ISRAEL JOURNAL OF MATHEMATICS, Vol. 70, No. 2, 1990

CATEGORICITY OVER P FOR FIRST ORDER T OR CATEGORICITY FOR $\varphi \in \mathscr{L}_{\omega_1\omega}$ CAN STOP AT $\aleph_k$ WHILE HOLDING FOR $\aleph_0, \ldots, \aleph_{k-1}$ †

BY

SAHARON SHELAH[a,††] AND BRADD HART[b,‡]

Department of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel; and

Department of Mathematics, University of California at Berkeley, Berkeley, California, USA

In the 1950's, Los conjectured that if T was countable first order theory in a language $\mathscr{L}$, then if it was categorical in some uncountable power it was categorical in all uncountable powers. In [7], Morley proved this. Buoyed by this success, more general forms of the Los conjecture were considered.

In [10], Shelah showed that if T was any first order theory categorical in some power greater than $|T|$ then T was categorical in all powers greater than $|T|$. Keisler took up the investigation of the $\mathscr{L}_{\omega_1\omega}$ case (see [5]) and gave a sufficient condition for the Morely analysis to work in this situation. Unfortunately, this condition was not necessary. (See the counter-example due to Marcus, [6].)

In [11] and [12], Shelah began the systematic investigation of the $\mathscr{L}_{\omega_1\omega}$ case. In [12], he identifies a class of $\mathscr{L}_{\omega_1\omega}$ sentences which he calls excellent and shows that if an $\mathscr{L}_{\omega_1\omega}$ sentence is excellent then that Los conjecture holds. (In [2], Hart shows that many other theorems which are analogs of those for first order theories also hold for excellent classes.) Furthermore, he shows that if GCH (or in fact much less) and φ is an $\mathscr{L}_{\omega_1\omega}$ sentence which is $\aleph_n$-categorical for all $n \in \omega$ then φ is excellent.

† To make Leo happy.

†† Partially supported by the BSF. Publication No. 323.

‡ Supported by a grant from the NSERC.

Received March 26, 1987 and in final revised form April 3, 1989

The question which naturally arises is, under suitable set theoretic assumptions, is categoricity in $\aleph_n$ for $n < k$ sufficient to prove full categoricity for a sentence in $\mathscr{L}_{\omega_1\omega}$.

The answer to this question must wait while we introduce another variant of the Los conjecture.

Suppose $\mathscr{L}$ is a relational language and $P \in \mathscr{L}$ is a unary predicate. If M is an $\mathscr{L}$-structure then $P(M)$ is the $\mathscr{L}$-structure formed as the substructure of M with domain $\{a : M \models P(a)\}$. Now suppose T is a complete first order theory in $\mathscr{L}$ with infinite models. Following Hodges, we define

DEFINITION 0.1. T is relatively λ-categorical if whenever $M, N \models T$, $P(M) = P(N)$, $|P(M)| = \lambda$ then there is an isomorphism $i : M \to N$ which is the identity on $P(M)$.

T is relatively categorical if it is relatively λ-categorical for every λ.

The notion of relative categoricity has been investigated by Gaifman ([1]), Hodges ([3] and [4]), Pillay ([8]) and Pillay and Shelah ([9]). In ([13]), Shelah gave a classification under some set theory.

Again the question arises whether the relative λ-categoricity of T for some $\lambda > |T|$ implies that T is relatively categorical.

In this paper, we provide an example, for every $k > 1$, of a theory T_k and an $\mathscr{L}_{\omega_1\omega}$ sentence φ_k so that T_k is not relatively $\beth_k$-categorical and φ_k is not $\beth_k$-categorical.

Then examples are due to Shelah. Harrington asked about the possibility of such examples in Chicago in December 1985 as he was not happy with the complexity of the classification. The examples provided $\mathscr{L}_{\omega_1\omega}$ sentences which were categorical but not excellent and so a proof of this fact was written up in [2].

The notation used is standard. $[A]^k$ will stand for all the k-element subsets of the set A. $\mathscr{P}^-(n)$ is the set of all subsets of n except n itself. $\coprod$ is used to represent the direct sum of groups and Π is used to represent the direct product of groups. Z_2 will represent the two element group. $2^{<\omega}$ will be used to represent the subgroup of eventually zero sequences in the abelian group $\Pi_\omega Z_2$ (written as 2^ω).

1. The example

We first describe the example informally. Fix a natural number k greater than one. There will be an infinite set I with $K = [I]^k$. There are constants c_n for

$n \in \omega$ and a predicate R containing all of them. R will be thought of as levels and we will refer to constants in R as standard levels. We fix Z_2, the abelian group, G, the direct sum of K-many copies of Z_2 and H, the direct sum of R-many copies of Z_2. In addition, all relevant projections onto Z_2 are available to us. All of this constitutes the P-part of the model.

Outside of this we have two types of objects. First, for every level $r \in R$ and every $u \in K$, we have a distinct copy of G. Via some connection between our fixed copy of G and this one we will be able to determine the sum of any three elements of G but we will have "lost" the zero. Second, for every $u \in K$ there will be a distinct copy of H in which we again have "lost" the zero.

We will be interested in the possibility of choosing elements from these copies of G and H to act as the zero in their respective groups. We won't put any more restraints on G's from non-standard levels so any element will do. However, for each $n \in \omega$, on the level corresponding to c_n, and for every $u \in K$, there will be a predicate connecting the copy of H corresponding to u and k of the copies of G on the nth level. It will be these predicates which make or break the categoricity by putting restraints on choices for the zeroes of the copies of G and H.

Convention 1. k will be a fixed natural number greater than one.

Now, more formally, we define the language for the example.

Definition 1.1. $\mathscr{L}$ will be the language that consists of
(1) unary predicates I, K, R, P, G^a, H^a,
(2) binary predicates $\in, H^b$,
(3) ternary predicates $\pi, \rho, +$ and G^b,
(4) a 4-ary predicate h,
(5) a 5-ary predicate g,
(6) a $k+1$-ary predicate Q_l for every $l < \omega$, and
(7) constants c_a for every $a \in Z_2 \cup \omega$.

We now describe the standard model on I.

Definiton 1.2. If I is an infinite set then the standard model on I denoted by M_I is the $\mathscr{L}$-structure with universe

$$I \dot\cup [I]^k \dot\cup \omega \dot\cup Z_2 \dot\cup \coprod_{[I]^k} Z_2 \dot\cup \coprod_{\omega} Z_2 \dot\cup \omega \times [I]^k \times \coprod_{[I]^k} Z_2 \dot\cup [I]^k \times \coprod_{\omega} Z_2$$

where the symbols of $\mathscr{L}$ are interpreted as follows:

(1) I is interpreted as I, K as $[I]^k$, R as ω, G^a as $\coprod_{[I]^k} Z_2$ *and* H^a as $\coprod_\omega Z_2$.
(2) the constants c_a are interpreted as a. That is, for example, $R(c_a)$ holds for every $a \in \omega$.
(3) $P(x)$ holds iff x is a constant or one of $I(x)$, $K(x)$, $G^a(x)$ or $H^a(x)$ holds.
(4) $G^b(l, u, x)$ holds iff $R(l)$, $K(u)$ and $x = (l, u, y)$ for some $y \in \coprod_{[I]^k} Z_2$.
(5) $H^b(u, x)$ holds iff $K(u)$ and $x = (u, y)$ for some $y \in \coprod_\omega Z_2$.
(6) $\in(x, y)$ holds iff $I(x)$, $K(y)$ and $x \in y$.
(7) $+(x, y, z)$ holds iff x, y and z are all in one of Z_2, $\coprod_{[I]^k} Z_2$ or $\coprod_\omega Z_2$ and $x + y = z$.
(8) $\pi(u, x, a)$ holds iff $K(u)$, $G^a(x)$ and $x(u) = a$, an element of Z_2.
(9) $\rho(l, x, a)$ holds iff $R(l)$, $H^a(x)$ and $x(l) = a$, an element of Z_2.
(10) $g(l, u, x, y, z)$ holds iff $R(l)$, $K(u)$, $G^a(x)$, $y = (l, u, a)$, $z = (l, u, b)$ (so $G^b(l, u, y)$ and $G^b(l, u, z)$) and $b = a + x$.
(11) $h(u, x, y, z)$ holds iff $K(u)$, $H^a(x)$, $y = (u, a)$, $z = (u, b)$ (so $H^b(u, y)$ and $H^b(u, z)$) and $b = a + x$.
(12) $Q_l(x_0, \ldots, x_k)$ holds iff $x_i = (c_l, u_i, y_i)$ with $G^b(c_l, u_i, x_i)$ for $i < k$ and $x_k = (u_k, z)$ with $H^b(u_k, x_k)$ where $u_0, \ldots, u_k$ are all the k-element subsets of some $(k+1)$-element subset of I and

$$\sum_{i<k} y_i(u_k) = z(c_l).$$

REMARKS. (1) In the previous definition, all of the direct sums used in the definition of the universe represent abelian groups. Hence on the right hand side of items (7), (10), (11) and (12), the addition mentioned is addition in the appropriate group.

(2) In item (12), each y_i is in $\coprod_{[I]^k} Z_2$ and u_k is in $[I]^k$ so $y_i(u_k)$ is in Z_2. z is in $\coprod_\omega Z_2$ and $c_l \in \omega$ so $z(c_l)$ is in Z_2. Hence, the displayed equality is comparing elements of Z_2.

Let's consider some of the sentences in $\mathcal{L}$ that the standard model satisfies. For a fixed infinite set I, M_I satisfies:
(1) I is an infinite set, K is the collection of k-element subsets of I and $\in$ is the membership relation between elements of I and elements of K.
(2) I, K, R, G^a, H^a are disjoint and their union together with the constants c_a for $a \in Z_2$ form P.
(3) $R(c_a)$ for every $a \in \omega$.
(4) $G^b(l, u, x)$ implies $R(l)$ and $K(u)$ and $H^b(u, x)$ implies $K(u)$.
(5) If x is not in P then either for some l and u, $G^b(l, u, x)$ or for some u,

Sh:323

$H^b(u, x)$ and for every $l \in R$ and $u, v \in K$, P, $H^b(u, -)$ and $G^b(l, v, -)$ are pairwise disjoint.

(6) If $\pi(u, a, z)$ then $K(u)$, $G^a(a)$ and z is one of the constants indexed by Z_2.

(7) If $\rho(l, b, z)$ then $R(l)$, $H^a(b)$ and z is one of the constants indexed by Z_2.

(8) If $g(l, u, a, v, w)$ then $R(l)$, $K(u)$, $G^a(a)$, $G^b(l, u, v)$ and $G^b(l, u, w)$.

(9) If $h(u, b, x, y)$ then $K(u)$, $H^a(b)$, $H^b(u, x)$ and $H^b(u, y)$.

(10) The constants c_a for $a \in Z_2$ together with $+$ have the group structure of Z_2.

(11) $+$ restricted to G^a gives a subgroup of $\Pi_K Z_2$ which contains $\amalg_K Z_2$ where the projections are given by π.

(12) $+$ restricted to H^a gives a subgroup of $\Pi_R Z_2$ which contains $\amalg_R Z_2$ where the projections are given by ρ.

(13) For every l in R and u in K, $G^b(l, u, -)$ is non-empty and for every l in R, u in K and x so that $G^b(l, u, x)$, $g(l, u, -, x, -)$ is a bijection from G^a onto $G^b(l, u, -)$. Moreover, $g(l, x, y, z)$ implies $g(l, u, x, z, y)$ and if $g(l, u, a, x, y)$ and $g(l, u, b, y, z)$ then $g(u, l, a + b, x, z)$ where $a + b$ is the unique c so that $+(a, b, c)$.

(14) For every u in K, $H^b(u, -)$ is non-empty and for every u in K and x so that $H^b(u, x)$, $h(u, -, x, -)$ is a bijection from H^a onto $H^b(u, -)$. Moreover, $h(u, x, y, z)$ implies $h(u, x, z, y)$ and if $h(u, a, x, y)$ and $h(u, b, y, z)$ then $h(u, a + b, x, z)$ where $a + b$ is the unique c so that $+(a, b, c)$.

(15) If $Q_l(x_0, \ldots, x_k)$ then for $i < k$, for some u_i in K, $G^b(c_l, u_i, x_i)$ and for some u_k in K, $H^b(u_k, x_k)$. Additionally, $u_0, \ldots, u_k$ are all the k-element subsets of some $(k + 1)$-element subset of I. If σ is a permutation of k then $Q_l(x_{\sigma(0)}, \ldots, x_{\sigma(k-1)}, x_k)$.

(16) If $Q_l(x_0, \ldots, x_k)$, $G^b(c_l, u, x_0)$, $H^b(v, x_k)$, $G^b(c_l, u, x_0')$ and $H^b(v, x_k')$ then $Q_l(x_0', \ldots, x_k)$ iff the v-projection of the unique element a so that $g(c_l, u, a, x_0, x_0')$ via π is 0 and $Q_l(x_0, \ldots, x_k')$ iff the c_l-projection of the unique element a so that $h(v, a, x_k, x_k')$ via ρ is 0.

(17) Suppose $l \in \omega$, u is in K and $i_0, \ldots, i_{n-1}$ are distinct elements of I not in u. For each $j < n$, let v_i^j for $1 \leqq i \leqq k$ be a list of the k-element subsets of $u \cup \{i_j\}$ besides u. If $G^b(c_l, v_i^j, x_i^j)$ for each $j < n$ and $i < k$ and $H^b(v_k^j, y_j)$ for every $j < n$ then

$$\exists x \bigwedge_{j<n} Q_l(x, x_1^j, \ldots, x_{k-1}^j, y_j).$$

(17) actually follows from the previous axioms but it is in the form that we will use it in Section 2. We make the following definition for the rest of the paper.

CONVENTION 2. Let T be the theory in $\mathscr{L}$ made up of the sentences enumerated (1)–(17) above.

The standard model satisfies some additional sentences in $\mathscr{L}_{\omega_1\omega}$. For any infinite set I, M_I satisfies:

(1) R contains only the constants indexed by ω.

(2) G^a is canonically isomorphic to $\coprod_K Z_2$.

(3) H^a is canonically isomorphic to $\coprod_\omega Z_2$.

CONVENTION 3. Let φ be the $\mathscr{L}_{\omega_1\omega}$ sentence which is the conjunction of T and the three sentences listed above.

REMARKS. (1) T is not complete, however we will show that it is relatively $\aleph_n$-categorical for all $n < k$.

(2) φ is the Scott sentence of any M_I where I is countable. This will follow from Section 2. Note that φ has arbitrarily large models.

2. Categoricity less than $\aleph_k$

In this section, we show that T is relatively $\aleph_n$-categorical for all $n < k$.

DEFINITION 2.1. Suppose $M \models T$, $W \subseteq \omega \times K(M) \cup K(M)$ and $f: W \to M$. Then f is called a solution for W if:

(1) $(l, u) \in W$ then $M \models G^b(c_l, u, f(l, u))$,

(2) if $u \in W$ then $M \models H^b(u, f(u))$, and

(3) if $u_0, \dots, u_k \in K(M)$ are all the k-element subsets of some fixed $(k+1)$-element subset of $I(M)$, $(l, u_i) \in W$ for all $i < k$ and $u_k \in W$ then

$$M \models Q_l(f(l, u_0), \dots, f(u_k)).$$

If $J \subseteq I(M)$ then f is called a J-solution if it is a solution for $\omega \times [J]^k \cup [J]^k$. f is called a solution if it is an $I(M)$-solution

REMARK. Note that the standard model for any I has a solution. Hence T (and φ) has arbitrarily large models with solutions.

LEMMA 2.2. *If $M, N \models T$, both M and N have solutions and $P(M) = P(N)$ then $M \cong N$ over $P(M)$.*

PROOF. Suppose f_M is a solution for M and f_N is a solution for N. We are really interested in those $G^b(u, M)$ and $G^b(l, u, N)$ where l is one of the constants in R. However, we must accommodate all l in R. Let

$$R^* = R(M) \setminus \{c_l : l \in \omega\}.$$

Extend f_M and f_N to include $R^* \times K(M)$ ($= R^* \times K(N)$) in their domains so that

$$M \models G^b(l, u, f_M(l, u)) \quad \text{and} \quad N \models G^b(l, u, f_N(l, u))$$

for all $(l, u) \in R^* \times K(M)$. Let j be a partial function from M to N so that j restricted to $P(M)$ is the identity, for every u, $j(f_M(u)) = f_N(u)$ and for every l and u, $j(f_M(l, u)) = f_N(l, u)$. We want to extend j to a function from M to N.

If $x \in M$ so that $M \models G^b(c_l, u, x)$ then there is a unique a so that

$$M \models g(c_l, u, a, f_M(l, u), x).$$

There is a unique $y \in N$ so that

$$N \models g(c_l, u, a, f_N(l, u), y).$$

Extend j so that $j(x) = y$.

We do a similar thing when $x \in M$, $M \models G^b(l, u, x)$ and $l \in R^*$.

If $x \in M$ so that $M \models H^b(u, x)$ then there is a unique a so that

$$M \models h(u, a, f_M(u), x).$$

There is a unique $y \in N$ so that

$$N \models g(u, a, f_N(u), y).$$

Extend j so that $j(x) = y$.

Using the fact that M and N satisfy T, it is not hard to show that j defines a function from M onto N. We want to show that it is an isomorphism. We'll check the hardest predicate, Q_l.

Suppose $M \models Q_l(x_0, \ldots, x_k)$ where

$$M \models G^b(c_l, u_i, x_i) \quad \text{for } i < k \quad \text{and} \quad M \models H^b(u_k, x_k).$$

Choose a_i for $i \leqq k$ so that $M \models g(c_l, u_i, f_M(l, u_i), x_i)$ for $i < k$ and

$$M \models h(u_k, a_k, f_M(u_k), x_k).$$

We know

$$M \models Q_l(f_M(l, u_0), \ldots, f_M(u_k))$$

since f_M is a solution. Suppose

$$M \models \pi(u_i, a_i, z_i) \quad \text{for } i < k \quad \text{and} \quad M \models \rho(c_l, a_k, z_k).$$

Then by using axioms (15) and (16) of T, we conclude that

$$\sum_{i<k} z_i = z_k$$

where the sum takes place in Z_2 and we identify the constants indexed by Z_2 with the elements they represent.

Since $P(M) = P(N)$, this happens in N as well and, since $N \models T$, we unravel the fact that f_N is a solution so $N \models Q_l(f_N(l, u_0), \ldots, f_N(u_k))$ to conclude that $N \models Q_l(y_0, \ldots, y_k)$ where $y_i = j(x_i)$ for $i \leqq k$.

A completely symmetric argument shows that if $N \models Q_l(j(x_0), \ldots, j(x_k))$ then $M \models Q_l(x_0, \ldots, x_k)$ so j is an isomorphism. □

LEMMA 2.3. *Suppose $M \models T$.*

(1) *If M is countable then M has a solution.*

(2) *If $A \subseteq B \subseteq I(M)$, B is countable and f is an A-solution then f can be extended to a B-solution.*

PROOF. The first follows from the second so we will prove the second.

Choose f' so that $f \subseteq f'$ and $\mathrm{dom}(f') = \mathrm{dom}(f) \cup [B]^k$ where, if $u \notin [A]^k$, then $M \models H^b(u, f'(u))$ and otherwise $f'(u)$ is arbitrary.

f' is a solution on its domain. To see this, note that if $i_0, \ldots, i_k \in B$ and $i_0 \notin A$ then, since $k > 1$, at least two k-element subsets of $\{i_0, \ldots, i_k\}$ are not in $[A]^k$. Hence, f' is a solution on its domain vacuously.

Now enumerate $\omega \times ([B]^k \setminus [A]^k)$ as $\{\langle l_i, u_i \rangle : i \in \omega\}$. We will define an increasing chain of functions f_n so that

(1) $f_0 = f'$,

(2) $\mathrm{dom}(f_n) = \mathrm{dom}(f') \cup \{\langle l_i, u_i \rangle : i < n\}$, and

(3) f_n is a solution on its domain.

If we accomplish this then $\bigcup f_n$ will provide a B-solution extending f.

Suppose we have defined f_n. We need to choose an a so that $M \models G^b(c_{l_n}, u_n, a)$ and which will be compatible with the demands of being a solution.

Say that a $(k+1)$-element subset v of B puts a constraint on u_n if $u_n \subseteq v$ and $k-1$ of the k-element subsets of v, say $w_1, \ldots, w_{k-1}$, are such that $\langle l_n, w_i \rangle \in \mathrm{dom}(f_n)$ for $i < k$. Note that since $u_n \not\subseteq A$, at least one of these w_i's must also not be a subset of A.

Now since only finitely many elements are enumerated before $\langle l_n, u_n \rangle$, there are only finitely many $(k+1)$-element subsets of B which put a constraint on

u_n. This is exactly the situation that axiom (17) of T was designed for, so we can find an a so that $f_{n+1} = f_n \cup \{\langle\langle l_n, u_n\rangle, a\rangle\}$ is a solution on its domain. □

COROLLARY 2.4. *φ is a complete $\mathcal{L}_{\omega_1\omega}$ sentence.*

PROOF. To see this, it suffices to see that if M and N are countable models of φ then $M \cong N$. But since M and N are models of φ, $P(M)$ and $P(N)$ are uniquely determined by φ so we may assume that $P(M) = P(N)$. By Lemma 2.3, M and N have solutions and hence, by Lemma 2.2, $M \cong N$. □

DEFINITION 2.5. Suppose $M \vDash T$, $A_\varnothing \subseteq I(M)$ and $a_0, \ldots, a_{m-1}$ are distinct elements of $I(M) \setminus A_\varnothing$. $\langle A_s, f_s : s \in \mathcal{P}^-(m)\rangle$ is a compatible $\aleph_n - \mathcal{P}^-(m)$-system of solutions if

(1) $\bigcup_{s \in \mathcal{P}^-(m)} A_s = A_\varnothing \cup \{a_0, \ldots, a_{m-1}\}$, $|A_\varnothing| \leqq \aleph_n$ and $A_s = A_\varnothing \cup \{a_t : t \in s\}$ for every $s \in \mathcal{P}^-(m)$,

(2) f_s is an A_s-solution for every $s \in \mathcal{P}^-(m)$,

(3) for every $s, t \in \mathcal{P}^-(m)$ if $s \subseteq t$ then $f_s \subseteq f_t$.

Using the notation from the definition, suppose $\langle A_s, f_s : s \in \mathcal{P}^-(m)\rangle$ is a compatible $\aleph_0 - \mathcal{P}^-(m)$-system with $m < k$. If

$$u \in \left[\bigcup_{s \in \mathcal{P}^-(m)} A_s\right]^k \setminus \bigcup_{s \in \mathcal{P}^-(m)} [A_s]^k$$

then $\{a_0, \ldots, a_{m-1}\} \subseteq u$. Since $m < k$, there is $b \in u \setminus \{a_0, \ldots, a_{m-1}\} \subseteq u$. If $c \in \bigcup_{s \in \mathcal{P}^-(m)} A_s \setminus u$ then

$$(u \setminus \{b\}) \cup \{c\} \notin \bigcup_{s \in \mathcal{P}^-(m)} [A_s]^k.$$

Hence, if $u \subsetneq v$ where v is any $(k+1)$-element subset of $\bigcup_{\mathcal{P}^-(m)} A_s$ then there is a k-element subset $u' \subseteq v$, $u \neq u'$ so that $u' \notin \bigcup_{\mathcal{P}^-(m)} A_s$ as well. Using this observation and a proof similar to the proof of Lemma 2.3, we obtain

LEMMA 2.6. *If $\langle A_s, f_s : s \in \mathcal{P}^-(m)\rangle$ is a compatible $\aleph_0 - \mathcal{P}^-(m)$-system with $m < k$ then there is $\bigcup_{s \in \mathcal{P}^-(m)} A_s$-solution f so that $f_s \subseteq f$ for every $s \in \mathcal{P}^-(m)$.*

We use this as the base step in the following lemma.

LEMMA 2.7. *If $\langle A_s, f_s : s \in \mathcal{P}^-(m)\rangle$ is a compatible $\aleph_n - \mathcal{P}^-(m)$-system with $m + n < k$ then there is $\bigcup_{s \in \mathcal{P}^-(m)} A_s$-solution f so that $f \subseteq f$ for every $s \in \mathcal{P}^-(m)$.*

Proof. We prove this by induction on n. If $n = 0$ then this is just Lemma 2.6. Suppose $n > 0$ and $A_s = A_\varnothing \cup \{b_t : t \in s\}$. Enumerate $A_\varnothing$, $\langle a_\beta : \beta < \aleph_n \rangle$ and let $A^\alpha_\varnothing = \{a_\beta : \beta < \alpha\}$. Now define $A^\alpha_s = A^\alpha_\varnothing \cup \{b_t : t \in s\}$ for every $s \in \mathscr{P}^-(m)$ and let f^α_s be the restriction of f_s to an A^α_s-solution. We wish to define g_α for every $\alpha < \aleph_n$ so that

(1) g_α is a $\bigcup_{s \in \mathscr{P}^-(m)} A^\alpha_s$-solution extending f^α_s for every $s \in \mathscr{P}^-(m)$,

(2) $g_\alpha \subseteq g_\beta$ for $\alpha < \beta < \aleph_n$.

Clearly, if we accomplish this then $\bigcup_{\alpha < \aleph_n} g_\alpha$ is the sought-after solution. But by taking unions at limit ordinals and using the induction hypothesis at successors we can easily satisfy these two conditions, so we are done. □

Lemma 2.8. *If $M \models T$ and $A \subseteq B \subseteq I(M)$ with $|B| < \aleph_{k-1}$ and f is an A-solution then f can be extended to a B-solution.*

Proof. Without loss of generality, $B = A \cup \{b\}$. We prove this lemma by induction on the cardinality of A. If A is countable then this is just Lemma 2.3. If $|A| = \aleph_n$ with $n > 0$ then enumerate A as $\langle a_\beta : \beta < \aleph_n \rangle$ and let $A_\alpha = \{a_\beta : \beta < \alpha\}$. Let f_α be the restriction of f to an A_α-solution. By induction, we define $A_\alpha \cup \{b\}$-solutions g_α extending f_α. If we have defined g_α, we use Lemma 2.7 in the case $m = 2$ to extend $g_\alpha \cup f_{\alpha+1}$ to an $A_{\alpha+1} \cup \{b\}$-solution. At limits we take unions and $\bigcup_{\alpha < \aleph_n} g_\alpha$ is a B-solution extending f. □

Theorem 2.9. *If $M \models T$ and $|M| < \aleph_k$ then M has a solution.*

Proof. By induction on the cardinality of M. If M is countable then this is Lemma 2.3. If $|M| = \aleph_n$ with $n > 0$ then we can choose N, $N \prec M$ with $|N| < \aleph_n$. By induction, N has a solution and, by using Lemma 2.8 repeatedly, we can extend it to a solution for M. □

Corollary 2.10. (1) *T is relatively $\aleph_n$-categorical for all $n < k$.*
(2) *φ is $\aleph_n$-categorical for all $n < k$.*

Proof. (1) Suppose M and N are models of T, $P(M) = P(N)$ and $|P(M)| = \aleph_n$ for some $n < k$. It follows that $|M| = |N| = \aleph_n$. By Theorem 2.9, M and N have solutions and so, by Lemma 2.2, $M \cong N$.

(2) Suppose M and N are models of φ and $|M| = |N| = \aleph_n$ for some $n < k$. $P(M)$ is uniquely determined by $I(M)$ and $P(N)$ is determined by $I(N)$. $|M| = |I(M)|$, so we may assume that $P(M) = P(N)$ and it follows then that $M \cong N$ by Theorem 2.9 and Lemma 2.2. □

3. The failure of full categoricity

In this section, we show that φ is not fully categorical.

Suppose $M \vDash \varphi$ and $I = I(M)$. Without loss of generality, we may assume that $K(M) = [I]^k$, $R(M) = \omega$, $G^\alpha(M) = \coprod_K Z_2$ and $H^a = \coprod_\omega Z_2$. Further, we may assume that the constants $c_l = l$ for $l \in \omega$ and $c_a = a$ for $a \in Z_2$. π, ρ and $+$ can also be assumed to be as in the standard model M_I.

Lemma 3.1. *If $M, N \vDash \varphi$, $M \subseteq N$ and N has a solution then M has a solution.*

Proof. Suppose that f is a solution for N. Fix some $g: \omega \times K(M) \to M$ so that

$$M \vDash G^b(l, u, g(l, u)) \qquad \text{for every } l \in \omega \text{ and } u \in K(M).$$

For $u \in K(M)$, let $c_{l,u}$ be such that

$$N \vDash g(l, u, c_{l,u}, g(l, u), f(l, u)).$$

Choose $d_{l,u}$ so that for every $v \in K(M)$ and $y \in Z_2$

$$M \vDash \pi(v, d_{l,u}, y) \qquad \text{iff } M \vDash \pi(v, c_{l,u}, y).$$

Define $f': \omega \times K(M) \cup K(M) \to M$ so that $f'(u) = f(u)$ for every $u \in K(M)$ and if $l \in \omega$ and $u \in K(M)$ then $f'(l, u) = z$ where $M \vDash g(l, u, d_{l,u}, g(l, u), z)$. To check that f' is a solution for M, suppose v is a $(k+1)$-element subset of $I(M)$ and $u_0, \ldots, u_k$ are all the k-element subsets of v. Fix $l \in \omega$.

$$N \vDash Q_l(f(l, u_0), \ldots, f(u_k)).$$

From above, we have

$$N \vDash g(l, u_i, c_{l,u_i}, d_{l,u_i}, f(l, u_i), f'(l, u_i)) \quad \text{for } i < k$$

and, by the choice of $d_{l,u}$,

$$(c_{l,u_i} + d_{l,u_i})(u_k) = 0 \quad \text{for all } i < k,$$

hence $M \vDash Q_l(f(l, u_0), \ldots, f(u_k))$. □

Lemma 3.2. *If $M \vDash \varphi$ and $\kappa > |M|$ then there is $N \vDash \varphi$ so that $|N| = \kappa$ and $M \subseteq N$.*

Proof. Let $I(N)$ be the disjoint union of $I(M)$ and κ. From our discussion at the beginning of the section, this defines the P-part of N. $P(M)$ will be subset of $P(N)$ except for $G^a(M)$. The small technical point here is that we have identified $G^a(N)$ with $\coprod_{K(N)} Z_2$. We will identify $x \in G^a(M)$ with $x' \in G^a(N)$ where $x'(u) = x(u)$ for all $u \in K(M)$ and $x'(u) = 0$ for all $u \in K(N) \setminus K(M)$. In this way, we embed $P(M)$ into $P(N)$.

Let's consider the other predicates. If $u \in K(M)$ then let $H^b(u, N) = H^b(u, M)$. If $u \in K(N) \setminus K(M)$, let $H^b(u, N) = 2^{<\omega}$. It is clear how to define h for N in a fashion appropriate for φ.

Let $J = \coprod_{K(N) \setminus K(M)} Z_2$. If $u \in K(M)$ and $l \in \omega$ then let $G^b(l, u, N) = G^b(l, u, M) \times J$ and identify $x \in G^b(l, u, M)$ with $(x, 0)$ where 0 is the identity in J. If $u \in K(N) \setminus K(M)$, let $G^b(l, u, N) = \coprod_{K(N)} Z_2$. We leave it to the reader to define a reasonable g.

It remains to define Q_l on N for each $l \in \omega$. Fix an arbitrary function $f: K(M) \to M$ so that

$$M \models H^b(u, f(u)) \quad \text{for all } u \in K(M).$$

f is needed only in case (3) below. Suppose v is a $(k+1)$-element subset of $I(N)$ and $u_0, \ldots, u_k$ are all the k-element subsets of v. Note that either $v \subseteq I(M)$ or at most one of the u_i's is a subset of $I(M)$. Further suppose $x_i \in G^b(l, u_i, N)$ for $i < k$ and $x_k \in H^b(u_k, N)$. There are a number of cases:

(1) $u_i \in K(M)$ for all i. Then $x_i = (x_i', a_i)$ for some $x_i' \in G^b(l, u_i, M)$ and $a_i \in J$ for $i < k$. Since $u_k \in K(M)$, let

$$Q_l(x_0, \ldots, x_k) \text{ hold in } N \text{ iff } M \models Q_l(x_0', \ldots, x_{k-1}', x_k).$$

(2) For only one $j < k$, $u_j \in K(M)$. $x_j = (x_j', a_j)$ for some $a_j \in J$. Let

$$Q_l(x_0, \ldots, x_k) \text{ hold in } N \text{ iff } \sum_{i<k} x_i(u_k) = x_k(l)$$

where $x_j(u_k)$ means $a_j(u_k)$.

(3) Only $u_k \in K(M)$. Choose c so that $M \models h(u_k, c, x_k, f(u_k))$. Let

$$Q_l(x_0, \ldots, x_k) \text{ hold in } N \text{ iff } M \models \sum_{i<k} x_i(u_k) = c(l).$$

(4) If none of the u_i's are in $K(M)$ then

$$Q_l(x_0, \ldots, x_k) \text{ hold in } N \text{ iff } \sum_{i<k} x_i(u_k) = x_k(l).$$

It is not hard to see that N defined in this way is a model of φ and with the appropriate identifications, $M \subseteq N$. □

Corollary 3.3. *If φ is not λ-categorical then it is not κ-categorical for any $\kappa > \lambda$.*

Proof. Any two models of φ of cardinality λ have isomorphic P-parts. Hence if φ is not λ-categorical there must be $M \models \varphi$, $|M| = \lambda$ so that M does not have a solution.

By Lemma 3.2, we can find $N \models \varphi$ and $M \subseteq N$ so that $|N| = \kappa$. If φ is κ-categorical then N has a solution since there is a model of φ of cardinality κ with a solution. But then by Lemma 3.1, M has a solution which is a contradiction. Hence φ is not κ-categorical. □

Definition 3.4. *Suppose $M \models \varphi$ and $i_0, \ldots, i_k$* are distinct elements of $I(M)$. Let $A = \omega \times ([\{i_0, \ldots, i_k\}]^k \setminus \{i_1, \ldots, i_k\})$ and f be a function with domain containing A so that

$$M \models G^b(l, u, f(l, u)) \quad \text{for all } (l, u) \in A.$$

Let

$$x_l^j = f(l, \{i_0, \ldots, i_{j-1}, i_{j+1}, \cdots, i_k\}) \quad \text{for } j \neq 0 \quad \text{and} \quad l < \omega$$

and choose $y \in H^b(\{i_1, \ldots, i_k\}, M)$. Define a function g as follows:

$$g(l) = \begin{cases} 0 & \text{if } M \models Q_l(x_l^0, \ldots, x_l^{k-1}, y), \\ 1 & \text{otherwise.} \end{cases}$$

The invariant for $i_0, \ldots, i_k$ via f is $g + 2^{<\omega}$, a coset of $2^{<\omega}$ in the abelian group 2^ω.

Lemma 3.5. *The definition of invariant given above is independent of the choice of y.*

Proof. Use the notation of the definition. Choose any y' so that

$$M \models H^b(\{i_1, \ldots, i_k\}, y').$$

Let $c \in H^a(M)$ be such that

$$M \models h(\{i_1, \ldots, i_k\}, c, y, y').$$

Let

$$g'(l) = \begin{cases} 0 & \text{if } M \models Q_l(x_l^0, \ldots, x_l^{k-1}, y'), \\ 1 & \text{otherwise.} \end{cases}$$

Now $g'(l) = g(l) + c(l)$ for all $l \in \omega$ and $c \in 2^{<\omega}$, so $g' + 2^{<\omega} = g + 2^{<\omega}$. □

If $m \in \omega$ and f, g are functions with the same domain, define the relation $\sim_m$ by

$$f \sim_m g \quad \text{iff } |\{x : f(x) \neq g(x)\}| < \aleph_m.$$

DEFINITION 3.6. Suppose $M \models \varphi$, $I \subseteq I(M)$ and $i_1, \ldots, i_k$ are distinct elements of $I(M) \setminus I$. Let f be a function with domain that contains

$$\omega \times ([I \cup \{i_1, \ldots, i_k\}]^k \setminus \{i_1, \ldots, i_k\})$$

so that

$$M \models G^b(l, u, f(l, u)) \quad \text{for all } (l, u) \in A.$$

The 0-invariant for $I, i_1, \ldots, i_k$ via f is the function g with domain I so that $g(a) =$ the invariant for $a, i_1, \ldots, i_k$ via f.

Suppose $0 < m < k$, $I \subseteq I(M)$ and $i_1, \ldots, i_{k-m}$ are distinct elements of $I(M) \setminus I$ and f is a function whose domain contains

$$A = \omega \times ([I \cup \{i_1, \ldots, i_{k-m}\}]^k \setminus \{u : \{i_1, \ldots, i_{k-m}\} \subseteq u\})$$

so that

$$M \models G^b(l, i, f(l, u)) \quad \text{for all } (l, u) \in A.$$

Let $I_0 \subseteq \cdots \subseteq I_{m-1} \subseteq I$ where $|I_i| = \aleph_i$. Choose a function f' so that the domain of f' contains

$$B = \omega \times ([I_{m-1} \cup \{i_1, \ldots, i_{k-m}\}|^k),$$

$f'(l, u) \in G^b(l, u, M)$ and f' and f agree on their common domain.

The m-invariant for $I, i_1, \ldots, i_{k-m}$ via $I_0, \ldots, I_{m-1}$ and f is the $\sim_m$-class of the function h with domain $I \setminus I_{m-1}$ where $h(a) =$ the $(m-1)$-invariant for I_{m-1} and $a, i_1, \ldots, i_{k-m}$ via $I_0, \ldots, I_{m-2}$ and $f' \cup f$.

LEMMA 3.7. *The definition of m-invariant above is independent of the choice of f'.*

PROOF. Note that by Lemma 3.5, the definition of 0-invariant is well-defined. Use the notation of the definition for m-invariant for $m > 0$. Choose any other applicable f''. Let

$$C = \bigcup \{v : \exists u \in K(M), l < \omega, c \in G^a(M) \text{ so that } (l, u) \in B,$$
$$c(v) \neq 0 \text{ and } M \models g(l, u, c, f'(l, u), f''(l, u))\}.$$

$|C| \leqq \aleph_{m-1}$ since $|B| = \aleph_{m-1}$ and if $a \in I \setminus (I_{m-1} \cup C)$ then the value of $h(a)$

is not affected by the choice of f'' instead of f'. Hence the $\sim_m$-class of h is well-defined. □

Suppose that I is an infinite set and $g:[I]^k \to 2^\omega/2^{<\omega}$. We will define the canonical structure M_g on I via g.

The P-part of M_g is the same as M_I. Moreover, so are the predicates G^b and g. However, $H^b(u, M_g) = \{u\} \times g(u)$ for all $u \in [I]^k$. We modify h so that

$$h(u, x, (u, y), (u, z)) \text{ holds in } M_g \text{ iff } x + y = z$$

where the addition takes place in 2^ω. (Note $2^{<\omega} \subseteq 2^\omega$.)

The definition of Q_l is identical to the one for M_I. It is not hard to show that M_g satisfies φ.

THEOREM 3.8. *Let λ be the least cardinal such that $\lambda^{\aleph_{k-1}} < 2^\lambda$. φ is not categorical in λ. In fact, there are 2^λ many non-isomorphic models of φ of cardinality λ.*

REMARK. Note that $\aleph_{k-1} < \lambda \leqq 2^{\aleph_k}$.

PROOF. Let $B_0 = \{f_a : a \in 2^\omega/2^{<\omega}\}$ where $f_a : \aleph_0 \to 2^\omega/2^{<\omega}$ so that $f_a(i) = a$ for all $i \in \aleph_0$. Define B_m inductively for $0 < m < k-1$. Suppose we have defined B_{m-1}. Let $C = \{h : h : \aleph_m \setminus \aleph_{m-1} \to B_{m-1}\}$. Let B_m be a maximal collection of $\not\sim_m$-equivalent elements in C. It is not hard to show that $|B_m| = 2^{\aleph_m}$.

Fix $A \subseteq B_{k-2}^{\aleph_{k-1}}$ of size λ. we wish to define a struture M^A in such a way as to be able to recover A. Let $I_A = \aleph_{k-2} \cup \aleph_{k-1} \times \aleph_{k-1} \cup A$. Choose $g_A : [I_A]^k \to 2^\omega/2^{<\omega}$ so that if $i_m \in \aleph_m \setminus \aleph_{m-1}$ for $0 < m < k-1$, $\alpha, \beta < \aleph_{k-1}$ and $a \in A$, then

$$g(\{a, (\alpha, \beta), i_{k-2}, \ldots, i_1\}) = a(\alpha)(i_{k-2}) \cdots (i_1)$$

and otherwise $g(u)$ is arbitrary. Let M^A be the canonical structure on I_A via g_A.

We try to recover A by looking at $(k-2)$-invariants. We need to fix certain functions for the rest of the argument. Let

$$\bar{f} : \omega \times K(M^A) \to M^A$$

be defined so that $\bar{f}(l, u) = (l, u, 0)$ where 0 is the identity element in $\amalg_{K(M^A)} Z_2$. Remember that $(l, u, 0)$ is a member of $G^b(l, u, M^A)$. Let f be the restriction of $\bar{f}$ to $\omega \times [\aleph_{k-2} \cup \aleph_{k-1} \times \aleph_{k-1}]^k$ and let h be the restriction of $\bar{f}$ to $\omega \times [\aleph_{k-2} \cup A]^k$.

CLAIM 3.9. *Suppose $m < k-1$ and $i_j \in \aleph_j \setminus \aleph_{j-1}$ for $m < j < k-1$. The*

Sh:323

m-invariant for $\aleph_m, i_m, \ldots, i_{k-2}, (\alpha,\beta)$, *a via* $\aleph_0, \ldots, \aleph_{m-1}$ *and* $\bar{f}$ *is the* $\sim_m$*-class of* $a(\alpha)(i_{k-2})\cdots(i_{m+1})$. (*If* $m=0$ *then* $a(\alpha)(i_{k-2})\cdots(i_1)$ *is the* 0-*invariant*.)

PROOF. Notice that $\bar{f}$ contains all possible domains required for calculating invariants. $\bar{f}$ essentially chooses the zero in all the $G^b(l,u,M^A)$'s.

We prove this claim by induction on m. Suppose the notation is as is in the claim. Choose

$$y \in a(\alpha)(i_{k-2})\cdots(i_1) = H^b(u, M^A)$$

where $u\{i_1,\ldots,i_{k-2},(\alpha,\beta),a\}$.

Since $\bar{f}$ chooses the zero in all $G^b(l,u,M^A)$'s, the value $y(l)$ determines the truth value of the appropriate instance of Q_l. This is independent of the choice of $i_0 \in \aleph_0$ so the 0-invariant is $a(\alpha)(i_{k-2})\cdots(i_1)$.

The induction step is similar. □

A consequence of the claim is that if $a \in A$ and $\alpha,\beta < \aleph_{k-1}$ then the $(k-2)$-invariant for $\aleph_{k-2}$, (α,β), a via $\aleph_0,\ldots,\aleph_{k-3}$ (if $k>3$) and $f \cup h$ is the $\sim_{k-2}$-class of $a(\alpha)$. The domain of h is too large however to allow us to say we have captured a.

So suppose we use some h' instead of h which agrees with f on their common domain. Then for any $a \in A$, the value of at most $\aleph_{k-2}$ many of the $(k-2)$-invariants calculated above would be affected. Hence to recover $a(\alpha)$, for every $\beta < \aleph_{k-1}$, calculate the $(k-2)$-invariant for I_{k-2}, (α,β), a via $\aleph_0,\ldots,\aleph_{k-3}$ and $f \cup h'$ for any h'. All but at most $\aleph_{k-2}$ of the $(k-2)$-invariants will agree and this $(k-2)$-invariant will be the $\sim_{k-2}$-class of $a(\alpha)$.

So by fixing $\aleph_{k-2} \cup \aleph_{k-1} \times \aleph_{k-1}$ and f we are able to recover A. We have fixed $\aleph_{k-1}$ elements then and there are 2^λ many possible A's, so 2^λ many of the M^A's are non-isomorphic since $\lambda^{\aleph_{k-1}} < 2^\lambda$. □

COROLLARY 3.10. (1) *φ is not* $2^{\aleph_{k-1}}$*-categorical*.

(2) *T is not relatively categorical*.

PROOF. The first is obvious from Theorem 3.8, the remark after it and Corollary 3.3. To see the second, notice that all the models built in the proof of Theorem 3.8 have isomorphic P-parts and are models of T. Hence T is not relatively categorical. □

REFERENCES

1. H. Gaifman, *Characterization of uniqueness and rigidity properties*, preprint.

2. Bradd Hart, *Some results in classification theory*, PhD thesis, McGill University, 1986.

3. W. Hodges, *Relative categoricity in abelian groups*, preprint.

4. W. Hodges, I. M. Hodkinson and H. D. Macpherson, *ω-categoricity, relative categoricity and coordinatisation*, preprint, 1987.

5. J. Keisler, *Model Theory for Infinitary Logic*, North-Holland, Amsterdam, 1971.

6. L. Marcus, *A prime minimal model with an infinite set of indiscernibles*, Isr. J. Math. **11** (1972), 180–183.

7. M. Morley, *Categoricity in power*, Trans. Am. Math. Soc. **114** (1965), 514–538.

8. A. Pillay, *$\aleph_0$-categoricity over a predicate*. Notre Dame J. Formal Logic **24** (1983).

9. A. Pillay and S. Shelah, *Classification theory over a predicate* 1,, Notre Dame J. Formal Logic **26** (1985).

10. S. Shelah, *Categoricity of uncountable theories*, in *Proceedings of the Tarski Symposium* (L. Henkin et al., eds.), Am. Math. Soc., 1974, pp. 187–203.

11. S. Shelah, *Categoricity in $\aleph_1$ of sentences in $\mathscr{L}_{\omega_1\omega}$*, Isr. J. Math. **20** (1975), 127–148.

12. S. Shelah, *Classification of non-elementary classes* 1: *the number of models of $\varphi \in \mathscr{L}_{\omega_1\omega}$* (a and b), Isr. J. Math. **46** (1983), 212–273.

13. S. Shelah, *Classification over a predicate II*, in *Around Classification Theory of Models*, Lecture Notes in Math. **1182**, Springer-Verlag, Berlin, 1986.