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Colouring and non-productivity of $\aleph_2$-C.C.

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Abstract

We prove that colouring of pairs from $\aleph_2$ with strong properties exists. The easiest to state (and quite a well-known) problem it solves is: there are two topological spaces with cellularity $\aleph_1$ whose product has cellularity $\aleph_2$; equivalently, we can speak of cellularity of Boolean algebras or of Boolean algebras satisfying the $\aleph_2$-c.c. whose product fails the $\aleph_2$-c.c. We also deal more with guessing of clubs.

Keywords: Colouring; Negative partition relations; Cellularity; Non productivity; Club guessing

0. Introduction

This paper is organized as follows: In Section 1 we prove $Pr_1(\aleph_1, \aleph_2, \aleph_2, \aleph_0)$ which is a much stronger result. In Section 2 we define a property implicit in Section 1, note what the proof in Section 1 gives, and look at the related implications for successor of singular non-strong limit and show that Pr_1 implies Pr_6. In Section 3 we improve some results mainly from [7], giving complete proofs. We show that for μ regular uncountable and $\chi < \mu$ we can find $\langle C_\delta : \delta < \mu^+, \mathrm{cf}(\delta) = \mu \rangle$ and functions h_δ, from C_δ onto χ, such that for every club E of μ^+ for stationarily many $\delta < \mu^+$ we have: $\mathrm{cf}(\delta) = \mu$ and for every $\gamma < \chi$ for arbitrarily large $\alpha \in \mathrm{nacc}(C_\delta)$ we have $\alpha \in E$, $h_\delta(\alpha) = \gamma$. Also if $C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \mu\}$ ($\alpha_{\delta,\varepsilon}$ increasing continuously in ε), we can demand that $\{\varepsilon < \mu : \alpha_{\delta,\varepsilon+1} \in E$ (and $\alpha_{\delta,\varepsilon} \in E)\}$ is a stationary subset of μ. In fact, for each $\gamma < \mu$, the set $\{\varepsilon < \mu : \alpha_{\delta,\varepsilon+1} \in E, \alpha_{\delta,\varepsilon} \in E$ and $f(\alpha_{\delta,\varepsilon+1}) = \gamma\}$ is a stationary subset of μ. We also deal with a parallel to the last version stated (but without f) to the case μ is singular and to the case μ is inaccessible. In Section 4 we prove that $\mathrm{Pr}_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ holds for regular λ.

For history, references and consequences see [5, AP1] and [5, Ch. III, Section 0].

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1. Retry at $\aleph_2$-c.c. not productive

1.1. Theorem. $Pr_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$.

1.2. Remark. (1) Is this hard? A posteriori it does not look so, but we have worked hard on it several times without success (worse: produced several false proofs). We thank Juhasz and Soukup for pointing out a gap.

(2) Remember that $Pr_1(\lambda, \mu, \theta, \sigma)$ means that there is a symmetric two-place function d from λ to θ such that *if* $\langle u_\alpha : \alpha < \mu \rangle$ satisfies

$$u_\alpha \subseteq \lambda,$$

$$|u_\alpha| < \sigma,$$

$$\alpha < \beta \Rightarrow u_\alpha \cap u_\beta = \emptyset,$$

and $\gamma < \theta$ *then* for some $\alpha < \beta$ we have

$$\zeta \in u_\alpha \quad \& \quad \xi \in u_\alpha \Rightarrow d(\zeta, \xi) = \gamma.$$

(3) If we are content with proving that there is a colouring with $\aleph_1$ colours, then we can simplify somewhat: in stage C we let $c(\beta, \alpha) = d_{\mathrm{sq}}(\rho_{h_1}(\beta, \alpha))$ and this shortens stage D.

Proof.

Stage A: First we define a preliminary colouring.

There is a function $d_{\mathrm{sq}} : {}^{\omega>}(\omega_1) \to \omega_1$ such that:

$\bigotimes$ *if* $A \in [\omega_1]^{\aleph_1}$ and $\langle (\rho_\alpha, \nu_\alpha) : \alpha \in A \rangle$ is such that $\rho_\alpha \in {}^{\omega>}\omega_1$, $\nu_\alpha \in {}^{\omega>}\omega_1$, $\alpha \in \mathrm{Rang}(\rho_\alpha) \cap \mathrm{Rang}(\nu_\alpha)$ and $\gamma < \omega_1$ *then* for some $\zeta < \xi$ from A we have: if ν', ρ' are subsequences of ν_ζ, ρ_ξ, respectively, and $\zeta \in \mathrm{Rang}(\nu')$, $\xi \in \mathrm{Rang}(\rho')$ then

$$d_{\mathrm{sq}}(\nu' {}^\frown \rho') = \gamma.$$

Proof of $\bigotimes$. Choose pairwise distinct $\eta_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$. Let $d_0 : [\omega_1]^2 \to \omega_1$ be such that:

(∗) if $n < \omega$ and $\alpha_{\zeta,\ell} < \omega_1$ for $\zeta < \omega_1$, $\ell < n$ are pairwise distinct and $\gamma < \omega_1$ then for some $\zeta < \xi < \omega_1$ we have $\ell < n \Rightarrow \gamma = d_0(\{\alpha_{\zeta,\ell}, \alpha_{\xi,\ell}\})$ (exists by [4, see (2.4), p. 176]; the n there is 2).

Define $d_{\mathrm{sq}}(\nu)$ for $\nu \in {}^{\omega>}(\omega_1)$ as follows. If $\ell g(\nu) \leqslant 1$ or ν is constant then $d_{\mathrm{sq}}(\nu)$ is 0. Otherwise, let

$$n(\nu) =: \max\{\ell g(\eta_{\nu(\ell)} \cap \eta_{\nu(k)}) : \ell < k < \ell g(\nu) \quad \text{and } \nu(\ell) \neq \nu(k)\} < \omega.$$

The maximum is on a non-empty set as $\ell g(\nu) \geqslant 2$ and ν is not constant; remember $\eta_\alpha \in {}^\omega 2$ were pairwise distinct so $\nu(\ell) \neq \nu(k) \Rightarrow \eta_{\nu(\ell)} \cap \eta_{\nu(k)} \in {}^{\omega>}2$ (is the largest common initial segment of $\eta_{v(\ell)}, \eta_{v(k)}$). Let $a(v) = \{(\ell,k) : \ell < k < \ell g(v)$ and $\ell g(\eta_{v(\ell)} \cap \eta_{v(k)}) = n(v)\}$ so $a(v)$ is non-empty and choose the (lexicographically) minimal pair (ℓ_v, k_v) in it. Lastly, let

$$d_{\mathrm{sq}}(v) = d_0(\{v(\ell_v), v(k_v)\}).$$

So d_{sq} is a function with the right domain and range. Now suppose we are given $A \in [\omega_1]^{\aleph_1}$, $\gamma < \omega_1$ and $\rho_\alpha, v_\alpha \in {}^{\omega >}(\omega_1)$ for $\alpha \in A$ such that $\alpha \in \mathrm{Rang}(\rho_\alpha) \cap \mathrm{Rang}(v_\alpha)$. We should find $\alpha < \beta$ from A such that $d_{\mathrm{sq}}(v' \hat{} \rho') = \gamma$ for any subsequences v', ρ' of v_α, ρ_β, respectively, such that $\alpha \in \mathrm{Rang}(v')$ and $\beta \in \mathrm{Rang}(\rho')$.

For each $\alpha \in A$ we can find $m_\alpha < \omega$ such that:

$(*)_0$ if $\ell < k < \ell g(v_\alpha \hat{} \rho_\alpha)$ and $(v_\alpha \hat{} \rho_\alpha)(\ell) \neq (v_\alpha \hat{} \rho_\alpha)(k)$ then $\eta_{(v_\alpha \hat{} \rho_\alpha)(\ell)} \restriction m_\alpha \neq \eta_{(v_\alpha \hat{} \rho_\alpha)(k)} \restriction m_\alpha$.

Next we can find $B \in [A]^{\aleph_1}$ such that for all $\alpha \in B$ (the point is that the values do not depend on α) we have:

(a) $\ell g(v_\alpha) = m^0$, $\ell g(\rho_\alpha) = m^1$,
(b) $a* = \{(\ell,k) : \ell < k < m^0 + m^1$ and $(v_\alpha \hat{} \rho_\alpha)(\ell) = (v_\alpha \hat{} \rho_\alpha)(k)\}$,
(c) $b* = \{\ell < m^0 + m^1 : \alpha = (v_\alpha \hat{} \rho_\alpha)(\ell)\}$,
(d) $m_\alpha = m^2$,
(e) $\langle \eta_{(v_\alpha \hat{} \rho_\alpha)(\ell)} \restriction m_\alpha : \ell < m^0 + m^1 \rangle = \bar{\eta}^*$,
(f) $\langle \mathrm{Rang}(v_\alpha \hat{} \rho_\alpha) : \alpha \in B \rangle$ is a Δ-system with heart w,
(g) $u* = \{\ell : (v_\alpha \hat{} \rho_\alpha)(\ell) \in w\}$ (so $u* \neq \{\ell : \ell < m^0 + m^1\}$ as $\alpha \in \mathrm{Rang}(v_\alpha \hat{} \rho_\alpha)$),
(h) $\alpha^*_\ell = (v_\alpha \hat{} \rho_\alpha)(\ell)$ for $\ell \in u*$,
(i) if $\alpha < \beta \in B$ then $\sup \mathrm{Rang}(v_\alpha \hat{} \rho_\alpha) < \beta$.

For $\zeta \in B$ let $\bar{\beta}^\zeta =: \langle (v_\zeta \hat{} \rho_\zeta)(\ell) : \ell < m^0 + m^1, \ell \notin u* \rangle$ and apply $(*)$, i.e. the choice of d_0. So for some $\zeta < \xi$ from B, we have

$$\ell < m^0 + m^1 \quad \& \quad \ell \notin u* \Rightarrow \gamma = d_0(\{(v_\zeta \hat{} \rho_\zeta)(\ell), (v_\xi \hat{} \rho_\xi)(\ell)\}).$$

We shall prove that $\zeta < \xi$ are as required (in $\otimes$). So let v', ρ' be subsequences of v_ζ, ρ_ξ (so let $v' = v_\zeta \restriction v_1$ and $\rho' = \rho_\xi \restriction v_2$) such that $\zeta \in \mathrm{Rang}(v'), \xi \in \mathrm{Rang}(\rho')$ and we have to prove $\gamma = d_{\mathrm{sq}}(v' \hat{} \rho')$. Let $\tau = v' \hat{} \rho'$, so $\tau = (v_\zeta \hat{} \rho_\xi) \restriction (v_1 \cup (m^0 + v_2))$ (in a slight abuse of notation, we look at τ as a function with domain $v_1 \cup (m^0 + v_2)$ and also as a member of ${}^{\omega >}(\omega_1)$ where $m + v =: \{m + \ell : \ell \in v\}$, of course). By the definition of d_{sq} it is enough to prove the following two things:

$(*)_1$ $n(v' \hat{} \rho') \geqslant m^2$ (see clause (d) and $(*)_0$ above),

$(*)_2$ for every $\ell_1, \ell_2 \in v_1 \cup (m^0 + v_2)$ we have $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m^2, \omega) \Rightarrow \gamma = d_0(\{\tau(\ell_1), \tau(\ell_2)\})$.

Proof of $(*)_1$. Let $\ell_1 \in v_1$ and $\ell_2 \in v_2$ be such that $v_\zeta(\ell_1) = \zeta$ and $\rho_\xi(\ell_2) = \xi$. So clearly ℓ_1, $m^0 + \ell_2 \in b*$ (see clause (c)) and $\eta_{\rho_\xi(\ell_2)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2 = \eta_{v_\zeta(\ell_1)} \restriction m^2$ (first equality as $\zeta, \xi \in B$ and $m_\zeta = m_\xi = m^2$ (see clauses (d) and (e)), second equality as $\eta_{\rho_\zeta(\ell_2)} = \eta_{v_\zeta(\ell_1)}$ since ℓ_1, $m^0 + \ell_2 \in b*$ (see clause (c)). But $\rho_\xi(\ell_2) = \xi \neq \zeta = v_\zeta(\ell_1)$, hence $\eta_{\rho_\xi(\ell_2)} \neq \eta_{\nu_\zeta(\ell_1)}$, so together with the previous sentence we have

$$m^2 \leqslant \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(m^0+\ell_2)}) < \omega.$$

Hence $n(\tau) \geqslant m^2$ as required in $(*)_1$.

Proof of $(*)_2$. If $\ell_1 < \ell_2$ are from v_1, by the choice of $m^2 = m_\zeta$, the proof is easy. Namely, if $(\ell_1, \ell_2) \in a(\tau)$ then $(\ell_1, \ell_2) \in a(\nu_\zeta)$ and $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) = \ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\nu_\zeta(\ell_2)}) < m_\zeta = m^2$. Similarly, if $\ell_1, \ell_2 \in m^0 + v^2$, by the choice of $m^2 = m_\xi$, it is easy to show that $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) < m^2$. So it is enough to prove:

$(*)_3$ assume $\ell_1 \in v_1$, $\ell_2 \in v_2$ and $\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) \in [m^2, \omega)$ then $\gamma = d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\})$.

Now the third assumption in $(*)_3$ means $\eta_{\nu_\zeta(\ell_1)} \restriction m^2 = \eta_{\rho_\xi(\ell_2)} \restriction m^2$ and as $\zeta, \xi \in B$ we know that $\eta_{\rho_\xi(\ell_2)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2$. Together we know that $\eta_{\nu_\zeta(\ell_1)} \restriction m^2 = \eta_{\rho_\zeta(\ell_2)} \restriction m^2$, hence by the choice of $m_\zeta = m^2$ necessarily $\eta_{\nu_\zeta(\ell_1)} = \eta_{\rho_\zeta(\ell_2)}$ so that $\nu_\zeta(\ell_1) = \rho_\zeta(\ell_2)$ and (see clause (b)) also $\nu_\xi(\ell_1) = \rho_\xi(\ell_2)$. So

$$d_0(\{\nu_\zeta(\ell_1), \rho_\xi(\ell_2)\}) = d_0(\{\nu_\zeta(\ell_1), \nu_\xi(\ell_1)\}).$$

The latter is the required γ provided that $\ell_1 \notin u^*$. Equivalently, $\nu_\zeta(\ell_1) \neq \nu_\xi(\ell_1)$ but otherwise also $\nu_\zeta(\ell_1) = \rho_\xi(\ell_2)$ so $\ell g(\eta_{\nu_\zeta(\ell_1)} \cap \eta_{\rho_\xi(\ell_2)}) = \omega$, contradicting the assumption of $(*)_3$ that $\ell g(\eta_{\tau(\ell_1)} \cap \eta_{\tau(\ell_2)}) \in [m^2, \omega)$ (so it is not equal to ω).
So we finish[1] proving $(*)_2$, hence $\otimes$.

Stage B: Like Stage A of the proof of [5, Ch. III, 4.4, p. 164]. (So for $\alpha < \beta < \omega_2$, α does not appear in $\rho(\beta, \alpha)$).

Stage C: Defining the colouring:

Remember that $\mathscr{S}^\alpha_\beta = \{\delta < \aleph_\alpha : \mathrm{cf}(\delta) = \aleph_\beta\}$.

For $\ell = 1,2$ choose $h_\ell : \omega_2 \to \omega_\ell$ such that $S^\ell_\alpha = \mathscr{S}^2_1 \cap h^{-1}_\ell(\{\alpha\})$ is stationary for each $\alpha < \omega_\ell$. For $\alpha < \omega_2$, let $A_\alpha \subseteq \omega_1$ be such that no one is included in the union of finitely many others.

For $\alpha < \beta < \omega_2$, let $\ell = \ell_{\beta,\alpha}$ be minimal such that

$$d_{\mathrm{sq}}(\rho_{h_1}(\beta, \alpha)) \in A_{\rho(\beta,\alpha)(\ell)}$$

and lastly let

$$c(\beta, \alpha) = c(\alpha, \beta) =: h_2((\rho(\beta, \alpha))(\ell_{\beta,\alpha})).$$

Stage D: Proving that the colouring works:

So assume that $n < \omega$, $\langle u_\alpha : \alpha < \omega_2 \rangle$ is a sequence of pairwise disjoint subsets of ω_2 of size n and $\gamma(*) < \omega_2$ and we should find $\alpha < \beta$ such that $c \restriction (u_\alpha \times u_\beta)$ is constantly $\gamma(*)$. Without loss of generality, $\alpha < \beta \Rightarrow \max(u_\alpha) < \min(u_\beta)$ and

[1] See alternatively Definition 2.2(1) and Claim 4.1.

$\min(u_\alpha) > \alpha$ and let $E = \{\delta : \delta$ a limit ordinal $< \omega_2$ and $(\forall\alpha)(\alpha < \delta \Rightarrow u_\alpha \subseteq \delta)\}$. Clearly, E is a club of ω_2. For each $\delta \in E \cap \mathscr{S}^2_1$, there is an $\alpha^*_\delta < \delta$ such that

$$\alpha \in [\alpha^*_\delta, \delta) \quad \& \quad \beta \in u_\delta \Rightarrow \rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha).$$

Also for $\delta \in \mathscr{S}^2_1$ let

$$\varepsilon_\delta =: \mathrm{Min}\left\{\varepsilon < \omega_1 : \zeta \in A_\delta \text{ but if } \alpha \in \bigcup_{\beta\in u_\delta} \mathrm{Rang}\,(\rho(\beta,\delta)) \right.$$
$$\left. (\text{so } \alpha > \delta) \text{ then } \varepsilon \notin A_\alpha \right\}.$$

Note that $\varepsilon_\delta < \omega_1$ is well defined by the choice of the A_α's. So, by Fodor's lemma, for some $\zeta^* < \omega_1$ and $\alpha^* < \omega_2$ we have that

$$W =: \{\delta \in S^2_{\gamma(*)} : \alpha^*_\delta = \alpha^* \text{ and } \varepsilon_\delta = \varepsilon^*\}$$

is stationary. Let h be a strictly increasing function from ω_2 into W such that $\alpha^* < h(\delta)$. By the demand on α^* (and W)

$$\bigoplus_0 \quad \alpha^* < \alpha < \delta \in W \quad \& \quad \beta \in u_\delta \Rightarrow \rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha).$$

Hence

$$\bigoplus_1 \quad \alpha^* < \alpha < \delta \in \mathscr{S}^2_1 \quad \& \quad \beta \in u_{h(\delta)}$$
$$\Rightarrow \mathrm{Min}\{\ell : \varepsilon^* \in A_{\rho(\beta,\alpha)(\ell)}\} = \mathrm{Min}\{\ell : \rho(\beta,\delta)(\ell) = h(\delta)\};$$

hence

$$\bigoplus_2 \quad \alpha^* < \alpha < \delta \in \mathscr{S}^2_1 \quad \& \quad \beta \in u_{h(\delta)}$$
$$\Rightarrow h_2(\rho(\beta,\delta)[\mathrm{Min}\{\ell : \varepsilon^* \in A_{\rho(\beta,\delta)(\ell)}\}]) = \gamma(*).$$

Let

$$E_0 =: \{\delta < \omega_2 : \delta \text{ a limit ordinal, } \delta \in E \text{ and}$$
$$\alpha < \delta \Rightarrow h(\alpha) < \delta \text{ (hence } \sup(u_{h(\alpha)}) < \delta)\}.$$

For each $\delta \in \mathscr{S}^2_1$ there is an $\alpha^{**}_\delta < \delta$ such that $\alpha^{**}_\delta > \alpha^*$ and

$$\alpha \in [\alpha^{**}_\delta, \delta) \quad \& \quad \beta \in u_{h(\delta)} \Rightarrow \rho(\beta,\delta)\hat{\ }\langle\delta\rangle \trianglelefteq \rho(\beta,\alpha).$$

For each $\gamma < \omega_1$, $\delta \mapsto \alpha^{**}_\delta$ is a regressive function on S^1_γ; hence for some $\alpha^{**}(\gamma) < \omega_2$ the set $S'_\gamma =: \{\delta \in S^1_\gamma \cap E_0 : \alpha^{**}_\delta = \alpha^{**}(\gamma)\}$ is stationary.

Let $\alpha^{**} = \sup\{\alpha^{**}(\gamma) + 1 : \gamma < \omega_1\}$ and note that $\alpha^{**} < \omega_2$. Let

$$E_1 =: \{\delta < \omega_2 : \text{for every } \gamma < \omega_1,\ \delta = \sup(S'_\gamma \cap \delta) \text{ and } \delta > \alpha^{**}\},$$

and note that E_1 is a club of $\aleph_2$ (and as $S'_\gamma \subseteq E_0$ clearly $E_1 \subseteq E_0$) and choose $\delta^* \in E_1 \cap S^2_{\gamma(*)}$. Then by induction on $i < \omega_1$ choose an ordinal ζ_i such that $\langle \zeta_i : i < \omega_1\rangle$ is strictly increasing with limit δ^* and $\zeta_i \in S'_i \setminus (\alpha^{**} + 1)$. We know that $\alpha < \zeta_i \Rightarrow u_\alpha \subseteq \zeta_i$

and $\alpha < \min(u_\alpha)$; hence for every $\alpha_i < \zeta_i$ large enough $(\forall \beta \in u_{\alpha_i})(\rho(\delta^*,\zeta_i)^\frown(\zeta_i) \trianglelefteq \rho(\delta^*,\beta))$.

Choose such $\alpha_i \in (\bigcup_{j<i} \zeta_j, \zeta_i)$. Lastly, for $i < \omega_1$ choose $\beta_i \in E \cap S'_i$ with $\beta_i > \delta^*$. Now for each $i < \omega_1$ for some $\xi(i) < \delta^*$,

$$\bigoplus_3 \quad \alpha \in (\xi(i), \delta^*) \quad \& \quad \beta \in u_{h(\beta_i)} \Rightarrow \rho(\beta,\delta^*)^\frown\langle\delta^*\rangle \trianglelefteq \rho(\beta,\alpha).$$

As $\delta^* = \bigcup_{i<\omega_1} \zeta_i$, without loss of generality $\xi(i) = \zeta_{j(i)}$, and $j(i)$ is (strictly) increasing with i and let $A =: \{\varepsilon < \omega_1 : \varepsilon$ a limit ordinal and $(\forall i < \varepsilon)(j(i) < \varepsilon)\}$. Clearly, A is a club of ω_1. Now putting all of this together we have the following:

- $(*)_1$ If $i(0) < i(1)$ are in $A, \alpha \in u_{\alpha_{i(1)}}, \beta \in u_{h(\beta_{i(0)})}$ then $\rho(\beta,\alpha) = \rho(\beta,\delta^*)^\frown\rho(\delta^*,\alpha)$. (Why? As $j(i(0)) < i(1)$, see $\bigoplus_3$).
- $(*)_2$ If $i < \omega_1$ then $\beta \in u_{h(\beta_i)} \Rightarrow i \in \mathrm{Rang}(\rho_{h_1}(\beta,\delta^*))$ (witnessed by β_i which belongs to this set by $\bigoplus_0 + \bigoplus_1$).
- $(*)_3$ If $i < \omega_1$ then $\alpha \in u_{\alpha_i} \Rightarrow i \in \mathrm{Rang}(\rho_{h_1}(\delta^*,\alpha))$ (witnessed by ζ_i which belongs to this set by the choice of α_i).
- $(*)_4$ If $i < \omega_1$ and $\beta \in u_{h(\beta_i)}$ then $\ell = \mathrm{Min}\{\ell : \varepsilon^* \in A_{\rho(\beta,\delta^*)(\ell)}\}$ is well defined and $h_2(\rho(\beta,\delta^*)(\ell)) = \gamma(*)$. (Why? By $\bigoplus_2$).

Now let ν_i, for $i < \omega_1$, be the concatenation of $\{\rho(\beta,\delta^*) : \beta \in u_{\beta_i}\}$ and ρ_i be the concatenation of $\{\rho(\delta^*,\alpha) : \alpha \in u_{\alpha_i}\}$. So we can apply $\otimes$ of Stage A to $\langle \nu_i, \rho_i : i < \omega_1\rangle$ and γ^* (its assumptions hold by $(*)_1 + (*)_2 + (*)_3$) and get that, for some $i < j < \omega_1$, we have $d_{\mathrm{sq}}(\nu'^\frown\rho') = \varepsilon^*$ whenever ν' is a subsequence of ν_i, ρ' a subsequence of ρ_j such that $i \in \mathrm{Rang}(\nu')$, $j \in \mathrm{Rang}(\rho')$. Now for $\beta \in u_{h(\beta_i)}$, $\alpha \in u_{\alpha_j}$ we have:

- (i) $\rho(\beta,\alpha) = \rho(\beta,\delta^*)^\frown\rho(\delta^*,\alpha)$ (see $(*)_1$);
- (ii) $\rho(\beta,\delta^*)$ is O.K. as ν'. (Why? Because it is a subsequence of ν_i (see the choice of ν_i) and i belongs to $\mathrm{Rang}(\rho(\beta,\delta^*))$ by $(*)_2$);
- (iii) $\rho(\delta^*,\alpha)$ is O.K. as ρ'. (Why? Because $\rho(\delta^*,\alpha)$ is a subsequence of ρ_j by the choice of ρ_j and j belongs to $\mathrm{Rang}(\rho(\delta^*,\alpha))$ by $(*)_3$).

Now by $(*)_4$ the colour $c(\beta,\alpha)$ is $\gamma(*)$ as required and get the desired conclusion. □

Remark. Can we get $Pr_1(\lambda^{+2},\lambda^{+2},\lambda^{+2},\lambda)$ for λ regulars by the above proof? If $\lambda = \lambda^{<\lambda}$ the same proof works (now $\mathrm{Dom}(d_{\mathrm{sq}}) = {}^{\omega>}(\lambda^+)$ and $\nu_\alpha, \rho_\alpha \in {}^{\lambda>}(\lambda^+)$).
See more in Section 2.

2. Larger cardinals: the implicit properties

More generally (than in the remark at the end of Section 1):

2.1. Definition. (1) $Pr_6(\lambda,\lambda,\theta,\sigma)$ means that there is a $d : {}^{\omega>}\lambda \to \theta$ such that: *if* $\langle (u_\alpha, v_\alpha) : \alpha < \lambda\rangle$ satisfies

$$u_\alpha \subseteq {}^{\omega>}\lambda, \quad v_\alpha \subseteq {}^{\omega>}\lambda,$$

$$|u_\alpha \cup v_\alpha| < \sigma,$$

$$\nu \in u_\alpha \cup v_\alpha \Rightarrow \alpha \in \mathrm{Rang}(\nu),$$

and $\gamma < \theta$ and E a club of λ *then* for some $\alpha < \beta$ from E we have

$$\nu \in u_\alpha \quad \& \quad \rho \in v_\beta \Rightarrow d(\nu\hat{\ }\rho) = \gamma.$$

(2) $Pr^6_S(\lambda, \lambda, \theta, \sigma)$ is defined similarly but $\alpha < \beta$ are required to be in $E \cap S$. $Pr^6_\tau(\lambda, \lambda, \theta, \sigma)$ means "for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) \geqslant \tau\}$ we have $Pr^6_S(\lambda, \lambda, \theta, \sigma)$". If τ is omitted, we mean $\tau = \sigma$. Lastly $Pr^6_{\text{nacc}}(\lambda, \lambda, \theta, \sigma)$ is defined similarly but demanding $\alpha, \beta \in \text{nacc}(E)$ and $Pr^-_6(\lambda, \lambda, \theta, \sigma)$ is defined similarly but $E = \lambda$.

2.2. Lemma. (0) *If* $Pr_6(\lambda, \lambda, \theta, \sigma)$ *and* $\theta_1 \leqslant \theta$ *and* $\sigma_1 \leqslant \sigma$ *then* $Pr_6(\lambda, \lambda, \theta_1, \sigma_1)$ *(and similar monotonicity properties for Definition* (2.1(2))*. Without loss of generality* $u_\alpha = v_\alpha$ *in Definition* 2.1.

(1) *If* $Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$, *then* $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$.

(2) *If* $Pr_6(\lambda^+, \lambda^+, \theta, \sigma)$, *so* $\theta \leqslant \lambda^+$ *then* $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$ *provided that*

(∗) *there is a sequence* $\bar{A} = \langle A_\alpha : \alpha < \lambda^{++} \rangle$ *of subsets of* θ *such that for every* $\alpha \in u \subseteq \lambda^{++}$ *with* u *of cardinality* $< \sigma$, *we have*

$$A_\alpha \backslash \bigcup \{A_\beta : \beta \in u, \beta \neq \alpha\} \neq \emptyset.$$

(3) *If* λ *is regular and* $\lambda = \lambda^{<\lambda}$ *then* $Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$.

(4) *In* [5, Ch. III, 4.7] *we can change the assumption accordingly.*

Proof. (0) Clear.

(1) By part (2) choosing $\theta = \lambda^+$, $\sigma = \lambda$ as (∗) holds as λ^+ is regular (so e.g. choose by induction on $\alpha < \lambda^{++}$, $A_\alpha \subseteq \lambda^+$ see unbounded non-stationary with $\beta < \alpha \Rightarrow |A_\alpha \cap A_\alpha| \leqslant \lambda$).

(2) Like the proof for $\aleph_2$, only now $\{\delta < \lambda^{++} : \text{cf}(\delta) = \lambda^+\}$ plays the role of $\mathscr{S}^2_1$ and let $h_1 : \lambda^{++} \to \theta$ and $h_2 : \lambda^{++} \to \lambda^{++}$ be such that for every γ and $\ell \in \{1,2\}$ such that $[\ell = 2 \Rightarrow \gamma < \lambda^{++}]$ and $[\ell = 1 \Rightarrow \gamma < \theta]$, the set $S^\ell_\gamma = \{\alpha < \lambda^{+2} : \text{cf}(\alpha) = \lambda^+ \text{ and } h_\ell(\alpha) = \gamma\}$ is stationary. Finally, if dq exemplifies $Pr_6(\lambda^+, \lambda^+, \theta, \sigma)$, then in defining c for a given $\alpha < \beta$, let $\ell_{\alpha,\beta}$ be the minimal ℓ such that $dq(\rho_{h_1}(\alpha, \beta))$ belongs to $A_{\rho_{h_1}(\alpha,\beta)(\ell)}$ and let $c(\beta, \alpha) = c(\alpha, \beta) = h_2(\rho(\beta, \alpha)(\ell_{\beta,\alpha}))$. Then in stage D, without loss of generality, $|u_\alpha| = \sigma_1 < \sigma$ for $\alpha < \lambda^+$ and continue as there, but after the definition of E_1 and choice of δ^* we do not choose ζ_i, α_i; instead we first continue choosing β_i, ξ_i for $i < \lambda^+$ as there is, without loss of generality, $\delta* = \bigcup_{i<\lambda^+} \xi(i)$. Only then we choose by induction on $i < \lambda^+$ the ordinal ζ_i such that: $\zeta_i \in S^\prime_i \backslash (\alpha** + 1)$, $\zeta_i > \sup[\{\xi(j) : j \leqslant i\} \cup \{\zeta_j : j < i\}]$ and then choose $\alpha_i < \zeta_i$ large enough (so no need of the club A of λ^+).

(3) As in the proof of 1.1, Stage A.

(4) Combine the proofs here and those in [5, Ch. III, 4.7] (and not used). □

This leaves some problems on Pr_1 open; e.g.

2.3. Question. (1) If $\lambda > \aleph_0$ is inaccessible, do we have $Pr_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ (rather than with $\sigma < \lambda$)?

(2) If $\mu > \aleph_0$ is regular (singular) and $\lambda = \mu^+$, do we have $Pr_1(\lambda^+, \lambda^+, \lambda^+, \mu)$? Clearly, yes, for the weaker version: c a symmetric two place function from λ^+ to λ^+ such that for every $\gamma < \lambda^+$ and pairwise disjoint $\langle u_\alpha : \alpha < \lambda^+\rangle$ with $u_\alpha \in [\lambda^+]^{<\lambda}$ we have

$$(\exists \alpha < \beta)\forall i \in u_\alpha \forall j \in u_\beta (\gamma \in \operatorname{Rang} \rho_c(j,i)).$$

See more in Section 4. Remember that we know $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$ for $\aleph_0 \leqslant \sigma < \lambda$ by [5, Ch. III, 4.7].

2.4. Claim. *Assume that μ is singular, $\lambda = \mu^+$, $2^\kappa > \mu > \kappa = \kappa^\theta, \theta = \mathrm{cf}(\theta) \geqslant \sigma + \mathrm{cf}(p)$ and $Pr_6(\theta, \theta, \theta, \sigma)$. Then $Pr_1(\mu^+, \mu^+, \theta, \sigma)$.*

Proof. Let $\bar{e} = \langle e_\alpha : \alpha < \lambda\rangle$ be a club system, $S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \theta\}$ stationary such that $\lambda \notin \mathrm{id}^a(\bar{e} \restriction S)$ and $\alpha \in e_\delta \Rightarrow \mathrm{cf}(\alpha) \neq \theta$ and

$$\begin{aligned} &\delta = \sup(\delta \cap S) \quad \& \quad \chi < \mu \\ &\quad \Rightarrow \delta = \sup(\{\alpha \in e_\delta : \mathrm{cf}(\alpha) > \chi + \sigma^+, \text{ so } \alpha \in \mathrm{nacc}(e_\delta)\}) \end{aligned}$$

and $\alpha \in e_\beta \cap S \Rightarrow e_\alpha \subseteq e_\beta$ (exists by [6, 2.10]). Let $\bar{f} = \langle f_\alpha : \alpha < \theta\rangle$, $f_\alpha : \mu^+ \to \kappa$ be such that every partial function g from μ^+ to κ (really, θ suffices) of cardinality $\leqslant \theta$ is included in some f_α (see [2] or [5, AP1.7]).

So for some $f = f_{\alpha(*)}$ we have the following:

(∗) for every club E of μ^+ for some $\delta \in S$ we have:

(a) $e_\delta \subseteq E$

(b) if $\chi < \mu$ and $\gamma < \theta$ then

$$\delta = \sup(\{\alpha \in \mathrm{nacc}(e_\delta) : f(\alpha) = \gamma \text{ and } \mathrm{cf}(\alpha) > \chi\}).$$

This actually proves $\mathrm{id}_p(\bar{e} \restriction S)$ is not weakly θ^+-saturated.

The rest is by combining the trick of [5, Ch. III, Section 4] (using first $\delta(*) \in S$ then some suitable $\alpha \in \mathrm{nacc}(e_{\delta(*)})$) and the proof for $\aleph_2$. □

2.5. Fact. *$Pr_1(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$ and $\mathrm{cf}([\lambda]^{<\mathrm{cf}\lambda}, \subseteq) = \lambda$ (which is trivial if $\lambda = \mathrm{cf}\lambda$) implies $Pr^6(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$.*

Remark. This is not totally immediate as in Pr_1 the sets are required to be pairwise disjoint.

Proof. Let $\kappa = \mathrm{cf}(\lambda)$ and $f_\alpha \in {}^\kappa\lambda$ for $\alpha < \lambda^+$ be such that $\alpha < \beta \Rightarrow f_\alpha <^*_{J^{bd}_\kappa} f_\beta$. Let $d : [\lambda^+]^2 \to \theta$ exemplifies $Pr_1(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$. For $\nu \in {}^{\omega>}(\lambda^+)$ we define $d^*_{\mathrm{sq}}(\nu)$ as follows.

If $\ell g(\nu) \leqslant 1$ or ν is constant, then $d^*_{\mathrm{sq}}(\nu) = 0$. So assume that $\ell g(\nu) \geqslant 2$ and ν is not constant.

For $\alpha < \beta < \lambda^+$ let $\mathbf{s}(\beta,\alpha) = \mathbf{s}(\alpha,\beta) = \sup\{i+1 : i < \kappa \text{ and } f_\alpha(i) \geqslant f_\beta(i)\}$,

$\mathbf{s}(\alpha,\alpha) = 0$,

$\mathbf{s}(\nu) = \max\{\mathbf{s}(\nu(\ell),\nu(k)) : \ell,k < \ell g(\nu) \text{ (so } \mathbf{s} \text{ is symmetric)}\}$,

$a(\nu) = \{(\ell,k) : \mathbf{s}(\nu(\ell),\nu(k)) = \mathbf{s}(\nu) \text{ and } \ell < k < \ell g(\nu)\}$.

As $\ell g(\nu) \geqslant 2$ and ν is not constant, clearly $a(\nu) \neq \emptyset$ and $a(\nu)$ is finite, so let (ℓ_ν, k_ν) be the first pair from $a(\nu)$ in lexicographical ordering.
Lastly, $d^*_{\mathrm{sq}}(\nu) = d(\{\nu(\ell_\nu), \nu(k_\nu)\})$.

Now we are given $\gamma < \theta$, a stationary $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \geqslant \mathrm{cf}(\lambda)\}$, $\langle u_\alpha : \alpha < \lambda^+\rangle$ (remember 2.2(0)), $|u_\alpha| < \mathrm{cf}(\lambda)$, $u_\alpha \subseteq {}^{\omega>}\lambda$ such that $\alpha \in \bigcap\{\mathrm{Rang}(\nu) : \nu \in u_\alpha\}$. Let $u'_\alpha = \bigcup\{\mathrm{Rang}(\nu) : \nu \in u_\alpha\}$ and $u''_\alpha = u'_\alpha \setminus \alpha$, and as $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ wlog for some $v \in [\lambda^+]^{<\kappa}$, we have $\alpha \in S \Rightarrow u'_\alpha \cap \alpha \subseteq v$. Without loss of generality for some stationary $S' \subseteq S$ and γ_0, β^* we have $\alpha \in S' \Rightarrow \gamma_0 = \min\{\gamma + 1 : \text{if } \beta_1 < \beta_2 \text{ are in } u'_\alpha \cup v \text{ then } f_{\beta_1} \restriction [\gamma, \mathrm{cf}(\lambda)) < f_{\beta_2} \restriction [\gamma, \mathrm{cf}(\lambda))\} < \kappa$ and $\sup(\bigcup\{u'_\alpha \cap \alpha : \alpha \in S'\}) < \beta^* < \lambda^+$. Now for some $\gamma_1 \in (\gamma_0, \mathrm{cf}(\lambda))$ and stationary $S'' \subseteq S'$ and $\gamma^* < \lambda$ we have

$$\alpha \in S'' \Rightarrow f_\alpha(\gamma_1) = \gamma^*.$$

Lastly, apply the choice of d. □

Remark. Instead $\kappa = \mathrm{cf}(\lambda, \mathrm{cf}[\lambda]^{<\kappa}, \subseteq) = \lambda$ we can use: $(*)'$ from below. Moreover, if $Pr_1(\lambda^+, \lambda^+, \theta, o)$, $\mathrm{cf}([\lambda]^{<\sigma}, \subseteq) = \lambda$ and $(*)'$ below, then $Pr^6(\lambda^+, \lambda^+, \theta, o)$ where $(*)'$ there is $\delta^* \leqslant \lambda$, and a sequence $\bar{A} = \langle A_\alpha \mid \alpha < \lambda^+\rangle$ of unbounded subsets of S^* such that if $\alpha \in u \in [\lambda^+]^{<\sigma}$, then $A_\alpha \cap \bigcup_{\beta \in u \setminus \langle\alpha\rangle} A_\beta$ is bounded in δ^*. The proof is as above.

3. Guessing clubs revisited

3.1. Claim. *Assume that $\lambda = \mu^+$, and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda$ and δ is divisible by $\lambda^2\}$ is stationary.*

(1) *There is a strict club system $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that $\lambda^+ \notin \mathrm{id}^p(\bar{C})$ and $(\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow \mathrm{cf}(\alpha) = \lambda)$; moreover, there are $h_\delta : C_\delta \to \mu$ such that for every club E of λ^+, for stationarily many $\delta \in S$,*

$$\bigwedge_{\zeta < \mu} \delta = \sup[h_\delta^{-1}(\{\zeta\}) \cap E \cap \mathrm{nacc}(C_\delta)].$$

(2) *If $\bar{C}$ is a strict S-system, $\lambda^+ \notin \mathrm{id}^p(\bar{C}, \bar{J})$, J_δ a λ-complete ideal on C_δ extending $J^{bd}_{C_\delta} + \mathrm{acc}(C_\delta)$ (with S, μ as above) then the parallel result holds for some $\bar{h} = \langle h_\delta : \delta \in S\rangle$ where h_δ is a function from C_δ to μ, i.e. we have for every club E of λ^+, for stationarily many $\delta \in S \cap \mathrm{acc}(E)$ for every $\gamma < \mu$ the set $\{\alpha \in C_\delta : h_\delta(\alpha) = \gamma$ and $\alpha \in E\}$ is $\neq \emptyset \bmod J_\delta$.*

3.2. Remark. (1) This improves [7, 3.1].

(2) Of course, we can strengthen (1) to:

$$\gamma < \mu \Rightarrow \delta = \sup\{\alpha \in C_\delta : h_\delta(\alpha) = \gamma \text{ and } \alpha \in E \text{ and } \alpha \in \text{nacc}(C_\delta) \text{ and } \sup(\alpha \cap C_\delta) \in E\}.$$

For example, for every thin enough club E of λ, $\bar{C}^E$ will serve where $C^E_\delta = C_\delta \cap E$ if $\delta \in \text{acc}(E)$ and $C^E_\delta = C_\delta$, otherwise. For Claim 3.1(2) we get slightly less: for some club E^* : (for every club $E \subseteq E^*$ for stationary maps $\delta \in S \cap \text{arc}(E)$ for every $\gamma < \mu$ we have) $\delta = \sup\{\alpha \in C_\delta : h_\delta(\alpha) = \gamma$ and $\alpha \in E$ and $\alpha \in \text{nacc}(C_\delta)$ and $\sup(\alpha \cap C_\delta \cap E^*) \in E\}$.

Proof. (1) Let $\langle C_\delta : \delta \in S\rangle$ be such that $\lambda^+ \notin \text{id}^p(\bar{C})$ and $[\alpha \in \text{nacc}(C_\delta) \Rightarrow \text{cf}(\delta) = \lambda]$ (such a sequence exists by [6, 2.4(3)]). Let $J_\delta = J^{bd}_{C_\delta} + \text{acc}(C_\delta)$. Now apply part (2).

(2) For each $\delta \in S$ let $\langle A^\alpha_\delta : \alpha \in C_\delta\rangle$ be a sequence of distinct non-empty subsets of μ to be chosen later. By induction on $\zeta < \lambda$ we try to define $E_\zeta, \langle Y^\zeta_\alpha : \alpha \in S\rangle$, $\langle Z^\zeta_{\alpha,\gamma} : \alpha \in E_\zeta \text{ and } \gamma < \mu\rangle$ such that

E_ζ is a club of λ^+, decreasing in ζ,

for $\gamma < \mu$,

$$Z^\zeta_{\delta,\gamma} = \{\alpha : \alpha \in E_\zeta \cap \text{nacc}(C_\delta) \text{ and } \gamma \in A^\alpha_\delta\},$$

$$Y^\zeta_\delta = \{\gamma < \mu : Z^\zeta_{\delta,\gamma} \neq \emptyset \bmod J_\delta\}.$$

$E_{\zeta+1}$ is such that

$$\{\delta \in S : Y^\zeta_\delta = Y^{\zeta+1}_\delta \text{ and } \delta \in \text{nacc}(E_{\zeta+1}) \text{ and } E_{\zeta+1} \cap \text{nacc}(C_\delta) \notin J_\delta\}$$

is not stationary and moreover disjoint to E_{3+1}, hence is empty.

If we succeed to define E_ζ, for each $\zeta < \lambda$, then $E =: \bigcap_{\zeta<\lambda} E_\zeta$ is a club of λ^+, and since $\lambda^+ \notin \text{id}^p(\bar{C})$, we can choose $\delta \in S$ such that $\delta = \sup(E \cap \text{nacc } C_\delta)$ and $E \cap \text{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$. Then as $\bigcup_{\gamma<\mu} Z^\zeta_{\delta,\gamma} \supseteq E \cap \text{nacc}(C_\delta)$ for each $\zeta < \lambda$ necessarily (by the requirement on J_δ) for some $\gamma < \mu$, $Z^\zeta_{\delta,\gamma} \neq \emptyset \bmod J_\delta$, hence $Y^\zeta_\delta \neq \emptyset$ so that $\langle Y^\zeta_\delta : \zeta < \lambda\rangle$ is a strictly decreasing sequence of subsets of μ, and since $\mu < \text{cf}(\mu^+) = \text{cf}(\lambda)$, we have a contradiction. So necessarily we will be stuck (say) for $\zeta(*) < \lambda$.

We still have the freedom of choosing A^α_δ for $\alpha \in C_\delta$.

Case 1: μ regular.

By induction on $\alpha \in C_\delta$ we can choose sets A^α_δ such that

(i) $A^\alpha_\delta \subseteq \mu$, $|A^\alpha_\delta| = \mu$, $\langle A^\alpha_\delta : \alpha \in C_\delta, \text{otp}(\alpha \cap C_\delta) < \mu\rangle$ are pairwise disjoint,

(ii) for $\beta \in C_\delta \cap \alpha$, $A^\alpha_\delta \cap A^\beta_\delta$ is bounded in μ,

(iii) if $\mu > \aleph_0$ then A^α_δ is non-stationary (just to clarify their choice).
There is no problem to carry out the induction.

We shall prove later that

(*) if $E \subseteq E_{\zeta(*)}$ is a club of $\lambda^+, \delta \in S \cap \mathrm{acc}(E)$ and $\delta = \sup(E \cap \mathrm{nacc}\, C_\delta)$ and $E \cap \mathrm{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$ then

$(**)_\delta$ for some $\alpha_\delta \in E \cap \mathrm{nacc}(C_\delta)$, the following set B_δ is unbounded in μ:

$$B_\delta = \{\gamma < \mu : \{\beta : \beta \in E \cap \mathrm{nacc}(C_\delta) \text{ and } \beta \neq \alpha_\delta \text{ and } \gamma = \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\} \neq \emptyset \bmod J_\delta\}.$$

Choose the minimal such that $\alpha_\delta = \alpha^E_\delta$ (for other δ's it does not matter, i.e. for those for which $\delta > \sup(E \cap \mathrm{nacc}(C_\delta))$ or $E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta) \in J_\delta$). Clearly, if $E' \supseteq E''$ and $\alpha^{E'}_\delta, \alpha^{E''}_\delta$ are defined then $\alpha^{E'}_\delta \leqslant \alpha^{E''}_\delta$. We shall choose a club $E^* \subseteq E_{\zeta(*)}$ of λ^+.

Now for any club E of λ^+ for stationarily many $\delta \in S \cap \mathrm{acc}(E^* \cap E)$, we have

$$\{\gamma < \mu : \{\alpha : \alpha \in E^* \cap E \cap E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta) \text{ and } \gamma \in A^\alpha_\delta\} \neq \emptyset \bmod J_\delta\} = Y^{\zeta(*)}_\delta$$

(this holds by the choice of $\zeta(*)$). Let the set of such $\delta \in S \cap \mathrm{acc}(E^* \cap E)$ be called $Z^{E^*}_E$. Now for each $\delta \in Z^{E^*}_E$, the set

$$B_\delta[E, E^*] =: \{\gamma < \mu : \{\beta : \beta \in E \cap E^* \cap E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta) \text{ and } \beta \neq \alpha^{E^*}_\delta \text{ and } \gamma = \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\} \neq \emptyset \bmod J_\delta\}$$

is necessarily unbounded in μ. So in the same way as we have got $E_{\zeta(*)}$ we can find club $E \subseteq E^* \subseteq E_{\zeta(*)}$ such that for any club $E \subseteq E^*$ of λ^+, for stationarily many $\delta \in Z^{E*}_E$, we, have $B_\delta[E, E_{\zeta(*)}] = B_\delta[E^*, E_{\zeta(*)}]$ and $\alpha^E_\delta = \alpha^{E^*}_\delta$ (note the minimality in the choice of α^E_δ so it can change $\leqslant \lambda + 1$ times; more elaborately if $\langle E^*_\zeta : \zeta < \lambda\rangle$ is a decreasing sequence of clubs and $\delta \in Z^{E^*_\zeta}_{E^*}$, where $E^* = \bigcap_{\zeta<\lambda} E^*_\zeta$, then $\langle \alpha^{E^*_\zeta}_\delta : \zeta < \lambda\rangle$ is increasing and bounded in C_δ (by $\alpha^{E^*}_\delta$), hence is eventually constant). Define $h_\delta : C_\delta \to \mu$ by $h_\delta(\beta) = \mathrm{otp}\left(B_\delta[E^*, E_{\zeta(*)}] \cap \sup(A^{\alpha_\delta}_\delta \cap A^\beta_\delta)\right)$ if $\beta \neq \alpha_\delta$ and $h_\delta(\beta) = 0$ if $\beta = \alpha_\delta$. Clearly $\langle h_\delta : \delta \in S \cap \mathrm{arc}(E^*)\rangle$ is as required.

Why does (*) hold?

If not, let $B = E \cap \mathrm{nacc}(C_\delta)$, so $\mathrm{otp}(B) = \lambda = \mu^+$ and $B \neq \emptyset \bmod J_\delta$, so for every $\alpha \in B$ we can find $\varepsilon_\alpha < \mu$ and $Y_{\alpha,\varepsilon} \in J_\delta$ (for $\varepsilon < \mu$) such that if $\xi \in B \setminus Y_{\alpha,\varepsilon} \setminus \{\alpha\}$ and $\varepsilon \in [\varepsilon_\alpha, \mu)$ then $\sup(A^\alpha_\delta \cap A^\xi_\delta) \neq \varepsilon$. Now let $Y_\alpha =: \bigcup\{Y_{\alpha,\varepsilon} : \varepsilon \in [\varepsilon_\alpha, \mu)\} \cup \{\alpha\}$ and note that $Y_\alpha \in J_\delta$. So for some $\varepsilon^* < \mu$, $B_1 =: \{\alpha \in B : \varepsilon_\alpha = \varepsilon^*\}$ is $\neq \emptyset \bmod J_\delta$. For each $\alpha \in B_1$ choose $\gamma_\alpha \in A^\delta_\alpha \setminus (\varepsilon^* + 1)$ (remember $|A^\delta_\alpha| = \mu$). So for some $\gamma^* < \mu$ the set $B_2 =: \{\alpha \in B_1 : \gamma_\alpha = \gamma^*\}$ is $\neq \emptyset \bmod J_\delta$. Let $\alpha^* = \mathrm{Min}(B_2)$, and for $\gamma \in [\gamma^*, \mu)$ we define $B_{\zeta,\gamma} = \{\alpha \in B_2 : \gamma = \sup(A^{\alpha^*}_\delta \cap A^\alpha_\delta\}$. So clearly $B_2 = \bigcup\{B_{\zeta,\gamma} : \gamma^* \leqslant \gamma < \mu\}$, hence for some $\gamma^{**} \in [\gamma^*, \mu)$ we have $B_{\zeta,\gamma^{**}} \neq \emptyset \bmod J_\delta$, hence γ^{**} contradicts the choice of $\varepsilon_{\alpha^*} = \varepsilon^*$.

Case 2: μ singular.

Let $\kappa = \mathrm{cf}(\mu)$, so by [5, Ch. II, Section 1] we can find an increasing sequence $\langle \lambda_i : i < \kappa\rangle$ of regular cardinals $> \kappa$ with limit μ such that $\lambda = \mu^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / J^{bd}_\kappa)$, Sh:572

and[2] let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifying this. Without loss of generality, $\bigcup_{j<i} \lambda_j < f_\alpha(i) < \lambda_i$. Let $g : \kappa \times \mu \times \kappa \times \mu \to \mu$ be one to one and onto, let $f^\delta_\alpha = f_{\mathrm{otp}(\alpha \cap C_\delta)}$ for $\alpha \in C_\delta$ and let $A^\delta_\alpha = \{g(i, f^\delta_\alpha(i), j, f^\delta_\alpha(j)) : i, j < \kappa\}$.

If $\delta = \sup(E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta))$ and $E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$ then (as J_δ is λ-complete) choose $Y_\delta \in J_\delta$ such that for each $i < \kappa$, $\varepsilon < \lambda_i$ we have

$(*)$ $(\exists \beta)[\beta \in E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta)\ \&\ \beta \notin Y_\delta\ \&\ f^\delta_\beta(i) = \varepsilon]$
$\Rightarrow \{\beta : \beta \in E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta)\ \&\ f^\delta_\beta(i) = \varepsilon\} \neq \emptyset \bmod J_\delta$.

Choose $i(\delta) < \kappa$ such that

$$B^0_\delta =: \{f^\delta_\beta(i(\delta)) : \beta \in E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta) \text{ and } \beta \notin Y_\delta\}$$

is unbounded in λ_i.

Let $\xi_\varepsilon = \xi^\delta_\varepsilon$ be the ε-th member of B^0_δ, for $\varepsilon < \kappa$. For each such $\varepsilon < \kappa$ for some $j_\varepsilon = j^\delta_\varepsilon \in (i(\delta) + 1 + \varepsilon, \kappa)$ we have $B^{1,\delta}_\varepsilon =: \{f^\delta_\beta(j_\varepsilon) : f^\delta_\beta(i(\delta)) = \xi^\delta_\varepsilon$ and $\beta \in E_{\zeta(*)} \cap \mathrm{nacc}(C_\delta)$ and $\beta \notin Y_\delta\}$ is unbounded in $\lambda_{j^\delta_\varepsilon}$.

Let $h_{\delta,\varepsilon}$ be a one to one function from $[\bigcup_{j<\varepsilon} \lambda_j, \lambda_\varepsilon)$ into $B^{1,\delta}_\varepsilon$.

Lastly, we define h_δ as follows:

if $\beta \in C_\delta$, $\varepsilon < \kappa$, $f^\delta_\beta(i(\delta)) = \xi^\delta_\varepsilon$ and $h_{\delta,\varepsilon}(\gamma) = f^\delta_\beta(j^\delta_\varepsilon)$
(so $\gamma \in [\bigcup_{j<\varepsilon} \lambda_j, \lambda_\varepsilon)$) *then* $h_\delta(\beta) = \gamma$

and $h_\delta(\beta) = 0$ otherwise. The rest is similar to the regular case. □

3.3. Claim. *If* $\lambda = \mu^+$, μ *regular uncountable, and* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ *is stationary*, then *for some strict S-club system* $\bar{C}$ *with* $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \mu\}$, *(where* $\alpha_{\delta,\zeta}$ *is strictly increasing continuously in* ζ*) for every club* $E \subseteq \lambda$ *for stationarily many* $\delta \in S$,

$$\{\zeta < \mu : \alpha_{\delta,\zeta+1} \in E\} \text{ is stationary (as a subset of } \mu).$$

3.4. Remark. (1) If $S \in I[\lambda]$ then without loss of generality we can demand (a) or we can demand (b) (but not necessarily both), where

(a) $X_\alpha = \{C_\delta \cap \alpha : \delta \in S$, is such that $\alpha \in \mathrm{nacc}(C_\delta)\}$ has cardinality $\leqslant \lambda$,

(b) $\alpha \in \mathrm{nacc}(C_\delta) \Rightarrow C_\alpha = C_\delta \cap \alpha$ but the conclusion is weakened to: for every club E of λ for stationarily many $\delta \in S$ the set $\{\zeta < \mu : (\alpha_{\delta,\zeta}, \alpha_{\delta,\zeta+1}) \cap E \neq \emptyset\}$ is stationary.

(2) In contrast to [7, 3.4], here we allow μ inaccessible.

(3) Clearly Claim 3.1(2) can be applied to the results of Claim 3.3, i.e. with

$$J_\delta = \{A \subseteq C_\delta : \{\zeta < \lambda : \alpha_{\delta,\zeta+1} \notin A\} \text{ is not stationary}\}.$$

Proof. We know that for some strict S-club system $\bar{C}^0 = \langle C^0_\delta : \delta \in S \rangle$ we have $\lambda \notin \mathrm{id}_p(\bar{C}^0)$ (see [6, 2.3(1)]). Let $C^0_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\}$ (increasing continuously in ζ). We shall prove below that for some sequence of functions $\bar{h} = \langle h_\delta : \delta \in S \rangle$, $h_\delta : \mu \to \mu$

[2] For the rest of this case "$\lambda = \mu^+$" is not used; also J^{bd}_κ can be replaced by any larger ideal.

we have:

$(*)_{\bar h}$ for every club E of μ^+ for stationarily many $\delta \in S \cap \mathrm{acc}(E)$, the following subset of μ is stationary:

$$A_E^{\delta,*} =: \{\zeta < \mu : \alpha_\zeta^\delta \in E \text{ and some ordinal in } \{\alpha_\xi^\delta : \zeta < \xi \leqslant h_\delta(\zeta)\} \text{ belongs to } E\}.$$

The proof now breaks into two parts.

Proving $(*)_{\bar h}$ suffices. For each club E of λ, let $Z_E =: \{\delta \in S : \delta = \sup(E \cap \mathrm{nacc}(C_\delta^0))\}$, and note that this set is a stationary subset of λ (by the choice of $\bar C^0$). For each such E and $\delta \in Z_E$ let $f_{\delta,E}$ be the partial function from μ to μ defined by

$$f_{\delta,E}(\zeta) = \mathrm{Sup}\{\xi : \zeta < \xi \leqslant h_\delta(\zeta) \text{ and } \alpha_\xi^\delta \in E\}.$$

So if there is no such ξ, then $f_{\alpha,E}(\zeta)$ is not well defined (i.e. if the supremum is on the empty set) but if $\xi = f_{\alpha,E}(\zeta)$ is well defined then $\alpha_\xi^\delta \in E, \xi \leqslant h_\delta(\zeta)$ (because α_ξ^δ is increasing continuously in ξ and E is a club of λ). Let $Y_E =: \{\delta \in Z_E : \mathrm{Dom}(f_{\delta,E})$ is a stationary subset of $\mu\}$. So by $(*)_{\bar h}$, we know that

$\bigoplus$ for every club E of μ^+ the set Y_E is a stationary subset of μ^+.

Also

$\bigotimes_1$ if $E_2 \subseteq E_1$ are clubs of μ^+ then $Z_{E_2} \subseteq Z_{E_1}$ and $Y_{E_2} \subseteq Y_{E_1}$ and for $\delta \in Y_{E_2}$, $\mathrm{Dom}(f_{\delta,E_2}) \subseteq \mathrm{Dom}(f_{\delta,E_1})$ and $\zeta \in \mathrm{Dom}(f_{\delta,E_2}) \Rightarrow f_{\delta,E_2}(\zeta) \leqslant f_{\delta,E_1}(\zeta)$.

We claim that

$\bigotimes_2$ for some club E_0 of μ^+ for every club $E \subseteq E_0$ of μ^+ for stationarily many $\delta \in S$ we have:
- (i) $\delta = \sup(E \cap \mathrm{nacc}\, C_\delta)$,
- (ii) $\{\zeta < \mu : \zeta \in \mathrm{Dom}(f_{E,\delta})$ (hence $\zeta \in \mathrm{Dom}\, f_{E_0,\delta}$) and $f_{E,\delta}(\zeta) = f_{E_0,\delta}(\zeta)\}$ is a stationary subset of μ.

If this fails, then for any club E_0 of λ there is a club $E(E_0) \subseteq E_0$ of λ, such that

$$A_{E_0} = \{\delta : \delta \in S, \delta = \sup(E(E_0) \cap \mathrm{nacc}(C_\delta)) \text{ and for some stationary subset } e_{E_0,\delta} \text{ of } \mu \text{ we have } \zeta \in e_{E_0,\delta} \cap \mathrm{Dom}(f_{E(E_0),\delta}) \Rightarrow f_{E(E_0),\delta}(\zeta) = f_{E_0,\delta}(\zeta)\}$$

is not a stationary subset of $\lambda = \mu^+$. By obvious monotonicity we can replace $E(E_0)$ by any club of μ^+ which is a subset of it, so, without loss of generality, $A_{E_0} = \emptyset$.

By induction on $n < \omega$ choose clubs E_n of μ^+ such that $E_0 = \mu^+$ and $E_{n+1} = E(E_n)$. Then $E_\omega =: \bigcap_{n<\omega} E_n$ is a club of μ^+ and, by $\bigoplus$ above, $Y_{E_\omega} \subseteq S$ is a stationary subset of λ, so we can choose a $\delta(*) \in Y_{E_\omega}$. So $f_{E_\omega,\delta(*)}$ has domain a stationary subset of μ (see the definition of Y_{E_ω}) and by $\bigotimes_1$ we know that $n < \omega \Rightarrow \mathrm{Dom}(f_{E_\omega,\delta(*)}) \subseteq \mathrm{Dom}(f_{E_n,\delta(*)})$. Also there is an $e_{E_n,\delta(*)}$, a club of μ, such that

$$\zeta \in e_{E_n,\delta(*)} \cap \mathrm{Dom}(f_{E_{n+1},\delta(*)}) \Rightarrow f_{E_{n+1},\delta(*)}(\zeta) < f_{E_n,\delta(*)}(\zeta)$$

(see the choice of $E_{n+1} = E(E_n)$, i.e. the function E and $\otimes_1$). So $e_{\delta(*)} =: \bigcap_{n<\omega} e_{E_n,\delta(*)}$ is a club of μ and, as $\mathrm{Dom}(f_{E_\omega,\delta(*)})$ is a stationary subset of μ, we can find $\zeta(*) \in e_{\delta(*)} \cap \mathrm{Dom}(f_{E_\omega,\delta(*)})$; hence $\zeta(*) \in \bigcap_{n<\omega} \mathrm{Dom}(f_{E_n,\delta(*)}) \cap \bigcap_{n<\omega} e_{E_n,\delta(*)}$, so that $\langle f_{E_n,\delta(*)}(\zeta(*)) : n < \omega\rangle$ is a well-defined strictly increasing ω-sequence of ordinals – a contradiction. So $\bigotimes_2$ cannot fail, and this gives the desired conclusion.

Proof of $(*)_{\bar h}$ *holds for some* $\bar h$. So assume that for no $\bar h$ does $(*)_{\bar h}$ hold, hence (by shrinking E) we can assume that for every $\bar h = \langle h_\delta : \delta \in S\rangle$, $h_\delta : \mu \to \mu$, for some club E for *every* $\delta \in S$, $A_E^{\delta,*}$ is not stationary (in μ). By induction on $n < \omega$, we define E_n, $\bar h^n = \langle h_\delta^n : \delta \in S\rangle$, $\bar e^n = \langle e_\delta^n : \delta \in S\rangle$, with E_n a club of λ, e_δ^n club of μ, $h_\delta^n : \mu \to \mu$ as follows.

Let $E_0 = \lambda$, $h_\delta^0(\zeta) = \zeta+1$ and $e_\delta^n = \mu$. If $E_0,\dots,E_n$, $\bar h^0,\dots,\bar h^n$, $\bar e^0,\dots,\bar e^n$ are defined, necessarily $(*)_{\bar h^n}$ fails, so for some club E_{n+1} of λ for every $\delta \in S \cap \mathrm{acc}(E_{n+1})$ there is a club $e_\delta^{n+1} \subseteq \mathrm{acc}(e_\delta^n)$ of μ, such that

$$\zeta \in e_\delta^{n+1} \Rightarrow \{\alpha_\xi^\delta : \zeta < \xi \leqslant h_\delta(\zeta)\} \cap E_{n+1} = \emptyset.$$

Choose $h_\delta^{n+1} : \mu \to \mu$ such that $(\forall \zeta < \mu)(h_\delta^n(\zeta) < h_\delta^{n+1}(\zeta))$ and if $\delta = \sup(E_{n+1} \cap \mathrm{nacc}(C_\delta^0))$ then $\zeta < \mu \Rightarrow \{\alpha_\xi^\delta : \zeta < \xi \leqslant h_\delta^{n+1}(\zeta)\} \cap E_{n+1} \neq \emptyset$. There is no problem to carry out this inductive definition. By the choice of $\bar C^0$, for some $\delta \in \mathrm{acc}(\bigcap_{n<\omega} E_n)$, we have $\delta = \sup(A')$, where $A' =: (\mathrm{acc} \bigcap_{n<\omega} E_n) \cap \mathrm{nacc}(C_\delta^0)$. Let $A \subseteq \mu$ be such that $A' = \{\alpha_\zeta^\delta : \zeta \in A\}$ (remember α_ζ^δ is increasing with ζ) and let ζ be the second member of $\bigcap_{n<\omega} e_\delta^n$. As A' is unbounded in δ, clearly A is unbounded in μ and $\bigcap_{n<\omega} e_\delta^n$ is a club of μ as $\mu = \mathrm{cf}(\mu) > \aleph_0$. Also as $A' \subseteq \mathrm{nacc}(C_\delta^0)$ clearly A is a set of successor ordinals (or zero).

Note that $\sup(e_n^\delta \cap \zeta)$ is well defined (as $\min(e_n^\delta) \leqslant \min(\bigcap_{n<\omega} e_n^\delta) < \zeta$) and $\sup(e_n^\delta \cap \zeta) < \zeta$ (as ζ is a successor ordinal). Now $\langle \sup(e_n^\delta \cap \zeta) : n < \omega\rangle$ is non-increasing (as e_δ^n decreases with n), hence for some $n(*) < \omega$ we have $n > n(*) \Rightarrow \sup(e_\delta^n \cap \zeta) = \sup(e_\delta^{n(*)} \cap \zeta)$ and call this ordinal ξ so that $\xi \in e_{n(*)+1}^\delta$ and $h_\delta^{n(*)}(\xi) = h_\delta^{n(*)+1}(\xi)$, so we get a contradiction for $n(*)+1$.

So $(*)_{\bar h}$ holds for some $\bar h$, which suffices, as indicated above. □

3.5. Discussion. (1) We can squeeze a little more, but it is not so clear if with much gain. So assume that

$(*)_0$ μ is regular uncountable, $\lambda = \mu^+$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}$ stationary, I an ideal on S, $\bar C = \langle C_\delta : \delta \in S\rangle$ a strict S-club system, $\bar J = \langle J_\delta : \delta \in S\rangle$ with J_δ an ideal on C_δ extending $J_{C_\delta}^{bd} + (\mathrm{acc}(C_\delta))$, such that for any club E of λ we have $\{\delta \in S : E \cap C_\delta \neq \emptyset \bmod J_\delta\} \neq \emptyset \bmod I$.

(2) If we imitate the proof of Claim 3.3 we get

$(*)_1$ if for $\delta \in S, J_\delta$ is not χ-regular (see the definition below) and $\chi \leqslant \mu$ then we can find $\bar e = \langle e_\delta : \delta \in S\rangle$ and $\bar g = \langle g_\delta : \delta \in S\rangle$ such that

$(*)'_1$ e_δ is a club of δ, $e_\delta \subseteq \mathrm{acc}(C_\delta)$, $g_\delta : \mathrm{nacc}(C_\delta)\backslash(\min(e_\delta)+1) \to e_\delta$ is defined by $g_\delta(\alpha) = \sup(e_\delta \cap \alpha)$ and for every club E of λ

$$\{\delta \in S : E \cap \mathrm{nacc}(C_\delta) \neq \emptyset \bmod J_\delta \text{ and}$$
$$\mathrm{Rang}(g_\delta \restriction (E \cap \mathrm{nacc}(C_\delta))) \text{ is a stationary subset of } \delta\} \neq \emptyset \bmod I.$$

(3) An ideal J on a set C is χ-regular if there is a set $A \subseteq C$, $A \neq \emptyset \bmod J$ and a function $f : A \to [\chi]^{<\aleph_0}$ such that $\gamma < \chi \Rightarrow \{x \in A : \gamma \notin f(x)\} = \emptyset \bmod J$. If $\chi = |C|$, we may omit it. (How do we prove $(*)'_1$? Try χ times E_ζ, $\langle e^\zeta_\delta : \delta \in S\rangle$ (for $\zeta < \chi$)).

(4) We can try to get results like Claim 3.1. Now

$(*)_2$ assume that $\lambda, \mu, S, I, \bar{C}, \bar{J}$ are as in $(*)_0$ and $\bar{e}, \bar{g}$ as in $(*)'_1$ and $\kappa < \mu$ and for $\delta \in S$, $J^0_\delta =: \{a \subseteq e_\delta : \{\alpha \in \mathrm{Dom}(g_\delta) : g(\alpha) \in a\} \in J_\delta\}$ is weakly normal and μ satisfies the condition from [6, Lemma 2.12]. *Then* we can find $h_\delta : e_\delta \to \kappa$ such that for every club E of λ, $\{\delta \in S :$ for each $\gamma < \kappa$ the set $\{\alpha \in \mathrm{nacc}(C_\delta) : h_\delta(g_\delta(\alpha)) = \gamma\}$ is $\neq \emptyset \bmod J_\delta\} \neq \emptyset \bmod I$.

(Why? For each $\delta \in S, \alpha \in \mathrm{acc}(e_\delta)$ choose a club $d_{\delta,\alpha} \subseteq e_\delta \cap \alpha$ such that for no club $d \subseteq e_\delta$ of δ do we have $(\forall \gamma < \delta)(\exists \alpha \in \mathrm{acc}(e_\delta))[d \cap \gamma \subseteq d_{\delta,\alpha})$. Now for every club E of λ let $S_E = \{\delta : E \cap \mathrm{nacc}(C_\delta) \neq \emptyset \bmod J_\delta$, and $g''_\delta(E \cap \mathrm{nacc}(C_\delta))$ is stationary$\}$ and for $\delta \in E$ and $\varepsilon < \mu$, we choose by induction on $\zeta < \kappa, \xi(\delta, \varepsilon)$ as the first $\xi \in e_\delta$ such that: $\xi > \bigcup_{\zeta<\varepsilon} \xi(\delta,\zeta)$ and $\{\alpha \in \mathrm{Dom}(g_\delta) : \alpha \in E$ and the ε-th member of $d_{\delta, g_\delta(\alpha)}$ is in the interval $[\bigcup_{\zeta<\varepsilon} \xi(\delta,\zeta), \xi)]\} \neq \emptyset \bmod J_\delta$.

(5) We deal below with successor of singulars and with inaccessibles, we can do parallel things.

3.6. Claim. *Suppose μ is a singular cardinal of cofinality $\kappa, \kappa \geqslant \aleph_0$ and $S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) = \kappa\}$ is stationary, and $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is an S-club system satisfying $\mu^+ \notin \mathrm{id}^p(\bar{C}, \bar{J}^{b[\mu]})$ where $\bar{J}^{b[\mu]} = \langle J^{b[\mu]}_{C_\delta} : \delta \in S\rangle$ and $J^{b[\mu]}_{C_\delta} =: \{A \subseteq C_\delta :$ for some $\theta < \mu$, we have $\delta > \sup\{\alpha \in A : \mathrm{cf}(\alpha) > \theta$ and $\alpha \in \mathrm{nacc}(C_\delta)\}\}$. Then we can find a strict λ-club system $\bar{e}^* = \langle e^*_\delta : \delta < \lambda\rangle$ such that*

$(*)$ for every club E of μ^+, for stationarily many $\delta \in S$, for every $\alpha < \delta$ and $\theta < \mu$ for some β we have

$(**)_{E,\beta}$ $\beta \in \mathrm{nacc}(C_\delta)$ and $\beta > \alpha$ and $\mathrm{cf}(\beta) > \theta$ and $\{\gamma \in e^*_\beta : \gamma \in E$ and $\min(e^*_\beta \backslash (\gamma + 1))$ belongs to $E\}$ is a stationary subset of β.

3.7. Remark. (1) We know that for the given μ and S there is $\bar{C}$ as in the assumption by [6, Section 2]. Moreover, if $\kappa > \aleph_0$ then there is such nice strict $\bar{C}$.

(2) Remember $J^{b[\mu]}_\delta = \{A \subseteq C_\delta :$ for some $\theta < \mu$ and $\alpha < \delta$ we have $(\forall \beta \in C_\delta)(\beta < \alpha \vee \mathrm{cf}(\beta) < \theta \vee \beta \in \mathrm{nacc}(C_\delta))\}$.

(3) We can worm $\alpha \in \mathrm{nacc}(C_\delta)$ in the definition of $J^{b[\mu]}_{C_\delta}$ if we weaken $\beta \in \mathrm{nacc}(C_\delta)$ to $\beta \in C_\delta$ in $(**)_{E,\beta}$.

Proof. Let $\bar{e} = \langle e_\beta : \beta < \lambda\rangle$ be a strict λ-club system where $e_\beta = \{\alpha^\beta_\zeta : \zeta < \mathrm{cf}(\beta)\}$ is a (strictly) increasing and continuous enumeration of e_β (with limit δ). Now we claim that for some $\bar{h} = \langle \bar{h}_\beta : \beta < \lambda, \beta \text{ limit}\rangle$ with h_β a function from e_β to e_β and $\bigwedge_{\alpha \in e_\beta} h_\beta(\alpha) > \alpha$, we have:

$(*)_{\bar{h}}$ for every club E of μ^+, for stationarily many $\delta \in S \cap \text{acc}(E)$, $A^\delta_E \notin J^{b[\mu]}_{C_\delta}$ where A^δ_E is the set of all $\beta \in C_\delta$ such that the following subset of e_β is stationary (in β):

$$\{\gamma \in e_\beta : \gamma \in E \text{ and } \min(e_\beta \backslash (\gamma + 1)) \in E\}.$$

The rest is like the proof of Claim 3.3 repeating κ^+ times instead of ω and using "$J^{b[\mu]}_{C_\delta}$ is $(\leqslant \kappa)$-based". □

3.8. Claim. *Suppose λ is inaccessible, $S \subseteq \lambda$ is a stationary set of inaccessibles, $\bar{C}$ an S-club system such that $\lambda \notin \text{id}^p(\bar{C})$. Then we can find $\bar{h} = \langle h_\delta : \delta \in S\rangle$ with $h_\delta : C_\delta \to C_\delta$, such that $\alpha < h(\alpha)$ and*

$(*)$ for every club E of λ, for stationarily many $\delta \in S \cap acc(E)$ we have that

$$\{\alpha \in C_\delta : \alpha \in E \text{ and } h(\alpha) \in E\} \text{ is a stationary subset of } \delta.$$

So for some $C'_\delta = \{\alpha_{\delta,\zeta} : \zeta < \delta\} \subseteq C_\delta, \alpha_{\delta,\zeta}$ increasing continuously in ζ we have $h(\alpha_{\delta,\zeta}) = \alpha_{\delta,\zeta+1}$.

Remark. Under quite mild conditions on λ and S there is $\bar{C}$ as required – see [6, 2.12, p. 134].

Proof. Like the proof of Claim 3.3.

3.9. Claim. *Let $\lambda = \text{cf}(\lambda) > \aleph_0$, $S \subseteq \lambda$ stationary, D a normal λ^+-saturated filter on λ, S is D-positive (i.e. $S \in D^+$, $\lambda \backslash S \notin D$).*

(1) *Assume that $\langle (C_\delta, I_\delta) : \delta \in S\rangle$ is such that*

(a) $C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathscr{P}(C_\delta)$,

(b) *for every club E of λ,*

$$\{\delta \in S : \textit{for some } A \in I_\delta \textit{ we have } \delta > \sup(A \backslash E)\} \in D^+.$$

Then for some stationary $S_0 \subseteq S, S_0 \in D^+$ we have

(b)$^+$ *for every club E of λ*

$$\{\delta \in S : \textit{for no } A \in I_\delta \textit{ do we have } \delta > \sup(A \backslash E)\} = \emptyset \bmod D.$$

(2) *Assume that $\langle \mathscr{P}_\delta : \delta \in S\rangle$ is such that (here really presaturated is enough)*

$(*)$ *for every D-positive $S_0 \subseteq S$ for some D-positive $S_1 \subseteq S_0$ and $\langle (C_\delta, I_\delta) : \delta \in S\rangle$ we have $(C_\delta, I_\delta) \in \mathscr{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta), I_\delta \subseteq \mathscr{P}(C_\delta)$ and for every club E of λ $\{\delta \in S_1 : \textit{for some } A \in I_\delta, \delta > \sup(A \backslash E)\} \neq \emptyset \bmod D$.*

Then

$(**)$ *for some $\langle (C_\delta, I_\delta) : \delta \in S\rangle$ we have $(C_\delta, I_\delta) \in \mathscr{P}_\delta, C_\delta \subseteq \delta = \sup(C_\delta)$, $I_\delta \subseteq \mathscr{P}(C_\delta)$ and for every club E of λ*

$$\{\delta \in S : \textit{ for no } A \in I_\delta, \delta > \sup(A \backslash E)\} = \emptyset \bmod D.$$

Remark. This is a straightforward generalization of [8, Ch. III, Section 6.2B]. Independently, Gitik found related results on generic extensions which were continued in [1, 3].

Proof. The same as the proofs cited above.

3.10. Lemma. *Suppose λ is regular uncountable and $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ is stationary. Then we can find $\langle (C_\delta, h_\delta, \chi_\delta) : \delta \in S \rangle$ and D such that*

(A) *D is a normal filter on λ^+,*

(B) *C_δ is a club of δ, say $C_\delta = \{\alpha_{\delta,\zeta} : \zeta < \lambda\}$, with $\alpha_{\delta,\zeta}$ increasing continuously in ζ,*

(C) *h_δ is a function from C_δ to $\chi_\delta, \chi_\delta \leqslant \lambda$,*

(D) *if $A \in D^+$ (i.e. $A \subseteq \lambda^+$ & $\lambda^+ \backslash A \notin D$) and E is a club of λ^+, then the following set belongs to D^+:*

$$B_{E,A} =: \{\delta : \delta \in A \cap S, \delta \in acc(E) \text{ and for each } i < \chi_\delta \\ \{\zeta < \lambda : \alpha_{\delta,\zeta+1} \in E \text{ and } h_\delta(\alpha_{\delta,\zeta}) = i \\ (\text{and } \alpha_{\delta,\zeta} \in E)\} \text{ is a stationary subset of } \lambda\}$$

(hence, for some $\alpha < \lambda^+$ and $\zeta < \lambda$, the set $B_{E,A,\alpha} =: \{\delta \in B_{E,A} : \alpha = \alpha_{\delta,\zeta}\}$ is in D^+).

(E) *If $\gamma < \lambda^+$ and χ satisfies one of the conditions listed below, then $S_{\gamma,\chi} = \{\delta \in S : \gamma = \min(C_\delta)$ and $\chi_\delta = \chi\} \in D^+$ where*

(α) *$\lambda = \chi^+$,*

(β) *λ is inaccessible not strongly inaccessible, $\chi < \lambda$ and there is T such that:*

(a) *T is a tree with $< \lambda$ nodes and a set Γ of branches, $|\Gamma| = \lambda$,*

(b)$'$ *if $T' \subseteq T, T'$ downward closed and $(\exists^\lambda \eta \in \Gamma)(\eta$ a branch of $T')$ then T' has an antichain of cardinality $\geqslant \chi$,*

(γ) *λ is inaccessible, not strongly inaccessible, and $\theta = \min\{\theta :$ for some $\chi < \lambda$ we have $\chi^\theta \geqslant \lambda\}$, and $\chi = \min\{\chi : \chi^\theta \geqslant \lambda$ and $\chi \geqslant \theta\}$.*

3.11. Remark. (1) We can replace λ^+ in Lemma 3.10 by any $\mu = \mathrm{cf}(\mu) > \lambda$, as if $\mu > \lambda^+$ we have even a stronger theorem. (2) We probably can add

(δ) *$\chi < \lambda$ and λ is strongly inaccessible, not ineffable; i.e. λ is Mahlo and we can find $\bar{A} = \langle A_\mu : \mu < \lambda$ is inaccessible$\rangle$, $A_\mu \subseteq \mu$ so that for no stationary $\Gamma \subseteq \{\mu < \lambda : \mu$ inaccessible$\}$ and $A \subseteq \lambda$ do we have: $\mu \in \Gamma \Rightarrow A_\mu = A \cap \mu$.*

Proof. Let for $\lambda = \mathrm{cf}(\lambda) > \aleph_0$,

$$\Theta = \Theta_\lambda = \{\chi \leqslant \lambda : \textit{if } S' \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\} \text{ is stationary} \\ \textit{then } \text{we can find } \langle (C_\delta, h_\delta) : \delta \in S' \rangle \text{ such that} \\ \text{(a)} \quad C_\delta \text{ is a club of } \delta \text{ of order type } \lambda,$$

(b) $h_\delta : C_\delta \to \chi$,
(c) for every club E of λ^+ for stationarily many $\delta \in S' \cap \mathrm{acc}(E)$ we have:
$i < \chi \Rightarrow B_E = \{\alpha \in C_\delta : \alpha \in E, h(\alpha) = i$ and $\min(C_\delta \backslash (\alpha + 1)) \in E\}$ is a stationary subset of $\delta\}$.

In 3.12 we show
$\bigotimes$ for each of the cases from clause (E), the χ belongs to Θ.

Proof of sufficiency of $\bigotimes$. We can partition S into λ^+ stationary sets so we can find a partition $\langle S_{\chi,\alpha} : \chi \in \Theta$ and $\alpha < \lambda^+\rangle$ of S into stationary sets. Without loss of generality, $\alpha \leqslant \min(S_{\chi,\alpha})$ and let $\langle (C^0_\delta, h^0_\delta) : \delta \in S_{\chi,\alpha}\rangle$ be as guaranteed by "$\chi \in \Theta$" for the stationary set $S_{\chi,\alpha}$. Now define C_δ, h_δ for $\delta \in S$ by:

C_δ is $C^0_\delta \cup \{\alpha\} \backslash \alpha$ *if* $\delta \in S_{\chi,\alpha}$ and $\alpha < \delta, h_\delta(\beta)$ is $h^0_\delta(\beta)$ if $\beta \in C_\delta \cap C^0_\delta$ and is zero otherwise. Of course, $\chi_\delta = \chi$ if $\delta \in S_{\chi,\alpha}$.
Lastly, let

$$D = \{A \subseteq \lambda^+ : \text{ for some club } E \text{ of } \lambda^+, \text{ for every } \delta \in S \cap \mathrm{acc}(E) \backslash A \text{ for some } i < \chi_\delta, \text{ the set } \{\beta \in C_\delta : \beta \in E, h_\delta(\beta) = i \text{ and } \min(C_\delta \backslash (\beta + 1)) \in E\} \text{ is not a stationary subset of } \delta\}.$$

So D and $\langle (C_\delta, h_\delta, \chi_\delta) : \delta \in S\rangle$ have been defined, and we have to check clauses (A)–(E).
Note that $\Theta \neq \emptyset$ and the proof which appears later does not rely on the intermediate proofs.

Clause (A): Suppose $A_\zeta \in D$ for $\zeta < \lambda$, so for each ζ there is a club E_ζ of λ^+, such that

(∗) if $\delta \in S_{\chi,\gamma}$ and $\delta \in S \cap \mathrm{acc}(E) \backslash A_\zeta$ then for some $i_\delta < \chi_\delta$ we have $\{\alpha \in C_\delta : \alpha \in E, \min(C_\delta \backslash (\alpha + 1)) \in E$ and $h_\delta(\alpha) = i_\delta\}$ is not stationary in δ.

Clearly, clubs of λ^+ belong to D. Clearly, $A \supseteq A_\zeta \Rightarrow A \in D$ (by definition), witnessed by the same E_ζ. Also $A' = A_0 \cap A_1 \in D$ as witnessed by $E = E_0 \cap E_1$. Lastly, $A = \triangle_{\zeta<\lambda} A_\zeta = \{\alpha < \lambda^+ : \alpha \in \bigcap_{\zeta<1+\alpha} A_\zeta\}$ belongs to D as witnessed by $E = \{\alpha < \lambda^+ : \alpha \in \bigcap_{\zeta<1+\alpha} E_\zeta\}$. Note that if $\delta \in S \cap \mathrm{acc}(E) \backslash A$ then for some $\zeta < \delta$

$$\delta \in S \cap \mathrm{acc}(E) \backslash A_\zeta \subseteq (S \cap \mathrm{acc}(E_\zeta) \backslash A_\zeta) \cup (1 + \zeta)$$

as $E_\zeta \backslash E$ is a bounded subset of δ included in $1 + \zeta$; so from the conclusion of (∗) for δ, A_ζ, E_ζ we get it for ζ, A, E.

Lastly, $\emptyset \notin D$; otherwise, let E be a club of λ^+ witnessing it, i.e. (∗) holds in this case. Choose $\chi \in \Theta$ and $\alpha = 0$ and use on it the choice of $\langle C^0_\delta : \delta \in S_{\chi,0}\rangle$ to show that for some $\delta \in S_{\chi,0} \subseteq S$ contradict the implication in $(*)$.

Clause (B): Trivial.

Clause (C): Trivial.

Clause (D): Note that we can ignore the "$\alpha_{\delta,\zeta} \in E$" as $\delta \in \mathrm{acc}(E)$ implies that it holds for a club of ζ's. Assume that $A \in D^+$ (for clause (D)) and E is a club of λ^+, which contradicts clause (D), so $B_{E,A} \notin D^+$; hence $\lambda^+ \backslash B_{E,A} \in D$. Also E witnessed that $\lambda^+ \backslash (A \backslash B_{E,A}) \in D$ by the definition of D. But by clause (A) we know that D is a filter on λ^+, so $(\lambda^+ \backslash B_{E,A}) \cap (\lambda^+ \backslash (A \backslash B_{E,A}))$ belongs to D, but this is the set $\lambda^+ \backslash B_{E,A} \backslash (A \backslash B_{E,A})$ which is (as $B_{E,A} \subseteq A$ by its definition) just $\lambda \backslash A$. So $\lambda \backslash A \in D$, hence $A \notin D^+$ – a contradiction.

Clause (E): By the proof of $\emptyset \notin D$ above, if $\chi \in \Theta$, also $S_{\chi,\alpha} \in D^+$, and by the definition of $\bar{C}, \bar{C} \restriction S_{\chi,\alpha}$ is as required. So it is enough to show

3.12. Claim. *If* $\chi < \lambda = \mathrm{cf}(\lambda)$ *and* χ *satisfies one of the clauses of Claim* 3.10, *then* $\chi \in \Theta$ *(from the proof of Claim* 3.10).

Proof.

Case (α): By Claim 3.1.

Case (β): Like the proof of Claim 3.1, for more details see [7, Section 3].

Case (γ): This is a particular case of case (β). Use $T = \bigcup_{\alpha<\theta} \alpha_\chi$, $\Gamma \subseteq^\theta \chi$ and we should check $(b)'$, we do it by cases: if $\chi > \theta$ and $\mathrm{cf}\chi = \chi$, necessarily for some $\alpha < \theta, |T' \cap^\alpha \chi| = \chi$. Similarly, if $\chi > \theta$ and $\chi > \mathrm{cf}\chi$ as wlog $v \in T' \Rightarrow |\{\eta \in \Gamma : v < \eta\}| = \lambda$. Lastly, if $\chi \leqslant \theta$, then $2^{<\theta} < \lambda$ and $(2^{<\theta})^{\mathrm{cf}(\theta)} = 2^\theta$ so θ is regular and it should be clear. □

More generally (see [7]):

3.13. Claim. *Let* $\lambda = \mathrm{cf}(\lambda) > \chi$. *A sufficient condition for* $\chi \in \Theta_\lambda$ *is the existence of some* $\zeta < \lambda^+$ *such that*

⊗ *in the following game of length* ζ, *second player has no winning strategy even for winning for at least one of* λ *boards: in the* ε*-th move first player chooses a function* $f_\varepsilon : \lambda \to \chi$ *and second player chooses* $\beta_\varepsilon < \chi$. *In the end, first player wins the play if* $\{\alpha < \lambda : \text{for every } \varepsilon < \gamma, f_\varepsilon(\alpha) \neq \beta_\varepsilon\}$ *is a stationary subset of* λ.

(If we weaken the demand in Θ_λ *from stationary to unbounded in* λ, *we can weaken it here too).*

4. More on Pr_6

4.1. Claim. $Pr_6(\lambda^+, \lambda^+, \lambda^+, \lambda)$ *for* λ *regular.*

Proof. We can find $h : \lambda^+ \to \lambda^+$ such that for every $\gamma < \lambda^+$ the set $S_\gamma =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda \text{ and } h(\delta) = \gamma\}$ is stationary, so $\langle S_\gamma : \gamma < \lambda \rangle$ is a partition of $S =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$. We can find $\bar{C}^\gamma = \langle C_\delta : \delta \in S_\gamma \rangle$ such that C_δ is a club of δ of order type λ. For any $v \in {}^{\omega>}(\lambda^+)$ we define:

(a) for $\ell < \ell g(\nu)$, if $\nu(\ell) \in S$ then let

$$a_\ell = a_{\nu,\ell} = \{\mathrm{otp}(C_{\nu(\ell)} \cap \nu(k)) : k < \ell g(\nu) \text{ and } \nu(k) < \nu(\ell)\},$$

(b) ℓ_ν is the $\ell < \ell g(\nu)$ such that
  (i) $\nu(\ell) \in S$,
  (ii) among those with $\sup(a_{\nu,\ell})$ is maximal, and
  (iii) among those with ℓ minimal,

(c) if ℓ_ν is well defined let $d(\nu) = h(\nu(\ell_\nu))$ otherwise let $d(\nu) = 0$.

Now suppose $\langle (u_\alpha, v_\alpha) : \alpha < \lambda^+ \rangle, \gamma < \lambda^+$ and E are as in Definition 2.1 and we shall prove the conclusion there. Let

$$E^* = \{\delta \in E : \delta \text{ is a limit ordinal and } \alpha < \delta \Rightarrow \delta > \sup[\bigcup\{\mathrm{Rang}(\eta) : \eta \in u_\alpha \cup v_\alpha\}]\}.$$

Clearly $E^* \subseteq E$ is a club of λ^+.

For each $\delta \in S_\gamma$ let

$$f_0(\delta) =: \sup[\delta \cap \bigcup\{\mathrm{Rang}(\nu) : \nu \in u_\delta \cup v_\delta\}].$$

As $\mathrm{cf}(\delta) = \lambda > |u_\alpha \cup v_\alpha|$ and the sequences are finite, clearly $f_0(\delta) < \delta$. Hence by Fodor's lemma for some $\xi^*, S^1_\gamma =: \{\delta \in S_\gamma : f_0(\delta) = \xi^*\}$ is a stationary subset of λ^+ (note that γ is fixed here). Let $\xi^* = \bigcup_{i<\lambda} a_{2,i}$ where $a_{2,i}$ is increasing with i and $|a_{2,i}| < \lambda$. So for $\delta \in S^1_\gamma$

$$f_1(\delta) = \mathrm{Min}\{i < \lambda : \delta \cap \bigcup\{\mathrm{Rang}(\nu) : \nu \in u_\delta \cup v_\delta\} \text{ is a subset of } a_{2,i}\}$$

is a well defined ordinal $< \lambda$ and hence for some $i^* < \lambda$ the set

$$S^2_\gamma =: \{\delta \in S^1_\gamma : f_1(\delta) = i^*\}$$

is a stationary subset of λ^+. For $\delta \in S^2_\gamma$ let

$$b_\delta =: \left\{\mathrm{otp}(C_\beta \cap \alpha) : \alpha < \beta \in S \text{ and both are in } a_{2,i^*} \cup \{\delta\} \cup \bigcup\{\mathrm{Rang}\,\nu : \nu \in u_\delta \cup v_\delta\}\right\}.$$

So b_δ is a subset of λ of cardinality $< \lambda$, and hence $\varepsilon_\delta =: \sup(b_\delta) < \lambda$ and hence for some ε^*

$$S^3_\gamma =: \{\delta \in S^2_\gamma : \varepsilon_\delta = \varepsilon^*\}$$

is a stationary subset of λ^+. Choose β^* such that

(∗) $\beta^* \in S^3_\gamma \cap E^*$ and $\beta^* = \sup(\beta^* \cap S^3_\gamma \cap E^*)$.

As C_{β^*} has order type λ (and is a club of β^*), for some $\alpha^* \in \beta^* \cap S^3_\gamma \cap E^*$ we have $\mathrm{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^*$.

We want to show that α^*, β^* are as required. Obviously, $\alpha^* < \beta^*, \alpha^* \in E$ and $\beta^* \in E$. So assume that $\nu \in u_{\alpha^*}, \rho \in v_{\beta^*}$ and we shall prove that $d(\nu\hat{\ }\rho) = \gamma$, which suffices.

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As $h(\beta^*) = \gamma$ (as $\beta^* \in S^3_\gamma \subseteq S_\gamma$) it suffices to prove that $(\nu\hat{}\rho)(\ell_{\nu\hat{}\rho}) = \beta^*$. Now for some ℓ_0, ℓ_1 we have $\nu(\ell_0) = \alpha^*, \rho(\ell_1) = \beta^*$ (as $\nu \in u_{\alpha^*}, \rho \in v_{\beta^*}$) and since $\mathrm{otp}(C_{\beta^*} \cap \alpha^*) > \varepsilon^*$, by the definition of $\ell_{\nu\hat{}\rho}$ it suffices to prove that

$\bigotimes$ *if* $\ell, k < \ell g(\nu\hat{}\rho), (\nu\hat{}\rho)(\ell) \in S, (\nu\hat{}\rho)(k) < (\nu\hat{}\rho)(\ell)$ *then*

(i) $\mathrm{otp}[C_{(\nu\hat{}\rho)(\ell)} \cap (\nu\hat{}\rho)(k)] \leqslant \varepsilon^*$ or

(ii) $(\nu\hat{}\rho)(\ell) = \beta^*$.

Assume that ℓ, k satisfy the assumption of $\otimes$ and we shall show its conclusion.

Case 1: If $(\nu\hat{}\rho)(\ell)$ and $(\nu\hat{}\rho)(k)$ belong to

$$a_{2,i^*} \cup \{\beta^*\} \cup \bigcup\{\mathrm{Rang}(\eta) : \eta \in u_{\beta^*} \cup v_{\beta^*}\}$$

then clause (i) holds because

(α) $\mathrm{otp}(C_{(\nu\hat{}\rho)(\ell)} \cap (\nu\hat{}\rho)(k)) \in b_{\beta^*}$ (see the definition of b_{β^*}) and

(β) $\sup(b_{\beta^*}) = \varepsilon_{\beta^*}$ (see the definition of ε_{β^*}) and

(γ) $\varepsilon_{\beta^*} = \varepsilon^*$ (as $\beta^* \in S^3_\gamma$ and see the choice of ε^* and S^3_γ).

Case 2: If $(\nu\hat{}\rho)(\ell)$ and $(\nu\hat{}\rho)(k)$ belong to

$$a_{2,i^*} \cup \bigcup\{\mathrm{Rang}(\eta) : \eta \in u_{\alpha^*} \cup v_{\alpha^*}\}$$

then the proof is similar to the proof of the previous case.

Case 3: No previous case.

So $(\nu\hat{}\rho)(\ell)$ and $(\nu\hat{}\rho)(k)$ are not in a_{2,i^*}, hence (as $\{\nu,\rho\} \subseteq (u_{\alpha^*} \cup v_{\beta^*})$, and $\{\alpha^*, \beta^*\} \subseteq S^2_\gamma \subseteq S^1_\gamma$)

$$m \in \{\ell, k\} \quad \& \quad m < \ell g(\nu) \Rightarrow (\nu\hat{}\rho)(m) = \nu(m) \geqslant \alpha^*,$$

$$m \in \{\ell, k\} \quad \& \quad m \geqslant \ell g(\nu) \Rightarrow (\nu\hat{}\rho)(m) = \rho(m - \ell g(\nu)) \geqslant \beta^*.$$

As $\beta^* \in E^*$ and $\beta^* > \alpha^*$ clearly $\sup(\mathrm{Rang}(\nu)) < \beta^*$, but also $(\nu\hat{}\rho)(k) < (\nu\hat{}\rho)(\ell)$ (see $\bigotimes$).

Together necessarily $k < \ell g(\nu)$, $\nu(k) \in [\alpha^*, \beta^*)$, $\ell \in [\ell g(\nu), \ell g(\nu) + \ell g(\rho))$ and $\rho(\ell - \ell g(\nu)) \in [\beta^*, \lambda^+)$. If $\rho(\ell) = \beta^*$ then clause (ii) of the conclusion holds. Otherwise necessarily $\nu(\ell) > \beta^*$, hence

$$\begin{aligned}\mathrm{otp}(C_{(\nu\hat{}\rho)(\ell)}) \cap (\nu\hat{}\rho)(k)) = {} & \mathrm{otp}(C_{\rho(\ell - \ell g(\nu))} \cap \nu(k)) \\ & \leqslant \mathrm{otp}(C_{\rho(\ell - \ell g(\nu))} \cap \beta^*) \leqslant \sup(b_{\beta^*}) \leqslant \varepsilon^*\end{aligned}$$

so clause (i) of $\otimes$ holds. □

Remark. Actually we now prove $Pr^6(\lambda^+, \lambda^+, \lambda^+, \lambda)$.

4.2. Conclusion. *For λ regular, $Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \lambda)$ holds.*

Proof. By Claim 4.1 and Lemma 2.2(1). □

4.3. Definition. (1) Let $Pr_6(\lambda, \theta, \sigma)$ means that for some Ξ, an unbounded subset of $\{\tau : \tau < \sigma, \tau$ is a cardinal (finite or infinite)$\}$, there is a $d : {}^{\omega>}(\lambda \times \Xi) \to \omega$ such that if $\gamma < \theta$ and $\tau \in \Xi$ are given and $\langle (u_\alpha, v_\alpha) : \alpha < \lambda \rangle$ satisfies

(i) $u_\alpha \subseteq {}^{\omega>}(\lambda \times \Xi) \setminus {}^{2\geqslant}(\lambda \times \Xi)$,
(ii) $v_\alpha \subseteq {}^{\omega>}(\lambda \times \Xi) \setminus {}^{2\geqslant}(\lambda \times \Xi)$,
(iii) $|u_\alpha| = |v_\alpha| = \tau$,
(iv) $\nu \in u_\beta \Rightarrow \nu(\ell g(\nu) - 1) = \langle \gamma, \tau \rangle$,
(v) $\rho \in u_\alpha \Rightarrow \rho(0) = \langle \gamma, \tau \rangle$,
(vi) $\eta \in u_\alpha \cup v_\alpha \Rightarrow (\exists \ell)(\eta(\ell) = \langle \alpha, \tau \rangle)$

then for some $\alpha < \beta$ we have

$$\nu \in u_\beta \quad \& \quad \rho \in v_\alpha \Rightarrow (\nu\hat{}\rho)[d(\nu\hat{}\rho)] = \langle \gamma, \tau \rangle.$$

(2) Let $Pr_6(\lambda, \sigma)$ means $Pr_6(\lambda, \lambda, \sigma)$.

4.4. Fact. *$Pr_6(\lambda, \lambda, \theta, \sigma), \theta \geqslant \sigma$ implies $Pr_6(\lambda, \theta, \sigma)$.*

Proof. Let c be a function from ${}^{\omega>}\lambda$ to θ exemplifying $Pr_6(\lambda, \lambda, \theta, \sigma)$. Let e be a one to one function from $\theta \times \Xi$ onto θ.

Now we define a function d from ${}^{\omega>}(\lambda \times \Xi)$ to ω:

$$d(\nu) = \text{ Min}\{\ell : c(\langle e(\nu(m)) : m < \ell g(\nu)\rangle) = e(\nu(\ell))\}. \qquad \square$$

4.5. Claim. *If $Pr_6(\lambda^+, \sigma)$, λ regular and $\sigma \leqslant \lambda$* then *$Pr_1(\lambda^{+2}, \lambda^{+2}, \lambda^{+2}, \sigma)$.*

Proof. Like the proof of Theorem 1.1.

4.6. Remark. Remember that by [6, 4.7], if $\mu > \text{cf}(\mu) + \sigma$, then $Pr_1(\mu^{+2}, \mu^{+2}, \mu^{+2}, \sigma)$.

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