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The monadic theory of $(\omega_2, <)$ may be complicated

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Summary. Assume ZFC is consistent then for every $B \subsetneqq \omega$ there is a generic extension of the ground world where B is recursive in the monadic theory of ω_2.

Introduction

The monadic language corresponding to first-order language L is obtained from L by adding variables for sets of elements, atomic formulas $x \in Y$, and the quantifier $(\exists Y)$. The monadic theory of a model M for L is the theory of M in the described monadic language when the set of variables are interpreted as arbitrary subsets of M[1]. Speaking about the monadic theory of an ordinal α, we mean the monadic theory of $\langle \alpha, < \rangle$. Gurevich, Magidor, and Shelah proved in [GMS] the following theorem:

Theorem. *Assume there is a weakly compact cardinal. Then there is an algorithm $n \rightarrow \psi_n$ such that ψ_n is a sentence in the monadic language of order and for every $B \subsetneqq \omega$ there is a generic extension of the ground world with $\{n : \omega_2 \models \psi_n\} = B$.*

Thus, there are continuum many possible monadic theories of ω_2 (is different universes) and for every $B \subsetneqq \omega$ there is a monadic theory of ω_2 (in some world) which is at least as complex as B.

Here we shall eliminate the assumption of the existence of a weakly compact cardinal and will prove the following theorem:

Theorem 1. *There is a set of sentences $\{\theta_n : n < \omega\}$ in the monadic language of order such that:*
if $V \models G.C.H$, then, for each $B \subsetneqq \omega$, there exists a forcing notion $P = P_B$, which is ω_1-closed, satisfies the $\aleph_3$-chain condition, preserves cardinals, cofinalities and the G.C.H and $|P| = \aleph_3$ such that $\Vdash_P \{n : (\omega_2, <) \models \theta_n\} = B$.

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[1] More details and Historical background can be found in [Gu]

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1 The sentences and the forcing notion

Notation. a) $S_i^2:(i=0,1)$ will be the sets $\{\alpha<\omega_2: cf(\alpha)=\omega_i\}$.

b) $S_n(n\leqq\omega)$ are pairwise disjoint stationary subsets of S_1^2 such that $\bigcup_{n\leqq\omega} S_n=S_1^2$.

1. Definition. (i) $\Phi_n(Y):=$"$Y\subseteqq S_1^2$, Y is stationary and for each function $h: Y\to\{0,\dots,n\}$ there is a function $g: S_0^2\to\{0,\dots,n\}$ such that: if $\delta\in Y$, then there is a club subset of $\delta\cap S_0^2$ in which g is constant and different from $h(\delta)$"
in this case we will say that g is a witness for h.

(ii) $\psi_n:=$"$\Phi_n(Y)$ and $\neg\Phi_{n-1}(Y)$ and for each stationary $Z\subseteqq Y$, $\neg\Phi_{n-1}(Z)$".

(iii) $\theta_n:=(\exists Y)[\psi_n(Y)]$.

It is easy to see that Φ_n, ψ_n, and θ_n are in the monadic language of order.

2. Definition. $\bar{Q}=\langle P_i, \underset{\sim}{Q}_i: i<\aleph_3\rangle$ is an iteration with support $<\aleph_2$ each Q_i is of the form Q_g where $g: S_n\to\{0,\dots,n\}$, $g\in V^{P_i}$, $n=n(i)<\omega$,

$Q_g:=\{f\mid$ *there is an ordinal* $\alpha<\aleph_2$ such that
1. $\mathrm{Dom} f=\alpha$,
2. $f:\alpha\to\{0,\dots,n\}$,
3. *if* $\underset{\sim}{\delta}\leqq\alpha$, $\underset{\sim}{\delta}\in S_n$, and $n\in A$ then there is a club subset E of $\underset{\sim}{\delta}$ on which f is constant and different from $g(\delta)\}$.

Q_g will be ordered by inclusion.

Moreover, if $i<\aleph_3$, $g\in V^{P_i}$, then there is a j, $i\leqq j<\aleph_3$ such that $Q_j=Q_g$.

2 Preserving cardinals and cofinality

3. Claim. *P is ω_1-closed.*

Proof. Easy.

4. Definition. (I) Let $S\subseteqq\aleph_2$. We will say that a model N is *suitable* for S if for a large enough χ, $N\prec(H(\chi),\varepsilon,<*)$, ($<*$ denotes a well ordering), $N\supset[N]^\omega$, $\|N\|=\aleph_1$, and $N\cap\omega_2\in S$.

(II) Let $P\in N$ be a forcing notion. $\langle p_\zeta:\zeta<\omega_1\rangle$ is a *generic sequence* for (N,P) if $p_\zeta\in P\cap N$, $P\models p_\zeta\leqq p_{\zeta+1}$ and for every dense open subset D of P which belongs to N, $D\cap\langle p_\zeta:\zeta<\omega_1\rangle\neq\emptyset$.

(III) We will say that P is *S-complete* if for every N, suitable for S such that $P\in N$, every generic sequence $\langle p_\zeta:\zeta<\omega_1\rangle$ for (N,P) has an upper bound is P.

4a. Observation. ($2^{\aleph_0}=\aleph_1$) for every $S\subseteqq S_1^2$, stationary, given a large enough χ and $X\in H(\chi)$ there exists a model N, suitable for S with $X\in N$.

5. Claim. *P is S_ω-complete (and in fact, S_n-complete for every $n\notin A$).*

Proof. Let N be suitable for S_ω $\bar{p}=\langle p_\zeta:\zeta<\omega_1\rangle\subseteqq P$ generic for (N,P), we will find an upper bound for the sequence. Define inductively on $j\in N\cap\aleph_3$ conditions q_j in P_j such that for every $\zeta<\omega_1$, $q_j\geqq p_\zeta\restriction_j$:
for $j+1$: Let $\underset{\sim}{r}$ be a P_j name for $\bigcup_\zeta p_\zeta(j)$. Since $\bar{p}$ is generic, $\Vdash_{P_j}\mathrm{Dom}\underset{\sim}{r}=\delta=N\cap\aleph_2$, therefore, since $\delta\in S_\omega$, $\Vdash_{P_j}\underset{\sim}{r}\in Q_j$ so $\underset{\sim}{r}$ is a condition.

Now let $g_{j+1}=q_j \vee \underset{\sim}{r}$ where $[q_j \vee \underset{\sim}{r}](\beta)=q_j(\beta)$ for $\beta<j$ and $[g_j \vee \underset{\sim}{r}](j)=\underset{\sim}{r}$. So g_{j+1} is a condition and it satisfies the requirements.
For j limit: take $\bigcup_{i<j} g_i$. Now $q_{\aleph_1}$ is the required upper bound. □

6. Corollary. *P does not add ω_1-sequences.*

Proof. Let $\underset{\sim}{c}$ be a P-name for an ω_1-sequence in V, $p \in P$ forcing it. It suffices to find a condition $q \geqq p$ such that $q \Vdash$ "$\underset{\sim}{c} \in V$".

Let N be suitable for S_ω, $P, p, \underset{\sim}{c} \in N$. N has $\aleph_1$ dense open sets $\{D_i : i<\omega_1, i=j+1\}$. We will construct inductively a sequence $\langle p_i : i<\omega_1\rangle$ generic for (N, P):

$p_0 = p$
p_{i+1}: take $r \geqq p_i$ such that $r \Vdash$ "$\underset{\sim}{c} \restriction_{i+1} \in V$" (an ω_1-complete forcing notion does not add new ω-sequences) and then $s \geqq r$ such that $s \in D_{i+1}$. Let $p_{i+1}=s$.
p_δ (δ a limit ordinal): use ω_1-completeness.
Clearly $\langle p_i : i<\omega_1\rangle$ is generic for (N, P) and by Claim 5, there exists an upper bound q for the sequence. So $q \geqq p$ and $q \Vdash$ "$\underset{\sim}{c} \in V$". □

7. Definition. A condition $p \in P$ will be called *real and rectangular* if there is a $\delta < \aleph_2$ s.t. for every $\beta \in \operatorname{Dom} p$, $p(\beta)$ is a function (not a name!) and $\operatorname{Dom} p(\beta)=\delta$.

8. Corollary. *For every $i<\aleph_3$ the set $\{p \in P_i$: is real and rectangular$\}$ is dense.*

Proof. Let $p \in P_\alpha$ be a condition, we have to find $q \geqq p$, q real and rectangular. Let N be suitable for $S_\omega, p, P_\alpha \in N$, $\delta = N \cap \omega_2$ (so $\delta \in S_\omega$), let $\{\alpha_i : i<\omega_1\}$ be the support of p. By Corollary 6, there's a real function extending every name $p(\alpha_i)$, therefore it's possible to build a sequence $\bar{q}=\langle q_i : i<\omega_1\rangle$ generic for (N, P_α) such that for every $i>j$, $q_i(\alpha_j)$ is a real function, and $q_0=p$. Let q be an upper bound for $\bar{q}$. Then, $q \restriction \delta$ is real and rectangular extending p, where $q \restriction \delta(i) := q(i) \restriction \delta$. □

9. Conclusion. *P satisfies the $\aleph_3$-chain condition.*

Proof. Take a set of conditions $\{p_i : i<\aleph_3\}$, we will find two compatible members. W.l.o.g., all the conditions are real and rectangular, moreover, we can assume they are all of height δ and (by the Δ-system theorem and G.C.H.) that $i \neq j \Rightarrow \operatorname{Dom} p_i \cap \operatorname{Dom} p_j$ is constant. But, assuming G.C.H., there are only $\aleph_2$ real and rectangular conditions with the same height and support. Therefore, there are two conditions p_i and p_j such that

$$p_i \restriction_{\operatorname{Dom} p_i \cap \operatorname{Dom} p_j} = p_j \restriction_{\operatorname{Dom} p_i \cap \operatorname{Dom} p_j}.$$

So they are compatible. □

10. Conclusion. *P preserves cardinals, the G.C.H. and cofinalities.*

Proof. Combine Claim 3, Corollary 6 and Conclusion 9. □

3 Preserving Stationarity

We shall prove that forcing with P does not destroy the stationarity of the sets S_n, using a construction similar to the construction in [SK] Lemma 2.8, and in [Sh3] but really simpler as in [Sh2] as S_ω is stationary.

11. Proposition. $\Vdash_P$"S_n *is a stationary subset of* ω_2".

Proof. Case (I) $n \notin A$:

Let $\underset{\sim}{C}$ a name for a club subset, $p \in P_\alpha$ forcing it. Let N be suitable for S_n, $\delta = N \cap \omega_1$, (so $\delta \in S_n$) $\underset{\sim}{C}$, p, $P_\alpha \in N$. We will find a condition $q \geqq p$ forcing $\delta \in \underset{\sim}{C}$. Let $\langle D_i : i < \omega_1, i = j+1 \rangle$ a sequence of the dense open subsets of P in N and $\langle \delta_i : i < \omega_1, \delta_i \in N \rangle$ an unbounded sequence in δ. We will construct a sequence of conditions $\bar{q} = \langle q_i : i < \omega_1, q_i \in N \cap P_\alpha \rangle$ inductively:

$$i = 0 : q_0 = p,$$

$$i = j+1 : q_i \geqq q_j,\ q_i \in D_j,\ q_i \Vdash (\exists x)(x \in \underset{\sim}{C} \,\&\, x > \delta_j),$$

i limit: take union.

So $\bar{q}$ is generic for (N, P_α) and therefore, by S_n-completeness it has an upper bound q. But q forces the existence of a subsequence of $\underset{\sim}{C}$, unbounded in δ so $q \Vdash \delta \in \underset{\sim}{C} \cap S_n$.

Case (II) $n \in A$:

Let $p \in P_\alpha$ forcing "$\underset{\sim}{C}$ is a club subset of ω_2". We will find a condition $q, q \geqq p$, $q \Vdash$ "$\underset{\sim}{C} \cap S_n \neq \emptyset$".

Let $\bar{N} = \langle N_\zeta : \zeta < \omega_1 \rangle$ an increasing continuous sequence of models, $N = \bigcup_{\zeta < \omega_1} N_\zeta$ such that:

(a) $N_\zeta \prec (H(\chi), \in, \underset{\sim}{C}, p, P_\alpha, \leqq_{P_\alpha}, \Vdash, <^*_\chi)$, $\quad \|N_\zeta\| = \aleph_1$,

(b) $\bar{N} \restriction_{[\zeta+1]} \in N_{\zeta+1}$, $\quad {}^\omega[N_{\zeta+1}] \subseteqq N_{\zeta+1}$,

(c) $\delta := N \cap \omega_2 \in S_n$,

(d) $\delta_{\zeta+1} := N_{\zeta+1} \cap \omega_2 \in S_\omega$.

Now let $A := \alpha \cap N = \langle \alpha_\zeta : \zeta < \omega_1 \rangle$ (and we can assume $\alpha_{\zeta+1} \in N_{\zeta+1}$) and $A_\zeta = \langle \alpha_\eta : \eta < \zeta \rangle$.

T_ζ will be the set of functions t such that:

(a) $\operatorname{Dom} t = A_\zeta$,

(b) for every $\alpha_\xi \in A$ $(\xi < \zeta)$, $\quad \operatorname{Dom}[t(\alpha_\xi)] = \{\delta_\eta : \xi \leqq \eta < \zeta\}$,

(c) $t(\alpha_\xi)$ is a constant function and equals a natural number $\leqq n(\alpha_\xi)$.

Note that $\zeta < \aleph_1 \Rightarrow |A_\zeta| \leqq \aleph_0$ and $|T_\zeta| \leqq \aleph_1$ and $T_{\zeta+1} \subseteqq N_{\zeta+1}$ and $T_{\zeta+1} \in N_{\zeta+1}$ and every $t \in T_\zeta$ is compatible with p.

Now define inductively $\bar{q}^\zeta = \langle q^\zeta_t : t \in T_\zeta \rangle$ with

(a) $q^\zeta_t \in P_\alpha$ real and rectangular and inducting t
($\operatorname{Dom} t \subseteqq \operatorname{Dom} q^\zeta_t$ and $t(\alpha) \subseteqq q^\zeta_t(\alpha)$ for $\alpha \in \operatorname{Dom} t$),

(b) $\bar{q}^{\zeta+1} \in N_{\zeta+1}$,

(c) $q^{\zeta+1}_t \Vdash$"there is an ordinal γ s.t. $\gamma \in \underset{\sim}{c}$, $\delta_{\zeta+1} > \gamma \geqq \delta_\zeta$",

(d) $\alpha_\eta \subseteqq \operatorname{Dom} q^{\zeta+1}_t$,

(e) $s \restriction_\beta = t \restriction_\beta \Rightarrow q^\zeta_t \restriction_\beta = q^\zeta_s \restriction_\beta$ for every $\beta \in A_\zeta$,

$\zeta = 0$: take $q^0_0 = p$.

ζ limit: take limit.

$\zeta=\eta+1$: Suppose we have defined $\bar{q}^\eta$ and remember that $|T_{\zeta+1}|\leqq\aleph_1$. Let $\langle t_i : i<\omega_1\rangle$ an enumeration of T_ζ, each member is taken $\aleph_1$ times. Choose by induction on i, $q_i\in N_\zeta$ such that

(1) $q_i\geqq q^\zeta_{s_i}$ where s_i is the only member in T_η satisfying $s_i\leqq t_i$,

(2) for every $\beta\in A_\zeta\cup\{\alpha\}$ and every $j<i$, if $t_j\restriction_\beta=t_i\restriction_\beta$ then $q_j\restriction_\beta=q_i\restriction_\beta$;

(3) for every t, the sequence $\langle q_i | t_i=t\rangle$ is generic for (N_ζ, P_α).

Now, for every t, the sequence $\langle q_i | t_i=t\rangle$ has an upper bound, and w.l.o.g. it is real and rectangular. So there is one in $H(\chi)$, choose the first one by $<^*_\chi$. It is easy to verify that the chosen upper bound satisfies a, c, d, and e. So there is a sequence in $H(\chi)$ with the properties of $\bar{q}^\zeta$ (take the first upper bounds for every $t\in T_\zeta$) and since $N_\zeta \prec H(\chi)$ there is one in N_ζ, the "first" one is the required $\bar{q}^\zeta$.

Having finished we will get a tree $T=T_{\omega_1}$ of functions and a tree T' of conditions "inducing" T, both of height ω_1.

We can correspond to each branch $b\subseteq T'$ a sequence $\eta\in{}^A\omega$ such that for every $q=q^\zeta_i\in b$ and $\beta\in\mathrm{Dom}(q)$, $\eta(\beta)=k$ iff $t(\beta)\equiv k$. In fact, the correspondence is $1-1$ if we restrict ourselves to sequences η with $\eta(\beta)\leqq n(\beta)$. Now define a P-name $\underset{\sim}{\eta}$ of a sequence in ${}^A\omega$ such that $\underset{\sim}{\eta}(\beta)=k \Rightarrow g(\delta)\neq k$ where $k\leqq n(\beta)$ and $Q_\beta=\underset{\approx}{Q}_g$. So $\underset{\sim}{\eta}(\beta)$ is a possible constant value for a member of $\underset{\approx}{Q}_\beta$ on a club subset of $\tilde{\delta}$. By the previous remark, $\underset{\sim}{\eta}$ can be viewed as a name of a branch in T'. It is easy to see that $\underset{\sim}{\eta}$ can be extended by a condition q and that $p\leqq q\Vdash\delta\in\underset{\sim}{C}\cap S_n$. □

12. Corollary. *For every $n\in A$, $\Vdash_P\Phi_n(S_n)$. Therefore for every $Y\subseteq S_n$ stationary, $\Vdash_P\Phi_n(Y)$.*

Proof. By 11. S_n is stationary, also, we have dealt with every possible function since because of the $\aleph_3$-chain condition, every P-name of a function is a P_i $(i<\aleph_3)$ name of one which has been taken care of. □

4 *B* is recursive in *M*th $(\omega_2, <)$

13. Proposition. *Suppose $V^P\models\Phi_n(Y)$, then $V^P\Vdash Y\subseteq\bigcup_{\substack{i\in B\\ i\leqq n}} S_i(\mathrm{mod}\,D_{\omega_2})$.*

Proof. By the $\aleph_3$-chain condition there is a P_α-name $\underset{\sim}{Y}$ such that $Y=\mathrm{Rel}(\underset{\sim}{Y}, Gp_\alpha)$. There is a $k\leqq\omega$ such that $Z:=Y\cap S_k$ is stationary.

Since $\Vdash_{P/p_\alpha}\Phi_n(\underset{\sim}{Y})$, also $\Vdash_{P/p_\alpha}\Phi_n(\underset{\sim}{Z})$ ($\underset{\sim}{Z}$ a name for Z). We will show that the only possible case is $k\leqq n$ and $k\in B$.

Case (I): Assume $k\in B$ but $k>n$.

W.l.o.g. $n(\alpha)=n$ [otherwise take $\alpha'>\alpha$ with $n(\alpha')=n$] the realization of the generic filter $\underset{\sim}{G}_\alpha\subseteq P_\alpha$ gives a function $f_\alpha:\omega_2\to\{0,\dots,n\}$. We will show that this function contradicts $\Phi_n(Z)$.

Otherwise there is a $p\in P$ and a P-name $\underset{\sim}{h}$ such that p forces: "$\underset{\sim}{h}$ is a witness for f_α and $\underset{\sim}{Z}$". Let N be suitable for Z, $\|N\|=\aleph_1$, $N\cap\omega_2=\delta\in Z$ (so $\delta\in S_k$), ${}^\omega[N]\subseteq N$, $\underset{\sim}{h}, \underset{\sim}{Z}, \alpha, G_{P_\alpha}\in N$. We will build a tree of conditions above p similarly to the construction in Proposition 11. Denote

$$A=N\cap\mathrm{supp}(p)\backslash\alpha=\{\alpha_\zeta:\zeta<\omega_1\},$$

(and we can assume w.l.o.g. that $\alpha_0 = \alpha + 1$ and that the sequence is increasing), then each branch of the tree can be viewed as a sequence $\bar{p}_\eta$, generic for (N, P) with $\eta \in {}^A k$. Denote the union of the sequence by p_η so p_η is a function. For each i such that $n(i) = k$ $p_\eta(i)$ is a function with domain δ, constant on a club subset of δ and equal there to $\eta(i)$. Moreover if $\eta \upharpoonright_i = \nu \upharpoonright_i$ then $p_\eta \upharpoonright_i = p_\nu \upharpoonright_i$.

In the places where $n(i) \neq k$ the value of $p_\eta(i)$ is not interesting and we will consider only the sequences η with $\eta(i) = 0$. Our aim now is to show that there is a branch that can be extended by n different conditions.

Now we will choose an increasing and continuous sequence of models $\bar{M} = \langle M_\zeta : \zeta < \omega_1 \rangle$ with:

(a) $N \prec M_0 \prec \langle H(\chi), \in, <^*_\chi \rangle, \quad \|M_\zeta\| = \aleph_1,$

(b) $\langle \bar{p}_\eta : \eta \in {}^A k \rangle \in M_0,$

(c) $M_{\zeta+1} \cap \omega_2 \in S_\omega,$

(d) $\bar{M} \upharpoonright_{\zeta+1} \in M_{\zeta+1},$

(e) ${}^\omega[M_{\zeta+1}] \subseteq M_{\zeta+1},$

Using $<^*_x$ we will choose inductively a sequence of sets of conditions $\langle q^l_{\alpha_l} : l \leq n \rangle$ and names of sequences $\underset{\sim}{\eta}_\zeta$ such that

(a) $q^l_{\alpha_\zeta} \in P/P_\alpha, \quad q^l_{\alpha_\zeta} \in M_{\zeta+1}, \quad \underset{\sim}{\eta}_\zeta \in {}_{A_\zeta}k, \quad \underset{\sim}{\eta}_\zeta \lhd \underset{\sim}{\eta}_{\zeta+1},$

(b) $q^l_{\alpha_\zeta}$ extends $p_{\eta \restriction A_\zeta}$ for every η with $\underset{\sim}{\eta} \lhd \eta$,

(c) q^l_ζ is real and rectangular and in every open and dense subset of P_ζ in M_ζ,

(d) $[q^l_{\alpha+1}(\alpha)](\delta) = l.$

Problems arise only when $n(\zeta) = k$ so suppose we have chosen $\langle q^l_\xi : \xi \leq \zeta \rangle$ and $\underset{\sim}{\eta}_\zeta$ and we want to choose $\underset{\sim}{\eta}_{\zeta+1}$. But each q^l_ζ rules out one possibility of extending $\underset{\sim}{\eta}_\zeta$ (i.e. it rules out one possible value for a function on a club subset of δ) so, $n+1$ possibilities are ruled out, but $k > n$ so at least one value is left to be chosen. In the end we will get a sequence $\underset{\sim}{\eta} \in {}^A k$ and conditions $\{q^l_{\aleph_3}\}_{l \leq n}$ each one of them above p_η and thus they all force the same value to $h \upharpoonright \delta$. (Every sequence $\langle q^l_\zeta : \zeta < \omega_1 \rangle$ can be extended by a condition $q^l_{\aleph_3}$).

Now, there is $0 \leq m \leq n$ such that $h^{-1}(\{m\})$ is a stationary subset of δ and $q^m_{\aleph_3}$ contradicts it since $q^m_{\aleph_3} \Vdash \underset{\sim}{f}_\alpha(\delta) = m$. So we have found $q^m_{\aleph_3} \geq p$ forcing "$\underset{\sim}{h}$ is not a witness for $\underset{\sim}{f}_\alpha$" a contradiction.

Case (II) $k \notin B$.

Follow the same construction. When choosing $\underset{\sim}{\eta}_\zeta$ no possibilities are ruled out so it should be slightly easier. □

14. Conclusion. $V^P \models \psi_n(Y)$ *for a stationary* Y *iff* $Y \subseteq S_n (\mathrm{mod}\, D_{\omega_2})$ *and* $n \in B$.

15. Conclusion. $\{n : V^P \models \phi_n\} = B$.

And this finishes the proof of Theorem 1.

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