mathrm{tr}(s) = 0$ then there is a one-step extension of the sequence $s$ avoiding every element of $b$.

The relevance of Definition 1 to our problem is explained in the following two lemmas. They show that the statement "a poset $P$ has a dense almost avoidable subset" is a good approximation of "$C_{\aleph_1} \lessdot P$".

Lemma 2. *Let $P$ be a poset of size $\kappa$ such that $C_\kappa \lessdot P$. Then $P$ has a dense almost avoidable subset.*

*Proof.* Let $P$ be an arbitrary poset of size $\kappa$, $P = \{p_\alpha : \alpha \in \kappa\}$, and suppose that $C_\kappa \lessdot P$. Choose a $P$-name $\dot{c}$ such that $P \Vdash$ "$\dot{c}: \kappa \to 2$ is $C_\kappa$-generic" and fix the induced embedding $e$ of $\mathbb{C}(\kappa)$ to $\mathrm{RO}(P)$. We define the set $D \subset P$ as follows: for each $\alpha \in \kappa$, we choose a condition $p'_\alpha \leqslant p_\alpha$ and a bit $i(\alpha) \in 2$ such that $p'_\alpha \Vdash$ "$\dot{c}(\alpha) = i(\alpha)$"; we set $D = \{p'_\alpha : \alpha \in \kappa\}$.

Now obviously the set $D$ is dense in $P$. We must show that $D$ is almost avoidable. To this aim, fix a condition $p \in P$. Definition 1 calls for a trace of $p$ in the set $D$. We choose a finite function $h \in C_\kappa$ with $h \leqslant \mathrm{proj}_{\mathbb{C}(\kappa)}(p)$ and set $\mathrm{tr}(p) = \{p'_\alpha : \alpha \in \mathrm{dom}(h)\}$.

To see that the set $\mathrm{tr}(p)$ has the required properties, let $b \subset D$ be a finite set disjoint from $\mathrm{tr}(p)$. So necessarily there is a finite set $d \subset \kappa$ disjoint from $\mathrm{dom}(h)$ such that $b = \{p'_\alpha : \alpha \in d\}$. Let $k \in C_\kappa$ be the function with $\mathrm{dom}(k) = \mathrm{dom}(h) \cup d$ and $k(\alpha) = h(\alpha)$ for $\alpha \in \mathrm{dom}(h)$ and $k(\alpha) = 1 - i(\alpha)$ for $\alpha \in d$. Since $k \leqslant h \leqslant \mathrm{proj}_{\mathbb{C}(\kappa)}(p)$, in the poset $P$ there must be a lower bound $p'$ of the conditions $p$ and $e(k)$. By the choice of the function $k$, necessarily $p' \perp p'_\alpha$ for $\alpha \in d$, and so $p' \leqslant p$ witnesses the statement of Definition 1 for $p$, $\mathrm{tr}(p)$, and $b$. ∎

Lemma 3. *Let $P$ be a poset of uniform density $\kappa$ with a dense almost avoidable subset. Then $C_\kappa \Vdash$ "$C_\kappa \lessdot \check{P}$".*

*Remark.* This is somewhat weaker than the straightforward converse of Lemma 2. Indeed, a complicated $\diamondsuit$-construction gives an example in $L$ of a poset $P$ of uniform density $\aleph_1$ with a dense almost avoidable subset such that $P$ does not add even one Cohen real. Thus the converse of Lemma 2 is not provable in ZFC even in the case $\kappa = \aleph_1$.

*Proof.* Let $P$ be a poset of uniform density $\kappa$ with a dense almost avoidable subset $D$. First, using the uniform density of $P$ we extract a system of $\kappa$ many disjoint maximal antichains of the set $D$.

Claim 4. *There is a system $\langle A_\gamma : \gamma \in \kappa \rangle$ of pairwise disjoint maximal antichains of the set $D$.*

*Proof.* We fix a bookkeeping device, a bijection $e: P \times \kappa \to \kappa$. By induction on $\alpha \in \kappa$, we construct a sequence $\langle p_\alpha : \alpha \in \kappa \rangle$ of pairwise distinct conditions in $D$ as follows. Given $\alpha \in \kappa$, $\alpha = e(p, \gamma)$, and the sequence $\langle p_\beta : \beta \in \alpha \rangle$, the condition $p_\alpha$ is any condition in the set $D$ which is less than $p$ and does not appear on the sequence $\langle p_\beta : \beta \in \alpha \rangle$. It is possible to choose such a condition since the set $D$, unlike the set $\{p_\beta : \beta \in \alpha\}$, is dense below the condition $p$.

By the construction, for $\gamma \in \kappa$ the sets $D_\gamma = \{p_\alpha : \alpha \in e''P \times \{\gamma\}\} \subset D$ are pairwise disjoint dense in $P$. The claim follows by choosing a maximal antichain $A_\gamma \subset D_\gamma$ for each $\gamma \in \kappa$. ▮C4

Fix a system $\langle A_\gamma : \gamma \in \kappa \rangle$ of antichains as in Claim 4. Thus each $A_\gamma \subset D$ is a maximal antichain of the poset $P$ by the density of $D$.

Definition 5. A forcing $Z$ is defined by $Z = \{z : z$ is a function with $\operatorname{dom}(z) \in [\bigcup_{\gamma \in \kappa} A_\gamma]^{<\aleph_0}$, $\operatorname{rng}(z) \subset 2\}$; order is by reverse extension.

*Explanation.* Essentially, we force a $P$-name for a $C_\kappa$-generic sequence $\langle \dot{c}_\gamma : \gamma \in \kappa \rangle$ by finite conditions. Given $\gamma \in \kappa$, the name $\dot{c}_\gamma$ will be a function from $A_\gamma$ to 2; for a condition $z \in Z$, the function $z \restriction A_\gamma$ is a finite piece of the future $\dot{c}_\gamma$.

Obviously, the forcing $Z$ is isomorphic to $C_\kappa$. Thus we will have proven the lemma once we show that $Z \Vdash$ "$C_\kappa \lessdot P$". If $H \subset Z$ is a generic filter and $\gamma \in \kappa$ then $\dot{c}_\gamma = \bigcup H \restriction A_\gamma$ is a $P$-name for an element of 2. We show that $Z \Vdash P \Vdash$ "$\langle \dot{c}_\gamma : \gamma \in \kappa \rangle$ is $C_\kappa$-generic". To this end, fix $z_0 \in Z$, $z_0 \Vdash$ "$\dot{E} \subset C_\kappa$ is open dense" and $p_0 \in P$. We find $z_1 \leqslant z_0$, $p_1 \leqslant p_0$ so that $z_1 \Vdash_Z p_1 \Vdash_P$ "$\langle \dot{c}_\gamma : \gamma \in \kappa \rangle$ meets $\dot{E}$", proving the lemma. Choose a trace $\operatorname{tr}(p_0)$ of $p_0$ in the dense set $D \subset P$ and let $d = \{\gamma \in \kappa : A_\gamma \cap \operatorname{tr}(p_0) \neq 0\}$; thus the set $d$ is finite. For the rest of the proof we adopt the following piece of notation: for two functions $h, k$ the symbol $h \circlearrowleft k$ stands for the unique function with domain $\operatorname{dom}(h) \cup \operatorname{dom}(k)$ which is equal to $k$ on $\operatorname{dom}(k)$ and equal to $h$ on $\operatorname{dom}(h) \setminus \operatorname{dom}(k)$.

Claim 6. *There are a condition $z_{1/2} \leqslant z_0$ in $Z$ and $h \in C_\kappa$ such that for any function $k : d \to 2$ we have $z_{1/2} \Vdash$ "$\check{h} \circlearrowleft \check{k} \in \dot{E}$".*

*Proof.* Let $n = |d|$ and $\langle k_j : j \in 2^n \rangle$ enumerate ${}^d 2$. By induction on $j \in 2^n + 1$ we construct $w_j \in Z$, $h_j \in C_\kappa$ so that

(1) $w_0 = z_0$, $h_0 = 0$

(2) $w_j$'s are decreasing in $Z$, $h_j$'s are decreasing in $C_\kappa$

(3) for $j \in 2^n$ we have $w_j \Vdash_Z$ "$\check{h}_{j+1} \circlearrowleft \check{k} \in \dot{E}$".

There is no problem in the induction. $z_{1/2} = w_{2^n}$, $h = h_{2^n}$ witness the statement of the claim. ▮C6

Pick $z_{1/2} \leqslant z_0$, $h \in C_\kappa$ as in the claim. By properties of the trace we can find $p_{1/2} \leqslant p$ so that for every $r \in \operatorname{dom}(z_{1/2}) \setminus \operatorname{tr}(p_0)$ we have $r \perp p_{1/2}$. We strengthen $p_{1/2}$ to $p_1$ such that for every $\gamma \in \operatorname{dom}(h)$ there is an element of the antichain $A_\gamma$ above $p_1$; denote this unique element by $p^\gamma$. Define a condition $w \in Z$ by $\mathrm{dom}(w) = \{p^\gamma : \gamma \in \mathrm{dom}(h)\}$, $w(p^\gamma) = h(\gamma)$ and set $z_1 = w \vec{\cup} z_{1/2}$. Thus $z_1 \in Z$ and moreover $z_1 \leqslant z_{1/2} \leqslant z_0$. The following claim completes the proof of the lemma:

Claim 7. $z_1 \Vdash_Z p_1 \Vdash_P$ *"the function* $\gamma \mapsto \dot{c}_\gamma$, $\gamma \in \mathrm{dom}(h)$ *is in* $\dot{E}$".

*Proof.* Comparing the function $\gamma \mapsto \dot{c}_\gamma$, $\gamma \in \mathrm{dom}(h)$ to $h$, we find that $z_1 \Vdash_Z p_1 \Vdash_P$ "$\dot{c}_\gamma \neq h(\gamma)$ implies $p^\gamma \in \mathrm{dom}(z_{1/2})$". By construction of $p_{1/2}$, $\{\gamma \in \omega_1 : p^\gamma \in \mathrm{dom}(z_{1/2})\} \subset d$. Therefore $z_1 \Vdash_Z p_1 \Vdash_P$ "$\dot{c}_\gamma = h(\gamma)$ for all $\gamma \in \mathrm{dom}(h) \setminus d$". By our choice of $h$ and $z_{1/2}$ we have $z_1 \Vdash_Z p_1 \Vdash_P$ "the function $\gamma \mapsto \dot{c}_\gamma$, $\gamma \in \mathrm{dom}(h)$ is in $\dot{E}$", i.e., the statement of the claim. ▮C7,L3

This brings us back to our original task. Fix a poset $P$ of uniform density $\aleph_1$. We construct a two-step iteration $Q = Q_0 * C_{\aleph_1} = Q_0 \times C_{\aleph_1}$. The forcing $Q_0$ serves to introduce a dense almost avoidable subset to $P$. By Lemma 3, we then have $Q \Vdash$ "$C_{\aleph_1} \lessdot \check{P}$". In the next section we show that in the special case of a productively c.c.c. poset $P$, the most optimistic variation of the above scenario works. In Section 3, we work on the general case, which is somewhat harder and technically more requiring. At the end of Section 3 we are finally in the position to prove the Main Theorem.

## 2. PRODUCTIVELY c.c.c. POSETS

Theorem 8. *Let P be a separative productively c.c.c. poset with uniform density* $\aleph_1$. *Then there is a c.c.c. forcing Q such that* $Q \Vdash$ "$C_{\aleph_1} \lessdot P$".

*Proof.* Fix a productively c.c.c. separative poset $P$ of uniform density $\aleph_1$. As we have seen in the previous section, we have to introduce a dense avoidable subset to $P$. To begin with, we stratify the poset a little. We fix a sequence $\langle r_\alpha : \alpha \in \omega_1 \rangle$ so that

(1) $\{r_\alpha : \alpha \in \omega_1\} \subset P$ is dense
(2) $\forall \beta \in \alpha \in \omega_1 \, r_\beta \not\leqslant r_\alpha$

together with a closed unbounded set $C \subset \omega_1$ with all $\alpha \in C$ satisfying

$$\langle \{r_\beta : \beta \in \alpha^{*C}\}, \{r_\beta : \beta \in \alpha\}, \leqslant \rangle \prec \langle \{r_\beta : \beta \in \omega_1\}, \{r_\beta : \beta \in \alpha\}, \leqslant \rangle. \qquad \text{(P1)}$$

This is quite easy after iterating some Skolem hull arguments. Let us remind the reader that for an ordinal $\nu$ and a set $X$ of ordinals, we use the notation $\nu^{*X} = \min(X \setminus (\nu + 1))$. The desired forcing $Q$ will be defined as an iteration $Q_0 * \dot{C}_{\aleph_1}$ of two c.c.c. forcings.

DEFINITION 9. $Q_0$ is the set of all functions $q$ satisfying the following:

(D9.1) $\mathrm{dom}(q) \in [C]^{<\aleph_0}$, $\forall \alpha \in \mathrm{dom}(q)\ q(\alpha) = \langle p^q_\alpha, g^q_\alpha \rangle$; if no confusion is possible we drop the superscript $q$

(D9.2) $\forall \alpha \in \mathrm{dom}(q)\ p_\alpha \in \{r_\beta : \alpha \leqslant \beta < \alpha^{*C}\}$, $g_\alpha \subset \mathrm{dom}(q) \cap \alpha$

(D9.3) $\forall \alpha \in \mathrm{dom}(q)\ \exists p'_\alpha \leqslant p_\alpha\ \forall \beta \in (\alpha \cap \mathrm{dom}(q)) \setminus g_\alpha\ p'_\alpha \perp p_\beta$.

Order is by reverse extension. We set $\bar{q} = \{p \in P : \exists \alpha \in \mathrm{dom}(q)\ p = p_\alpha\}$.

*Explanation.* So this is a rather straightforward try at forcing a dense almost avoidable subset $D \subset P$ with finite conditions. For $q \in Q_0$, the set $\bar{q}$ is a finite piece of the future set $D$. In the generic extension, we will need to produce a trace of $p^q_\alpha$ in $D$. This is the role of $g^q_\alpha$: we shall set $\mathrm{tr}(p^q_\alpha) = \{p^q_\alpha\} \cup \{p^q_\beta : \beta \in g^q_\alpha\}$. Note that it is enough to produce traces for a dense set of conditions in $P$.

LEMMA 10. *$Q_0$ is c.c.c.*

*Proof.* Assume for contradiction that $\{q_\xi : \xi \in \omega_1\}$ is an antichain in $Q_0$; without loss of generality $|q_\xi| = n$ for all $\xi \in \omega_1$ for some fixed $n \in \omega$. Applying $\Delta$-system argument to $\{\mathrm{dom}(q_\xi) : \xi \in \omega_1\}$ and using the pigeonhole principle repeatedly we can obtain $a \in [\omega_1]^{<\aleph_0}$, $q \in Q_0$, $\mathrm{dom}(q) = a$ and a set $A \subset \omega_1$ of full cardinality so that for every $\xi < \nu$ in $A$ we have $q_\xi \cap q_\nu = q$ and $\max(\mathrm{dom}(q_\xi)) < \min(\mathrm{dom}(q_\nu) \setminus a)$. Note that now no confusion is possible with the notation $p_\alpha = p^{q_\xi}_\alpha$ if $\alpha \in \mathrm{dom}(q_\xi) \setminus a$ for some $\xi \in A$, since this $\xi$ is unique.

CLAIM 11. *For each $\xi \in A$ and each $\alpha \in \mathrm{dom}(q_\xi) \setminus a$, there is a condition $p'_\alpha \leqslant p_\alpha$ with the following properties*:

(C11.1) *$p'_\alpha \leqslant p_\alpha$ witnesses* (D9.3) *for $\alpha$ and $q_\xi$*

(C11.2) *for each $\delta \in \mathrm{dom}(q_{\xi^{*A}}) \setminus a$ we have $p'_\alpha \perp p_\delta$.*

*Proof.* Fix $\xi \in A$ and $\alpha \in \mathrm{dom}(q_\xi) \setminus a$ as required in the Claim. First we choose a condition $p'^0_\alpha \leqslant p_\alpha$ witnessing (D9.3) for $q_\xi$ and $\alpha$. By the elementarity properties of $C$ (P1) we can require that $p'^0_\alpha \in \{r_\beta : \beta \in \alpha^{*C}\}$. Now let $\delta_0 < \delta_1 < \cdots < \delta_i < \cdots$, $i < n - |a|$, be a list of all ordinals in $\mathrm{dom}(q_{\xi^{*A}}) \setminus a$. By induction on $i \leqslant n - |a|$ we build $p'^i \in P$ so that

(1) $p'^0 \geqslant p'^1 \geqslant \cdots$

(2) $p'^i \in \{r_\beta : \beta \in \delta_i^{*C}\}$

(3) $p'^{i+1} \perp p_{\delta_{i+1}}$ for $i < n$.

$p'^0 = p'^0_\alpha$ already satisfies all of (1)–(3). Given $p'^i$, $i < n - |a|$, we can choose $p'^{i+1} \leqslant p'^i$ as required since by (2) and the choice of $\langle r_\beta : \beta \in \omega_1 \rangle$ we have

$p'^i \not\leqslant p_{\delta_{i+1}}$. Note that $p_{\delta_{i+1}} \in \{r_\beta : \delta_{i+1} \leqslant \beta < \delta_{i+1}^{*C}\}$. To make (2) hold for $i+1$ we use (P3) again and find $p'^{i+1} \in \{r_\beta : \delta_{i+1} \leqslant \beta < \delta_{i+1}^{*C}\}$.

We set $p'_\alpha = p'^{n-|a|}$. Thus $p'_\alpha \leqslant p'^0_\alpha$ is still a witness of (D9.3) for $q_\xi$ and $\alpha$ and moreover $p'_\alpha \perp p_\delta$ for all $\delta \in \mathrm{dom}(q_{\xi^{*A}}) \setminus a$. ▮C11

Fix a sequence of $p'_\alpha$'s for $\alpha \in \mathrm{dom}(q_\xi) \setminus a$, $\xi \in A$ as in the claim. Let $B \subset A$ be a set of cardinality $\aleph_1$ such that for all $\xi \in B$ we have $\xi^{*B} > \xi^{*A}$. For each ordinal $\xi \in B$, let $\langle \alpha_{i,\xi} : i < n - |a| \rangle$ be an increasing list of all ordinals in $\mathrm{dom}(q_\alpha) \setminus a$. The collection $\{\langle p'_{\alpha_{i,\xi}} : i < n - |a| \rangle : \xi \in B\}$ is not an antichain in $P^{n-|a|}$ since the poset $P$ is productively c.c.c. and the collection in question is indexed by the uncountable set $B$. Thus we may pick ordinals $\xi < \nu$ in $B$ so that $p'_{\alpha_{i,\xi}}$ is compatible with $p'_{\alpha_{i,\nu}}$ for all $i < n - |a|$.

Claim 12. *The conditions $q_{\xi^{*A}}, q_\nu$ are compatible in $Q_0$.*

*Proof.* Set $\mu = \xi^{*A}$ and $q = q_\mu \cup q_\nu$. We need to verify that $q \in Q_0$; then $q$ is the needed lower bound of $q_\mu, q_\nu$, providing the claim. The only difficulty here is checking (D9.3) for $q$. We split into two cases: $\alpha \in \mathrm{dom}(q_\mu)$ and $\alpha \in \mathrm{dom}(q_\nu) \setminus a$. In the former case, $p'_\alpha$ witnessing (D9.3) for $g_\mu$ and $\alpha$ will do, since the only new values for $q$ as compared to $q_\mu$ are above $\alpha$. In the latter case, we find $i < n - |a|$ with $\alpha = \alpha_{i,\nu}$ and set $p''_\alpha$ to be a common lower bound of $p'_{\alpha_{i,\xi}}$ and $p'_{\alpha_{i,\nu}}$, which exists by the choice of $\xi < \nu$. We claim that $p''_\alpha \leqslant p_\alpha$ witnesses (D9.3) for $q$ and $\alpha$:

(1) Let $\beta \in (\mathrm{dom}(q_\nu) \cap \alpha) \setminus g^{q_\nu}_\alpha$. Then $p_\beta \perp p'_\alpha$ and as $p''_\alpha \leqslant p'_\alpha$ we have $p_\beta \perp p''_\alpha$ as well.

(2) Let $\beta \in (\mathrm{dom}(q_\nu) \setminus a$. Then by construction of $p'_{\alpha_{i,\xi}}$ (Claim 2.8) we have $p_\beta \perp p'_{\alpha_{i,\xi}}$ and as $p''_\alpha \leqslant p'_{\alpha_{i,\xi}}$ we conclude that $p_\beta \perp p''_\alpha$ again.

All relevant $\beta$'s from the second universal quantifier in (D9.3) for $q$ and $\alpha$ have been checked. The claim follows. ▮C12

By the choice of $B$ we have that $\xi^{*A} < \nu$ and so Claim 12 stands in direct contradiction with our assumption of $\{q_\xi : \xi \in \omega_1\}$ being an antichain. ▮L10

The forcing $Q_0$ as above is actually even productively c.c.c. since its definition from $\langle r_\alpha : \alpha \in \omega_1 \rangle$ and $C$ is absolute, and "productive c.c.c." of the poset $P$ is preserved under c.c.c. forcings.

Fix a generic filter $G \subset P$ and work in $V[G]$. We define a set $D \subset P$ by $D = \bigcup \{\bar{q} : q \in G\}$.

Lemma 13. *The set $D \subset P$ is dense almost avoidable in $P$.*

*Proof.* As for the density of $D$, work in $V$ for a moment. Let $q_0 \in Q_0$ and $p \in P$. Choose $\delta \in C$, $\delta > \max(\mathrm{dom}(q_0))$ so that there is $\alpha \in \delta$ with $r_\alpha \leqslant p$. By elementarity properties of $C$ (P1) there is $\beta$, $\delta \leqslant \beta < \delta^{*C}$ with

$r_\beta \leqslant r_\alpha$. We set $q_1 = q_0 \cup \{\langle \delta, \langle r_\beta, \mathrm{dom}(q_0)\rangle\rangle\}$. We have that $q_1 \in Q_0$, $q_1 \leqslant q_0$ and $q_1 \Vdash$ "there is an element of $\dot{D}$ below $\check{p}$". The density of the set $D \subset P$ follows by a genericity argument.

As for the almost avoidability, let $p \in P$. We shall produce a trace of $p$ in the set $D$ with the required properties. There is $q_0 \in G$ and $\alpha \in \mathrm{dom}(q_0)$ such that $p_\alpha^{q_0} \leqslant p$. We claim that the trace $\mathrm{tr}(p) = \{p_\alpha^{q_0}\} \cup \{p_\xi^{q_0} : \xi \in g_\alpha^{q_0}\}$ does the trick. To see this, choose $b \in [D]^{<\aleph_0}$ disjoint from $\mathrm{tr}(p)$. One can find $q_1 \leqslant q_0$, $q_1 \in G$ with $b \subset \overline{q_1}$. Notice that $p_\alpha^{q_0} = p_\alpha^{q_1}$ and $g_\alpha^{q_0} = g_\alpha^{q_1}$. Choose $p' \leqslant p_\alpha$ witnessing (D9.3) for $q_1, \alpha$. By elementarity properties of the set $C$ (P1) there is such $p'$ in $\{r_\beta : \beta \in \alpha^{*C}\}$. Now we repeat the process from Claim 11 to get $p'' \leqslant p'$ which is incompatible with all $p_\delta^{q_1}$, for $\delta \in \mathrm{dom}(q_1) \setminus (\alpha + 1)$; such $p''$ will be incompatible with all elements of $\bar{q}_1$ except those in $\mathrm{tr}(p)$. It follows that $p \geqslant p_\alpha^{q_0} \geqslant p'' \perp r$ for all $r$ in $b$. Therefore $p''$ witnesses the desired property of $\mathrm{tr}(p)$ with respect to $b$. ▮L13

Note that in $V[G]$, the poset $P$ still has uniform density $\aleph_1$. The reason is that this is expressible by the first-order statement "for no ordinals $\alpha, \beta < \omega_1$ the set $\{r_\xi : \xi < \alpha\}$ is dense below $r_\beta$", whose truth value is absolute between $V$ and $V[G]$. So we can use Lemma 3, finishing the proof of Theorem 8. The forcing we have been looking for is $Q_0 * \dot{C}_{\aleph_1} = Q_0 \times C_{\aleph_1}$. ▮T8

COROLLARY 14. *Under* $\mathrm{MA}_{\aleph_1}$ *the algebra* $\mathbb{C}(\aleph_1)$ *embeds into all c.c.c. algebras of uniform density* $\aleph_1$.

*Proof.* Assume $\mathrm{MA}_{\aleph_1}$ and choose a separative c.c.c. poset $P$ of uniform density $\aleph_1$. Without loss of generality the underlying set of $P$ is $\omega_1$. By [12] the poset $P$ is $\sigma$-centered and so by Theorem 3 there is a c.c.c. $Q$ with $Q \Vdash$ "$C_{\aleph_1} \lessdot \check{P}$". Choose a large regular cardinal $\kappa$ and a model $M \prec \langle H_\kappa, \in, P, Q\rangle$ with $\omega_1 \subset M$, $|M| = \aleph_1$. The poset $Q \cap M$ is c.c.c. and so we can use MA to get a filter $G \subset Q \cap M$ which meets all sets in $\{D \cap M : D \in M,\ D \subset Q \text{ dense}\}$, since by elementarity all of these sets are dense in $Q \cap M$. Let $i : M \to \bar{M}$ be the transitive collapse of $M$, $\bar{G} = i''G$. Then $i \restriction (P \cup C_{\aleph_1}) = \mathrm{id}$ and $\bar{G} \subset i(Q)$ is $\bar{M}$-generic. By our choice of $Q$ and the elementarity of $M$ we have $\bar{M}[\bar{G}] \models$ "$i(C_{\aleph_1}) = C_{\aleph_1} \lessdot i(P) = P$". The following claim completes the proof of the corollary.

CLAIM 15. *The statement* $C_{\aleph_1} \lessdot P$ *is upwards absolute*; *that is, if* $M \subset N$ *are two transitive models of rich fragments of set theory,* $\aleph_1^M = \aleph_1^N$, $P \in M$, *and* $M \models$ "$C_{\aleph_1} \lessdot P$" *then* $N$ *models the same statement.*

*Proof.* We use an alternative characterization of regular embedding: $C_{\aleph_1} \lessdot P$ if there is a function $e : C_{\aleph_1} \to \mathrm{RO}(P)^+$ preserving incompatibility such that for every $p \in P$ there is $h \in C_{\aleph_1}$ such that for any $k \in C_{\aleph_1}$ with

$k \leqslant h$ the value $e(k)$ is compatible with $p$ in $\mathrm{RO}(P)$. So we have such $e$ in $M$. Now $C^M_{\aleph_1} = C^N_{\aleph_1}$ and $\mathrm{RO}(P)^M \subset \mathrm{RO}(P)^N$ is dense; thus properties of $e$ survive in $N$, showing that $N \models C_{\aleph_1} \lessdot P$. ▮C15, Co14

## 3. THE GENERAL CASE

In the case of a general poset $P$, we cannot succeed with the scenario outlined in the previous section. The forcing $Q$ defined there has a dense subset of size $\aleph_1$, and that is just too simple to work:

LEMMA 16. *Let $P$ be a $\sigma$-closed poset and let $Q$ be a forcing of size $\aleph_1$ preserving $\aleph_1$. Then $Q \Vdash$ "$P$ is $\aleph_0$-distributive".*

*Proof.* Let the posets $P$, $Q$ be as in the assumption of the lemma. Let $Q \Vdash$ "$\langle \dot{D}_i : i < \omega \rangle$ is a system of open dense subsets of $P$". We fix a bookkeeping device, a bijection $e: Q \times \omega \to \omega_1$ and construct a descending sequence $\langle p_\alpha : \alpha \in \omega_1 \rangle$ of conditions in $P$ by induction as follows:

(1) $p_0 = 1_P$

(2) for $\alpha = \beta + 1$, where $\beta = e(q, i)$, we find a condition $q' \leqslant q$ in $Q$ and a condition $p \leqslant p_\beta$ in $P$ such that $q' \Vdash$ "$\check{p} \in \dot{D}_i$". We set $p_\alpha = p$.

(3) for $\alpha$ limit we set $p_\alpha \in P$ to be any lower bound of the chain $\langle p_\beta : \beta < \alpha \rangle$.

By the construction, $Q \Vdash$ "$\forall i < \omega\ \exists \alpha \in \omega_1\ \check{p}_\alpha \in \dot{D}_i$". Since the forcing $Q$ preserves $\aleph_1$, we have that $Q$ forces that "for every $i < \omega$, let $\alpha_i \in \omega_1$ be the least ordinal such that $p_{\alpha_i} \in \dot{D}_i$. Then $\dot{\alpha} = \sup_{i<\omega} \alpha_i$ is less than $\omega_1$. Therefore $p_{\dot{\alpha}} \in \bigcap_{i<\omega} \dot{D}_i$ and $\bigcap_{i<\omega} \dot{D}_i \neq 0$."

The previous argument relativized to any $Q \restriction q$ and $P \restriction p$, where $q \in Q$ and $p \in P$, gives the lemma. ▮L16

Under the Continuum Hypothesis there exists a $\sigma$-closed poset $P$ of size $\aleph_1$, and, as shown in Lemma 16, the forcing $Q$ as defined in the previous section cannot force $C_{\aleph_1} \lessdot P$. Tracing the problem, we conclude that $Q_0$, the first component of the forcing $Q$, collapses $\aleph_1$. However, we are still able to modify the forcing $Q_0$ so that we get

THEOREM 17. *For any separative partial order $P$ of uniform density $\aleph_1$ there is a proper, $\omega_2$-p.i.c. forcing $Q$ such that $Q \Vdash$ "$C_{\aleph_1} \lessdot \check{P}$". Moreover, if GCH holds then we can find such $Q$ of size $\aleph_2$.*

Here, $\omega_2$-p.i.c. is one of the strong forms of $\aleph_2$-c.c. introduced by the first author [8]. It will be instrumental for iteration purposes later.

The proof strategy will be the same as for Theorem 8. Given the poset $P$, we construct a mild forcing $Q_0$ which introduces a dense almost avoidable set $D \subset P$. Then by Lemma 3, the forcing $Q = Q_0 \times C_{\aleph_1}$ will be as desired. Now our $Q_0$ will be almost the same as in the previous section, only modified by side conditions in the spirit of [11]. Now every side conditions argument consists of three ingredients: a finite conditions construction, here supplied by the poset $Q_0$ from the previous section; matrices of models as in Definitions 18, 19; and a certain notion of transcendence as in Definition 20. We start with disclosing the matrices of models.

Let $\kappa$ be an uncountable regular cardinal and fix $\ll$, a well-ordering of $H_\kappa$. Also, choose one distinguished element $\Delta$ of $H_\kappa$.

Definition 18. We say that $\mathfrak{m}$ is a matrix of models if the following conditions are satisfied:

(D18.1) $\mathfrak{m}$ is a function, $\mathrm{dom}(\mathfrak{m}) \in [\omega_1]^{<\aleph_0}$ and for each $\alpha \in \mathrm{dom}(\mathfrak{m})$ the value $\mathfrak{m}(\alpha)$ is a finite set of isomorphic countable submodels of $\langle H_\kappa, \in, \ll, \Delta \rangle$

(D18.2) for each $\alpha < \beta$, both in $\mathrm{dom}(\mathfrak{m})$, we have $\forall N \in \mathfrak{m}(\alpha)\ \exists M \in \mathfrak{m}(\beta)\ N \in M$

(D18.3) for each $\alpha < \beta$, both in $\mathrm{dom}(\mathfrak{m})$, we have $\forall M \in \mathfrak{m}(\beta)\ \exists N \in \mathfrak{m}(\alpha)\ N \in M$.

We consider the set $\mathfrak{M}$ of all matrices of models to be ordered by $\geqslant$, the reverse coordinatewise extension. That is, $\mathfrak{n} \geqslant \mathfrak{m}$ if $\mathrm{dom}(\mathfrak{n}) \subset \mathrm{dom}(\mathfrak{m})$ and for each $\alpha \in \mathrm{dom}(\mathfrak{n})$ we have $\mathfrak{n}(\alpha) \subset \mathfrak{m}(\alpha)$.

The poset $\mathfrak{M}$ is a subset of $H_\kappa$ and it is not necessarily separative. Its definition has three parameters: the cardinal $\kappa$, the well-ordering $\ll$, and the distinguished element $\Delta$. The following definition is motivated by some technical considerations. For a detailed treatment, see [14].

Definition 19. Let $M \prec \langle H_\kappa, \in, \ll, \Delta \rangle$ be a countable model and let $\mathfrak{m} \in \mathfrak{M}$ be such that $M \in \mathfrak{m}(M \cap \omega_1)$. Then we define the following notions:

(D19.1) $\mathrm{pr}_M(\mathfrak{m})$, the projection of $\mathfrak{m}$ into $M \cap \mathfrak{M}$. This is the function defined by $\mathrm{dom}(\mathfrak{n}) = \mathrm{dom}(\mathfrak{m}) \cap M$ and $N \in \mathfrak{n}(\alpha)$ iff there are models $N = N_0 \in N_1 \in \cdots \in N_k = M$ such that $N_i \in \mathfrak{m}(\alpha_i)$, where $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_k = M \cap \omega_1$ is an increasing list of all ordinals in $\mathrm{dom}(\mathfrak{m})$ between $\alpha$ and $M \cap \omega_1$.

(D19.2) A matrix $\mathfrak{m}$ is said to be $M$-full if for each $\alpha \in M \cap \mathrm{dom}(\mathfrak{m})$ and each $N \in \mathfrak{m}(\alpha) \setminus \mathrm{pr}_M(\mathfrak{m})(\alpha)$ there is $\bar{M} \in \mathfrak{m}(M \cap \omega_1)$ such that $N \in \bar{M}$ and $i(N) \in \mathrm{pr}_M(\mathfrak{m})(\alpha)$, where $i\colon \bar{M} \to M$ is the unique isomorphism of $\bar{M}$ and $M$.

Obviously, $\mathrm{pr}_M(\mathfrak{m}) \in \mathfrak{M} \cap M$. The idea behind this definition is that $\mathrm{pr}_M(\mathfrak{m})$ should be a matrix in $M$ which grasps all the information about $\mathfrak{m}$ understandable from within $M$.

Now here is the promised notion of transcendence over a countable submodel $M \prec H_\kappa$. Parallel definitions appeared before in Todorčević's work.

DEFINITION 20. Let $P$ be a separative partially ordered set. A set $R \subset P$ is small if for every $a \in [R]^{\aleph_0}$ there is $b \in [P]^{<\aleph_0}$ such that for every $r \in a$ there is $p \in b$ with $r \geqslant p$.

Some elementary observations: principal filters in the poset $P$ are small; and a small set cannot contain an infinite antichain. A good example of a small set is a cofinal branch in a tree of height $\omega_1$. Obviously, the set of all small subsets of $P$ is an ideal. The idea behind Definition 20 is that if the poset $P$ is complicated enough, the small sets cannot capture the structure of $P$. This is recorded in the following:

LEMMA 21. *Assume that a poset $P$ has no countable locally dense subsets and let $\mathfrak{I}$ denote the $\sigma$-ideal on $P$ generated by the small subsets of $P$. Then for every $R \in \mathfrak{I}$ the set $P \setminus R \subset P$ is dense; in other words, for every $p \in P$ the set $P \restriction p$ is $\mathfrak{I}$-positive.*

*Remark.* Say that a condition $p \in P$ is "transcendental" over a countable model $M \prec H_\kappa$ if $p \notin \bigcup \{R \in M\colon R$ is a small subset of $P\}$. Then the lemma says that there is a dense set of conditions in the poset $P$ "transcendental" over $M$, provided that $P$ has uniform density $\aleph_1$.

*Proof.* The proof is by contradiction. Assume that $p \in P$, $R \in \mathfrak{I}$, $R = \bigcup_{i \in \omega} R_i$, and $P \restriction p \subset R$, where the sets $R_i \subset P$ are small. To simplify the notation we assume that $p = 1$. There are two cases:

(1) There is a c.c.c. forcing $Q$ such that $Q \Vdash$ "$\check{P}$ is not c.c.c.". Choose such $Q$ and a $Q$-name $\dot{A}$ such that $\Vdash$ "$\dot{A} \subset P$ is an uncountable antichain". As $Q$ preserves $\aleph_1$, we can find $q \in Q$, $i \in \omega$ so that $q \Vdash$ "$\dot{A} \cap \check{R}_i$ is uncountable". So $R_i$ contains infinite antichains in a generic extension; therefore it must contain such an antichain in the ground model (the tree of finite sequences of pairwise incompatible elements of $R_i$ is ill-founded). So the set $R_i$ is not small, contradiction.

(2) Otherwise. In particular, $P$ is productively c.c.c. We fix a large enough regular cardinal $\kappa$ and build a sequence $\langle\langle M_\alpha, p_\alpha\rangle\colon \alpha \in \omega_1\rangle$ so that $M_\alpha \prec H_\kappa$ is a countable model, $p_\alpha \in P$, $P$, $R_i$, $\{\langle M_\beta, p_\beta\rangle\colon \beta \in \alpha\} \in M_\alpha$ and for no $r \in P \cap M_\alpha$ we have $r \leqslant p_\alpha$. This is possible as $P \cap M_\alpha \subset P$ is not dense by our assumption on $P$. Take $i \in \omega$ such that $A = \{\alpha \in \omega_1 \colon p_\alpha \in R_i\}$

is uncountable. Since $R_i \subset P$ is small, for each $\alpha \in \omega_1$ we can find a finite collection $\{r_{\alpha,j}: j < n_\alpha\} \subset P$ so that

$$\forall \beta \in A \cap \alpha \qquad \exists j < n_\alpha \qquad p_\beta \geqslant r_{\alpha,j}. \tag{P2}$$

By elementarity we may and will assume that $\{r_{\alpha,j}: j < n_\alpha\} \subset M_\alpha$. From the construction of $p_\alpha$'s we can then conclude that $P_{\alpha^{*A}} \not\geqslant r_{\alpha,j}$ for $\alpha \in \omega_1$, $j < n_\alpha$. By separativity we can strengthen all $r_{\alpha,j}$ so that they are incompatible with $p_{\alpha^{*A}}$. This preserves the property (P2) of the system $\{r_{\alpha,j}: j < n_\alpha,\ \alpha \in \omega_1\}$ even though now $r_{\alpha,j}$ may be outside $M_\alpha$. Fix $n \in \omega$ and an uncountable set $B \subset \omega_1$ so that for all $\alpha \in B$ we have $n_\alpha = n$ and $\alpha^{*B} > \alpha^{*A}$. Remembering that the poset $P$ is assumed to be productively c.c.c., the collection $\{\langle r_{\alpha,j}: j < n\rangle: \alpha \in B\} \subset P^n$ is not an antichain in $P^n$ and we can choose $\xi < \nu$ in $B$ with $r_{\xi,j}, r_{\nu,j}$ compatible for all $j < n$. By (P2) there is $j < n$ so that $p_{\xi^{*A}} \geqslant r_{\nu,j}$. However, $r_{\xi,j}$ is both incompatible with $p_{\xi^{*A}}$ and compatible with $r_{\nu,j}$, contradiction. ▮L21

Finally, we are ready to define the forcing $Q_0$ introducing a dense almost avoidable subset to a given poset $P$. Fix a poset $P$ of uniform density $\aleph_1$. Without loss of generality we may assume that the universe of $P$ is $\omega_1$. Furthermore, set $\kappa = \omega_2$, $\Delta = P$ and fix a wellordering $\ll$ of $H_\kappa$. Below, the set $\mathfrak{M}$ of matrices of models will be computed using these parameters.

Definition 22. A forcing notion $Q_0$ is defined as the set of all pairs $\langle q, \mathfrak{m}\rangle$ for which

(D22.1) $q$ and $\mathfrak{m}$ are finite functions with the same domain, which is a finite subset of $\omega_1$

(D22.2) for every $\alpha \in \mathrm{dom}(q)$ the value $q(\alpha)$ is a pair $\langle p^q_\alpha, g^q_\alpha\rangle$ where, if no confusion can result, we can drop the superscript $q$

(D22.3) $\forall \alpha \in \mathrm{dom}(q)\ p_\alpha \in P$ and $g_\alpha \subset \mathrm{dom}(q) \cap \alpha$

(D22.4) $\forall \alpha \in \mathrm{dom}(q)\ \exists p'_\alpha \leqslant p_\alpha\ \forall \beta \in (\mathrm{dom}(q) \cap \alpha) \setminus g_\alpha\ p'_\alpha \perp p_\beta$

(D22.5) $\mathfrak{m}$ is a matrix of models, i.e., $\mathfrak{m} \in \mathfrak{M}$

(D22.6) for every $\alpha < \beta$ both in $\mathrm{dom}(q) = \mathrm{dom}(\mathfrak{m})$ and for every $N \in \mathfrak{m}(\beta)$ we have $p_\alpha \in N$

(D22.7) for every $\alpha \in \mathrm{dom}(q)$, for every $N \in \mathfrak{m}(\alpha)$ and for every small set $R \subset P$ in $N$, we have $p_\alpha \notin R$.

The order is defined by $\langle q_0, \mathfrak{m}_0\rangle \geqslant \langle q_1, \mathfrak{m}_1\rangle$ if $q_0 \subset q_1$ and $\mathfrak{m}_0 \geqslant_{\mathfrak{M}} \mathfrak{m}_1$. For a condition $\langle q, \mathfrak{m}\rangle \in Q_0$, we set $\bar{q} = \{p \in P: \exists \alpha \in \mathrm{dom}(q)\ p = p^q_\alpha\}$.

*Explanation.* This may look complicated but in fact it is not. In a condition $\langle q, \mathfrak{m}\rangle$, the $q$ part is exactly like an element of $Q_0$ in the previous section, except that it ignores any stratification of the poset $P$. The properties (D22.2, 3, 4) describe just this fact. The matrix $\mathfrak{m}$ is just the control device described in Definition 18. Here it is tied to $q$ by (D22.6, 7). The transcendence requirement (D22.7) is the main technical point in the construction.

As it was the case in the previous section, the forcing $Q_0$ serves to introduce a dense almost avoidable subset $D$ to $P$. The set $\bar{q}$ is a finite piece of the future set $D$, and the trace of $p_\alpha$ will be obtained as $\mathrm{tr}(p^q_\alpha) = \{p^q_\alpha\} \cup \{p^q_\beta : \beta \in g^q_\alpha\}$.

We start with a simple preliminary lemma.

Lemma 23. *If $\langle q, \mathfrak{m}\rangle \in Q_0$ and $M \prec \langle H_\kappa, \in, \ll, P\rangle$ are such that $M \in \mathfrak{m}(M \cap \omega_1)$ then there is a condition $\langle q, \mathfrak{n}\rangle \in Q_0$ such that $\langle q, \mathfrak{n}\rangle \leqslant \langle q, \mathfrak{m}\rangle$ and $\mathfrak{n}$ is $M$-full.*

*Proof.* Fix a condition $\langle q, \mathfrak{m}\rangle \in Q_0$. The $M$-full matrix $\mathfrak{n} \leqslant \mathfrak{m}$ will be built so that $\mathrm{dom}(\mathfrak{m}) = \mathrm{dom}(\mathfrak{n})$. We shall start with $\mathfrak{m}$; then we gradually add some new models to the values $\mathfrak{m}(\alpha)$, $\alpha \in \mathrm{dom}(\mathfrak{m}) \cap M$, preserving properties (D18.1, 2, 3), (D22.6, 7) at each step. After finitely many steps, an $M$-full system $\mathfrak{n} \leqslant \mathfrak{m}$ will emerge.

Let $\alpha_0 < \alpha_1 < \cdots < \alpha_k = M \cap \omega_1$ be an increasing list of all ordinals in $\mathrm{dom}(\mathfrak{m})$ below $M \cap \omega_1$ inclusive. Let $N \in \mathfrak{m}(\alpha_j) \setminus \mathrm{pr}_M(\mathfrak{m})(\alpha_j)$ be a model, for some $j < k$. Then by using (D18.3) repeatedly, we can find an $\in$-chain $N_0 \in N_1 \in \cdots \in N_j = N \in N_{j+1} \in \cdots \in N_k$ such that $N_l \in \mathfrak{m}(\alpha_l)$, all $l \leqslant k$. Let $i: N_k \to M$ be the isomorphism. We throw all models $i(N_l)$ into $\mathfrak{n}(\alpha_l)$, for $l < k$. It is readily checked that this operation preserves properties (D18.1, 2, 3), (D22.6, 7); for example, $i(N_l)$ is isomorphic to $N_l$ via $i \restriction N_l$ and if $l_1 < l_2$ then $i(N_{l_1}) \in i(N_{l_2})$. We repeat this procedure for all models $N \in \mathfrak{m}(\alpha_j) \setminus \mathrm{pr}_M(\mathfrak{m})(\alpha_j)$. The reader can check that the resulting matrix $\mathfrak{n}$ is as required. ▮L23

Now we are in a position to demonstrate that the forcing $Q_0$ is as mild as needed for Theorem 17. The following proof is much like some arguments in [11].

Lemma 24. *$Q_0$ is proper.*

*Proof.* Choose a large regular cardinal $\lambda$, a condition $\langle q_0, \mathfrak{m}_0\rangle \in Q_0$, and a countable submodel $M \prec H_\lambda$ with $q_0, \mathfrak{m}_0, \kappa, \ll, P$ in $M$. We shall produce a master condition $\langle q_1, \mathfrak{m}_1\rangle \leqslant \langle q_0, \mathfrak{m}_0\rangle$ for the model $M$. Find $p \in P \setminus \bigcup \{R \in M : R \subset P \text{ small}\}$; there is a dense set in $P$ of these due to Lemma 21. We define $q_1 = q_0 \cup \{\langle M \cap \omega_1, \langle p, \mathrm{dom}(q_0)\rangle\rangle\}$ and $\mathfrak{m}_1 = \mathfrak{m}_0 \cup \{\langle M \cap \omega_1, \{M \cap H_\kappa\}\rangle\}$.

CLAIM 25. $\langle q_1, \mathfrak{m}_1\rangle \in Q_0$, $\langle q_1, \mathfrak{m}_1\rangle \leqslant \langle q_0, \mathfrak{m}_0\rangle$.

We must verify that $\langle q_1, \mathfrak{m}_1\rangle$ is a master condition for the model $M$. So for any maximal antichain $A$ of $Q_0$ in $M$, the set $A \cap M$ should be predense below $\langle q_1, \mathfrak{m}_1\rangle$. To prove this, let $A \in M$ be a maximal antichain of $Q_0$ and choose a condition $\langle q_2, \mathfrak{m}_2\rangle \leqslant \langle q_1, \mathfrak{m}_1\rangle$. By eventually strengthening the condition, we can assume that there is an element $x$ of $A$ above it and $\mathfrak{m}_2$ is $M \cap H_\kappa$-full (Lemma 23). We shall show that the element $x$ belongs actually to $A \cap M$, finishing the proof of properness. We define a condition $\langle q_3, \mathfrak{m}_3\rangle \geqslant \langle q_2, \mathfrak{m}_2\rangle$, a sort of projection of $\langle q_2, \mathfrak{m}_2\rangle$ to the model $M$. So, let $q_3 = q_2 \restriction M$ and $\mathfrak{m}_3 = \mathrm{pr}_{M \cap H_\kappa} \mathfrak{m}_2$.

CLAIM 26. $\langle q_3, \mathfrak{m}_3\rangle \in M \cap Q_0$, $\langle q_2, \mathfrak{m}_2\rangle \leqslant \langle q_3, \mathfrak{m}_3\rangle$.

The task now is to carefully extend the condition $\langle q_3, \mathfrak{m}_3\rangle$ within $M$ to $\langle q_4, \mathfrak{m}_4\rangle$ which has an element of $A$ above it and is still compatible with $\langle q_2, \mathfrak{m}_2\rangle$. Let $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ be an increasing list of $\mathrm{dom}(q_2) \setminus \mathrm{dom}(q_3)$; thus $\alpha_0 = M \cap \omega_1$. For $0 \leqslant i \leqslant n$ we put $p_{\alpha_i} = p^{q_2}_{\alpha_i}$.

DEFINITION 27. For all $x \in [P]^{<\aleph_0}$ simultaneously by induction on $i \in \omega$ we define sets $x^{(i)} \subset P$:

(D27.1) $x^{(0)} = \{p \in P\colon \exists \langle q_4, \mathfrak{m}_4\rangle \leqslant \langle q_3, \mathfrak{m}_3\rangle\ \bar{q}_4 = \bar{q}_3 \cup x \cup \{p\}$ and there is an element of $A$ above $\langle q_4, \mathfrak{m}_4\rangle\}$.

(D27.2) $x^{(i+1)} = \{p \in P\colon (x \cup \{p\})^{(i)}$ is not small$\}$.

Note that the collection $\{x^{(i)}\colon x \in [P]^{<\aleph_0}, i \in \omega\}$ is in $M \cap H_\kappa$.

CLAIM 28. *The set* $0^{(n)}$ *is not small in the poset* $P$.

*Proof.* By contradiction. Assume the set is small. By induction on $0 \leqslant i \leqslant n$ we prove that

(1) $Z_i = \{p_{\alpha_0}, p_{\alpha_1}, ..., p_{\alpha_j}, ..., j < i\}^{(n-i)}$ is small in $P$

(2) $p_{\alpha_i} \notin \{p_{\alpha_0}, p_{\alpha_1}, ..., p_{\alpha_j}, ..., j < i\}^{(n-i)}$,

which wil be a contradiction for $i = n$, as $p_{\alpha_n} \in \{p_{\alpha_0}, p_{\alpha_1}, ..., p_{\alpha_j}, ..., j < n\}^{(0)}$, as witnessed by $\langle q_2, \mathfrak{m}_2\rangle$. Now for $i = 0$ we have $0^{(n)}$ is small by the assumption and $p_{\alpha_0} = p \notin 0^{(n)} \in M \cap H_\kappa$ by the choice of $p$. Now given (1) and (2) for $i < n$, by (D27.2) we immediately get that the set $Z_{i+1}$ is small, i.e., (1) for $i+1$. Now by (D18.2) for the system $\mathfrak{m}_2$ we find a model $N \in \mathfrak{m}_2(\alpha_{i+1})$ with $M \cap H_\kappa \in N$. Then $Z_{i+1} \in N$ and by (D22.7) $p_{\alpha_{i+1}} \notin Z_{i+1}$, i.e., (2) for $i+1$. This completes the argument. ▮C28

We proceed with the construction of $\langle q_4, \mathfrak{m}_4\rangle$. For $0 \leqslant j \leqslant n$ fix $p'_{\alpha_j} \leqslant p_{\alpha_j}$ witnessing (D22.4) for $q_2$. By induction on $0 \leqslant i \leqslant n$ we build $r_i, p'^{i}_{\alpha_j}$, $0 \leqslant j \leqslant n$ so that

(1) $r_i \in P \cap M$, $p_{\alpha_j} \geqslant p'_{\alpha_j} \geqslant p'^{0}_{\alpha_j} \geqslant p'^{1}_{\alpha_j} \geqslant \cdots \geqslant p'^{i}_{\alpha_j}$ is a decreasing sequence of elements of $P$ for all $0 \leqslant j \leqslant n$

(2) Let $Z_i = \{r_0, r_1, ..., r_k, ..., k < i\}^{(n-i)}$. Then $Z_i \subset P$ is not small and $r_i \in Z_i \cap M$

(3) $p'^{i}_{\alpha_j} \perp r_i$ for all $0 \leqslant j \leqslant n$.

To construct $r_0$ recall that the set $0^{(n)}$ is not small. Thus there is $a \in [0^{(n)}]^{\aleph_0} \cap M$ witnessing that. We have $a \subset M \cap H_\kappa$ and as $\{p'_{\alpha_j}: 0 \leqslant j \leqslant n\}$ does not bound all elements of $a$ we can choose $r_0 \in a$ with $r_0 \ngeqslant p'_{\alpha_j}$ for all $0 \leqslant j \leqslant n$. By the separativity of the poset $P$ there are $p'_{\alpha_j} \geqslant p'^{0}_{\alpha_j}$ with $r_0 \perp p'^{0}_{\alpha_j}$. By (D27.2) the set $\{r_0\}^{(n-1)}$ is not small. The induction step from $i < n$ to $i+1$ is carried out similarly with $p'^{i}_{\alpha_j}$ in place of $p'_{\alpha_j}$ and $Z_i$ in place of $0^{(n)}$.

The induction have been carried out up to $n$ we have $r_n \in \{r_0, r_1, ..., r_k, ..., k < n\}^{(0)}$ and so by (D27.1) applied in $M$ there exists a condition $\langle q_4, \mathfrak{m}_4\rangle \in M$ such that $\langle q_4, \mathfrak{m}_4\rangle \geqslant \langle q_3, \mathfrak{m}_3\rangle$, $\bar{q}_4 = \bar{q}_3 \cup \{r_i: 0 \leqslant i \leqslant n\}$ and there is an element of $A \cap M$ above it.

Claim 29. *The conditions $\langle q_2, \mathfrak{m}_2\rangle$ and $\langle q_4, \mathfrak{m}_4\rangle$ are compatible.*

*Proof.* We shall produce a lower bound of $\langle q_5, \mathfrak{m}_5\rangle \in Q_0$ of the two conditions. First, we define $q_5 = q_2 \cup q_4$. It is easy to see that $q_5$ is a function satisfying (D22.2, 3) and such that $q_5 \restriction M = q_4$. We must check the property (D22.4). There are two cases:

(1) $\alpha \in \mathrm{dom}(q_5) \cap M$ (i.e. $\alpha \in \mathrm{dom}(q_4)$). In this case, the element $p'_\alpha \leqslant p_\alpha$ witnessing (D22.4) for $q_4$ will do even for $q_5$, since $q_5 \restriction (\alpha+1) = q_4 \restriction (\alpha+1)$.

(2) $\alpha \in \mathrm{dom}(q_2) \backslash M$ (i.e. $\alpha \geqslant M \cap \omega_1$ and $\alpha \in \mathrm{dom}(q_2)$). Then $\alpha = \alpha_j$ for some $j \leqslant n$. We claim that $p''_\alpha = p'_{\alpha_j} \leqslant p_\alpha$ witnesses (D22.4) for $\alpha$ and $q_5$. To see this, choose an ordinal $\beta$ in $(\mathrm{dom}(q_5) \cap \alpha) \backslash g_\alpha$. Only two things can happen here. Either $\beta \in \mathrm{dom}(q_4)$. In this case already $p'_{\alpha_j} \leqslant p_\alpha$ as fixed above is incompatible with $p_\beta$; since $p''_\alpha \leqslant p'_{\alpha_j}$ we then have $p''_\alpha \perp p_\beta$ as well. Or, $\beta \in \mathrm{dom}(q_4) \backslash \mathrm{dom}(q_2)$. Then by the above construction, $p_\beta = r_i$ for some $i \leqslant n$ and consequently $p'^{i}_{\alpha_j} \leqslant p_\alpha$ is incompatible with $p_\beta = r_i$. Since $p''_\alpha \leqslant p'^{i}_{\alpha_j}$ we have $p''_\alpha \perp p_\beta$ as well. All relevant $\beta$'s in the second universal quantifier in (D22.4) have been checked and (D22.4) follows.

We still have to define the matrix $\mathfrak{m}_5$. Here is the place where we use the $M \cap H_\kappa$-fullness of the matrix $\mathfrak{m}_2$. We shall have $\mathrm{dom}(\mathfrak{m}_5) = \mathrm{dom}(\mathfrak{m}_2) \cup \mathrm{dom}(\mathfrak{m}_4)$; the description of the values $\mathfrak{m}_5(\alpha)$ splits into two cases:

(1) if $\alpha \in \mathrm{dom}(\mathfrak{m}_5) \setminus M$ (i.e. $\alpha \geqslant M \cap \omega_1$ and $\alpha \in \mathrm{dom}(\mathfrak{m}_2)$) then $\mathfrak{m}_5(\alpha) = \mathfrak{m}_2(\alpha)$

(2) if $\alpha \in \mathrm{dom}(\mathfrak{m}_5) \cap M$ (i.e. $\alpha \in \mathrm{dom}(\mathfrak{m}_4)$) then $\mathfrak{m}_5(\alpha) = \mathfrak{m}_4(\alpha) \cup \{i(N): N \in \mathfrak{m}_2(\alpha)$ and $i: M \to \bar{M}$ is an isomorphism with $\bar{M} \in \mathfrak{m}_2(M \cap \omega_1)\}$.

The reader can verify that the function $\mathfrak{m}_5$ is in $\mathfrak{M}$ and satisfies the conditions (D22.6, 7). The $M \cap H_\kappa$-fullness of the matrix $\mathfrak{m}_2$ together with our construction of $\mathfrak{m}_5$ ensures that $\mathfrak{m}_5 \leqslant \mathfrak{m}_2$, $\mathfrak{m}_4$ as desired. ▮C29

Now $A \subset Q_0$ is an antichain and its elements above the conditions $\langle q_2, \mathfrak{m}_2 \rangle$ and $\langle q_4, \mathfrak{m}_4 \rangle$ must be identical. However, the unique element of $A$ above $\langle q_4, \mathfrak{m}_4 \rangle \in M$ is in $M$ by elementarity, and we have finished the proof of Lemma 24. ▮L24

Lemma 30. *$Q_0$ has $\omega_2$-p.i.c.*

We remind the reader what this is all about.

Definition 31 [7, Chap. VIII, Sect. 2]. A forcing $Q$ has $\omega_2$-p.i.c. (properness isomorphism condition) if for all regular cardinals $\lambda$ and every $\Delta \in H_\lambda$, *for every* $\gamma < \delta < \omega_2, q_0, h, M_\gamma, M_\delta$ countable submodels of $\langle H_\lambda, \in, \Delta \rangle$ with $\gamma \in M_\gamma$, $\delta \in M_\delta$, $Q \in M_\gamma \cap M_\delta$, $M_\gamma \cap \gamma = M_\delta \cap \delta$, $M_\gamma \cap \omega_2 \subset \delta$, $q \in Q \cap M_\gamma$, $i: M_\gamma \to M_\delta$ an isomorphism which is identity on $M_\gamma \cap M_\delta$ *there is* $q_1 \leqslant q_0$, a master condition for $M_\gamma$ such that $q_1 \Vdash$ "$i''(\check{M}_\gamma \cap \dot{G}) = \check{M}_\delta \cap \dot{G}$". A condition $q_1$ as above is called a symmetric master condition for $M_\gamma, M_\delta$.

Intuitively, we want the isomorphism $i$ to extend in the generic extension to an isomorphism $\hat{\imath}: M_\gamma[G] \to M_\delta[G]$ in the most natural way: we want to set $\hat{\imath}(\tau/G) = i(\tau)/G$. The condition $q_1$ forces that this will indeed be an isomorphism. Perhaps at least one rather trivial example is in order: any proper forcing $Q$ of cardinality $\aleph_1$ has $\omega_2$-p.i.c. This is because in any two models as in the Definition we obtain $i \restriction Q \cap M_\gamma = id$. Therefore, every master condition $q_1$ for the model $M_\gamma$ will have the requred "symmetricity" property.

The point in such a strange property of the forcing $Q$ is that granted the Continuum Hypothesis, an $\omega_2$-p.i.c. forcing $Q$ has $\aleph_2$-c.c. and preserves the Continuum Hypothesis. In fact, this is even true for short iterations of $\omega_2$-p.i.c. forcings:

*Fact* 32 [7, Ch. VIII, §2]. Assume CH. If $\langle P_\alpha: \alpha \leqslant \omega_2, \dot{Q}_\alpha: \alpha < \omega_2 \rangle$ is a countable support iteration of forcings such that for each $\alpha < \omega_2$ we have $P_\alpha \Vdash$ "$\dot{Q}_\alpha$ has $\omega_2$-p.i.c." then

(F32.1) $\forall \alpha \in \omega_2 P_\alpha$ has $\omega_2$-p.i.c. and $P_\alpha \Vdash$ "$CH$"

(F32.2) $P_{\omega_2}$ has $\aleph_2$-c.c.

*Proof of Lemma* 30. We show a little more general statement than that of Definition 31. Choose a large regular cardinal $\lambda$, a condition $\langle q_0, \mathfrak{m}_0\rangle \in Q_0$ and two isomorphic countable submodels $M_0, M_1 \prec \langle H_\lambda, \in, P\rangle$ such that $Q_0, \ll, \kappa$ are in both of them and $\langle q_0, \mathfrak{m}_0\rangle \in M_0$. Let $i\colon M_0 \to M_1$ be an isomorphism, $i(P)=P$, $i(\ll)=\ll$. We shall produce the desired symmetric master condition $\langle q_1, \mathfrak{m}_1\rangle \leqslant \langle q_0, \mathfrak{m}_0\rangle$ for the two models.

First, we pick $p \in P$ which does not belong to any small subset of $P$ which is in $M_0 \cup M_1$. There is a dense set of these due to Lemma 21. Now we set $q_1 = q_0 \cup \{M_0 \cap \omega_1, \ \langle p, \mathrm{dom}(q_0)\rangle\rangle\}$. We construct $\mathfrak{m}_1$ as the unique function such that $\mathrm{dom}(\mathfrak{m}_1) = \mathrm{dom}(\mathfrak{m}_0) \cup \{M_0 \cap \omega_1\}$ and the values are defined as follows: for $\alpha \in \mathrm{dom}(\mathfrak{m}_0)$ we set $\mathfrak{m}_1(\alpha) = \mathfrak{m}_0(\alpha) \cup i(\mathfrak{m}_0(\alpha))$ and for $\alpha = M_0 \cap \omega_1$ we set $\mathfrak{m}_1(\alpha) = \{M_0 \cap H_\kappa, \ M_1 \cap H_\kappa\}$. The following is immediate:

CLAIM 33. $\langle q_1, \mathfrak{m}_1\rangle \in Q_0$, $\langle q_1, \mathfrak{m}_1\rangle \leqslant \langle q_0, \mathfrak{m}_0\rangle$.

We claim that $\langle q_1, \mathfrak{m}_1\rangle$ is the desired symmetric master condition for $M_0, M_1$. Obviously, $\langle q_1, \mathfrak{m}_1\rangle$ is a master condition for $M_0$ since it is stronger than the master condition described in Lemma 24. We must verify that $\langle q_1, \mathfrak{m}_1\rangle \Vdash$ "$i''(\check{M}_0 \cap \dot{G}) = \check{M}_1 \cap \dot{G}$". We prove that $\langle q_1, \mathfrak{m}_1\rangle \Vdash$ "$i''(\check{M}_0 \cap \dot{G}) \subset \check{M}_1 \cap \dot{G}$"; the proof of the opposite inclusion is parallel. So let $x \in M_0 \cap Q_0$ and let $\langle q_2, \mathfrak{m}_2\rangle \leqslant \langle q_1, \mathfrak{m}_1\rangle$ be a condition such that $\langle q_2, \mathfrak{m}_2\rangle \Vdash$ "$\check{x} \in \dot{G}$". We shall obtain a condition $\langle q_2, \mathfrak{m}_3\rangle$ such that $\langle q_3, \mathfrak{m}_3\rangle \Vdash$ "$i(\check{x}) \in \dot{G}$". By a genericity argument, this will complete the proof. Now by eventually strengthening the condition $\langle q_2, \mathfrak{m}_2\rangle$ we can assume that $\langle q_2, \mathfrak{m}_2\rangle \leqslant x$.

CLAIM 34. $M_0 \models$ "*there is a matrix* $\mathfrak{n}$ *such that* $\mathfrak{n} \leqslant \mathrm{pr}_{M \cap H_\kappa} \mathfrak{m}_2$ *and* $\langle q_2 \restriction M_0, \mathfrak{n}\rangle \leqslant x$".

*Proof.* Notice first that the parameters of the formula–the system $\mathrm{pr}_{M \cap H_\kappa} \mathfrak{m}_2$, the condition $x$ and the finite function $q_2 \restriction M_0$-are all in the model $M_0$. The claim follows from the elementarity of $M_0$, since the formula is witnessed in $H_\lambda$ by the matrix $\mathfrak{m}_2 \restriction M_0$. ▮C34

Now let a system $\mathfrak{n}$ be as in the Claim. We define a matrix $\mathfrak{m}_3$ by $\mathrm{dom}(\mathfrak{m}_3) = \mathrm{dom}(\mathfrak{m}_2)$, for $\alpha \in \mathrm{dom}(\mathfrak{m}_2) \cap M_0$ we set $\mathfrak{m}_3(\alpha) = \mathfrak{m}_2(\alpha) \cup i(\mathfrak{n}(\alpha))$ and for $\alpha \in \mathrm{dom}(\mathfrak{m}_2) \setminus M_0$ we set $\mathfrak{m}_3(\alpha) = \mathfrak{m}_2(\alpha)$.

CLAIM 35. $\mathfrak{m}_3$ *is a matrix of models and* $\langle q_2, \mathfrak{m}_3\rangle \in Q_0$.

We claim that $\langle q_2, \mathfrak{m}_3\rangle$ is the desired condition. First, obviously $\langle q_2, \mathfrak{m}_3\rangle \leqslant \langle q_2, \mathfrak{m}_2\rangle$. Second, we have $\langle q_2, \mathfrak{m}_3\rangle \leqslant i(x)$: this is because $i(q_2 \restriction M_0) = q_2 \restriction M_0$ and so $\langle q_2, \mathfrak{m}_2\rangle \leqslant i(\langle q_2 \restriction M_0, \mathfrak{n}\rangle \leqslant i(x)$ by the isomorphism properties of $i$. ▮L30

Now we proceed exactly as in the previous Section. Choose a generic filter $G \subset Q_0$ and in $V[G]$, define a set $D \subset P$ by $D = \bigcup \{\bar{q}: \langle q, \mathfrak{m} \rangle \in G$, for some coherent system $\mathfrak{m}\}$.

LEMMA 36. *The set $D \subset P$ is dense almost avoidable in $P$.*

*Proof.* This is almost exactly the same as Lemma 13. We show why the set $D$ is dense in the poset $P$. Fix $p \in P$ and a condition $\langle q_0, \mathfrak{m}_0 \rangle \in Q_0$. We shall produce a condition $\langle q_1, \mathfrak{m}_1 \rangle \leqslant \langle q_0, \mathfrak{m}_0 \rangle$ such that $\langle q_1, \mathfrak{m}_1 \rangle \Vdash$ "there is an element of $\dot{D}$ below $\check{p}$". The density of $D$ will then follow from a genericity argument. So we choose a large regular cardinal $\lambda$ and a countable elementary submodel $M \prec H_\lambda$ such that $Q, P, p, \langle q_0, \mathfrak{m}_0 \rangle$ are all in $M$. By Lemma 21, there is $p' \leqslant p$ in the poset $P$ such that $p'$ does not belong to any small subset of $P$ which is in the model $M$. We set $q_1 = q_0 \cup \{\langle M \cap \omega_1, \ \{p', \mathrm{dom}(q_0)\} \rangle\}$ and $\mathfrak{m}_1 = \mathfrak{m}_0 \cup \{\langle M \cap \omega_1, \ \{M \cap H_\kappa\} \rangle\}$. Obviously, the condition $\langle q_1, \mathfrak{m}_1 \rangle$ is as desired. ▮L36

The proof of Theorem 17 is now finished as in the previous section with $Q = Q_0 * \dot{C}_{\aleph_1}$. We must verify that the forcing $Q = Q_0 * \dot{C}_{\aleph_1}$ has the required properties. As in Lemma 3 $Q \Vdash$ "$C_{\aleph_1} \lessdot P$". The forcing $Q$ is proper $\omega_2$-p.i.c. since it is an interation of two forcings. The last thing to check is the size of $Q$. The forcing $Q$ as an iteration may be large, but it has a dense subset isomorphic to $Q_0 \times C_{\aleph_1}$. Now under GCH, I have $|H_{\aleph_2}| = \aleph_2$ and so $|Q_0| = \aleph_2$. As a result, the forcing $Q$ has a dense subset of size $\aleph_2 \cdot \aleph_1 = \aleph_2$. Theorem 17 has been proven. ▮T17

COROLLARY 37. *Under the Proper Forcing Axiom, every complete Boolean algebra of uniform density $\aleph_1$ contains a complete subalgebra isomorphic to $\mathbb{C}(\aleph_1)$.*

To simplify the proof of this, we fist we prove the following multipurpose Lemma.

LEMMA 38. *The Proper Forcing Axiom implies that for every proper forcing notion $Q$, a regular large enough cardinal $\kappa$ and a distinguished element $\Delta \in H_\kappa$ there are a model $M$ so that $M \prec H_\kappa$, $\omega_1 \subset M$, $Q$, $\Delta \in M$ and a filter $G \subset M \cap Q$ which is $M$-generic over $Q$. That is, for every dense set $D \subset Q$ which is in $M$, we obtain $G \cap D \neq 0$.*

*Remark.* A similar statement for $\mathrm{MA}_{\aleph_1}$ and c.c.c. forcings is virtually trivial, since c.c.c.-ness of $Q$ is inherited by $Q \cap M$: first, choose a model $M$ of cardinality $\aleph_1$ and then apply $\mathrm{MA}_{\aleph_1}$ to $Q \cap M$ and all the dense sets of $Q$ in $M$. However, properness is not usually inherited to arbitrary subposets and we need an additional twist to complete the argument.

*Proof.* Choose a proper forcing $Q$, a large regular cardinal $\kappa$ and $\Delta \in H_\kappa$. There is a function $f: H_\kappa^{<\omega} \to H_\kappa$ such that if a set $M \subset H_\kappa$ is closed under $f$, then $M$ is already a submodel of $\langle H_\kappa, \in, \Delta, Q \rangle$. So let us choose such a function $f$. By induction on $n \in \omega$ we define simultaneously for all sets $a \in [H_\kappa]^{<\aleph_0}$ and all conditions $q \in Q$ the following finite sets $a^{(n,q)} \subset H_\kappa$:

(1) $a^{(0,q)} = a$.

(2) The induction step from $n$ to $n+1$ is conducted as follows: we set $b = a^{(n,q)} \cup \{x \in Q$: there is a maximal antichain $A \subset Q$ such that $A \in a^{(n,q)}$, $x \in A$ and $q \leqslant x\}$. Then we define $a^{(n+1,q)} = b \cup f''b^{<\omega}$.

For an integer $n$ and a set $a \in [H_\kappa]^{<\aleph_0}$ we define a subset $D_{n,\alpha}$ of the forcing $Q$ by $D_{n,a} = \{q \in Q$: for every $i < n$ and every maximal antichain $A \subset Q$ with $A \in a^{(i,q)}$ there is $x \in A$ such that $q \leqslant x\}$.

CLAIM 39. *The sets $D_{n,a} \subset Q$ are open dense in $Q$.*

*Proof.* The openness of $D_{n,a}$ follows straight from its definition. Note that if $q \in D_{n,a}$ and $r \leqslant q$ then we have $a^{(i,r)} = a^{(i,q)}$ for all integers $i < n$. To show that the set $D_{n,a} \subset Q$ is dense, fix $q \in Q$ and by induction on $i \leqslant n+1$ build a decreasing sequence $q(0) \geqslant q(1) \geqslant \cdots \geqslant q(i) \geqslant \cdots$ so that

(1) $q = q(0)$

(2) $q(i+1) \in D_{i,a}$.

This is easily done, since at each step we have to meet only finitely many antichains. The above observation makes sure that by passing to stronger conditions we do not destroy the work done so far. The $q(n+1) \in D_{n,a}$ and $q(n+1) \leqslant q$ and the argument is complete. ▮C39

Now by the Proper Forcing Axiom there is a filter $H \subset Q$ meeting all the sets in the family $\{D_{n,a}: n \in \omega,\ a \in [\omega_1]^{<\aleph_0}\}$. We define a function $g: H_\kappa \to H_\kappa$ by

(1) If $A \subset Q$ is a maximal antichain such that $H \cap A \neq 0$ then $g(A) =$ the unique element of $H \cap A$.

(2) Otherwise the function $g$ is just identity.

Let $M$ be the closure of $\omega_1$ under the functions $f, g$. So $M \prec H_\kappa$, $Q, \Delta \in M$ and $\omega_1 \subset M$. The following Claim completes the proof of the Lemma.

CLAIM 40. *Let $G = M \cap H$. Then $G \subset Q \cap M$ is an $M$-generic filter.*

*Proof.* For the genericity of $G$, it is enough to prove that for any maximal antichain $A \subset Q$ in $M$, We have $G \cap A \neq 0$. So fix an antichain $A \in M$. Since the model $M$ is chosen as a closure, there are a set $a \in [\omega_1]^{<\aleph_0}$ and an integer $n$ such that $A$ belongs to the closure of $a$ under $f, g$ and is obtained after $n$ successive applications of the functions $f$ or $g$. By the genericity of the filter $H \subset Q$, there is a condition $q \in H \cap D_{n+1,a}$. By the definition of the set $D_{n+1,a}$, the antichain $A$ belongs to the finite set $a^{(n,q)}$ and the condition $q$ has an element $x$ of $A$ above it. Since $q \leqslant x$ and $q \in H$, we have $x \in H \cap A$. Since the model $M$ is closed under the function $g$, we have $x \in M$ and so $x \in G$.

We should verify that $G$ is a filter on $Q \cap M$. Upwards closure follows from the same property of the filter $H$. If $q$ and $r$ are two conditions in $G$, then there is a lower bound of these two conditions in $H$, but not *a priori* in $G$. To remedy this defect we use the previous paragraph: by elementarity, in the model $M$ there is a maximal antichain $A \subset Q$ such that for $x \in A$, either $r \perp x$ or $q \perp x$ or $x \leqslant q, r$. By the above argument, there is $x \in A$ with $x \in G \subset H$. But this $x$ must be compatible with both $q, r$ (since $H$ is a filter) and so it falls into the third category. Thus there is a lower bound of $q$ and $r$ in $G$ and $G$ is a filter. ▮C40,L38

The proof of Corollary 37 is now finished in the same fashion as the argument for Corollary 14.

MAIN THEOREM. *If ZFC set theory is consistent then so is ZFC + "every complete Boolean algebra of uniform density $\aleph_1$ contains a complete subalgebra isomorphic to $\mathbb{C}(\aleph_1)$".*

*Proof.* The hard work has been done. The proof is now a routine iteration argument using Theorem 17 to deal with one algebra at a time. We give only a outline of the argument, since we believe that a reader who could bear with us up to here can easily provide the details. The scrupulous reader is advised to check with [7] for every detail.

We start with a model of ZFC + GCH and set up a countable support iteration

$$\langle P_\alpha : \alpha \leqslant \omega_2, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$$

such that $P_{\omega_2} \Vdash$ "every complete Boolean algebra of uniform density $\aleph_1$ contains a complete subalgebra isomorphic to $\mathbb{C}(\aleph_1)$". We shall have

(1) the iterands are proper $\omega_2$-p.i.c. forcings of size $\aleph_2$.

Using a suitable bookkeeping device $\langle \tau_\alpha : \alpha \in \omega_2 \rangle$ we shall browse through all potential $P_{\omega_2}$-names $\tau_\alpha$ for separative posets of uniform density $\aleph_1$ whose universe is $\omega_1$. At all intermediate stages $\alpha < \omega_2$ we shall have

(2) $P_\alpha$ is a proper $\aleph_2$-c.c. forcing notion of size $\aleph_2$

(3) $P_\alpha \Vdash$ "GCH"

These two properties hold true for any countable support iteration with property (1)–see Fact 32. So it will be possible, using Theorem 17 in $V^{P_\alpha}$, to pick a $P_\alpha$-name $\dot{Q}_\alpha$ for a proper $\omega_2$-p.i.c. forcing of size $\aleph_2$ so that

(4) $P_\alpha \Vdash$ "if $\tau_\alpha$ is a separative poset of uniform density $\aleph_1$ then $\dot{Q}_\alpha \Vdash C_{\aleph_1} \lessdot \tau_\alpha$". For the final forcing $P_{\omega_2}$, the following will be true:

(5) $P_{\omega_2}$ is a proper $\aleph_2$-c.c. forcing–this holds by (1) and Fact 32.

(6) $|P_{\omega_2}| = \aleph_2$—this is because the forcing $P_{\omega_2}$ is a direct limit of the forcings $P_\alpha$, $\alpha < \omega_2$ of size $\aleph_2$.

The properties (5), (6) make it possible to choose that suitable bookkeeping device $\langle \tau_\alpha : \alpha < \omega_2 \rangle$. Now by (5), $P_{\omega_2}$ does not collapse cardinals. We must verify that $P_{\omega_2} \Vdash$ "every complete Boolean algebra of uniform density $\aleph_1$ contains a complete subalgebra isomorphic to $\mathbb{C}(\aleph_1)$". So let $\tau$ be a $P_{\omega_2}$-name such that $P_{\omega_2} \Vdash$ "$\tau$ is a separative poset of uniform density $\aleph_1$ with universe $\omega_1$". Then for some $\alpha < \omega_2$ we shall have that $\tau = \tau_\alpha$, $\tau$ is a $P_\alpha$-name and $P_\alpha \Vdash$ "$\tau$ is a separative poset of uniform density $\aleph_1$ with universe $\omega_1$". By (4) we have $P_{\alpha+1} \Vdash C_{\aleph_1} \lessdot \tau$. As a result, $P_{\omega_2} \Vdash$ "the poset $C_{\aleph_1}$ regularly embeds into every separative poset $\tau$ of uniform density $\aleph_1$" and the Main Theorem is proven. ▮MT

## 4. TOWARDS HIGHER DENSITIES

A natural question arises immediately upon seeing results à la Theorem 5: Is it possible to repeat such a feat for cardinalities higher than $\aleph_1$? We are very pessimistic about such a possibility; already the $\aleph_2$ case seems to present unsurmountable difficulties. The following Theorem is the best negative result we can find in ZFC:

THEOREM 41. *There is a separative partially ordered set $P$ of uniform density $\aleph_{\omega+1}$ such that $C_{\aleph_{\omega+1}}$ does not embed into it.*

It should be remarked that if e.g. the cardinals are the same in $V$ as in $L$, the constructible universe, then we can find a poset as in Theorem 41 already in $L$.

*Proof.* We shall need the following two facts from pcf theory.

LEMMA 42. *Let $\langle \kappa_n : n \in \omega \rangle$ be an increasing sequence of regular cardinals with* $\operatorname{tcf}(\prod_{n\in\omega} \kappa_n) \bmod \text{fin} = \lambda$ *as witnessed by a modulo finite increasing and* *cofinal sequence* $\langle f_\beta : \beta < \lambda \rangle \subset \prod_{n\in\omega} \kappa_n$. *Then there are ordinals* $\beta_0 < \beta_1 < \lambda$ *such that for all* $n \in \omega$ *we have* $f_{\beta_0}(n) \leqslant f_{\beta_1}(n)$.

The proof is supplied below.

*Fact* 43 ([1]). There is $\langle \kappa_n : n \in \omega \rangle$, an increasing sequence of regular cardinals $< \aleph_\omega$ with $\mathrm{tcf}(\prod_{n\in\omega} \kappa_n) \bmod \mathrm{fin} = \aleph_{\omega+1}$.

Now fix $\langle \kappa_n : n \in \omega \rangle$, an increasing sequence of regular cardinals $< \aleph_\omega$ with $\mathrm{tcf}(\prod_{n\in\omega} \kappa_n) \bmod \mathrm{fin} = \aleph_{\omega+1}$ and a modulo finite increasing and cofinal sequence $\langle f_\beta : \beta < \omega_{\omega+1} \rangle \subset \prod_{n\in\omega} \kappa_n$. We are ready to define our partially ordered set $P$:

Definition 44. The partially ordered set $P$ is a set of all pairs $\langle s, f \rangle$ such that there are an integer $m$ with $s \in \prod_{n\in m} \kappa_n$ and an ordinal $\beta < \omega_{\omega+1}$ with $f = f_\beta \}$.

The order is defined by $\langle s^0, f^0 \rangle \geqslant \langle s^1, f^1 \rangle$ if $s^0 \subset s^1$, $\forall n \in \mathrm{dom}(s^1) \setminus \mathrm{dom}(s^0)\ s^1(n) > f^0(n)$ and $\forall n \notin \mathrm{dom}(s^1)\ f^1(n) \geqslant f^0(n)$.

*Explanation.* So we add a function $\prod_{n\in\omega} \kappa_n$ which modulo finite dominates all the $f_\beta$'s. The $s$ part of a condition in $P$ is just a finite piece of this function.

We prove now that the poset $C_{\aleph_{\omega+1}}$ does not embed into $P$. Actually, more is true: if $c : \omega^V_{\omega+1} \to 2$ is a funcion in the generic extension by $P$, then there is an infinite set $A \subset \omega_{\omega+1}$ in the ground model such that $c \restriction A$ is in the ground model again. Consequently, the function $c$ cannot be $C_{\aleph_{\omega+1}}$-generic over the ground model.

So let $p \in P$, $p \Vdash$ "$\dot{c} : \omega^V_{\omega+1} \to 2$ is a function". We choose a sequence $\{ \langle s_\alpha, f_{\beta_\alpha} \rangle, i_\alpha : \alpha \in \omega_{\omega+1} \}$ such that the following conditions are satisfied:

(1) for each ordinal $\alpha \in \omega_{\omega+1}$ we have $\langle s_\alpha, f_{\beta_\alpha} \rangle \in P$, $i_\alpha \in 2$

(2) for each ordinal $\alpha \in \omega_{\omega+1}$ we have $\langle s_\alpha, f_{\beta_\alpha} \rangle \Vdash$ "$\dot{c}(\alpha) = i_\alpha$"

(3) for ordinals $\xi < \nu < \omega_{\omega+1}$ we have $\beta_\xi < \beta_\nu$.

This is easily done. Now there are a set $S \subset \omega_{\omega+1}$ of full cardinality and a finite sequence $s$ such that for every ordinal $\alpha \in S$ the constructed $s_\alpha$ is just $s$. We define the following partition $h$ of $S^2$: for ordinals $\xi < \nu$ both in $S$ we set $h(\xi, \nu) = 0$ if there is an integer $n$ such that $f_{\beta_\xi}(n) > f_{\beta_\nu}(n)$; otherwise, we let $h(\xi, \nu) = 1$. By the Erdős–Dushnik–Miller theorem, we can have two cases:

(1) There is a set $T \subset S$ of cardinality $\aleph_{\omega+1}$ homogeneous in 0. But this cannot happen since then the sequence $\langle f_{\beta_\alpha} : \alpha \in T \rangle \subset \prod_{n\in\omega} \kappa_n$ contradicts Lemma 42. Notice that this sequence is indeed cofinal in $\prod_{n\in\omega} \kappa_n$ since by (3) above, the set $\{ \beta_\alpha : \alpha \in T \}$ is cofinal in $\omega_{\omega+1}$.

(2) There is a set $A \subset S$ of ordertype $\omega + 1$ homogeneous in 1.

Since the first case leads to a contradiction, the second case must happen. But then, if $\alpha = \max(A)$ and $\xi \in A$, we have by the definition of the poset $P$ that $\langle s_\xi, f_{\beta_\xi} \rangle \geqslant \langle s_\alpha, f_{\beta_\alpha} \rangle$. As a result, $\langle s_\alpha, f_\beta \rangle \Vdash$ "for every $\xi \in A$, we have $\dot{c}(\xi) = i_\xi$" and the argument is complete, since the condition $\langle s_\alpha, f_{\beta_\alpha} \rangle \leqslant p$ decides the values of $\dot{c}$ on an infinite set $A$ as desired. This leaves us with the last thing to demonstrate, namely Lemma 42.

*Proof of Lemma* 42. The proof is quite technical and is modeled after Todorcevic's proof of a similar fact about unbounded sequences of functions in ${}^\omega\omega$[11]. Fix $\gamma < \lambda$ such that $\{s \in \bigcup_{m \in \omega} \prod_{n \in m} \kappa_n \colon \exists \beta < \gamma s \subset f_\beta\} = \{s \in \bigcup_{m \in \omega} \prod_{n \in m} \kappa_n \colon \exists \beta < \lambda s \subset f_\beta\}$. This is possible since $\lambda > \sup\langle \kappa_n \colon n \in \omega \rangle$ is regular. We choose an integer $n_0$ and a set $S \subset \lambda$ of full cardinality so that for every $n \geqslant n_0$ and for every $\beta \in S$ we have $f_\beta(n) \geqslant f_\gamma(n)$. Define $T = \{s \in \bigcup_{m \in \omega} \prod_{n \in M} \kappa_n \colon |\{\beta \in S \colon s \subset f_\beta\}| = \lambda\}$. So $T$ is a tree of height $\omega$. By induction on $n \in \omega$ simultaneously for all $s \in T$ we define set $A(s, n)$:

(1) $A(s, 0) = \{t \in T \colon s \subset t, lth(t) = lth(s) + 1\}$

(2) $A(s, n+1) = \{t \in T \colon s \subset t,\ lth(t) = lth(s) + 1,\ |A(t, n)| = \kappa_{lth(t)}\}$.

Claim 45. *There is $s \in T$ such that for all $n \in \omega$ $|A(s, n)| = \kappa_{lth(s)}$.*

*Proof of the Claim.* By contradiction. Assume that the Claim is false and for any sequence $s \in T$ define $o(s) = \min\{n \in \omega \colon |A(s, n)| < \kappa_{lth(s)}\}$. Choose $\delta \in \lambda$ such that for all $s \in \bigcup_{m \in \omega} \prod_{n \in m} \kappa_n \setminus T$ we have $\{\beta \in S \colon s \subset f_\beta\} \subset \delta$. We define a function $g \in \prod_{n \in \omega} \kappa_n$ by:

$$g(n) = \max\left\{f_\delta(n), \sup\left\{t(n) \colon t \in \prod_{m \in n+1} \kappa_m \cap T \text{ and } t \in A(t \restriction n, o(t \restriction n))\right\}\right\}$$

This is well-defined as the sets $A(s, o(s))$ are small. Now by the cofinality of the sequence $\langle f_\beta \colon \beta \in S \rangle$ one can find an ordinal $\beta \in S$ and integer $n_1$ such that for all $n \geqslant n_1$ $f_\beta(n) \geqslant g(n)$. By our choice of the ordinal $\delta$ we have that $f_\beta$ is a path through $T$. It can be easily verified now that the sequence of integers $\langle o(f_\beta \restriction n) \colon n \geqslant n_1 \rangle$ is strictly decreasing before it hits 0 for the first time. Let $n_2 \geqslant n_1$ be such that $o(f_\beta \restriction n_2) = 0$. So $|A(f_\beta \restriction n_2, 0)| < \kappa_{n_2}$ and since $f_\beta \restriction n_2 + 1 \in A(f_\beta \restriction n_2, 0)$ we obtain $f_\beta(n_2) < g(n_2)$, contradicting our choice of $n_1$. ▌C45

To complete the proof of Lemma 42, choose a sequence $s \in T$ as in Claim 45. By our choice of $\gamma$, there is an ordinal $\beta_0 < \gamma$ such that $s \subset f_{\beta_0}$. Since $f_{\beta_0}$ is modulo finite less than $f_\gamma$ we can find an integer $n_1 \geqslant n_0$ such that $\forall n \geqslant n_1\ f_{\beta_0}(n) \leqslant f_\gamma(n)$. Set $m = \mathrm{lth}(s)$ and choose by induction finite sequences $s = s_m \subset s_{m+1} \subset \cdots \subset s_{n_1}$ so that:

(1) $s_j \in T$, $\mathrm{lth}(s_j) = j$

(2) $s_{j+1} \in A(s_j, n_1 - j)$, $s_{j+1}(j) \geqslant f_{\beta_0}(j)$.

This is possible since by induction on $j$, $m \leqslant j \leqslant n_1$ one can verify that $|A(s_j, n_1 - j)| = \kappa_j$. Now pick $\beta_1 \in S$ with $s_{n_1} \subset f_{\beta_1}$. We claim that the ordinals $\beta_0 < \beta_1$ exemplify the statement of the Lemma.

So we should show that for $n \in \omega$, $f_{\beta_0}(n) \leqslant f_{\beta_1}(n)$. There are three cases. If $n < \mathrm{lth}(s)$ then actually $f_{\beta_0}(n) = f_{\beta_1}(n)$. For $\mathrm{lth}(s) \leqslant n < n_1$ the desired inequality follows from (2) above and for $n \geqslant n_1$ the inequality holds since $f_{\beta_0}(n) \leqslant f_\gamma(n) \leqslant f_{\beta_1}(n)$ (remember $n \geqslant n_1 \geqslant n_0$). The argument is complete. ▮L42,T41

## 5. OPEN PROBLEMS

There are several questions related to the Main Theorem left open in this paper. The first two concern the structure of the real line in the resulting model.

*Problem* 46. Assume that $\mathbb{C}(\aleph_1)$ embeds into every algebra of uniform density $\aleph_1$. Does it follow that $2^{\aleph_0} = \aleph_2$?

*Problem* 47. Assume that $\mathbb{C}(\aleph_1)$ embeds into every algebra of uniform density $\aleph_1$. Does it follow that there is a Cohen real over $L$?

Section 4 provides definite limitations for the possibility of obtaining results à la Theorem 5 for higher densities than $\aleph_1$. In the positive direction we can ask (motivated by [FMS]):

*Problem* 49. Is it consistent that $\mathbb{C}(\kappa)$ embeds into every separative partial order in $L$ of uniform density $\kappa$? Is it implied by $0^{\#}$?

The following questions can hopefully inspire further development of our techniques for the $\aleph_1$ case:

*Problem* 50. Is it consistent that the following are equivalent for a separative poset $P$ of size $\aleph_1$:

(1) $P$ is nowhere c.c.c.

(2) $P$ adds a closed unbounded subset of $\omega_1$ with no infinite subset in the ground model.

*Problem* 51 (Laver). Is it true in ZFC (or is it consistent or does it follow from PFA) that every forcing of size $\aleph_1$ adds a Souslin tree?

The last question can be extended into an infinite scheme of problems, replacing "adding a Souslin tree" by other combinatorial consequences of adding a single Cohen real. Many of them would have a positive answer if the following problem does.

*Problem* 52. Is it consistent (does it follow from PFA) that every forcing $P$ of size $\aleph_1$ embeds $C_{\aleph_0}$ as a factor, i.e., $P$ can be written as $P = C_{\aleph_0} \times Q$ for some forcing $Q$ in the ground model?[1]

Our proofs do not say anything about the way $C_{\aleph_1}$ is embedded into the forcings we are working with, and this is no accident: it is provable in ZFC that a c.c.c. $\kappa$-generated algebra does not embed any $\lambda$-generated algebra as a factor, if $\kappa < \lambda$. Now $C_{\aleph_1}$ is $\aleph_1$-generated, and it is easy to construct a c.c.c. $\aleph_0$-generated algebra of uniform density $\aleph_1$ in ZFC. So this algebra, even though it may have to embed $\mathbb{C}_{\aleph_1}$, it can never embed it as a factor. This argument fails for $\mathbb{C}_{\aleph_0}$ and that leads to Problem 52.

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[1] Answered negatively by the first author.