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THE BOUNDED PROPER FORCING AXIOM

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Abstract. The bounded proper forcing axiom BPFA is the statement that for any family of $\aleph_1$ many maximal antichains of a proper forcing notion, each of size $\aleph_1$, there is a directed set meeting all these antichains.

A regular cardinal κ is called Σ_1-reflecting, if for any regular cardinal χ, for all formulas φ, "$H(\chi) \vDash$'φ'" implies "$\exists\delta < \kappa$, $H(\delta) \vDash$'φ'".

We investigate several algebraic consequences of BPFA, and we show that the consistency strength of the bounded proper forcing axiom is exactly the existence of a Σ_1-reflecting cardinal (which is less than the existence of a Mahlo cardinal).

We also show that the question of the existence of isomorphisms between two structures can be reduced to the question of rigidity of a structure.

Introduction. The proper forcing axiom has been successfully employed to decide many questions in set-theoretic topology and infinite combinatorics. See [Ba 1] for some applications, and [Sh b] and [Sh f] for variants.

In the recent paper [Fu], Fuchino investigated the following two consequences of the proper forcing axiom:

(a) If a structure $\mathfrak{A}$ of size $\aleph_1$ cannot be embedded into a structure $\mathfrak{B}$, then such an embedding cannot be produced by a proper forcing notion.

(b) If two structures $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic, then they cannot be made isomorphic by a proper forcing notion.

He showed that (a) is in fact equivalent to the proper forcing axiom, and asked if the same is true for (b).

In this paper we find a natural weakening of the proper forcing axiom, the "bounded" proper forcing axiom, and show that it already implies property (b) above.

We then investigate the consistency strength of this new axiom. While the exact consistency strength of the proper forcing axiom is still unknown (but large, see [To]), it turns out that the bounded proper forcing axiom is equiconsistent to a rather small large cardinal.

For notational simplicity we will, for the moment, only consider forcing notions which are complete Boolean algebras. See 0.4 and 4.6.

We begin by recalling the forcing axiom in its usual form: For a forcing notion P, $\mathrm{FA}(P, \kappa)$ is the following statement:

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Whenever $\langle A_i : i < \kappa\rangle$ is a family of maximal antichains of P, then there is a filter $G^* \subseteq P$ meeting all A_i.

If $\underset{\sim}{f}$ is a P-name for a function from κ to the ordinals, we will say that $G^* \subseteq P$ *decides* $\underset{\sim}{f}$ if for each $i < \kappa$ there is a condition $p \in G^*$ and an ordinal α_i such that $p \Vdash \underset{\sim}{f}(i) = \alpha_i$. (If G^* is directed, then this ordinal must be unique, and we will write $\underset{\sim}{f}[G^*]$ for the function $i \mapsto \alpha_i$.) Now it is easy to see that the FA(P, κ) is equivalent to the following statement:

Whenever $\underset{\sim}{f}$ is a P-name for a function from κ to the ordinals, then there is a filter $G^* \subseteq P$ which decides $\underset{\sim}{f}$.

This characterization suggests the following weakening of the forcing axiom:

0.1. Definition. Let P be a forcing notion, and let κ and λ be infinite cardinals. BFA(P, κ, λ) is the following statement: Whenever $\underset{\sim}{f}$ is a P-name for a function from κ to λ then there is a filter $G^* \subseteq P$ which decides $\underset{\sim}{f}$, or equivalently: Whenever $\langle A_i : i < \kappa\rangle$ is a family of maximal antichains of P, each of size $\leq \lambda$, then there is a filter $G^* \subseteq P$ which meets all A_i.

0.2. *Notation.* (1) BFA(P, λ) is BFA(P, λ, λ), and BFA(P) is BFA(P, ω_1).

(2) If $\mathscr{E}$ is a class or property of forcing notions, we write BFA$(\mathscr{E})$ for $\forall P \in \mathscr{E}$ BFA(P), etc.

(3) BPFA = the bounded proper forcing axiom = BFA(proper).

Also, we use ☺ to denote the end of a proof, and we write 😐 when we leave a proof to the reader.

0.3. Remark. For the class of ccc forcing notions we get nothing new: BFA(ccc, λ) is equivalent to Martin's axiom MA(λ), i.e., FA(ccc, λ). 😐$_{0.3}$

0.4. Remark. If the forcing notion P is not a complete Boolean algebra but an arbitrary poset, then it is possible that P does not have any small antichains, so it could satisfy the second version of BFA(P) vacuously. The problem with the first definition, when applied to an arbitrary poset, is that a filter on ro(P) which interprets the P-name (= ro(P)-name) $\underset{\sim}{f}$ does not necessarily generate a filter on P. So for the moment our offical definition of BFA(P) for arbitrary posets P will be

$$\text{BFA}(P) \; :\Leftrightarrow \text{BFA}(\text{ro}(P)).$$

In 4.4 and 4.5 we will find an equivalent (and more natural?) definition BFA$'(P)$ which does not explicitly refer to ro(P).

The contents of the paper are as follows. In §1 we investigate connections between BFA and Fuchino's axioms. In §2 we define the concept of a Σ_1-reflecting cardinal, and we show that from a model with such a cardinal we can produce a model for the bounded proper forcing axiom. In §3 we describe a (known) forcing notion which we will use in §4, where we complement our consistency result by showing that a Σ_1-reflecting cardinal is necessary: If BPFA holds, then $\aleph_2$ must be Σ_1-reflecting in L.

We will use gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{M}, \ldots$ for structures (= models of a first order language), and the corresponding latin letters $A, B, M, \ldots$ for the underlying universes. Thus, a model $\mathfrak{A}$ will have the universe A, and if $A' \subseteq A$ then we let $\mathfrak{A}'$ be the submodel (possibly with partial functions) with universe A', etc.

All theorems of this paper are due to the second author. We are grateful to Jörg Brendle for pointing out a mistake in an earlier version of §1.

§**1. Fuchino's problem and other applications.** Let $\mathscr{E}$ be a class of forcing notions.

1.1. DEFINITION. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures for the same first order language, and let $\mathscr{E}$ be a class (or property) of forcing notions. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *$\mathscr{E}$-potentially isomorphic* ($\mathfrak{A} \simeq_{\mathscr{E}} B$) iff there is a forcing $P \in \mathscr{E}$ such that $\Vdash_P$ "$\mathfrak{A} \simeq \mathfrak{B}$". $\mathfrak{A} \simeq_P \mathfrak{B}$ means $\mathfrak{A} \simeq_{\{P\}} \mathfrak{B}$.

1.2. DEFINITION. We say that a structure $\mathfrak{A}$ is *nonrigid* if it admits a nontrivial automorphism. We say that $\mathfrak{A}$ is *$\mathscr{E}$-potentially nonrigid* if there is a forcing notion $P \in \mathscr{E}$ such that $\Vdash_P$ "$\mathfrak{A}$ is nonrigid".

We say that $\mathfrak{A}$ has an $\mathscr{E}$-potential nontrivial endomorphism if there is a forcing notion $P \in \mathscr{E}$ such that

$$\Vdash_P \text{"there is a homomorphism } f : \mathfrak{A} \to \mathfrak{A} \text{ which is not the identity".}$$

1.3. DEFINITION. (1) $\mathrm{PI}(\mathscr{E}, \lambda)$ is the statement: Any two $\mathscr{E}$-potentially isomorphic structures of size λ are isomorphic.

(2) $\mathrm{PA}(\mathscr{E}, \lambda)$ is the statement: Any $\mathscr{E}$-potentially nonrigid structure of size λ is nonrigid.

(3) $\mathrm{PE}(\mathscr{E}, \lambda)$ is the statement: For any structure $\mathfrak{A}$ of size at most λ, if $\mathfrak{A}$ has an $\mathscr{E}$-potential nontrivial endomorphism, then $\mathfrak{A}$ has a nontrivial endomorphism.

$\mathrm{PI}(\mathscr{E}, \lambda)$ was defined by Fuchino [Fu]. It is clear that

$$\mathrm{FA}(\mathscr{E}, \lambda) \Rightarrow \mathrm{BFA}(\mathscr{E}, \lambda) \Rightarrow \mathrm{PI}(\mathscr{E}, \lambda) \;\&\; \mathrm{PA}(\mathscr{E}, \lambda) \;\&\; \mathrm{PE}(\mathscr{E}, \lambda)$$

for all $\mathscr{E}$, and Fuchino asked if $\mathrm{PI}(\mathscr{E}, \lambda)$ implies $\mathrm{FA}(\mathscr{E}, \lambda)$, in particular for the cases $\mathscr{E} =$ ccc, $\mathscr{E} =$ proper, and $\mathscr{E} =$ stationary-preserving.

In the next sections we will show that for $\mathscr{E} =$ proper, the first implication cannot be reversed, by computing the exact consistency strength of BPFA and comparing it to the known lower bounds for the consistency strength of PFA.

1.4. THEOREM. *For any forcing notion P and for any λ, we have*:

$$\mathrm{BFA}(P, \lambda) \Leftrightarrow \mathrm{PE}(P, \lambda) \Rightarrow \mathrm{PA}(P, \lambda) \Leftrightarrow \mathrm{PI}(P, \lambda).$$

PROOF OF PI $\Rightarrow$ PA. We will only give a proof under the additional assumption that we have not only $\mathrm{PI}(P)$ but also $\mathrm{PI}(P_p)$ for all $p \in P$, where P_p is the set of all elements of P which are stronger that p.

Let $\mathfrak{M}$ be a potentially nonrigid structure. So there is a P-name $\underset{\sim}{f}$ such that

$$\Vdash_P \text{"}\underset{\sim}{f} \text{ is a nontrivial automorphism of } \mathfrak{M}\text{".}$$

We can find a condition $p \in P$ and two elements $a \neq b$ of $\mathfrak{M}$ such that

$$p \Vdash_P \text{"}\underset{\sim}{f}(a) = b\text{".}$$

Since we can replace P by P_p, we may assume that p is the weakest condition of P.

So we have that $(\mathfrak{M}, a)$ and $(\mathfrak{M}, b)$ are potentially isomorphic. Any isomorphism from $(\mathfrak{M}, a)$ to $(\mathfrak{M}, b)$ is an automorphism of $\mathfrak{M}$ mapping a to b, so we are done. ☺$_{\mathrm{PI}\Rightarrow\mathrm{PA}}$

1.5 SETUP. Let P be a complete Boolean algebra, and let $(A_i : i \in I)$ be a system of λ many maximal antichains of size λ. We may assume that this is a directed system, i.e., for any $i, j \in I$ there is a $k \in I$ such that A_k refines both A_i and A_j. So if we write $i < j$ for "A_j refines A_i", then $(I, <)$ becomes a partially ordered upwards directed set. (We say that A *refines* B if each element of A is stronger than some unique element of B, or in the Boolean sense if there is a partition $A = \dot{\bigcup}_{b \in B} A_b$ of the set A satisfying $\forall b \in B\ \sum_{a \in A_b} a = b$.)

Assuming $\mathrm{PE}(P, \lambda)$, we will find a filter(base) meeting all the sets A_i.

1.6. DEFINITION. (a) Let M be the disjoint union of the sets A_i.

(b) For $i \in I, z \in A_i$ let $R_{i,z} = \{(x, y) : x, y \in A_i, x = y \text{ or } x = z\}$.

(c) If $i < j$, then there is a "projection" function h_i^j from A_j to A_i: For $p \in A_j$, $h_i^j(p)$ is the unique element of A_i which is compatible with (and in fact weaker than) p.

1.7. FACT. (1) *The functions h_i^j commute, i.e., if $i < j < k$ then $h_i^k = h_i^j \circ h_j^k$.*

(2) *If $i < j$ and $p \in A_j$, then p is stronger than $h_i^j(p)$.* ☹$_{1.7}$

Now let $\mathfrak{M} = (M, (A_i)_{i \in I}, (R_{i,z})_{i \in I, z \in A_i}, (h_i^j)_{i \in I, j \in I, i < j})$, where we treat the sets $A_i, R_{i,z}, h_i^j$ as relations on M.

1.8. DEFINITION. Let $G \subseteq P$ be a filter which meets all the sets A_i, say $G \cap A_i = \{y_i(G)\}$. Define $f_G : M \to M$ as follows: If $x \in F_i$, then $f_G(x) = x * y_i(G)$ (here $* = *_i$ is the group operation on F_i).

1.9. FACT. *If G is a filter which meets all sets A_i, then f_G is an endomorphism of $\mathfrak{M}$.* ☹$_{1.8}$

So $\mathfrak{M}$ has a potential nontrivial endomorphism. So by $PA(P, \lambda)$ we know that $\mathfrak{M}$ really has such an endormorphism.

Finally we will show how a nontrivial endomorphism of $\mathfrak{M}$ defines a filter G^* meeting all the sets A_i.

Let $F : \mathfrak{M} \to \mathfrak{M}$ be an endomorphism which is not the identity. Let $y_0 = f(x_0) \neq x_0, x_0 \in A_{i_0}$. We claim that

(1) For all $j \geq i_0$, $F \restriction A_j$ is not the identity.

(2) For all $j \geq i_0$, $F \restriction A_j$ is constant, say with value p_j.

(3) The set $\{p_j : j \geq i_0\}$ generates a filter G_F meeting all sets A_i.

PROOF. (1) If $h_{i_0}^j(x) = x_0$, then $h_{i_0}^j(F(x)) = y_0$, so $F(x) \neq x$.

PROOF. (2) Let $x \in A_j$, $F(x) \neq x$. Then for all $y \in A_j$ we have $(x, y) \in R_{j,x}$, so $(F(x), F(y)) \in F_{j,x}$, and we must have $F(x) = F(y)$. So F is constant on A_j.

PROOF. (3) If $j \geq i \geq i_0$, then $h_i^j(p_j) = p_i$ (since F is a homomorphism), and p_j is stronger than p_i. Since the set $\{j \in I : j \geq i_0\}$ is directed, also $\{p_j : j \geq i_0\}$ is directed. For any $i \in I$ there is $j \geq i$ satisfying $j \geq i_0$, so $A_i \cap G_f \supseteq \{h_i^j(p_j)\}$. ☺$_{1.4}$

For Theorem 1.11 below we need the following definitions.

1.10. DEFINITION. A *tree* on a set X is a nonempty set T of finite sequences of elements of X which is closed under restrictions, i.e., if $\eta : k \to X$ is in T and $i < k$, then also $\eta \restriction i \in T$. The *tree ordering* $\leq_T$ is given by the subset (or extension) relation: $\eta \leq \nu$ iff $\eta \subseteq \nu$ iff $\exists i : \eta = \nu \restriction i$.

For $\eta \in T$ let $\mathrm{Suc}_T(\eta) := \{x \in X : \eta^\frown x \in T\}$.

For $A \subseteq T$ and $\eta \in T$ we let $\mathrm{rk}(\eta, A)$ be the rank of η with respect to A, i.e., the rank of the (inverse) tree ordering on the set

$$\{\nu : \eta \leq \nu \in T,\ \forall \nu' : \eta \leq \nu' < \nu \Rightarrow \nu' \notin A\}.$$

In other words, $\mathrm{rk}(\eta, A) = 0$ iff $\eta \in A$, $\mathrm{rk}(\eta, A) = \infty$ iff there is an infinite branch of T starting at η which avoids A, and $\mathrm{rk}(\eta, A) = \sup\{\mathrm{rk}(\nu, A) + 1 : \nu$ a direct successor of $\eta\}$ otherwise.

1.11. THEOREM. *For any two structures $\mathfrak{A}$ and $\mathfrak{B}$ there is a structure $\mathfrak{C} = \mathfrak{C}(\mathfrak{A}, \mathfrak{B})$ such that, in any extension $V' \supseteq V$ of the universe, $V' \models$ "$\mathfrak{A} \simeq \mathfrak{B} \leftrightarrow \mathfrak{C}$ is not rigid".*

PROOF. Without loss of generality we take $|A| \leq |B|$, and $\mathfrak{A}$ and $\mathfrak{B}$ are structures in a purely relational language $\mathscr{L}$. We may also assume that $A \cap B = \varnothing$.

We will say that a tree T on $A \cup B$ "codes A" iff the following three conditions hold.

(1) $\mathrm{Suc}_T(\eta) \in \{A, B\}$ for all $\eta \in T$.

(2) When $T^A := \{\eta \in T : \mathrm{Suc}_T(\eta) = A\}$, the ranks $\mathrm{rk}(\eta, T^A)$ are less than ∞ for all $\eta \in T$.

(3) The function $\eta \mapsto \mathrm{rk}(\eta, T^A \setminus \{\eta\})$ is one-to-one on T^A.

Such a tree can be constructed inductively as $T = \bigcup_n T_n$, where the T_n are well-founded trees, each T_{n+1} end-extends T_n, and all nodes in $T_{n+1} - T_n$ are from B except those at the top (i.e., those whose immediate successors will be in $T_{n+2} - T_{n+1}$). Because we have complete freedom in what the rank of the tree ordering for each connected component of $T_{n+1} - T_n$ should be (and because all the T_n have size $=|B|$), we can arrange to satisfy (1), (2), and (3).

Moreover, we can find trees T_0 and T_1, both coding A, such that

(4) $\mathrm{Suc}_{T_0}(\varnothing) = A, \mathrm{Suc}_{T_1}(\varnothing) = B$.

We will replace the roots $(\varnothing)$ of the trees T_0 and T_1 by some new and distinct objects $\varnothing_0$ and $\varnothing_1$. So the trees T_0 and T_1 will be disjoint (by (4)).

Now define the structure $\mathfrak{C}$ as follows: We let $C = T_0 \cup T_1$.

The underlying language of $\mathfrak{C}$ will be the language $\mathscr{L}$ plus an additional binary relation symbol $\leq$, which is to be interpreted as the tree order. Whenever R is an n-ary relation in the language $\mathscr{L}$, we interpret R in $\mathfrak{C}$ by

$$\begin{aligned} R^{\mathfrak{C}} := \{(\eta^\frown a_1, \ldots, \eta^\frown a_n) : \eta \in T_0 \cup T_1, \text{ and } \\ \mathrm{Suc}(\eta) = A \Rightarrow (a_1, \ldots, a_n) \in R^{\mathfrak{A}}, \\ \mathrm{Suc}(\eta) = B \Rightarrow (a_1, \ldots, a_n) \in R^{\mathfrak{B}}\}. \end{aligned}$$

Now work in any extension $V' \supseteq V$. First assume that $f : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism. We will define a map $g : T_0 \to T_1$ such that the map $g \cup g^{-1}$ is a (nontrivial) automorphism of $\mathfrak{C}$.

We define g inductively as follows:

(a) $g(\varnothing_0) = \varnothing_1$.

(b) If $\mathrm{Suc}_{T_0}(\eta) = \mathrm{Suc}_{T_1}(g(\eta))$, then $g(\eta^\frown a) = g(\eta)^\frown a$.

(c) Otherwise, $g(\eta^\frown a) = g(\eta)^\frown f(a)$ or $g(\eta^\frown a) = g(\eta)^\frown f^{-1}(a)$, as appropriate.

It is easy to see that $g \cup g^{-1}$ will then be a nontrivial automorphism.

Now assume conversely that $g\colon \mathfrak{C} \to \mathfrak{C}$ is a nontrivial automorphism. Recall that the tree ordering is a relation on the structure $\mathfrak{C}$, so it must be respected by g.

First assume that there are $i, j \in \{0, 1\}$ and an η such that

$$(*) \qquad \eta \in T_i, \quad g(\eta) \in T_j, \qquad \mathrm{Suc}_{T_i}(\eta) \neq \mathrm{Suc}_{T_j}(g(\eta)).$$

So, without loss of generality, $\mathrm{Suc}_{T_i}(\eta) = A$ and $\mathrm{Suc}_{T_j}(g(\eta)) = B$. Now define a map $f\colon A \to B$ by requiring $g(\eta^\frown a) = g(\eta)^\frown f(a)$, and check that f must be an isomorphism.

Now we show that we can always find i, j, η as in $(*)$. If not, then we can first see that g respects T_0 and T_1, i.e., $g(\eta) \in T_0$ iff $\eta \in T_0$. Next, our assumption implies that the function $g \restriction T_0$ respects the set T_0^A, i.e., $\eta \in T_0^A$ iff $g(\eta) \in T_0^A$. Hence for all $\eta \in T_0^A$, we have $\mathrm{rk}(\eta, T_0^A) = \mathrm{rk}(g(\eta), T_0^A)$, so (by condition (3) above) $g(\eta) = \eta$ for all $\eta \in T_0^A$. Since every $\nu \in T_0$ can be extended to some $\eta \in T_0^A$ and g respects $<$, we must have $g(\nu) = \nu$ for all $\nu \in T_0$. The same argument shows that also $g \restriction T_1$ is the identity. ☺$_{1.11}$

1.12. Remarks on other applications. Which other consequences of PFA (see, e.g., [Ba 1]) are already implied by BPFA? On the one hand it is clear that if PFA is only needed to produce a sufficiently generic function from ω_1 to ω_1, then the same proof should show that BPFA is a sufficient assumption. For example:

BPFA implies "all $\aleph_1$-dense sets of reals are isomorphic".

On the other hand, as we will see in the next section, the consistency strength of BPFA is quite weak. So BPFA cannot imply any statement which needs large cardinals, such as "there is an Aronszajn tree on $\aleph_2$". In particular, BPFA does not imply PFA.

We do not know if BPFA already decides the size of the continuum, but Woodin has remarked that the bounded *semi*proper forcing axiom implies $2^{\aleph_0} = \aleph_2$.

§2. The consistency of BPFA.

2.1. Definition. For any cardinal χ, $H(\chi)$ is the collection of sets which are hereditarily of cardinality $< \chi$: Letting $\mathrm{trcl}(x)$ be the transitive closure of x, $\mathrm{trcl}(x) = \{x\} \cup \bigcup x \cup \bigcup\bigcup x \cup \cdots$, we have

$$H(\chi) = \{x\colon |\mathrm{trcl}(x)| < \chi\},$$

(usually we require χ to be regular).

2.2. Definition. Let κ be a regular cardinal. We say that κ is "reflecting" or, more precisely, Σ_1-*reflecting*, if:

For any first order formula φ in the language of set theory, for any $a \in H(\kappa)$:
IF there exists a regular cardinal $\chi \geq \kappa$ such that $H(\chi) \vDash \varphi(a)$,
THEN there is a cardinal $\delta < \kappa$ such that $a \in H(\delta)$ and $H(\delta) \vDash \varphi(a)$.

2.3. REMARK. (1) We may require δ to be regular without changing the concept of "Σ_1-reflecting".

(2) We can replace "for all χ" by "for unboundedly many χ".

PROOF. (1) Assume that $H(\chi) \vDash \varphi(a)$, χ regular. Choose some large enough χ_1 such that $H(\chi) \in H(\chi_1)$ and χ_1 is a successor cardinal. So $H(\chi_1) \vDash$ "$\exists\chi$, χ regular, $H(\chi)$ exists, and $H(\chi) \vDash$ '$\varphi(a)$'". We can find a (successor) $\delta_1 < \kappa$ such that $H(\delta_1) \vDash$ "$\exists\delta$, δ regular, $H(\delta) \vDash$ '$\varphi(a)$'". So δ is really regular.

(2) If $\chi < \chi_1$ then $H(\chi) \vDash$ "φ" iff $H(\chi_1) \vDash$ "$H(\chi) \vDash$ 'φ'". ☺$_{2.3}$

2.4. REMARK. It is easy to see that if κ is reflecting, then κ is a strong limit, hence inaccessible. Applying Σ_1 reflection, we get that κ is hyperinaccessible, etc. ☺$_{2.4}$

2.5. REMARK. (1) There is a closed unbounded class C of cardinals such that every regular $\kappa \in C$ (if there are any) is Σ_1 reflecting. So if "∞ is Mahlo", then there are many Σ_1-reflecting cardinals.

(2) If κ is reflecting, then $L \vDash$ "κ is reflecting".

PROOF. (1) For any set a and any formula φ let

$$f'(a,\varphi) = \min\{\chi \in \mathrm{RCard}\colon H(\chi) \vDash \varphi(a)\}$$

(where RCard is the class of regular cardinals, and we define $\min\varnothing = 0$). Now let $f\colon \mathrm{RCard} \to \mathrm{RCard}$ be defined by $f(\alpha) = \sup\{f'(a,\varphi)\colon \varphi$ a formula, $a \in H(\alpha)\}$ and let $C = \{\delta \in \mathrm{Card}\colon \forall\alpha \in \mathrm{RCard} \cap \delta\ f(\alpha) < \delta\}$.

(2) is also easy. ☺$_{2.5}$

Our main interest in this concept is its relativization to L. In this context we recall the following fact:

2.6. FACT. *Assume $V = L$. Then for all (regular) cardinals $\chi, H(\chi) = L_\chi$.* ☺$_{2.6}$

2.7. FACT. *Assume $P \in H(\lambda)$ is a forcing notion, and $\chi > 2^{2^\lambda}$ is regular. Then:*

(1) *For any P-name $\underset{\sim}{x}$ there is a P-name $\underset{\sim}{y} \in H(\chi)$ such that $\Vdash_P$ "$\underset{\sim}{x} \in H(\chi) \Rightarrow \underset{\sim}{x} = \underset{\sim}{y}$". (And conversely, if $\underset{\sim}{x} \in H(\chi)$, then $\Vdash_P$ "$\underset{\sim}{x} \in H(\chi)$".)*

(2) *If $\underset{\sim}{x} \in H(\chi)$ and $\varphi(\cdot)$ is a formula, then*

$$\Vdash \text{“}H(\chi) \vDash \varphi(\underset{\sim}{x})\text{”} \Leftrightarrow \text{“}H(\chi) \vDash \text{‘}\Vdash \varphi(\underset{\sim}{x})\text{’”}.$$

PROOF. (1) is by induction on the rank of $\underset{\sim}{x}$ in V^P, and (2) uses (1). ☺$_{2.7}$

2.8. FACT. *Let P be a forcing notion, $P \in H(\lambda)$, and $\chi > 2^{2^\lambda}$ regular. Then P is proper iff $H(\chi) \vDash$ "P is proper".*

2.9. LEMMA. *Assume that κ is reflecting, $\lambda < \kappa$ is a regular cardinal, and $\mathfrak{A}$ and $\mathfrak{B}$ are structures in $H(\lambda)$. If there is a proper forcing notion P such that $\Vdash_P$ "$\mathfrak{A} \simeq \mathfrak{B}$", then there is such a (proper) forcing notion in $H(\kappa)$.*

PROOF. Fix P, and let χ be a large enough regular cardinal. So $H(\chi) \vDash$ "P proper, $P \in H(\mu), (2^{2^\lambda})$ exists". Also, there is a P-name $\underset{\sim}{f} \in H(\chi)$ such that $\Vdash_P$ "$H(\chi) \vDash$ '$\underset{\sim}{f}\colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism'", so, by 2.7(2), $H(\chi) \vDash$ "$\Vdash_P$ '$\underset{\sim}{f}\colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism'".

Now we use the fact that κ is reflecting. We can find $\delta < \kappa$, $\delta > \lambda$, and $\chi' \in H(\delta)$ such that $H(\delta) \vDash$ "$\exists\nu\, \exists Q \in H(\nu)$, Q proper, $\exists \underset{\sim}{g} \Vdash_Q$ '$\underset{\sim}{g}\colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism', and (2^{2^λ}) exists." So this Q is really proper, and Q forces that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. ☺$_{2.9}$

2.10. FACT. *If κ is reflecting, and $P \in H(\kappa)$ is a forcing notion, then $\Vdash_P$ "κ is reflecting".*

PROOF. Let $P \in H(\lambda), \lambda < \kappa$. Assume that $p \Vdash$ "$H(\chi) \models$ '$\varphi(\underset{\sim}{a})$', $\underset{\sim}{a} \in H(\kappa)$". We may assume that $\underset{\sim}{a} \in H(\kappa)$. By 2.7 we have $H(\chi) \models$ "$p \Vdash$ '$\varphi(\underset{\sim}{a})$'", so there is a $\delta < \kappa$, $\delta > \lambda$, such that $H(\delta) \models$ "$p \Vdash$ '$\varphi(\underset{\sim}{a})$'", hence $p \Vdash$ "$H(\delta) \models$ '$\varphi(\underset{\sim}{a})$'". δ is a cardinal in V^P, because $|P| < \lambda < \delta$. ☺$_{2.10}$

2.11. THEOREM. *If "there is a reflecting cardinal" is consistent with ZFC, then also* PI(*proper*) (*and hence* BPFA, *by* 1.4) *is consistent with ZFC.*

PROOF (Short version). We will use a CS iteration of length κ, where κ reflects. All intermediate forcing notions will have hereditary size $< \kappa$. By a bookkeeping argument we can take care of all possible structures on ω_1. If in the intermediate model there is a proper forcing notion which forces a nontrivial endomorphism, then there is such a forcing notion of size $< \kappa$, so we continue.

PROOF (More detailed version). Assume that κ reflects. We define a countable support iteration $(P_i, Q_i : i < \kappa)$ of proper forcing notions and a sequence $\langle \underset{\sim}{\mathfrak{M}}_i : i < \kappa\rangle$ with the following properties for all $i < \kappa$:

(1) $P_i \in H(\kappa)$.

(2) Q_i is a P_i-name, $\Vdash_{P_i}$ "Q_i is proper, $Q_i \in H(\kappa)$".

(3) $\Vdash_{P_i} 2^{\aleph_1} < \kappa$. (This follows from (1) and (2).)

(4) $\underset{\sim}{\mathfrak{M}}_i$ is a name for a structure on ω_1.

(5) $\Vdash_{P_i}$ "if there is a proper forcing notion of size $< \kappa$ forcing a nontrivial endomorphism of $\underset{\sim}{\mathfrak{M}}_i$, then Q_i is such a forcing notion".

With the usual bookkeeping argument we can also ensure that

(6) Whenever $\underset{\sim}{\mathfrak{M}}$ is a P_i-name for a structure on ω_1 for some i, then there are unboundedly (or even stationarily) many $j > i$ with $\Vdash_j$ "$\underset{\sim}{\mathfrak{M}}_j = \underset{\sim}{\mathfrak{M}}$".

From (1) we also get the following two properties:

(7) $P_\kappa \models \kappa$-cc.

(8) Whenever $\underset{\sim}{\mathfrak{M}}$ is a P_κ-name for a structure on ω_1, then there are $i < \kappa$ and a P_i-name $\underset{\sim}{\mathfrak{M}}'$ such that $\Vdash_\kappa \underset{\sim}{\mathfrak{M}} = \underset{\sim}{\mathfrak{M}}'$.

From these properties we can now show $\Vdash_\kappa$ BPFA. P_κ is proper, so ω_1 is not collapsed. Let p be a condition, and let $\underset{\sim}{\mathfrak{M}}$ be a P_κ-name for a structure on ω_1, and assume that

$$p \underset{\kappa}{\Vdash} \text{"}\underset{\sim}{Q} \text{ proper }, \underset{\underset{\sim}{Q}}{\Vdash} \underset{\sim}{\mathfrak{M}} \text{ has a nontrivial endomorphism"},$$

where $\underset{\sim}{Q}$ is a P_κ-name. So by (8) we may assume that, for some large enough $i < \kappa$, $\underset{\sim}{\mathfrak{M}}$ is a P_i-name and $p \in P_i$. By (6) we may assume without loss of generality that $\underset{\sim}{\mathfrak{M}} = \underset{\sim}{\mathfrak{M}}_i$. Now letting R be the P_i-name $(P_\kappa/G_i) * \underset{\sim}{Q}$, we get

$$p \underset{i}{\Vdash} \text{" } \underset{R}{\Vdash} \underset{\sim}{\mathfrak{M}} \text{ has a nontrivial endomorphism"}.$$

But by 2.10, $\Vdash_i$ "κ is reflecting", so by the definition of Q_i and by 2.9 we get that $p \Vdash_{i+1} \underset{\sim}{\mathfrak{M}}$ has a nontrivial endomorphism. ☺$_{2.11}$

2.12. REMARK. Since 2.8 is also true with "proper" replaced by "semiproper", we similarly get that the consistency of a Σ_1-reflecting cardinal implies the consistency of the bounded semiproper forcing axiom. ☺$_{2.12}$

§3. Sealing the ω_1-branches of a tree. In this section we will define a forcing notion which makes the set of branches of an ω_1-tree absolute.

3.1. DEFINITION. Let T be a tree of height ω_1. We say that $B \subseteq T$ is an ω_1-*branch* if B is a maximal linearly ordered subset of T and has order type ω_1.

3.2. LEMMA. *Let T be a tree of height ω_1. Assume that every node of T is on some ω_1-branch, and that there are at uncountably many ω_1-branches. (These assumptions are just to simplify the notation.) Then there is a proper forcing notion P'_T (in fact, P'_T is a composition of finitely many σ-closed and ccc forcing notions) forcing the following:*

(1) *T has $\aleph_1$ many ω_1-branches, i.e., there is a function $b: \omega_1 \times \omega_1 \to T$ such that each set $B_\alpha = \{b(\alpha, \beta): \beta < \omega_1\}$ is an end segment of a branch of T (enumerated in its natural order), and every ω_1-branch is (modulo a countable set) equal to one of the B_α's, and the sets B_α are pairwise disjoint.*

(2) *There is a function $g: T \to \omega$ such that, for all $s < t$ in T, if $g(s) = g(t)$ then there is some (unique) $\alpha < \omega_1$ such that $\{s, t\} \subseteq B_\alpha$.*

The proof consists of two parts. In the first part (3.3) we show that we may without loss of generality assume that T has exactly $\aleph_1$ many branches. This observation is a special case of a theorem of Mitchell [Mi, 3.1].

In the second part we describe the forcing notion which works under the additonal assumption that T has only $\aleph_1$ many branches. This forcing notion is essentially the same as the one used by Baumgartner in [Ba 2, §8].

3.3. FACT. *Let T be a tree of height ω_1, $\kappa > |T|$, and let R_1 be the forcing notion adding κ many Cohen reals. In V^{R_1}, let R_2 be a σ-closed forcing notion. Then every branch of T in $V^{R_1 * R_2}$ is already in V^{R_1} (and in fact already in V).*

Hence, if we take R_2 to be the Levy collapse of the number of branches of T to $\aleph_1$ (with countable conditions), T will have at most $\aleph_1$ many branches in $V^{R_1 * R_2}$.

PROOF. Assume that $\underset{\sim}{b}$ is a name of a new branch. So the set

$$T_{\underset{\sim}{b}} := \{t \in T: \exists p \in R_2 \ p \Vdash t \in \underset{\sim}{b}\}$$

is (in V^{R_1}) a perfect subtree of T. In particular, there is an order-preserving function $f: 2^{<\omega} \to T_{\underset{\sim}{b}}$. Since κ was chosen big enough, we can find a real $c \in 2^\omega \cap V^{R_1}$ which is not in $V[f]$. Now note that T' is σ-closed, so there is $t^* \in T$ such that $\forall n \ f(c \restriction n) \leq t^*$. But this implies that

$$c = \bigcup \{s \in 2^{<\omega}: f(s) \leq t^*\}$$

can be computed in $V[f]$, a contradiction. ☺$_{3.3}$

Now we describe a forcing notion P'_T which works under the assumption that T has not more than $\aleph_1$ branches. In the general case we can then use the forcing $P_T = R_1 * R_2 * P'_T$.

3.4. DEFINITION. Let T be a tree of height ω_1 with $\aleph_1$ many ω_1-branches $\{B_i: i < \omega_1\}$, and assume that each node of T is on some ω_1-branch. Let $B'_j = B_j \setminus \bigcup_{i<j} B_i$

and $x_j = \min(B'_j)$, so that the sets B'_j are disjoint end segments of the branches B_j, and they form a partition of T. Let $A = \{x_i : i < \omega_1\}$.

The forcing "sealing the branches of T" is defined as $P'_T = \{f : f$ a finite function from A to ω, and if $x < y$ are in $\mathrm{dom}(f)$, then $f(x) \neq f(y)\}$.

3.5. LEMMA. *P'_T satisfies the countable chain condition. (In fact, much more is true: If $\langle p_i : i < \omega_1\rangle$ are conditions in P, then there are uncountable sets $S_1, S_2 \subseteq \omega_1$ such that whenever $i \in S_1$ and $j \in S_2$, then p_i and p_j are compatible. See* [Sh f, XI].)

PROOF. Essentially the same as in [Ba 2, 8.2]. ☺$_{3.5}$

To conclude the proof of 3.2, note that any generic filter G on P'_T induces a generic $f_G : A \to \omega$. Let $g_G : T \to \omega$ be defined by $g_G(y) = f_G(x_i)$ for all $y \in B_i$. This function g_G fulfills the requirement 3.2(2). ☺$_{3.2}$

§**4. BPFA and reflecting cardinals are equiconsistent.** In this section we will prove

4.1. THEOREM. *If* BPFA *holds, then the cardinal $\aleph_2$ (computed in V) is Σ_1-reflecting in L.*

Before we start the proof of this theorem, we exhibit some general properties of "sufficiently generic" filters.

First a remark on terminology. When we consider BFA(P, λ), then by "for all sufficiently generic $G^* \subseteq P$, $\varphi(G^*)$ holds" we mean: "there is a P-name $\underset{\sim}{f} : \lambda \to \lambda$ such that, whenever a filter G^* interprets $\underset{\sim}{f}$, then $\varphi(G^*)$ will hold". A description of the name $\underset{\sim}{f}$ can always be deduced from the context. Instead of a single name $\underset{\sim}{f}$ we usually have a family of λ many names.

The first lemma shows that from any sufficiently generic filter we can correctly compute the first order theory (that is, the part of it which is forced), or, equivalently, the first order diagram, of any small structure in the extension.

4.2. LEMMA. *Let P be a forcing notion, $\Vdash_P$ "$\mathfrak{M}$ is a structure with universe λ with λ many relations $(\underset{\sim}{R}_i : i < \lambda)$". Assume* BFA$(P, \lambda)$. *Then for every sufficiently generic filter $G^* \subseteq P$, letting $\mathfrak{M}^* = (\lambda, \underset{\sim}{R}_i[G^*])_{i<\lambda}$, (where $\underset{\sim}{R}_i[G^*] := \{(x_1, \ldots, x_k) \in \lambda^k : \exists p \in G^*\ p \Vdash \mathfrak{M} \vDash \underset{\sim}{R}_i(x_1, \ldots, x_k)\}$) we have:*

Whenever φ is a closed formula such that $\Vdash_P \mathfrak{M} \vDash \varphi$, then $\mathfrak{M}^* \vDash \varphi$.

PROOF. Let χ be a large enough cardinal, and let N be an elementary submodel of $H(\chi)$ of size λ containing all the necessary information (i.e., $\lambda \subseteq N$, $(P, \leq) \in N$, $(\underset{\sim}{R}_i : i < \lambda) \in N$). By BFA$(P, \lambda)$ we can find a filter $G^* \subseteq P$ which decides all P-names of elements of $\mathfrak{M}$ which are in N and all first order statements about $\mathfrak{M}$, i.e.,

(1) For all $\underset{\sim}{\alpha} \in N$, if $\Vdash_P$ "$\underset{\sim}{\alpha} \in \lambda$" then there is $\beta \in \lambda$ and $p \in G^*$ such that $p \Vdash_P$ "$\underset{\sim}{\alpha} = \check{\beta}$".

(2) For all $\alpha_1, \ldots, \alpha_k \in \lambda$ and all formulas $\varphi(x_1, \ldots, x_k)$ there is $p \in G^*$ such that either $p \Vdash$ "$\mathfrak{M} \vDash \varphi(\alpha_1, \ldots, \alpha_k)$" or $p \Vdash$ "$\mathfrak{M} \vDash \neg\varphi(\alpha_1, \ldots, \alpha_k)$".

We now claim that, for every formula $\varphi(x_1, \ldots, x_k)$ and every $\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_k \in N$, if $\Vdash_P$ "$\mathfrak{M} \vDash \varphi(\underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_k)$", then $\mathfrak{M}^* \vDash \varphi(\underset{\sim}{\alpha}_1[G^*], \ldots, \underset{\sim}{\alpha}_k[G^*])$. We assume that φ is in prefix form, so in particular negation signs appear only before atomic formulas. The proof is by induction on the complexity of φ, starting from atomic and negated atomic formulas. We will only treat the case $\varphi = \exists x\ \varphi_1$. So assume that $\Vdash_P \mathfrak{M} \models \exists x\ \varphi_1(x, \underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_k)$. We can find a name $\underset{\sim}{b} \in N$ such that $\Vdash_P \mathfrak{M} \models \varphi_1(\underset{\sim}{b}, \underset{\sim}{\alpha}_1, \ldots, \underset{\sim}{\alpha}_k)$, so by the induction hypothesis we get

$$\mathfrak{M}^* \models \varphi_1(\underset{\sim}{b}[G^*], \underset{\sim}{\alpha}_1[G^*], \ldots, \underset{\sim}{\alpha}_n[G^*]). \qquad ☺_{4.2}$$

4.3. Remark. In a sense the previous lemma characterizes "sufficiently generic" filters. More precisely, the following is (trivially) true: Let P be a complete Boolean algebra, let $\Vdash_P \underset{\sim}{f}: \lambda \to \lambda$, and let $\mathfrak{M} = (\lambda, \underset{\sim}{f})$, where we treat $\underset{\sim}{f}$ as a relation. For any ultrafilter $G^* \subseteq P$ the model $\mathfrak{M}^* = (\lambda, \underset{\sim}{f}[G^*])$ is well-defined. Since $\underset{\sim}{f}$ is forced to be a function, we have $\Vdash_P$ "$\mathfrak{M} \models$ '$\forall \alpha\ \exists \beta\ (\alpha, \beta) \in \underset{\sim}{f}$' ". Clearly G^* "decides" $\underset{\sim}{f}$ (as a function) iff $\mathfrak{M}^*$ satisfies the same $\forall\exists$ statement. $☺_{4.3}$

This last remark suggests the following easy characterization of BFA(P):

4.4. Definition. Let P be an arbitrary forcing notion, not necessarily a complete Boolean algebra. If $\underset{\sim}{f}$ is a P-name of a function from λ to λ, then let the "(forced) diagram" of $\underset{\sim}{\mathfrak{M}} = (\lambda, \underset{\sim}{f})$ be defined by

$$D^{\Vdash}(\underset{\sim}{M}) = D^{\Vdash}(\underset{\sim}{f}) = \{(\varphi, \alpha_1, \ldots, \alpha_n) : \varphi(x_1, \ldots, x_n) \text{ a first order formula,}$$
$$\alpha_1, \ldots, \alpha_n \in \lambda\ \Vdash_P \varphi(\alpha_1, \ldots, \alpha_n)\}.$$

The "open (forced) diagram" $D^{\Vdash}_{\text{open}}(\underset{\sim}{f})$ is defined similarly, but φ ranges only over quantifier-free formulas.

4.5. Definition. For any forcing notion P let BFA$'(P, \lambda)$ be the statement

$$\text{BFA}'(P, \lambda) = \text{Whenever } \underset{\sim}{f}: \lambda \to \lambda \text{ is a } P\text{-name of a function,}$$
$$\text{then there is a function } f^* \text{ such that } (\lambda, f^*) \models D^{\Vdash}_{\text{open}}(\underset{\sim}{f}).$$

4.6. Fact. *For any forcing notion P, BFA(P, λ) iff BFA$'(P, \lambda)$.*

Proof. BFA$'(P, \lambda)$ is clearly equivalent to BFA$'(\text{ro}(P), \lambda)$. The same is true (by definition) for BFA. So we may assume without loss of generality that P is a complete Boolean algebra. It is clear that BFA$(P, \lambda) \Rightarrow$ BFA$'(P, \lambda)$.

Conversely, if f^* is a function as in BFA$'$, then we claim that the set $\{[\![\underset{\sim}{f}(\alpha) = f^*(\alpha)]\!] : \alpha \in \lambda\}$ generates a filter on P (where $[\![\varphi]\!]$ denotes the Boolean value of a closed statement φ). Indeed, otherwise there are ordinals $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ such that

$$f^*(\alpha_1) = \beta_1\ \&\ \cdots\ \&\ f^*(\alpha_n) = \beta_n$$

but the Boolean value

$$[\![\underset{\sim}{f}(\alpha_1) = \beta_1\ \&\ \cdots\ \&\ \underset{\sim}{f}(\alpha_n) = \beta_n]\!]$$

is 0. This contradicts the fact that f^* witnesses BFA$'(P, \lambda)$. $☺_{4.6}$

After this digression we now continue our preparatory work for the proof of Theorem 4.1. Our next lemma shows that a generic filter will not only reflect first order statements about small structures, but will also preserve their well-foundedness.

4.7. Lemma. *Assume that $\Vdash_P$ "$\underset{\sim}{\mathfrak{M}} = (\lambda, \underset{\sim}{E})$ is a well-founded structure, λ is a cardinal". Assume that $\text{cf}(\lambda) > \omega$, and assume that BFA$(P, \lambda)$ holds. Then for every sufficiently generic filter $G^* \subseteq P$ we have that $\mathfrak{M}^* := (\lambda, \underset{\sim}{E}[G^*])$ is well-founded.*

(We will use this lemma only for the case where P is proper and $\lambda = \omega_1$.)

PROOF. For each $\alpha < \lambda$ let $\underset{\sim}{r}_\alpha$ be the name of the canonical rank function on $(\alpha, \underset{\sim}{E})$, i.e.,

$$\Vdash_P \text{“}\operatorname{dom}(\underset{\sim}{r}_\alpha) = \alpha,\ \forall\beta < \alpha\ \underset{\sim}{r}_\alpha(\beta) = \sup\{\underset{\sim}{r}_\alpha(\gamma) + 1 \colon \gamma E \beta\}\text{”}.$$

As $\Vdash_P$ “λ is a cardinal”, we have $\Vdash_P$ “$\operatorname{rng}(\underset{\sim}{r}_\alpha) \subseteq \lambda$”, so any sufficiently generic filter G^* will interpret all the functions $\underset{\sim}{r}_\alpha$. Applying Lemma 4.2 to the structure $(\alpha, \underset{\sim}{E}[G^*], \underset{\sim}{r}_\alpha[G^*])$, we see that $\underset{\sim}{r}_\alpha[G^*]$ is indeed a rank function witnessing that $(\alpha, \underset{\sim}{E}[G^*])$ is well-founded. Since $\operatorname{cf}(\lambda) > \omega$, this now implies that $(\lambda, \underset{\sim}{E}[G^*])$ is also well-founded. ☺$_{4.7}$

We now start the proof of 4.1. The definitions in the following paragraphs will be valid throughout this section.

Assume BPFA. Let $\kappa := \aleph_2$. We will show that κ is reflecting in L. It is clear that κ is regular in L.

4.8. *Claim. Without loss of generality we may assume*:

(1) $0^\#$ *does not exist, i.e., the covering lemma holds for* L.

(2) $\aleph_2^{\aleph_1} = \aleph_2$.

(3) *There is* $A \subseteq \aleph_2$ *such that whenever* $X \subseteq \mathrm{Ord}$ *is of size* $\leq \aleph_1$, *then* $X \in L[A]$.

Proof. (1) if $0^\#$ exists, then $L_\kappa \prec L$, and it is easy to see that this implies that κ is a reflecting cardinal in L.

(2) Let $P = \operatorname{Levy}(\aleph_2, \aleph_2^{\aleph_1})$, i.e., members of P are partial functions from $\aleph_2$ to $\aleph_2^{\aleph_1}$ with bounded domain. Since P does not add new subsets of $\aleph_1$ and P is proper, also V^P will satisfy PE(proper, $\aleph_1$). Also $\aleph_2^V = \aleph_2^{V^P}$ and $V^P \models \aleph_2^{\aleph_1} = \aleph_2$, so we can without loss of generality work in V^P instead of V.

(3) By (2) we can find a set $A \subseteq \aleph_2$ such that $\aleph_2^{L[A]} = \aleph_2$ and every function from $\aleph_1$ to $\aleph_2$ is already in $L[A]$. By (1), every set X of ordinals of size $\leq \aleph_1$ can be covered by a set $Y \in L$, $|Y| = \aleph_1$. Let $j \colon Y \to \operatorname{otp}(Y)$ be order preserving; then $j[X] \in L[A]$, $j \in L$, so $X \in L[A]$.

PROOF OF 4.1. Let $\varphi(x)$ be a formula, $a \in L_\kappa$, and assume that $\chi > \kappa$, $L_\chi \models \varphi(a)$, and χ is a regular cardinal in L. We have to find an L-cardinal $\chi' < \kappa$ such that $a \in L_{\chi'}$ and $L_{\chi'} \models \varphi(a)$.

By 2.3, we may assume that χ is a cardinal in $L[A]$ or even in V.

Informal outline of the proof. We will define a forcing notion P. In V^P we will construct a model $\mathfrak{M} = (M, \in, \chi, x, \dots) \prec V^P$ of size $\aleph_1$ containing all necessary information. This model has an isomorphic copy $\overline{\mathfrak{M}}$ with underlying set ω_1. We will find a “sufficiently generic” filter G^* which will “interpret” $\overline{\mathfrak{M}}$ as $\mathfrak{M}^*$. By 4.7 we may assume that $\mathfrak{M}^* = (\omega_1, E^*, \chi^*, \dots)$ will be well-founded, so we can form its transitive collapse $\mathfrak{M}' = (M', \in, \chi', \dots)$. By 4.2 we have that $\mathfrak{M}' \models$ “χ' is a cardinal in L”, i.e., χ' is a cardinal in $L_{M' \cap \mathrm{Ord}}$. The main point will be to show that any filter on our forcing notion P will code enough information to enable us to conclude that χ' is really a cardinal of L.

4.9. DEFINITION OF THE FORCING NOTIONS Q_0 AND Q_1. Let Q_0 be the Levy-collapse of $L_\chi[A]$ to $\aleph_1$, i.e. the set of countable partial functions from ω_1 to $L_\chi[A]$ ordered by extension.

In V^{Q_0} let T be the following tree: Elements of T are of the form

$$(\langle \mu_i : i < \alpha \rangle, \langle f_{ij} : i \leq j < \alpha \rangle)$$

(we will usually write them as $\langle \mu_i, f_{ij} : i \leq j < \alpha \rangle$), where the μ_i are ordinals less than χ, the f_{ij} are a system of commuting order-preserving embeddings, and $\alpha < \omega_1$. T is ordered by the relation "is an initial segment of".

If B is a branch of T (in V^{Q_0}, or in any bigger universe) of length δ, then B defines a directed system $\langle \mu_i, f_{ij} : i \leq j < \delta \rangle$ of well-orders. We will call the direct limit of this system $(\gamma_B, <_B)$. In general this may not be a well-order, but it is clear that if the length of B is ω_1, then $(\gamma_B, <_B)$ will be a well-order.

Let $Q_1 = P_T$ be the forcing "sealing the ω_1-branches of T" described in 3.2. We let $P = Q_0 * Q_1$. So P is a proper order, in fact it is a finite iteration of σ-closed and ccc partial orderings.

4.10. Definition. In V^P we define a model $\mathfrak{M}$ as follows. Let Ω be a large enough regular cardinal of V. Let $(M, \in)$ be an elementary submodel of $(H(\Omega)^{V^P}, \in)$ of size $\aleph_1$ containing all necessary information; in particular, $M \supseteq L_\chi[A]$. We now expand $(M, \in)$ to a model $\mathfrak{M} = (M, \in, \chi, A, \dots)$ by adding the following functions, relations and constants:

- a constant for each element of L_ξ (where ξ is chosen such that $a \in L_\xi$);
- relations M_0 and M_1 which are interpreted as $M \cap H(\Omega)^V$ and $M \cap H(\Omega)^{V^{Q_0}}$, respectively;
- constants χ, A, κ, T, g, b (b is the function enumerating the branches of T from 3.2, and g is the specializing function $g: T \to \omega$, also from 3.2);
- a function $c: \chi \times \omega_1 \to \chi$ such that, for all $\delta < \chi$; if $\mathrm{cf}(\delta) = \aleph_1$, then $c(\delta, \cdot): \omega_1 \to \delta$ is increasing and cofinal in δ.

Since M, the underlying set of $\mathfrak{M}$, is of cardinality $\aleph_1$, we can find an isomorphic model

$$\overline{\mathfrak{M}} = (\omega_1, \overline{E}, \overline{\chi}, \dots).$$

In V we have names for all the above: $\underset{\sim}{\overline{\mathfrak{M}}}, \underset{\sim}{\overline{E}}$, etc. Now let G^* be a sufficiently generic filter, i.e., G^* will interpret all these names. Writing E^* for $\underset{\sim}{\overline{E}}[G^*]$, etc., and letting $\mathfrak{M}^* = (\omega_1, E^*, \chi^*, \dots)$, we may by 4.7 and 4.2 assume that the following holds:

4.11. Fact. (1) (ω_1, E^*) *is well-founded.*

(2) *if ψ is a closed formula such that $\Vdash_P$ "$\mathfrak{M} \models \psi$", then $\mathfrak{M}^* \models \psi$.* ☺$_{4.11}$

4.12. Main definition. We let

$$\mathfrak{M}' = (M', \in, \chi', \dots)$$

be the Mostowski collapse of $\mathfrak{M}^*$. This is possible by 4.11(1). $\mathfrak{M}'_0 = (M'_0, \in)$ and $\mathfrak{M}'_1 = (M'_1, \in)$ will be "inner models" of $\mathfrak{M}'$.

Note. We will now do several computations and absoluteness arguments involving the universes $V, L[A'], \mathfrak{M}', L[A']^{\mathfrak{M}'} = L_{M' \cap \mathrm{Ord}}[A']$, etc. By default, all set-theoretic functions, quantifiers, etc., are to be interpreted in V, but we will often also have to consider relativized notions, like $\mathfrak{M}' \models$ "$L[A'] \models$ '$\cdots$' " (which is of course equivalent to $L_{M' \cap \mathrm{Ord}}[A] \models$ '$\cdots$'), or $\mathrm{cf}^{L[A']}$, etc.

Note that $\mathfrak{M}' \models$ "$L[A'] \models$ '$\kappa' = \aleph_2$' " so we get $\aleph_1^{\mathfrak{M}'} = \aleph_1^V$.

We will finish the proof of 4.1 with the following two lemmas:

4.13. LEMMA. *$a \in L_{\chi'}, L_{\chi'} \subseteq M'$, and $L_{\chi'} \vDash \varphi(a)$.*

4.14. LEMMA. *$L \vDash \chi'$ is a cardinal.*

Proof of 4.13. Since $\chi' + 1 \subseteq M'$ and $\mathfrak{M}'$ satisfies a large fragment of ZFC, we have $L_{\chi'} \subseteq M'$ and $L_{\chi'} \in M'$. For each $y \in L_\xi$ let c_y be the associated constant symbol; then by induction (using 4.11(2)) it is easy to show that $y = c_y^{\mathfrak{M}'}$ for all $y \in L_\xi$. Since $\Vdash_P$ "$\mathfrak{M} \vDash [L_\chi \vDash$ '$\varphi(a)$']", we thus have $\mathfrak{M}' \vDash$ "$L_{\chi'} \vDash$ '$\varphi(a)$' ". But $L_{\chi'} \subseteq M'$, so $L_{\chi'} \vDash \varphi(a)$.

So we are left with proving 4.14. In $L[A']$ let μ be the cardinality of χ', and (again in $L[A']$) let ν be the successor of μ. We will prove 4.14 by showing the following fact:

4.15. LEMMA. *$\nu \subseteq M'$.*

Proof of 4.14 (using 4.15). In fact we show that 4.15 implies that χ' is a cardinal even in $L[A']$: If not, then $\mu < \chi'$, and since ν is a cardinal in $L[A']$ we can find a $\gamma < \nu$ such that $L_\gamma[A'] \vDash$ "there is a function from μ onto χ'". By 4.15, $\gamma \in M'$, so by the well-known absoluteness properties of L we have $L_\gamma[A'] \subseteq M'$, so $\mathfrak{M}' \vDash$ "$L[A'] \vDash$ 'χ' is not a cardinal' ". But we also have $\Vdash_P \mathfrak{M} \vDash$ "$L[A] \vDash$ 'χ IS a cardinal' ", so we get a contradiction to 4.11(2). ☺$_{4.14}$

Proof of 4.15. We will distinguish two cases, according to what the cofinality of μ is.

Case 1. $\mathrm{cf}(\mu) = \aleph_0$. (This is the "easy" case, for which we do not need to know anything about the forcing Q_1 other than that it is proper, so the class $\{\delta\colon \mathrm{cf}(\delta) = \aleph_0\}$ is the same in V, V^{Q_0}, V^P, and $L[A]$.) We start our investigation of Case 1 with the following remark:

4.16. FACT. (1) *For all δ, if $\mathrm{cf}^{L[A]}(\delta) > \aleph_0$, then $\mathrm{cf}(\delta) > \aleph_0$.*

(1) $\Vdash_P$ "*For all $\delta < \chi$, if $\mathrm{cf}^{L[A]}(\delta) > \aleph_0$, then $\mathrm{cf}(\delta) = \aleph_1$*".

(2) *If $\mathfrak{M}' \vDash \mathrm{cf}^{L[A']}(\mu) > \aleph_0$, then $\mathfrak{M}' \vDash \mathrm{cf}(\mu) = \aleph_1$.*

(3) *If $\mathfrak{M}' \vDash$ "$\mathrm{cf}(\mu) = \aleph_1$", then $\mathrm{cf}(\mu) = \aleph_1$.*

Proof. (1) By the choice of A, (see 4.8(3)).

(2) Use (1) and the fact that P is proper, hence does not cover old uncountable sets by new countable sets.

(3) Use (2) and 4.2.

(4) If $\mathfrak{M}' \vDash$ "$\mathrm{cf}(\mu) = \aleph_1$", then the fucntion $c'(\mu, \cdot)$ is increasing and cofinal in μ. (Recall that $\omega_1^V = \omega_1^{\mathfrak{M}'}$.)

4.17. *Conclusion.* Since $\mathrm{cf}(\mu) = \aleph_0$, from (3) and (4) we get

$$\mathfrak{M}' \vDash \text{"}L[A'] \vDash \text{'}\mathrm{cf}(\mu) = \aleph_0\text{'"}.$$

Let $\mathfrak{M}' \vDash$ "ν_1 is the $L[A']$-successor of μ". We will show that $\nu_1 = \nu$. This suffices, because M' is transitive.

So assume that $\nu_1 < \nu$. Working in $L[A']$, we have $|[\mu]^{\aleph_0}| = \nu$ and $|L_{\nu_1}[A']| < \nu$, so we can find a $y \in [\mu]^{\aleph_0}$ such that $y \in L_\gamma[A'] \setminus L_{\nu_1}[A']$ for some $\gamma < \nu$. Working in V, let $L_\gamma[A'] = \bigcup_{i<\omega_1} X_i$, where $\langle X_i\colon i < \omega_1\rangle$ is a continuous increasing chain of elementary countable submodels of $L_\gamma[A']$, with $y, A' \in X_0$. In $\mathfrak{M}'_1 = (M'_1, \in)$ we can find a continuous increasing sequence $\langle Y_i\colon i < \omega_1\rangle$ of countable elementary submodels of $L_\mu[A']$ with $\bigcup_{i<\omega_1} Y_i = L_\mu[A']$ and $A' \in Y_0$. We can find an i such that $X_i \cap L_\mu[A'] = Y_i$.

Let $j\colon (X_i, \in, A, Y_i) \to (L_{\hat\gamma}[\hat A], \in \hat A, L_{\hat\mu}[\hat A])$ be the collapsing isomorphism.

Now note that $Y_i = X_i \cap L_\mu[A']$ is a transitive subset of X_i, so $j \restriction Y_i$ is exactly the Mostowski collapse of $(Y_i, \in)$, so $j \restriction Y_i \in M_1'$ and $\hat A \in M_1'$. Hence also $j(y) \in L_{\hat\gamma}[\hat A] \subseteq M_1'$, so we can compute

$$y = \{\alpha\colon (j \restriction Y_i)(\alpha) \in j(y)\}$$

in $\mathfrak{M}_1'$. Hence $y \in M_1'$. But

$$\mathfrak{M}' \models \text{“}[\mu]^{\aleph_0} \cap M_1' = [\mu]^{\aleph_0} \cap M_0' = [\mu]^{\aleph_0} \cap L[A']\text{”}$$

(the first equality holds because Q_0 is a σ-closed forcing notion, the second because of our assumption 4.8(3)).

Hence $\mathfrak{M}' \models y \in L[A']$, so $\mathfrak{M}' \models y \in L_{\nu_1}[A']$, a contradiction to our choice of ν. ☺$_{4.15,\text{ Case 1}}$

Case 2. $\mathrm{cf}(\mu) = \aleph_1$. Let $\gamma < \nu$. We have to show that $\gamma \in M'$. Since $L[A'] \models |\gamma| = \mu$, in $L[A']$ we can find an increasing sequence $\langle A_\xi\colon \xi < \mu\rangle, \gamma = \bigcup_{\xi<\mu} A_\xi$, where each A_ξ has cardinality $< \mu$ in $L[A']$. Let α_ξ be the order type of A_ξ; then the inclusion map from A_ξ into A_ζ naturally induces an order-preserving function $f_{\xi\zeta}\colon \alpha_\xi \to \alpha_\zeta$. Let $B = \langle \alpha_\xi, f_{\xi\zeta}\colon \xi \le \zeta < \mu\rangle$, and write $B \restriction \beta$ for $\langle \alpha_\xi, f_{\xi\zeta}\colon \xi \le \zeta < \beta\rangle$. Clearly the direct limit of this system is a well-ordered set of order type γ.

So B is in $L[A']$, but we can moreover show that each initial segment $B \restriction \beta$ is already in $L_\mu[A']$. This follows from the fact that each such initial segment can be canonically coded by a bounded subset of μ.

Since $L_\mu[A'] \subseteq L_{\chi'}[A'] \subseteq M_1'$, we know that $B \restriction \beta$ is in M_1' for all $\beta < \mu$. In M_1' let $\langle \xi_i\colon i < \omega_1\rangle$ be an increasing cofinal subsequence of μ. Let $\beta_i = \alpha_{\xi_i}$ and $h_{ij} = f_{\xi_i,\xi_j}$. Note that the direct limit of the system $\langle \beta_i, h_{ij}; i \le j < \omega_1\rangle$ is still a well-ordered set of order type γ.

So for each $\delta < \omega_1$ we know that the sequence $b_\delta := \langle \beta_i, f_{ij}\colon i \le j \le \delta\rangle$ is in M_1', and $\mathfrak{M}_1' \models b_\delta \in T'$.

Now we can (in V) find an uncountable set $C \subseteq \omega_1$ and a natural number n such that for all $\delta \in C$ we have $g'(b_\delta) = n$. Now recall the characteristic property of g (see 3.2) and hence of g' (by 4.11): for each $\delta_1 < \delta_2$ in C we have a unique branch $B_\alpha' = \{b'(\alpha, \beta)\colon \beta < \omega_1\}$ with $\{b_{\delta_1}, b_{\delta_2}\} \subseteq B_\alpha$. A priori this α depends on δ_1 and δ_2, but since $B_\alpha \cap B_\beta = \varnothing$ for $\alpha \ne \beta$ we must have the same α for all $\delta \in C$.

So the sequence $\langle b_\delta\colon \delta \in C\rangle$ is cofinal on some branch B_α' which is in $\mathfrak{M}'$. So we get that γ, the order type of the limit of this system, is also in M'.

☺$_{4.15,\text{ Case 2}}$ ☺$_{4.1}$

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