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There are no very meager sets in the model in which both the Borel Conjecture and the dual Borel Conjecture are true

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We show that the model for the simultaneous consistency of the Borel Conjecture and the dual Borel Conjecture given in [4] actually satisfies a stronger version of the dual Borel Conjecture: there are no uncountable very meager sets.

1 Introduction

A set $X \subseteq 2^\omega$ is called *strong measure zero* if for each sequence $(\varepsilon_n)_{n<\omega}$ of positive real numbers there is a sequence $([s_n])_{n<\omega}$ of basic clopen sets such that $\mu([s_n]) < \varepsilon_n$ for each $n < \omega$ and $X \subseteq \bigcup_{n<\omega}[s_n]$. The following statement is known as the *Borel Conjecture* (BC): "Every strong measure zero set is countable".

Galvin, Mycielski and Solovay [3] proved the following characterization of strong measure zero sets: $X \subseteq 2^\omega$ is strong measure zero if and only if every comeager (dense G_δ) set contains a translate of X. This gives rise to the following dual notion (introduced by Příkrý):

> A set $X \subseteq 2^\omega$ is called *strongly meager* ($X \in \mathcal{SM}$) if every set of Lebesgue measure 1 contains a translate of X.

The *dual Borel Conjecture* (dBC) is the statement: "Every strongly meager set is countable".

Under CH, both BC and dBC fail. Laver and Carlson showed that BC and dBC are consistent, respectively [7, 2]. In [4], the consistency of BC + dBC was proved (i.e., consistently, BC and dBC hold simultaneously). The purpose of this paper is to show that the model given in [4] actually satisfies a stronger version of dBC. Here, we call this model $\mathbf{M}_{\mathsf{BC+dBC}}$ (which is the "final model" $V^{\mathbb{R}*\mathbf{P}_{\omega_2}}$ in [4]).

We have to assume familiarity with [4], since we strengthen the result from the paper by modifying the respective arguments. In particular, we use the concept of "Janus forcing" introduced there (cf. [4, § 2.A]). Each time we use (or modify) a lemma of [4], we are going to explicitly say so.

2 Very meager sets

Note that $X \subseteq 2^\omega$ is strongly meager if and only if for each null set $Z \subseteq 2^\omega$ there is a real $t \in 2^\omega$ such that $X \subseteq t + (2^\omega \setminus Z)$. The following weaker[1] notion was defined in Kysiak's master thesis (cf. [5, Definicja 5.4]; in Polish):

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[1] Actually, both the notion of strongly meager and the notion of very meager (cf. Definition 2.1) can be considered to be a "category analogue" of the notion of strong measure zero. This is because the two respective "versions" of strong measure zero coincide: according to the Galvin-Mycielski-Solovay characterization mentioned in the introduction, X is strong measure zero if and only if for each meager set

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Definition 2.1 A set $X \subseteq 2^\omega$ is *very meager* ($X \in \mathcal{VM}$) if for each null set $Z \subseteq 2^\omega$ there is a countable set $T \subseteq 2^\omega$ such that $X \subseteq T + (2^\omega \setminus Z)$.

Clearly, $\mathcal{SM} \subseteq \mathcal{VM}$, i.e., every strongly meager set is very meager; it is easy to see that the collection $\mathcal{VM}$ of very meager sets forms a σ-ideal; in particular, the σ-ideal generated by the strongly meager sets (denoted by $\sigma\langle\mathcal{SM}\rangle$) is contained in $\mathcal{VM}$. Under CH, the collection of strongly meager sets does not form an ideal (cf. [1]).

The following is unknown:

Question 2.2 Do the σ-ideal generated by the strongly meager sets and the collection of very meager sets coincide in general?

Note that even without knowing the answer to this question, it might be the case that the dual Borel Conjecture implies that all very meager sets are countable.

The notion of very meager does not appear often in the literature; the only published reference for the definition we are aware of is [6, Definition 2.4]. A quite similar notion was considered by Scheepers (cf. [8]):[2]

Definition 2.3 A set $X \subseteq 2^\omega$ is *piecewise strongly meager* ($X \in \mathcal{PSM}$) if for each countable sequence $C_0, C_1, C_2, \ldots$ of closed nowhere dense subsets of 2^ω with $\lim_{n\to\infty} \mu(2^\omega \setminus C_n) = 0$, there is a countable sequence $t_0, t_1, t_2, \ldots$ of elements of 2^ω such that $X \subseteq \bigcup_{n\in\omega}(t_n + C_n)$.

The collection of piecewise strongly meager sets forms a σ-ideal containing all strongly meager sets (as the collection of very meager sets does). Moreover, it is not hard to show that every piecewise strongly meager set is very meager.

Altogether, we have the following inclusions: $\mathcal{SM} \subseteq \sigma\langle\mathcal{SM}\rangle \subseteq \mathcal{PSM} \subseteq \mathcal{VM}$.

For the proof of the main result, we prepare ourselves with a lemma yielding an equivalent formulation of being (not) very meager.

Lemma 2.4 *Let $X \subseteq 2^\omega$, and let $Z \subseteq 2^\omega$ be a null set. Then the following are equivalent:*

1. *for each countable set $T \subseteq 2^\omega$, we have $X \not\subseteq T + (2^\omega \setminus Z)$,*
2. *whenever $\bigcup_{\ell\in\omega} X_\ell = X$ is a partition of X, there exists an $\ell \in \omega$ such that $X_\ell + Z = 2^\omega$.*

P r o o f. An easy computation shows the equivalence of the two assertions (even for arbitrary $Z \subseteq 2^\omega$). □

We say that X *is not very meager witnessed by* Z if Lemma 2.4 (1) or Lemma 2.4 (2) holds (for X and Z). By definition, X is not very meager if and only if there is a null set Z such that X is not very meager witnessed by Z.

In § 3.1 we are going to use the formulation of Lemma 2.4 (2), whereas in § 3.2 the one of Lemma 2.4 (1).

3 No uncountable very meager sets in $\mathbf{M}_{\mathsf{BC+dBC}}$

We strengthen the result of [4] by showing that the following stronger version of dBC indeed holds[3] in the model $\mathbf{M}_{\mathsf{BC+dBC}}$:

Theorem 3.1 *In the model $\mathbf{M}_{\mathsf{BC+dBC}}$ for $\mathrm{Con}(\mathsf{BC}+\mathsf{dBC})$, we have $\mathcal{VM} = [2^\omega]^{\leq\aleph_0}$, i.e., every very meager set is countable.*[4]

The proof of Theorem 3.1 is a slight variant of the proof in [4] that dBC holds in $\mathbf{M}_{\mathsf{BC+dBC}}$: in § 3.1 we show how to adapt [4, Lemma 2.9], and in § 3.2 we present the necessary modifications in [4, § 5].

$M \subseteq 2^\omega$ there is a real $t \in 2^\omega$ such that $X \subseteq t + (2^\omega \setminus M)$. One could define that a set X is "very null" if for each meager set $M \subseteq 2^\omega$ there is a countable set $T \subseteq 2^\omega$ such that $X \subseteq T + (2^\omega \setminus M)$. But actually it is not too hard to prove that these two properties are equivalent (cf. [5, Stwierdzenie 5.6]; in Polish), i.e., the notions of strong measure zero and very null coincide.

[2] In [8], the definition is given for subsets of the real line $\mathbb{R}$; we give the analogous definition for 2^ω.

[3] The second author thanks Marcin Kysiak for asking him this question (at the Winterschool 2011 in Hejnice, Czech Republic). Kysiak wondered whether the model $\mathbf{M}_{\mathsf{BC+dBC}}$ could be the "first model" in which the σ-ideal generated by the strongly meager sets is strictly smaller than the collection of very meager sets. Theorem 3.1 shows that it is not.

[4] Let us point out that the result looks like an "asymmetric strengthening" of $\mathrm{Con}(\mathsf{BC}+\mathsf{dBC})$, only being concerned with dBC. However, the respective "stronger version of BC" is equivalent to BC (and therefore holds in $\mathbf{M}_{\mathsf{BC+dBC}}$ anyway): this is due to the fact that the two respective "versions" of strong measure zero coincide; cf. footnote 1.

3.1 Janus forcing kills very meager sets

We adapt the lemma in [4] for "killing strongly meager sets" to the setting of "killing very meager sets". The original [4, Lemma 2.9] reads:

Lemma 3.2 *If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar\ell^*$, and $\underset{\sim}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underset{\sim}{R}$ forces that X is not strongly meager witnessed by the null set $\underset{\sim}{Z}_\nabla$.*

Recall that we have a fixed increasing sequence $\bar\ell^* = (\ell^*_i)_{i\in\omega}$ and B^*, and that whenever we say "(very) thin" we mean "(very) thin with respect to $\bar\ell^*$ and B^*" (cf. [4, Definition 1.22]).

The adapted lemma reads as follows (and will be used in § 3.2 to obtain the strong version of dBC in the final model):

Lemma 3.3 *If X is not thin, $\mathbb{J}$ is a countable Janus forcing based on $\bar\ell^*$, and $\underset{\sim}{R}$ is a $\mathbb{J}$-name for a σ-centered forcing notion, then $\mathbb{J} * \underset{\sim}{R}$ forces that X is not very meager witnessed by the null set $\underset{\sim}{Z}_\nabla$.*

Proof. Let $\underset{\sim}{c}$ be a $\mathbb{J}$-name for a function $\underset{\sim}{c} : \underset{\sim}{R} \to \omega$ witnessing that $\underset{\sim}{R}$ is σ-centered.

Assume towards a contradiction that $(p, r) \in \mathbb{J} * \underset{\sim}{R}$ forces the opposite, i.e., X is very meager witnessed by the null set $\underset{\sim}{Z}_\nabla$. According to Lemma 2.4 (2), we can fix $(\mathbb{J} * \underset{\sim}{R})$-names $(\underset{\sim}{\xi}_\ell)_{\ell<\omega}$ and "partition labels" $(\underset{\sim}{\ell}_x)_{x\in X}$ (i.e., the name $\underset{\sim}{\ell}_x$ tells us which part of the partition of X the element x belongs to) such that $(p, r) \Vdash (\forall x \in X)\ \underset{\sim}{\xi}_{\underset{\sim}{\ell}_x} \notin x + \underset{\sim}{Z}_\nabla$. By definition of $\underset{\sim}{Z}_\nabla$, we get

$$(p, r) \Vdash (\forall x \in X)\,(\exists n \in \omega)\,(\forall i \geq n)\ \underset{\sim}{\xi}_{\underset{\sim}{\ell}_x}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}^\nabla_i.$$

For each $x \in X$ we can find $(p_x, r_x) \leq (p, r)$ and natural numbers $n_x \in \omega$, $m_x \in \omega$ and $\ell_x \in \omega$ such that p_x forces that $\underset{\sim}{c}(r_x) = m_x$, and that

$$(p_x, r_x) \Vdash \underset{\sim}{\ell}_x = \ell_x$$

and

$$(p_x, r_x) \Vdash (\forall i \geq n_x)\ \underset{\sim}{\xi}_{\ell_x}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}^\nabla_i.$$

So $X = \bigcup_{p\in\mathbb{J}, m\in\omega, n\in\omega, \ell\in\omega} X_{p,m,n,\ell}$, where $X_{p,m,n,\ell}$ is the set of all x with $p_x = p$, $m_x = m$, $n_x = n$, $\ell_x = \ell$. (Note that $\mathbb{J}$ is countable, so the union is countable.) As X is not thin, there is some p^*, m^*, n^*, ℓ^* such that $X^* := X_{p^*,m^*,n^*,\ell^*}$ is not very thin.

In the preceding paragraph, we see the (result of the) essential modification of the original proof: the partition $X = \bigcup_{p\in\mathbb{J}, m\in\omega, n\in\omega, \ell\in\omega} X_{p,m,n,\ell}$ replaces the partition $X = \bigcup_{p\in\mathbb{J}, m\in\omega, n\in\omega} X_{p,m,n}$. Starting with the next paragraph, the proof is literally the same as the original proof of [4, Lemma 2.9] (with $\underset{\sim}{\xi}$ replaced by $\underset{\sim}{\xi}_{\ell^*}$).

We get for all $x \in X^*$:

$$(p^*, r_x) \Vdash (\forall i \geq n^*)\ \underset{\sim}{\xi}_{\ell^*}\restriction L_i \notin x\restriction L_i + \underset{\sim}{C}^\nabla_i. \tag{1}$$

Since X^* is not very thin, there is some $i_0 \in \omega$ such that for all $i \geq i_0$

$$\text{the (finite) set } X^*\restriction L_i \text{ has more than } B^*(i) \text{ elements.} \tag{2}$$

Due to the fact that $\mathbb{J}$ is a Janus forcing (cf. [4, Definition 2.5 (3)]), there are arbitrarily large $i \in \omega$ such that there is a core condition $\sigma = (A_0, \ldots, A_{i-1}) \in \nabla$ with

$$\frac{|\{A \in \mathcal{A}_i : \sigma^\frown A \not\leq_{\mathbb{J}} p^*\}|}{|\mathcal{A}_i|} \geq \frac{2}{3}. \tag{3}$$

Fix such an i larger than both i_0 and n^*, and fix a condition σ satisfying (3).

We now consider the following two subsets of $\mathcal{A}_i$:

$$\{A \in \mathcal{A}_i : \sigma^\frown A \not\leq_{\mathbb{J}} p^*\} \ \text{ and } \ \{A \in \mathcal{A}_i : X^*\restriction L_i + A = 2^{L_i}\}. \tag{4}$$

By (3), the relative measure (in $\mathcal{A}_i$) of the left one is at least $\frac{2}{3}$; due to (2) and the definition of $\mathcal{A}_i$ according to [4, Corollary 2.2], the relative measure of the right one is at least $\frac{3}{4}$; so the two sets in (4) are not disjoint, and we can pick an A belonging to both.

Clearly, $\sigma^\frown A$ forces (in $\mathbb{J}$) that $\underset{\sim}{C}_i^\nabla$ is equal to A. Fix $q \in \mathbb{J}$ witnessing $\sigma^\frown A \not\perp_{\mathbb{J}} p^*$. Then

$$q \Vdash_{\mathbb{J}} X^*\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + A = 2^{L_i}. \tag{5}$$

Since p^* forces that for each $x \in X^*$ the color $\underset{\sim}{c}(r_x) = m^*$, we can find an r^* which is (forced by $q \leq p^*$ to be) a lower bound of the *finite* set $\{r_x : x \in X^{**}\}$, where $X^{**} \subseteq X^*$ is any finite set with $X^{**}\restriction L_i = X^*\restriction L_i$.

By (1),

$$(q, r^*) \Vdash \underset{\sim}{\xi}_{\ell^*}\restriction L_i \notin X^{**}\restriction L_i + \underset{\sim}{C}_i^\nabla = X^*\restriction L_i + \underset{\sim}{C}_i^\nabla,$$

contradicting (5). □

3.2 The strong version of the dual Borel Conjecture in the final model

We now show how to modify the respective part of [4, § 5]. The original [4, Lemma 5.2] reads:

Lemma 3.4 *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not* strongly meager.

The adapted lemma reads as follows:

Lemma 3.5 *Let $X \in V$ be an uncountable set of reals. Then $\mathbb{R} * \mathbf{P}_{\omega_2}$ forces that X is not* very meager.

Note that (as the lemmas in [4, § 5.A]) the lemma shows the stronger version of dBC *only* for sets in the *ground model V*. However, the way to conclude that "there are no uncountable very meager sets in the final model $V^{\mathbb{R}*\mathbf{P}_{\omega_2}}$" from our Lemma 3.5 is completely analogous to the way that "there are no uncountable strongly meager sets in the final model $V^{\mathbb{R}*\mathbf{P}_{\omega_2}}$" was concluded from [4, Lemma 5.2] (i.e., Lemma 3.4) in [4] (using the "factor lemma", cf. [4, § 5.B]). So we do not repeat the arguments given there, i.e., Lemma 3.5 finishes the proof of Theorem 3.1.

P r o o f o f L e m m a 3.5. The proof is parallel to the one of [4, Lemma 5.2] in [4, § 5.A] (and therefore also to the one of [4, Lemma 5.1]):

(1) Fix any even $\alpha < \omega_2$ (i.e., an ultralaver position) in our iteration. The Janus forcing $\mathbf{Q}_{\alpha+1}$ adds a (canonically defined code for a) null set $\dot{Z}_\nabla$. (Cf. [4, Definition 2.6] and [4, Fact 2.7].)

In the following, when we consider various Janus forcings $\mathbf{Q}_{\alpha+1}$, $Q_{\alpha+1}$, $Q^x_{\alpha+1}$, we treat $\dot{Z}_\nabla$ not as an actual name, but rather as a definition which depends on the forcing used.

(2) It is enough to show that "X is not very meager witnessed by $\dot{Z}_\nabla$" holds in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension; i.e., $X \nsubseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ holds for every countable set $T \subseteq 2^\omega$ (cf. Lemma 2.4 (1)).

Assume towards a contradiction that we have $X \subseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ for some fixed countable $T \subseteq 2^\omega$ (in the $\mathbb{R} * \mathbf{P}_{\omega_2}$-extension). We can fix a β with $\alpha < \beta < \omega_2$ such that T already exists in the $\mathbb{R} * \mathbf{P}_\beta$-extension; note that $X \subseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ holds there as well (by absoluteness). So we can fix a condition $(x, p) \in \mathbb{R} * \mathbf{P}_{\omega_2}$ and an $\mathbb{R} * \mathbf{P}_\beta$-name $\underset{\sim}{\dot{T}}$ of a countable set of reals such that

$$(x, p) \Vdash X \subseteq \underset{\sim}{\dot{T}} + (2^\omega \setminus \dot{Z}_\nabla). \tag{6}$$

(3) Using the dense embedding $j_{\omega_2} : \mathbf{P}'_{\omega_2} \to \mathbf{P}_{\omega_2}$, we may replace (x, p) by a condition $(x, p') \in \mathbb{R} * \mathbf{P}'_{\omega_2}$. According to [4, Fact 4.14] (recall that we know that $\mathbf{P}_{\omega_2}$ satisfies ccc) and [4, Lemma 4.15] (note that [4, Lemma 4.15] allows for countably many reals, so it is no problem to apply it to our name $\underset{\sim}{\dot{T}}$ of a *countable* set of reals) we can assume that p' is already a P^x_β-condition p^x and that $\underset{\sim}{\dot{T}}$ is (forced by x to be the same as) a P^x_β-name $\dot{T}^x$ in M^x.

Actually, at this point it was crucial that we have argued via the formulation of Lemma 2.4 (1) for "X is not very meager witnessed by $\dot{Z}_\nabla$": the *countable* set T can easily (and absolutely) be coded by a single real (and hence "captured" by a condition of the preparatory forcing $\mathbb{R}$), whereas a partition of the uncountable set X (or equivalently: the corresponding equivalence relation, or a coloring inducing the partition) is inherently an uncountable object, which can not easily be coded by a real. So we would run into trouble if we had used the formulation of Lemma 2.4 (2) (i.e., "there exists a partition of X ...").

(4) We construct (in V) an iteration $\bar{P}$ in the following way:

(a) Up to α, we take an arbitrary alternating iteration into which x embeds. In particular, P_α again forces that X is still uncountable.
(b1) Let Q_α be any ultralaver forcing (over Q^x_α). Then Q_α forces that X is not thin (cf. [4, Corollary 1.24]).
(b2) Let $Q_{\alpha+1}$ be a countable Janus forcing. So $Q_{\alpha+1}$ forces "X is not very meager witnessed by $\dot{Z}_\nabla$". Here we apply our adapted lemma, i.e., we use Lemma 3.3 instead of [4, Lemma 2.9].
(c) We continue the iteration in a σ-centered way. I.e., we use an almost FS iteration over x of ultralaver forcings and countable Janus forcings, using trivial Q_ζ for all $\zeta \notin M^x$; cf. [4, Lemma 3.17].
(d) So P_β still forces that "X is not very meager witnessed by $\dot{Z}_\nabla$", i.e., $X \nsubseteq T + (2^\omega \setminus \dot{Z}_\nabla)$ for each countable $T \subseteq 2^\omega$ (recall Lemma 2.4). This is again due to Lemma 3.3 (instead of [4, Lemma 2.9]).
So in particular, it is forced that $X \nsubseteq \dot{T}^x + (2^\omega \setminus \dot{Z}_\nabla)$.

As usual, we pick a countable $N \prec \mathbf{H}(\chi^*)$ containing everything and ord-collapse $(N, \bar{P})$ to $y \le x$. (Cf. [4, Fact 4.4].) Set $X^y := X \cap M^y$ (the image of X under the collapse). By elementarity, M^y thinks that (a)–(d) above holds for $\bar{P}^y$ and that X^y is uncountable. Note that $X^y \subseteq X$.

(5) As always, this gives a contradiction: Let G be $\mathbb{R}$-generic over V and contain y, and let H_β be $\mathbf{P}_\beta$-generic over $V[G]$ and contain p; then $M^y[H^y_\beta]$ thinks that $X^y \nsubseteq T + (2^\omega \setminus Z_\nabla)$ (where T is $\dot{T}^x$ evaluated by H^y_β); so there is an $x \in X^y$ which is not in $T + (2^\omega \setminus Z_\nabla)$; but $x \in X^y \subseteq X$, and $\underset{\sim}{\dot{T}}$ is forced to be the same as $\dot{T}^x$ (cf. (3)), contradicting (6). □

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