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UNCOUNTABLE SATURATED STRUCTURES HAVE THE SMALL INDEX PROPERTY

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ABSTRACT

We prove the following theorem. Let $\mathfrak{M}$ be an uncountable saturated structure of cardinality $\lambda = \lambda^{<\lambda}$ and assume that G is a subgroup of Aut$(\mathfrak{M})$ whose index is less than or equal to λ. Then there exists a subset A of cardinality strictly less than λ such that every automorphism of $\mathfrak{M}$ leaving A pointwise fixed is in G.

1. *Introduction and notations*

In this paper, $\mathfrak{M}$ will be a saturated structure of cardinality $\lambda = \lambda^{<\lambda}$ in a language $\mathscr{L}$ of cardinality less than λ. The group of automorphisms of $\mathfrak{M}$ will be denoted by Aut$(\mathfrak{M})$. The aim of this paper is to prove the following theorem.

THEOREM 1. *Assume that G is a subgroup of* Aut$(\mathfrak{M})$ *of index less than or equal to λ. Then there exists a subset A of cardinality strictly less than λ such that every automorphism of $\mathfrak{M}$ leaving A pointwise fixed is in G.*

If the conclusion of this theorem holds, we say that Aut$(\mathfrak{M})$, or simply $\mathfrak{M}$, has the small index property. Various theorems related to the small index property (with additional hypothesis on $\mathfrak{M}$, for countable or uncountable $\mathfrak{M}$) have been proved. See **[1]** for more information and for the relevance of the small index property in the countable case. In **[4]**, it is explained that Theorem 1 above was the last piece needed to prove that, from the automorphism group of one uncountable saturated model, one can reconstruct most of the theory. See also **[2]**, where the same tree technique (invented by S. Shelah) is used to prove the small index property for some countable structures.

We fix a language L and a complete theory T; the structures that we shall consider will be models of T. If $\mathfrak{M}_0$ is a structure, we shall denote by Aut$(\mathfrak{M}_0)$ the group of automorphisms of $\mathfrak{M}_0$, and if A is a subset of $\mathfrak{M}_0$, then the subgroup of automorphisms of $\mathfrak{M}_0$ leaving A pointwise fixed will be denoted by $\mathrm{Aut}_A(\mathfrak{M}_0)$.

If $\alpha \in \mathrm{Aut}(\mathfrak{M})$ and $A \subseteq \mathfrak{M}$, then $\alpha[A]$ will denote the image of A by α. We shall also consider sequences of automorphisms. We generalise for these sequences the notions of extension, restriction and conjugation: if $\gamma = (g_i; i \in I)$ is a sequence of automorphisms of $\mathfrak{M}_1$ and $\mathfrak{M}_0 \prec \mathfrak{M}_1 \prec \mathfrak{M}_2$, then $\gamma \restriction \mathfrak{M}_0 =_{\mathrm{def}} (g_i \restriction \mathfrak{M}_0; i \in I)$; if $\alpha \in \mathrm{Aut}(\mathfrak{M}_1)$, then $\alpha \cdot \gamma \cdot \alpha^{-1} =_{\mathrm{def}} (\alpha \cdot g_i \cdot \alpha^{-1}; i \in I)$; a sequence of automorphisms γ' of $\mathfrak{M}_2$ is an extension of γ if $\gamma' \restriction \mathfrak{M}_1 = \gamma$.

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The group $\mathrm{Aut}(\mathfrak{M})$ is naturally equipped with a topological structure: a basis of open sets is

$$\{g\cdot \mathrm{Aut}_A(\mathfrak{M}); A \subseteq \mathfrak{M}, \mathrm{card}(A) < \lambda \text{ and } g \in \mathrm{Aut}(\mathfrak{M})\}.$$

So Theorem 1 states that a subgroup G of $\mathrm{Aut}(\mathfrak{M})$ of index less than or equal to λ is open.

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2. *Existentially closed and generic automorphisms*

Several notions of generic automorphism have already been considered, in **[5]** and in [3], for example, and have been proved to be very useful. To show an existence theorem, some kind of amalgamation property for automorphisms was needed. We shall prove this amalgamation property over the so-called existentially closed automorphisms. In fact, for our purpose, automorphisms will not be sufficient, and we need to work with sequences of automorphisms instead.

Fix an arbitrary set I. For each $i \in I$, choose, once for all, a unary function symbol $\mathfrak{a}_i$, and let V_I be the theory expressing that each of these symbols is interpreted by an automorphism. If $\mathfrak{M}_0$ is a model of T and $\gamma = (g_i; i \in I)$ is a sequence of automorphisms of $\mathfrak{M}_0$, we shall denote by $T_0(\gamma)$ the following set of sentences in the language $L(\mathfrak{M}_0) \cup \{\mathfrak{a}_i; i \in I\}$:

$$T_0(\gamma) = \Delta(\mathfrak{M}_0) \cup V_I \cup \{\mathfrak{a}_i(a) = b; i \in I, a \in \mathfrak{M}_0, b \in \mathfrak{M}_0 \text{ and } g_i(a) = b\},$$

where $\Delta(\mathfrak{M}_0)$ is the complete diagram of $\mathfrak{M}_0$, that is,

$$\Delta(\mathfrak{M}_0) = \{\phi(\bar{a}); \phi \in \mathscr{L}, \bar{a} \text{ is a finite sequence from } \mathfrak{M}_0 \text{ and } \mathfrak{M}_0 \models \phi(\bar{a})\}.$$

$T(\gamma)$ will be the set of finite conjunctions of formulae in $T_0(\gamma)$. It should be clear that a model of $T(\gamma)$ is (isomorphic to) an elementary extension of $\mathfrak{M}_0$, say $\mathfrak{M}_1$, and for each $i \in I$, an interpretation of $\mathfrak{a}_i$, which is an automorphism of $\mathfrak{M}_1$ extending g_i. Conversely, if γ_0 and γ_1 are sequences of automorphisms of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ respectively, where $\mathfrak{M}_0 \prec \mathfrak{M}_1$ and γ_1 extends γ_0, then $T(\gamma_0) \subseteq T(\gamma_1)$.

DEFINITION 2. Let $\mathfrak{M}_0$ be a model of T and $\gamma = (g_i; i \in I)$ be a sequence of automorphisms of $\mathfrak{M}_0$. We say that γ is existentially closed if, whenever $\mathfrak{M}_1 \succ \mathfrak{M}_0$, $\theta = (h_i; i \in I)$ extends γ and $\phi(\bar{a}, \bar{a}_1) \in T(\theta)$, where $\bar{a}$ is a finite sequence from $\mathfrak{M}_0$ and $\bar{a}_1$ is a finite sequence from $\mathfrak{M}_1$, there exists $\bar{b}$ in $\mathfrak{M}_0$ such that $\phi(\bar{a}, \bar{b}) \in T(\gamma)$.

It is easy to prove that existentially closed sequences of automorphisms exist, but we shall not do that now because this naturally appears later as a corollary. The nice fact about those existentially closed sequences is that amalgamation holds for their extensions.

THEOREM 3. *Let $\gamma = (g_i; i \in I)$ be an existentially closed sequence of automorphisms of $\mathfrak{M}_0$, and $\gamma^1 = (g_i^1; i \in I)$ and $\gamma^2 = (g_i^2; i \in I)$ be two extensions of γ over $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Then there exists a third extension $\gamma^3 = (g_i^3; i \in I)$ over $\mathfrak{M}_3$ and elementary embeddings α_1 and α_2 from $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively into $\mathfrak{M}_3$ such that $\alpha_2 \restriction \mathfrak{M}_0 = \alpha_3 \restriction \mathfrak{M}_0 = \mathrm{id}(\mathfrak{M}_0)$, and γ^3 extends both $\alpha_1 \cdot \gamma^1 \cdot \alpha_1^{-1}$ and $\alpha_2 \cdot \gamma^2 \cdot \alpha_2^{-1}$.*

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Proof. Without loss of generality, we assume that $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathfrak{M}_0$. It is enough to prove that $T(\gamma^1) \cup T(\gamma^2)$ is consistent. By compactness, it suffices to prove that, if $\bar{a}$, $\bar{a}_1$ and $\bar{a}_2$ are finite sequences from $\mathfrak{M}_0$, $\mathfrak{M}_1 - \mathfrak{M}_0$ and $\mathfrak{M}_2 - \mathfrak{M}_0$ respectively, and $\phi_1(\bar{a}, \bar{a}_1) \in T(\gamma^1)$ and $\phi_2(\bar{a}, \bar{a}_2) \in T(\gamma^2)$, then $\phi_1(\bar{a}, \bar{a}_1) \wedge \phi_2(\bar{a}, \bar{a}_2)$ is consistent. But, since γ is existentially closed, there exist $\bar{b}_1$ and $\bar{b}_2$ in $\mathfrak{M}_0$ such that $\phi_1(\bar{a}, \bar{b}_1) \in T(\gamma)$ and $\phi_2(\bar{a}, \bar{b}_2) \in T(\gamma)$, and this gives a model of $\phi_1(\bar{a}, \bar{a}_1) \wedge \phi_2(\bar{a}, \bar{a}_2)$.

The proof of the next lemma, expressing that existentially closed sequences are closed under unions, is straightforward.

LEMMA 4. *Let I be a set, $(X, \leqslant)$ a totally ordered set, and for each $x \in X$, $\mathfrak{M}_x$ a model of T and $\gamma^x = (g_i^x; i \in I)$ an existentially closed sequence of automorphisms of $\mathfrak{M}_x$, such that, for any x and y in X, if $x \leqslant y$, then $\mathfrak{M}_x \prec \mathfrak{M}_y$ and γ^y extends γ^x. Then $\gamma = \bigcup_{x \in X} \gamma^x = (\bigcup_{x \in X} \gamma_i^x; i \in I)$ is an existentially closed sequence of automorphisms of $\mathfrak{N} = \bigcup_{x \in X} \mathfrak{M}_x$.*

DEFINITION 5. Let $\mathfrak{M}_0$, $\mathfrak{M}_1$, $\mathfrak{M}_2$ be such that $\mathfrak{M}_0 \prec \mathfrak{M}_1$, $\mathfrak{M}_0 \prec \mathfrak{M}_2$, and let $\gamma_0 = (g_i^0; i \in I)$, $\gamma_1 = (g_i^1; i \in I)$ and $\gamma_2 = (g_i^2; i \in I)$ be sequences of automorphisms of $\mathfrak{M}_0$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively, γ_1 and γ_2 extending γ_0. Then we say that γ_1 and γ_2 are compatible over $\mathfrak{M}_0$ if there exist $\mathfrak{N} \succ \mathfrak{M}_0$, elementary maps f_1 and f_2 from $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively into $\mathfrak{N}$, and $\kappa = (k_i; i \in I)$ a family of automorphisms of $\mathfrak{N}$, such that $f_1 \restriction \mathfrak{M}_0 = f_2 \restriction \mathfrak{M}_0 = \mathrm{id}(\mathfrak{M}_0)$ and κ extends both $f_1 \cdot \gamma_1 \cdot f_1^{-1}$ and $f_2 \cdot \gamma_2 \cdot f_2^{-1}$.

In the case where $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathfrak{M}_0$, then γ_1 and γ_2 as above are compatible if and only if $T(\gamma_1) \cup T(\gamma_2)$ is consistent. Theorem 3 just states that two extensions of an existentially closed sequence of automorphisms are compatible.

We recall that $\mathfrak{M}$ is a saturated model of cardinality λ. We denote

$$\Theta = \{\mathfrak{M}_0; \mathfrak{M}_0 \prec \mathfrak{M} \text{ and } \mathrm{card}(\mathfrak{M}_0) < \lambda\}.$$

DEFINITION 6. (1) Assume that $\mathrm{card}(I) < \lambda$, and let $\gamma = (g_i; i \in I)$ be a sequence of automorphisms of $\mathfrak{M}$. We say that γ is generic if, whenever $\mathfrak{M}_0 \in \Theta$ is such that $g_i[\mathfrak{M}_0] = \mathfrak{M}_0$ for all $i \in I$, $\mathfrak{M}_1 \succ \mathfrak{M}_0$ is such that $\mathrm{card}(\mathfrak{M}_1) < \lambda$, and $\theta = (h_i; i \in I)$ is a sequence of automorphisms of $\mathfrak{M}_1$ extending $\gamma \restriction \mathfrak{M}_0$, one of the two following cases arises:

(i) there exists α, an elementary embedding from $\mathfrak{M}_1$ into $\mathfrak{M}$ such that $\alpha \restriction \mathfrak{M}_0 = \mathrm{id}(\mathfrak{M}_0)$ and γ extends $\alpha \cdot \theta \cdot \alpha^{-1}$;

(ii) γ and θ are incompatible.

(2) If $\mathrm{card}(I) \geqslant \lambda$, then $\gamma = (g_i; i \in I)$ is generic if every subsequence of length less than λ is generic.

Generic sequences are existentially closed. In fact, we have more.

PROPOSITION 7. *Let $\gamma = (g_i; i \in I)$ be a generic sequence of automorphisms of $\mathfrak{M}$ and $\mathfrak{M}_0 \in \Theta$ be such that $g_i[\mathfrak{M}_0] = \mathfrak{M}_0$ for all $i \in I$ and $(\mathfrak{M}_0, g_i \restriction \mathfrak{M}_0)_{i \in I}$ is an elementary $\mathscr{L} \cup \{\mathfrak{a}_i; i \in I\}$-substructure of $(\mathfrak{M}, g_i)_{i \in I}$. Then $\gamma \restriction \mathfrak{M}_0 = (g_i \restriction \mathfrak{M}_0; i \in I)$ is existentially closed.*

Proof. Notice that $(g_i \restriction \mathfrak{M}_0; i \in I)$ is existentially closed if, for all subsets I' of I of cardinality less than λ, $(g_i \restriction \mathfrak{M}_0; i \in I')$ is existentially closed. So, we may assume that card$(I) < \lambda$. Let $\mathfrak{M}_1$ be an elementary extension of $\mathfrak{M}_0$, and $\theta = (h_i; i \in I)$ be a sequence of automorphisms of $\mathfrak{M}_1$ extending $\gamma \restriction \mathfrak{M}_0$. Let $\phi(\bar{a}, \bar{a}_1)$ be a formula in $T(\theta)$, where $\bar{a}$ is a sequence in $\mathfrak{M}_0$ and $\bar{a}_1$ a sequence in $\mathfrak{M}_1$. We want to find a sequence $\bar{c}$ in $\mathfrak{M}_0$ such that $\phi(\bar{a}, \bar{c})$ belongs to $T(\gamma \restriction \mathfrak{M}_0)$. Without loss of generality, we may assume that $\mathfrak{M} \cap \mathfrak{M}_1 = \mathfrak{M}_0$ and that the cardinality of $\mathfrak{M}_1$ is less than λ; since $(\mathfrak{M}_0, g_i \restriction \mathfrak{M}_0)_{i \in I} \prec (\mathfrak{M}, g_i)_{i \in I}$, it is enough to find a sequence $\bar{c}$ in $\mathfrak{M}$ such that $\phi(\bar{a}, \bar{c}) \in T(\gamma)$. We refer to Definition 6: if there exists α, an elementary embedding from $\mathfrak{M}_1$ into $\mathfrak{M}$ such that $\alpha \restriction \mathfrak{M}_0 = \mathrm{id}(\mathfrak{M}_0)$ and γ extends $\alpha \cdot \theta \cdot \alpha^{-1}$, then it suffices to choose $\bar{c} = \alpha(\bar{a}_1)$. Let us show that the other case cannot arise, that is, that $T(\gamma) \cup T(\theta)$ is consistent. By compactness, it suffices to show that if $\psi(\bar{b}_0, \bar{b}_1)$ is a formula in $T(\theta)$ and $\xi(\bar{b}_0, \bar{b})$ is a formula in $T(\gamma)$, where $\bar{b}_0$ is a sequence from $\mathfrak{M}_0$, $\bar{b}_1$ a sequence from $\mathfrak{M}_1 - \mathfrak{M}_0$ and $\bar{b}$ a sequence from $\mathfrak{M} - \mathfrak{M}_0$, then $\psi(\bar{b}_0, \bar{b}_1) \wedge \xi(\bar{b}_0, \bar{b})$ is consistent. Since

$$(\mathfrak{M}, g_i)_{i \in I} \models \exists \bar{v}\, \xi(\bar{b}_0, \bar{v}),$$

there exists a sequence $\bar{c}$ in $\mathfrak{M}_0$ such that $\xi(\bar{b}_0, \bar{c})$ belongs to $T(\gamma \restriction \mathfrak{M}_0)$, and also to $T(\theta)$, so in $T(\theta)$ we have both $\xi(\bar{b}_0, \bar{c})$ and $\psi(\bar{b}_0, \bar{b}_1)$.

There are two nice facts about generic sequences of automorphisms. The first, which is stated in the next proposition, is that two of them are conjugate as soon as they agree on an existentially closed model. The second, which will be proved in the next section, is that there are plenty of them.

PROPOSITION 8. *Let $\mathfrak{M}_0 \in \Theta$, I a set of cardinality less than λ and $\gamma = (g_i; i \in I)$ an existentially closed family of automorphisms of $\mathfrak{M}_0$. Assume that $\theta = (h_i; i \in I)$ and $\kappa = (k_i; i \in I)$ are two generic families of automorphisms of $\mathfrak{M}$, both extending γ. Then there exists $f \in \mathrm{Aut}_{\mathfrak{M}_0}(\mathfrak{M})$ such that $\theta = f \cdot \kappa \cdot f^{-1}$.*

Proof. We use the back and forth method. Let $(a_\alpha; \alpha < \lambda)$ be an enumeration of $\mathfrak{M}$, and we define, by induction on $\alpha < \lambda$, a model $\mathfrak{M}_\alpha \in \Theta$ and an elementary map f_α from $\mathfrak{M}_\alpha$ into $\mathfrak{M}$ in such a way that:

(i) if α is even, then, for all $\beta < \alpha$, $a_\beta \in \mathfrak{M}_\alpha$;

(ii) if α is odd, then, for all $\beta < \alpha$, $a_\beta \in f_\alpha[\mathfrak{M}_\alpha]$;

(iii) the applications $\alpha \to \mathfrak{M}_\alpha$ and $\alpha \to f_\alpha$ are increasing and continuous;

(iv) for all $\alpha < \lambda$ and $i \in I$, $h_i[\mathfrak{M}_\alpha] = \mathfrak{M}_\alpha$ and $\theta \restriction \mathfrak{M}_\alpha$ is an existentially closed sequence of automorphisms of $\mathfrak{M}_\alpha$;

(v) for all $\alpha < \lambda$, $f_\alpha \cdot (\theta \restriction \mathfrak{M}_\alpha) \cdot f_\alpha^{-1} = \kappa \restriction f_\alpha[\mathfrak{M}_\alpha]$ (so $\kappa^\alpha =_{\mathrm{def}} \kappa \restriction f_\alpha[\mathfrak{M}_\alpha]$ is an existentially closed sequence of automorphisms of $f_\alpha[\mathfrak{M}_\alpha]$).

We start with $\mathfrak{M}_0$ and f_0 equal to the identity on $\mathfrak{M}_0$. There are no choices and no problems at limit steps (see Lemma 4). So we assume, for example, that $\alpha + 1$ is even, and we show how to construct $\mathfrak{M}_{\alpha+1}$ and $f_{\alpha+1}$.

Let $\mathfrak{M}_{\alpha+1}$ be an element of Θ containing $\mathfrak{M}_\alpha$, $a_{\alpha-1}$ and a_α, such that

$$(\mathfrak{M}_{\alpha+1}, h_i \restriction \mathfrak{M}_{\alpha+1}; i \in I) \prec (\mathfrak{M}, h_i; i \in I).$$

Such a model exists by Löwenheim–Skolem (we have assumed that card$(I) < \lambda$) and, by Proposition 7, $\theta \restriction \mathfrak{M}_{\alpha+1}$ is existentially closed. We can extend f_α to $\mathfrak{M}_{\alpha+1}$: there exist $\mathfrak{N} \succ f_\alpha[\mathfrak{M}_\alpha]$ and g an isomorphism from $\mathfrak{M}_{\alpha+1}$ onto $\mathfrak{N}$ extending f_α, and we may assume that $\mathfrak{N} \cap \mathfrak{M} = f_\alpha[\mathfrak{M}_\alpha]$. Call $\nu = (g \cdot (h_i \restriction \mathfrak{M}_{\alpha+1}) \cdot g^{-1}; i \in I)$. Then ν is a sequence of automorphisms of $\mathfrak{N}$ extending κ^α. Since κ^α is existentially closed, $T(\kappa) \cup T(\nu)$ is consistent, and, since κ is generic, there exists an elementary map g_1 from $\mathfrak{N}$ into $\mathfrak{M}$, which is the identity on $f_\alpha[\mathfrak{M}_\alpha]$ such that κ extends $g_1 \cdot \nu \cdot g_1^{-1}$, which is equal to $g_1 \cdot g \cdot (\theta \restriction \mathfrak{M}_{\alpha+1}) \cdot g^{-1} \cdot g_1^{-1}$. So it suffices to define $f_{\alpha+1} = g_1 \cdot g$.

Finally, $f = \bigcup_{\alpha<\lambda} f_\alpha$ is the required automorphism of $\mathfrak{M}$.

3. *Collecting a large number of generics*

Our aim is Proposition 10, which will be used in the construction of the tree in the next section.

LEMMA 9. *Let $(g_{i,j}; i \in I, j \in J)$ be a matrix of automorphisms of $\mathfrak{M}$, where* $\operatorname{card}(I) \leqslant \lambda$ *and* $\operatorname{card}(J) \leqslant \lambda$, *and $(\mathfrak{M}_i : i \in I)$ be a sequence of elements of Θ. Then there exists $(h_i; i \in I)$ such that*

(i) *for all $i \in I$, $h_i \in \operatorname{Aut}_{\mathfrak{M}_i}(\mathfrak{M})$;*

(ii) *for all applications $\delta : I \to J$, the family $(h_i \cdot g_{i,\delta(i)}; i \in I)$ is generic.*

Proof. Set

$$X = \{(I_0, \mathfrak{N}_0, \mathfrak{N}_1, \delta, (k_i; i \in I_0)); I_0 \subseteq I, \operatorname{card}(I_0) < \lambda, \delta \text{ is an application from } I_0 \text{ into } J, \mathfrak{N}_0 \prec \mathfrak{N}_1 \in \Theta, k_i \in \operatorname{Aut}(\mathfrak{N}_1), k_i[\mathfrak{N}_0] = \mathfrak{N}_0\}.$$

We remark that $\operatorname{card}(X) = \lambda$ (here we use the hypothesis $\lambda^{<\lambda} = \lambda$). Let $(x_\alpha; \alpha < \lambda)$ be an enumeration of X. We define by induction on $\alpha < \lambda$ a family $(\mathfrak{M}^i_\alpha; i \in I)$ of elements of Θ and a family $H_\alpha = (h^i_\alpha; i \in I)$, where $h^i_\alpha \in \operatorname{Aut}(\mathfrak{M}^i_\alpha)$, in such a way that:

(i) $\mathfrak{M}^i_0 = \mathfrak{M}_i$ and h^i_0 is the identity on $\mathfrak{M}_i$;

(ii) for all $i \in I$, the applications $\alpha \to \mathfrak{M}^i_\alpha$ and $\alpha \to h^i_\alpha$ are increasing and continuous;

(iii) for all $i \in I$, $\bigcup_{\alpha<\lambda} \mathfrak{M}^i_\alpha = \mathfrak{M}$ and $h^i =_{\text{def}} \bigcup_{\alpha<\lambda} h^i_\alpha \in \operatorname{Aut}(\mathfrak{M})$;

(iv) for all applications δ from I to J, the family $(h_i \cdot g_{i,\delta(i)}; i \in I)$ is generic.

There is no problem for (i) and (ii). To ensure (iii) and (iv), we proceed as follows. Assume that $x_\alpha = (I_0, \mathfrak{N}_0, \mathfrak{N}_1, \delta, (k_i; i \in I_0)) \in X$.

- Find first a model $\mathfrak{M}' \in \Theta$, and for each $i \in I_0$, $m_\alpha \in \operatorname{Aut}(\mathfrak{M}')$ such that:

$$\mathfrak{N}_0 \prec \mathfrak{M}';$$

$$\text{for all } i \in I_0,\ \mathfrak{M}^i_\alpha \prec \mathfrak{M}';$$

$$\text{for all } i \in I_0,\ g_{i,\delta(i)}[\mathfrak{M}'] = \mathfrak{M}';$$

$$\text{for all } i \in I_0,\ m_\alpha \text{ extends } h^i_\alpha.$$

(The two last conditions are achieved by using a Löwenheim–Skolem argument.)

- If the families $(m_\alpha \cdot (g_{i,\delta(i)} \restriction \mathfrak{M}'); i \in I_0)$ and $(k_i; i \in I_0)$ are not compatible over $\mathfrak{N}_0$, then define, for all $i \in I_0$, $\mathfrak{M}^i_{\alpha+1} = \mathfrak{M}'$ and $h^i_{\alpha+1} = m_\alpha$.
- If the families $(m_\alpha \cdot (g_{i,\delta(i)} \restriction \mathfrak{M}'); i \in I_0)$ and $(k_i; i \in I_0)$ are compatible over $\mathfrak{N}_0$, then we can find, for all $i \in I_0$, $\mathfrak{M}^i_{\alpha+1} \succ \mathfrak{M}'$ and $h^i_{\alpha+1} \in \operatorname{Aut}(\mathfrak{M}^i_{\alpha+1})$ which are such that there exists an elementary map f from $\mathfrak{N}_1$ into $\mathfrak{M}^i_{\alpha+1}$ which extends the identity on $\mathfrak{N}_0$ and such that, for all $i \in I_0$, $h^i_{\alpha+1} \cdot g_{i,\delta(i)} \restriction \mathfrak{M}^i_{\alpha+1}$ extends $f \cdot k_i \cdot f^{-1}$.
- If $i \notin I_0$, then $\mathfrak{M}^i_{\alpha+1} = \mathfrak{M}^i_\alpha$ and $h^i_{\alpha+1} = h^i_\alpha$.

We consider now a subgroup G of $\operatorname{Aut}(\mathfrak{M})$ whose index is not bigger than λ. We assume, toward a contradiction, that G is not open.

PROPOSITION 10. *There exists a generic sequence* $\mathscr{F} = (g_i; i \in I)$ *such that*:

(1) *for all* $\mathfrak{M}_0 \in \Theta$ *and* $h \in \operatorname{Aut}(\mathfrak{M}_0)$, *the set* $\{i \in I; g_i \upharpoonright \mathfrak{M}_0 = h \text{ and } g_1 \notin G\}$ *has cardinality* λ;

(2) *the set* $\{i \in I; g_i \in G\}$ *has cardinality* λ.

Proof. We consider the set

$$X = \{(\mathfrak{M}_0, f); \mathfrak{M}_0 \in \Theta, f \in \operatorname{Aut}(\mathfrak{M}_0)\}.$$

It has cardinality λ (because $\lambda = \lambda^{<\lambda}$), and we can construct a set I_0 of cardinality λ and a sequence $((\mathfrak{M}_i, f_i); i \in I_0)$ of elements of X in such a way that, for all $(\mathfrak{M}_0, f) \in X$, the set

$$\{i \in I_0; (\mathfrak{M}_i, f_i) = (\mathfrak{M}_0, f)\}$$

has cardinality λ. Let I_1 be any set of cardinality λ disjoint with I_0, and $I = I_0 \cup I_1$; let J be any set of cardinality λ.

For all $i \in I$ and $j \in J$, we define $g_{i,j} \in \operatorname{Aut}(\mathfrak{M})$ in such a way that the following hold.

(1) If $i \in I_0$, then $g_{i,j} \upharpoonright \mathfrak{M}_i = h_i$. Moreover, we demand that the set $\{g_{i,j}; j \in J\}$ meets at least two classes modulo G. This is possible because, as G is not open, none of its classes contains a non-empty open set.

(2) If $i \in I$, then the set $\{g_{i,j}; j \in J\}$ meets all classes modulo G. This is possible because the index of G is not bigger than λ.

Now define arbitrary $\mathfrak{M}_i$ for $i \in I_1$, and apply Lemma 9: we obtain a family $(h_i; i \in I)$ satisfying the conditions (i) and (ii) of this lemma. Then we choose an application δ from I into J in such a way that:

- if $i \in I_0$, $g_{i,\delta(i)}$ is not in the class of h_i^{-1};
- if $i \in I_1$, $g_{i,\delta(i)}$ is in the class of h_i^{-1}.

The family $(h_i \cdot g_{i,\delta(i)}; i \in I_0)$ satisfies our requirements.

4. *Construction of the tree*

Denote by $\mathscr{S} = 2^{<\lambda}$ the set of sequences of 0 and 1 of length less than λ, and $\mathscr{S}^* = \{s \in \mathscr{S};$ the length of s is successor$\}$. We start with a generic family $\mathscr{F}$ as in Proposition 10 and an enumeration $(a_\alpha; \alpha < \lambda)$ of $\mathfrak{M}$. We construct, by induction on $s \in \mathscr{S}$, a model $\mathfrak{M}_s \in \Theta$, an automorphism $g_s \in \operatorname{Aut}(\mathfrak{M}_s)$ and, if $s \in \mathscr{S}^*$, automorphisms h_s and k_s in $\operatorname{Aut}_{\mathfrak{M}_s}(\mathfrak{M})$ in such a way that the following conditions are satisfied:

(1) the maps $s \to \mathfrak{M}_s$ and $s \to g_s$ are increasing and continuous;

(2) for all $s \in \mathscr{S}$, $h_{s,0} \in G$ and $h_{s,1} \notin G$;

(3) for all $s \in \mathscr{S}$, $k_{s,0} = k_{s,1}$;

(4) for all $s \in \mathscr{S}$, for all $t \in \mathscr{S}^*$ such that $t \leqslant s$, $h_t[\mathfrak{M}_s] = \mathfrak{M}_s$, and $(h_t \upharpoonright \mathfrak{M}_s; t \leqslant s$ and $t \in \mathscr{S}^*)$ is existentially closed;

(5) for all $s \in \mathscr{S}$ and $t \in \mathscr{S}^*$ such that $t \leqslant s$, $g_s \cdot (h_t \upharpoonright \mathfrak{M}_s) \cdot g_s^{-1} = k_t \upharpoonright \mathfrak{M}_s$;

(6) for all $s \in \mathscr{S}$ and $\beta < \operatorname{lgh}(s)$, $a_\beta \in \mathfrak{M}_s$;

(7) for all s, the families $(h_t; t \leqslant s, t \in \mathscr{S}^*)$ and $(k_t; t \leqslant s, t \in \mathscr{S}^*)$ are both sequences of distinct elements of $\mathscr{F}$ (and so they are generic).

If s is the null sequence, define $\mathfrak{M}_s$ arbitrarily and $g_s = \operatorname{id}(\mathfrak{M}_s)$. For limit s, there are no choices and no problems.

Assume that everything has been defined for s. First, let $h_{s,0}$ be an element in $\mathscr{F} \cap G$ not in $\{h_t; t \in \mathscr{S}^*, t \leqslant s\}$. Extend g_s to $g \in \operatorname{Aut}(\mathfrak{M})$ in such a way that $g h_t g^{-1} = k_t$ for all $t \in \mathscr{S}^*$, $t \leqslant s$. To do that, first extend g_s arbitrarily to $g' \in \operatorname{Aut}(\mathfrak{M})$. Then the families $(g' \cdot h_t \cdot g'^{-1}; t \leqslant s, t \in \mathcal{S}^*)$ and $(k_t; t \leqslant s, t \in \mathcal{S}^*)$ are generic, and they agree on $\mathfrak{M}_s$. Moreover, $(g' \cdot h_t \cdot g'^{-1} \restriction \mathfrak{M}_s; t \leqslant s, t \in \mathcal{S}^*)$ is existentially closed, so, by Proposition 8, there exists $g'' \in \mathrm{Aut}_{\mathfrak{M}_s}(\mathfrak{M})$ such that for all $t \leqslant s$, $t \in \mathcal{S}^*$, $k_t = g'' \cdot g' \cdot h_t \cdot g'^{-1} \cdot g''^{-1}$. Take $g = g'' \cdot g'$.

Now, choose $\mathfrak{M}_{s,0} = \mathfrak{M}_{s,1}$ in such a way that: (a) it is closed for h_t, for $t \leqslant (s, 0)$ and for g; (b) it contains $\mathfrak{M}_s$ and a_α, where α is the length of s; (c) $(\mathfrak{M}_{s,0}, h_t \restriction \mathfrak{M}_{s,0}; t \leqslant (s,0))$ is an elementary substructure of $(\mathfrak{M}, h_t; t \leqslant (s,0))$.

Set $g_{s,0} = g_{s,1} = g \restriction \mathfrak{M}_{s,0}$ and $h_{s,1}$ an element of $\mathcal{F}$ extending $h_{s,0} \restriction \mathfrak{M}_{s,0}$, not in G and not in $\{h_t; t \in \mathcal{S}^*, t \leqslant s\}$, and $k_{s,0} = k_{s,1}$ an element of $\mathcal{F}$ extending $g \cdot (h_{s,0} \restriction \mathfrak{M}_{s,0}) \cdot g^{-1}$ not in $\{k_t : t \in \mathcal{S}^*, t \leqslant s\}$.

We are now able to reach a contradiction: for each $\sigma \in 2^\lambda$, let $g_\sigma = \bigcup_{s<\sigma} g_s$. Then $g_\sigma \in \mathrm{Aut}(\mathfrak{M})$ and for all $t < \sigma$, $t \in \mathcal{S}^*$, $g_\sigma \cdot h_t \cdot g_\sigma^{-1} = k_t$. Assume that σ and τ are two distinct elements of 2^λ; let s be their largest common initial segment, and assume, without loss of generality, that $(s, 0) < \sigma$ and $(s, 1) < \tau$. Then

$$g_\sigma \cdot h_{s,0} \cdot g_\sigma^{-1} = k_{s,0} = k_{s,1} = g_\tau \cdot h_{s,1} \cdot g_\tau^{-1},$$

thus

$$h_{s,0} = g_\sigma^{-1} \cdot g_\tau \cdot h_{s,1} \cdot g_\tau^{-1} \cdot g_\sigma,$$

and since $h_{s,0} \in G$ and $h_{s,1} \notin G$, $g_\sigma^{-1} \cdot g_\tau \notin G$: G has index 2^λ in $\mathrm{Aut}(\mathfrak{M})$.

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