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THE BOREL CONJECTURE

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We show the Borel Conjecture is consistent with the continuum large.

0. Introduction

In this work we consider the problem of the Borel Conjecture with large continuum. This problem has been one of the most interesting in the post-Cohen era of set theory. The consistency of the Borel Conjecture was proved by Laver [8], from the consistency of Zermelo–Fraenkel set theory.

The importance of the Laver result is two-fold: On the one hand it gives a connection between abstract set theory and problems in analysis and on the other hand his solution contains the first use of countable support iterated forcing (this will produce such deep developments as the Proper Forcing Axiom). However, the Laver solution was not general, in fact in his model the cardinality of the continuum is equal to $\aleph_2$. Further analysis of the technique of 'countable support iteration' proves that it is impossible to enlarge the continuum farther than $\aleph_2$ using such techniques. The fact is that in any extension by a ω_1-iteration with countable support of non-trivial forcing notions, the continuum has cardinality $\aleph_1$. Also a finite support iteration forcing does not help for getting a model for the Borel Conjecture, because Cohen reals are added in every ω-limit. These facts show us that the problem of the Borel Conjecture with the continuum bigger than $\aleph_2$ may be a really hard problem.

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For many years the experts were thinking that a solution for this problem should bring new ideas about forcing iteration. Unfortunately this did not happen: We will show in Section 3 that adding random reals (by the measure algebra) to the Laver model preserves the Borel Conjecture. This clearly is enough in order to enlarge the continuum to every cardinal of cofinality bigger than ω and preserve the Borel Conjecture. The proof presented in Section 3 works for a large class of forcing extensions, and the only two properties used are 'the Laver property' and 'the Laver condition'. Woodin proved, in 1981, that for a model having $\diamondsuit_{\aleph_1}$, adding ω_2 Laver reals followed by random reals yields a model for the Borel Conjecture. This result was never published. Judah (Ihoda) and Shelah [5] proved the results in Sections 1 and 2, and get a different proof of what Woodin [11] had proved, and this proof is presented in Section 3. This proof does not use diamond at all, and is shorter.

In Section 1 we will develop a way to take the support in the limit stages. We will call this 'mixed support iteration' and we will show that this 'mixed support iteration' satisfies the usual properties required for an iteration framework. We will show that the mixed support satisfies the Laver condition and if in each coordinate we are using Ramsey ultrafilters then the mixed support satisfies the Laver property.

In Section 2 we will show that if a model satisfies CH, then there is a partially ordered set, forcing the Borel Conjecture. This forcing notion is given by using the technology introduced in Section 1. Good test problems can be found in [4].

We finish this section by giving some definitions and facts about strong measure zero sets.

0.1. Definition. A set $X \subseteq \mathbb{R}$ has *strong measure* zero iff for every $\langle \varepsilon_i : i < \omega \rangle \in (\mathbb{R}^+)^\omega$ there is $\langle x_i : i < \omega \rangle \in (\mathbb{R})^\omega$ such that

$$X \subseteq \bigcup_{i<\omega} (x_i - \varepsilon_i, x_i + \varepsilon_i).$$

In order to produce strong measure zero sets, the following definition was used.

0.2. Definition. A set $X \subseteq \mathbb{R}$ is a *generalized Luzin set* if for every meager set M, $|X \cap M| < 2^{\aleph_0}$.

0.3. Fact. (a) CH *implies there are generalized Luzin sets of size* $2^{\aleph_0}$.

(b) *If* $2^{\aleph_0}$ *is regular and* X *is a generalized Luzin set, then* X *has strong measure zero.*

Proof. See [4]. □

But there is a weaker assumption which produces uncountable measure zero sets, which was given by Rothberger in the 40's.

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0.3. Definition. (a) Let $F = \langle f_i : i < \kappa \rangle \in (\omega^\omega)^\kappa$. We say that F is an *unbounded family* if for every $g \in \omega^\omega$ there is $i < \kappa$ such that for infinitely many $n \in \omega$, $g(n) < f_i(n)$.

(b) Let $\mathfrak{b}$ be the minimal κ such that there is an unbounded family of cardinality κ.

0.4. Theorem (Rothberger). $\mathfrak{b} = \aleph_1 \Rightarrow$ *there is an uncountable strong measure zero set.*

Proof. This proof was supplied by A. Miller. We say that X is concentrated on $\mathbb{Q}$ iff for every $\mathcal{U}$, open, if $\mathcal{U} \supseteq \mathbb{Q}$ then $X - \mathcal{U}$ is countable. Clearly X concentrated on $\mathbb{Q}$ implies X strong measure zero. Let $\langle f_i : i < \omega_1 \rangle$ be an unbounded family, w.l.o.g. $i < j$ implies $(\exists n\, \forall m > n)(f_i(m) < f_j(m))$ (this means $f_i <^* f_j$). Identify ω^ω with $P \subseteq [0, 1]$, where P are the irrational numbers. If $\mathcal{U} \subseteq [0, 1]$ is an open set and $\mathbb{Q} \subseteq \mathcal{U}$, then $K = [0, 1] - \mathcal{U} \subseteq P$ is compact. Therefore there is $g \in \omega^\omega$ such that $K \subseteq \{f : f \leq^* g\}$ but $\{i : f_i \leq^* g\}$ is countable. □

0.5. Definition. *Borel's Conjecture* holds iff every strong measure zero set is countable.

1. The Laver condition

We will give the definition of a strong measure zero set of reals when instead of the real line we consider 2^ω, the set of ω-sequences of 0's and 1's. It is well-known that any result about strong measure zero sets in 2^ω may be translated to a result on strong measure zero sets in the usual representation of the real line (see [1, §9]).

1.0. Definition. A set $X \subseteq 2^\omega$ has *strong measure* zero if and only if for every $f \in \omega^\omega$ there exists $g \in (2^{<\omega})^\omega$ such that

(i) for every $n \in \omega$, $g(n) \in 2^{f(n)}$;

(ii) for every $x \in X$ there exists infinitely many $n \in \omega$ such that $x \restriction f(n) = g(n)$.

Clearly every countable set of reals has strong measure zero. A good reference for strong measure zero sets is [4].

The Borel Conjecture is the statement that says that every strong measure zero set is countable. The Borel Conjecture fails when the continuum hypothesis holds and Laver [8] proved:

1.2. Theorem (Laver). Cons(ZF) $\Rightarrow$ Cons(ZFC + *Borel Conjecture* + $2^{\aleph_0} = \aleph_2$).

We will give our version of this theorem in Section 2. There are essentially two technical devices involved in the proof of Theorem 1.2 and in the proof of similar theorems. They are the Laver condition and the Laver property.

1.3. Definition. Suppose P is a forcing notion and the members of P have the form (s, p), where $s \in [\kappa]^{<\omega}$ for some cardinal κ. Then we say that P has the *Laver condition* if and only if for every P-sentence φ and for every (s, p) in P, there exists q such that $(s, p) \leqslant_P (s, q) \in P$ and

$$(s, q) \Vdash \text{``}\varphi\text{''} \quad \text{or} \quad (s, q) \Vdash \text{``}\neg\varphi\text{''}.$$

1.4.1. Definition. Let P be a forcing notion. We say that P has the *Laver property* if and only if for every P-name f for a function from ω to ω and for every $p \in P$, and for every $h, g \in \omega^\omega$, if

(i) $h(n) \to \infty$ and
(ii) $p \Vdash_P$ "$f(n) < g(n)$" then there exist $q \in P$, $p \leqslant q$ and $F \in (\omega^{<\omega})^\omega$ such that
(iii) $|F(n)| \leqslant h(n)$ and
(iv) "$q \Vdash f(n) \in F(n)$".

The Laver real forcing has both the Laver condition and the Laver property, the Mathias real forcing also satisfies both definitions (see [8] and [1]). Shelah [10] has proved that the Laver property is preserved under countable support iterated forcing. Clearly this property is not preserved under finite support iteration. In [7] we introduce a forcing notion $P(D)$. This forcing was implicit in a paper of A. Louveau and in a paper of A. Blass. We recall the definition of $P(D)$ and some facts on this forcing notion.

1.4.2. Definition. If D is an ultrafilter over ω, let $P(D)$ be the following partially ordered set:

(i) $p \in P(D)$ iff $p \subseteq \omega^{<\omega}$ is a tree and there exists $s \in p$, called the stem of p, such that for every $t \in p$, $t \subseteq s$ or $s \subseteq t$ and $\{n \in \omega : t^\frown\langle n\rangle \in p\} \in D$.
(ii) If $p, q \in P(D)$ we say that $p \leqslant q$ iff $q \subseteq p$.
(iii) Clearly we can identify a $P(D)$-generic object with an infinite subset of ω (= the generic branch $= \bigcap\{p : p \in G_{P(D)}\}$.
(iv) If $p \in P(D)$ then $s(p)$ is the stem of p.
(v) A condition $p \in P(D)$ is pure if $s(p) = \langle\ \rangle$.
(vi) We say that q is a pure extension of p, $p \leqslant_{\text{pr}} q$, if $p \leqslant q$ and $s(p) = s(q)$.
(vii) Note that D may be a filter and the definition of $P(D)$ makes sense.
(viii) If $s \in \omega^{<\omega}$ and $p \in P(D)$, then we define $p^{[s]} = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$.

1.5. Theorem [7]. (a) *If $x \in [\omega]^\omega$ is $P(D)$-generic over V, then for every $a \in D$, $x \subseteq^* a$ ($|x \setminus a| < \aleph_0$) and for every $y \in [x]^y$, y is $P(D)$-generic over V (so x is a Ramsey real).*

(b) *For every $P(D)$-sentence φ, and for every $p \in P(D)$ there exists $q \in P(D)$ such that*

$$p \leqslant q \Vdash \text{“}\varphi\text{”} \quad or \quad p \leqslant q \Vdash \text{“}\neg\varphi\text{”}$$

and the stem of p is equal to the stem of q.

(c) *If P_D is the Silver forcing notion using a Ramsey ultrafilter D, then $a \in [\omega]^\omega$ is P_D-generic over V iff a is $P(D)$-generic over V.*

(d) *If $a \in [\omega]^\omega$ is $P(D)$-generic and P_D-generic over V, then D is a Ramsey ultrafilter (where P_D is the 'usual' forcing notion for shooting a real 'through' the ultrafilter D).*

(e) *If $a \in [\omega]^\omega$ is $P(D)$-generic and D is an ultrafilter and $\pi\colon [\omega]^2 \to 2$ is in V, then there exists $n \in \omega$ such that $|\pi''[a - n]^2| = 1$.*

1.6. Corollary. *If D is an ultrafilter, then $P(D)$ has the Laver condition.*

In Section 2 we will prove the following

1.7. Theorem. *If D is an ultrafilter in V and $X \in V$ is an uncountable set of reals, then*

$$V^{P(D)} \models \text{“}X \text{ does not have strong measure zero”}.$$

Unfortunately $P(D)$ does not have, in general, the Laver property. In order to obtain a $P(D)$ having the Laver property we need to introduce the notion of Ramsey filter.

1.8. Definition. Let D be an ultrafilter on ω, then D is a *Ramsey ultrafilter* iff for every

$$\pi\colon [\omega]^2 \to 2$$

there exists $a \in D$ such that $|\pi''[a]^2| = 1$.

1.9. Theorem (Canjar [2]). *The following are equivalent:*

(a) *Every filter of cardinality less than $2^{\aleph_0}$ can be extended to a Ramsey ultrafilter.*

(b) *The real line is not the union of less than $2^{\aleph_0}$ meager sets.*

1.10. Lemma (see [1]). *If D is a Ramsey ultrafilter, then $P(D)$ has the Laver property.*

1.11. Definition. The *Ramsey number* 'r' is the minimal cardinal κ satisfying: There exists $\langle \pi_i\colon i < \kappa \rangle$ such that

(a) $\pi_i\colon [\omega]^2 \to 2$ for $i < \kappa$;

(b) for every $a \in [\omega]^\omega$ there exists $i < \kappa$ such that for no $n \in \omega$, $\|\pi_i''[a - n]^2\| = 1$.

Such a family $\langle \pi_i\colon i < \kappa \rangle$ is called a *Ramsey family*.

1.12. Corollary. *Forcing with $P(D)$, when D is an ultrafilter, destroys Ramsey families.*

Proof. By Theorem 1.5(e). □

We are trying to avoid the countable support iteration as well as the finite support iteration. Therefore the next stage is to define our framework for the iteration. More information can be found in [3].

1.13. Definition. We define by induction on $\alpha \geq 1$ when $\langle \mathbb{P}_i; \mathbf{Q}_j: i, j < \alpha \rangle$ is a system of mixed support iteration of $P(D)$:

$\alpha = 1$: $\mathbb{P}_0$ is the trivial forcing notion (i.e. $\mathbb{P}_0 = \{\emptyset\}$) and there exists $\mathbf{D}$ (a $\mathbb{P}_0$-name) such that

$$\Vdash_{\mathbb{P}_0} \text{“}\mathbf{D} \text{ is an ultrafilter on } \omega \text{ and } \mathbf{Q}_0 = P(\mathbf{D})\text{”}.$$

$\alpha = \beta + 1$: $\langle \mathbb{P}_i; \mathbf{Q}_j: i, j < \beta \rangle$ is a system of mixed support iteration of $P(D)$ and $\mathbb{P}_\beta$ is defined by $p \in \mathbb{P}_\beta$ iff

(i) $\operatorname{dom}(p) = \beta$,

(ii) $|\{\gamma < \beta: p \restriction \gamma \nVdash_{\mathbb{P}_\gamma} \text{“}\phi_{\mathbf{Q}_\gamma} = p(\gamma)\text{”}\}| \leq \aleph_0$,

(iii) $|\{\gamma < \beta: p \restriction \gamma \nVdash_{\mathbb{P}_\gamma} \text{“}\phi_{\mathbf{Q}_\gamma} \leq_{\mathrm{pr}} p(\gamma)\}| < \aleph_0$,

(iv) For every $\gamma < \beta$, $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}p(\gamma) \in \mathbf{Q}_\gamma\text{”}$.

We call P_β the mixed support limit of $\langle P_\alpha, \mathbf{Q}_\alpha: \alpha < \beta \rangle$. And we let $p \leq_{\mathrm{pr}} q$ mean $p \leq q$ and for every $\beta < \alpha$, $q \restriction \beta \Vdash_{\mathbb{P}_\gamma} \text{“}p(\beta) \leq_{\mathrm{pr}} q(\beta)\text{”}$. Now we give the definition of $\mathbf{Q}_\beta$: There is $\mathbf{D}$ (a $\mathbb{P}_\beta$-name) such that $\Vdash_{\mathbb{P}_\beta} \text{“}\mathbf{D}$ is an ultrafilter on ω and $\mathbf{Q}_\beta = P(\mathbf{D})$”. The ordering $\leq_{\mathbb{P}_\beta}$ on $\mathbb{P}_\beta$ is defined by: $p \leq_{\mathbb{P}_\beta} q$ iff $\forall \gamma < \beta\, q \restriction \Vdash \text{“}p(\gamma) \leq_{\mathbf{Q}_\gamma} q(\gamma)\text{”}$. W.l.o.g. we say in this case that $\mathbb{P}_\beta$ is a β-stage mixed support iteration of $P(D)$.

$\alpha = \bigcup \alpha \neq 0$: $\langle \mathbb{P}_i; \mathbf{Q}_j; i, j < \alpha \rangle$ is a system of mixed support iteration of $P(D)$ iff for each $\beta < \alpha$, $\langle \mathbb{P}_i; \mathbf{Q}_j; i, j < \beta \rangle$ is a system of mixed support iteration of $P(D)$.

The following facts may be proved by following the arguments given in [1, §5].

1.14. Fact. *If $\langle P_i; \mathbf{Q}_j: i, j < \beta + 1 \rangle$ is a mixed support iteration of $P(D)$ then*

$$P_{\beta+1} \cong P_\beta * \mathbf{Q}_\beta.$$

1.15. Fact. *If $\langle P_i; \mathbf{Q}_j: i, j < \alpha + 1 \rangle$ is a mixed support iteration of $P(D)$ then:*

(i) *If $\beta < \alpha$ then $P_\beta \lessdot P_\alpha$ (i.e., every maximal antichain of P_β is maximal antichain of P_α).*

(ii) *Therefore there exists $\mathbf{Q}$ (a P_β-name) such that $P_\beta * \mathbf{Q} \cong P_\alpha$ and $\Vdash_{P_\beta}$ “$\mathbf{Q}$ is isomorphic to a mixed support iteration of $P(D)$ of length $\alpha - \beta$”.*

(iii) *If $\alpha_0 < \alpha_1 < \cdots < \alpha = \bigcup \alpha_n$, P_α is the mixed support limit of $\langle P_\beta, \mathbf{Q}_\beta: \beta < \alpha \rangle$, P is the mixed support limit of $\langle P_{\alpha_n}, P_{\alpha_{n+1}}/P_{\alpha_n}: n < \omega \rangle$, then $P \approx P_\alpha$.*

Also when P_α is a mixed support iteration of $P(D)$ we can restrict ourselves to those conditions $p \in P_\alpha$ which are simple (see [10]), this means that the names used in p are simple names. These conditions are dense in P_α.

We will finish this section by showing that if P_α is a mixed support iteration of $P(D)$ and in each stage D is a Ramsey ultrafilter, then P_α has the Laver property.

1.16. Main Claim. *Let $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \alpha \rangle$ be an iteration of mixed support. If $\boldsymbol{\tau}$ is a P_α-name and $\Vdash_{P_\alpha}$ "$\boldsymbol{\tau} \in \{0, 1\}$" and $p \in P_\alpha$, then there are $q \in P_\alpha$ and $t \in \{0, 1\}$ such that*

(i) $p \leqslant_{\text{pr}} q \in P_\alpha$,

(ii) $q \Vdash \boldsymbol{\tau} = t$.

Proof. This will be proved by induction on α.

Case $\alpha = 0$. Trivial.

Case $\alpha = \beta + 1$. Let $G_\beta \subseteq P_\beta$ be generic over V. In $V[G_\beta]$, $p(\beta) \in \mathbf{Q}_\beta[G_\beta]$, $\boldsymbol{\tau}/G_\beta$ is a $\mathbf{Q}_\beta[G_\beta]$-name, therefore there are s, $q(\beta)$ such that

$$s(q(\beta)) = s(p(\beta)), \qquad q(\beta) \in \mathbf{Q}_\beta[G_\beta],$$
$$q(\beta) \Vdash_{\mathbf{Q}_\beta[G_\beta]} \text{“}\boldsymbol{\tau} = s\text{”}, \qquad s \in \{0, 1\}.$$

Hence we have P_β-names $\boldsymbol{s}$ and $\boldsymbol{q}(\beta)$ for s and $q(\beta)$ respectively. Apply the induction hypothesis to $\bar{Q} \restriction \beta$ for $\boldsymbol{s}$ and $p \restriction \beta$ and we get $s' \in \{0, 1\}$ and $q \restriction \beta \in P_\beta$ such that

$$p \restriction \beta \leqslant_{\text{pr}} q \restriction \beta \Vdash_{P_\beta} \boldsymbol{s} = s'.$$

Thus $q \restriction \beta \cup \{\langle \beta, \boldsymbol{q}(\beta) \rangle\}$ and s' are as required.

Case $\text{cof}(\alpha) = \omega$. By Fact 1.15(iii), w.l.o.g. $\alpha = \omega$, $p = \langle p_n : n < \omega \rangle$, p_n is a P_n-name of a member of $\mathbf{Q}_n$. We define q_n such that:

(i) q_n is P_n-name of a member of Q_n.

(ii) $\Vdash_{P_n}$ "$p_n \leqslant_{\text{pr}} q_n$".

(iii) In V^{P_n}, q_n decides $\boldsymbol{s}_n$, where: for $G_{n+1} \subseteq P_{n+1}$ generic over V, $\boldsymbol{s}_n$ is $i + 1$ iff there is $r \in P_\omega / G_{n+1}$ such that $\text{Dom}\, r = [n + 1, \omega)$, $P_\omega / G_n \models$ "$p \restriction [n + 1, \omega) \leqslant_{\text{pr}} r$" and $r \Vdash_{P_\omega / G_{n+1}}$ "$\boldsymbol{\tau} = \hat{i}$", with i minimal under those conditions; otherwise (i.e., there is no such k) $\boldsymbol{s}_n = 0$ (actually q_n is a P_n-name of a member of $\mathbf{Q}_n[G_n]$).

Now $q = \langle q_n : n < \omega \rangle \in P_\omega$, $p \leqslant_{\text{pr}} q$; clearly there exist r, $q \leqslant r \in P_\omega$, and $l < 2$ such that

$$r \Vdash \text{“}\boldsymbol{\tau} = \hat{l}\text{”}.$$

There is some m such that $m \leqslant n < \omega$ implies $r \restriction \{n\}$ is pure. Hence $p \restriction [m, \omega) \leqslant_{\text{pr}} r \restriction [m, \omega]$. We can prove by downward induction on $j \leqslant m$ that for some $l > 0$ we have

$$(r \restriction m) \cup \{q_m\} \Vdash \text{“}\boldsymbol{s}_m = l\text{”}.$$

For $m = 0$ we finish, by definition of $\boldsymbol{s}_m$.

Case $\text{cof}(\alpha) > \omega$. Then for some $\beta < \alpha$, τ is a P_β-name. $\square$

1.17. Fact. *If $\bar{Q} = \langle P_i; Q_i: i < \alpha \rangle$ is a mixed support iteration of $P(D)$, when for every coordinate D is a Ramsey ultrafilter, then P_α = the mixed limit of $\bar{Q}$ has the Laver property.*

Proof. Because in each coordinate we are forcing with a Ramsey ultrafilter, we know that every coordinate satisfies the Laver property. In such a case the most natural proof for the limit case should be a 'preservation under mixed support iteration' as in [10, VI §1.6]. But in this specific case the most simple argument is given by a 'fusion argument', as in [1, §7.1]. Essentially we need to show [1, §9.6], and for this we use our 1.16, and the fact that in every coordinate D is a Ramsey ultrafilter (this is necessary!). We leave the deails to the reader. □

2. The first model for the Borel Conjecture

2.0. Lemma. *Let $\langle f_\alpha: \alpha < \omega_1 \rangle$ be in V a sequence of distinct reals and let D be an ultrafilter in V and $P \in V^{P(D)}$ be such that*

$$V^{P(D)} \models \text{"}P \text{ has the Laver property"}.$$

Then

$$V^{P(D)*P} \models \text{"}\langle f_\alpha: \alpha < \omega_1 \rangle \text{ does not have strong measure zero"}.$$

Proof. Suppose the conclusion fails. Working in $V^{P(D)}$ let $\langle n_k: k < \omega \rangle$ be the increasing sequence of natural numbers given by the Ramsey real; then there exists $\langle \boldsymbol{\eta}_k: k < \omega \rangle$ such that

(i) $\Vdash_P$ "$\boldsymbol{\eta}_k \in {}^{n_k}2$",

(ii) $\Vdash_P$ "$(\forall_\alpha < \omega_1\, \exists^\infty k < \omega)(f_\alpha \restriction n_k = \boldsymbol{\eta}_k)$".

Then, in $V^{P(D)}$, there exists $\langle \bar{\eta}_k: k < \omega \rangle$ such that for every $k < \omega$ we have

$$\bar{\eta}_k \subseteq {}^{n_k}2 \quad \text{and} \quad |\bar{\eta}_k| = k^2$$

and

$$\Vdash_P \text{"}\boldsymbol{\eta}_k \in \bar{\eta}_k\text{"}$$

(all this is possible because P has the Laver property). The sequences $\langle \bar{\eta}_k: k < \omega \rangle$, $\langle \eta_k: k < \omega \rangle$ belong to $V^{P(D)}$, therefore the problem is now a problem that involves only $P(D)$.

Now we pass to V, and here we have $\langle \bar{\boldsymbol{\eta}}_k: k < \omega \rangle$ and $\langle \boldsymbol{n}_k: k < \omega \rangle$ being $P(D)$-names, and we have

$$(*) \qquad \Vdash_{P(D)} \text{"}(\forall_\alpha\, \exists^\infty k\, \exists \eta \in \bar{\boldsymbol{\eta}}_k)(f_\alpha \restriction \boldsymbol{n}_k = \eta)\text{"}.$$

We will show that $(*)$ is false. This gives the proof of the lemma.

Let N be a countable elementary substructure of $H(2^{2^{\aleph_0}}, \in, \leqslant)$ containing D, $\langle \bar{\boldsymbol{\eta}}_k; k < \omega \rangle$, $\langle \boldsymbol{n}_k: k < \omega \rangle$, $\langle f_\alpha: \alpha < \omega_1 \rangle$.

We will prove that the following holds:

(**) Let $\alpha \notin N$ be a countable ordinal; then for every $p \in P(D) \cap N$, and for every $\rho \in p$, $s(p) \subseteq \rho$, there exists $A \in D$ such that for every $m \in A$, there exists $q_m \in N \cap P(D)$, $p^{[\rho^\wedge\langle m\rangle]} \leqslant_{\mathrm{pr}} q_m$ and $q_m \Vdash_{P(D)}$ "$f_\alpha \restriction m \neq \bar{\boldsymbol{\eta}}_{|\rho|}$". (Note that $q_m \Vdash$ "$n_{|\rho|} = m$".)

By proving (**), and using induction on the levels of the tree it is possible to build a $(N, P(D))$-generic condition q satisfying

$$q \Vdash (\forall k)(f_\alpha \restriction \boldsymbol{\eta}_k \notin \bar{\boldsymbol{\eta}}_k)$$

in contradiction with (*).

In order to show (**) let $p \in N \cap P(D)$, and $\rho \in p$ be given. Let A be such that

$$A = \{m \colon \rho^\wedge\langle m\rangle \in p\};$$

then $A \in D \cap N$. For every $m \in A$ let q_m in $P(D) \cap N$, and $\eta^m_{|\rho|} \subseteq {}^m 2$ be such that

$$|\bar{\eta}^m_{|\rho|}| = |\rho|^2, \qquad q_m \Vdash_{P(D)} \text{"}\bar{\eta}^m_{|\rho|} = \bar{\boldsymbol{\eta}}_{|\rho|}\text{"}, \qquad p^{[\rho^\wedge\langle m\rangle]} \leqslant_{\mathrm{pr}} q_m$$

(for this use the Laver condition of $P(D)$).

Set $A_0 = \{m \colon f_\alpha \restriction m \notin \bar{\eta}^m_{|\rho|}\}$.

If $A_0 \in D$ we are done. If $A_0 \notin D$, then set

$$H = \{\beta \colon (\exists A \in D)(\forall m \in A)(f_\beta \restriction m \in \bar{\eta}^m_{|\rho|})\}.$$

Clearly the parameters in the definition of H belong to N, therefore H belongs to N.

Claim. *H is finite.*

Proof. If not, fix $l > |\bar{\eta}_{|\rho|}| = |\rho|^2$, and $\alpha_1, \alpha_2, \ldots, \alpha_l$ in H and $A_1, \ldots, A_l$ in D witnessing this. Let $m \in A_1 \cap \cdots \cap A_l$ be such that for every $l_1 < l_2 < l$, $f_{\alpha_{l_1}} \restriction m \neq f_{\alpha_{l_2}} \restriction m$. Then by hypothesis on α_j, $1 \leqslant j \leqslant l$,

$$f_{\alpha_j} \restriction m \in \eta^m_{|\rho|}$$

and this implies that there exist $\alpha_j \neq \alpha_k$ such that $f_{\alpha_j} \restriction m = f_{\alpha_k} \restriction m$, a contradiction □ (Claim)

Then the Claim implies that $H \subset N$, and if $A_0 \notin D$ then $\alpha \in H \subseteq N$, a contradiction. □

2.1. Theorem. *Let Q be a forcing notion satisfying: for every Q-name, for a sequence $\langle f_\alpha \colon \alpha < \omega_1\rangle$ of distinct real numbers there exists P and a P-name $\boldsymbol{D}$ of an ultrafilter such that*

(i) *$\langle f_\alpha \colon \alpha < \omega_1\rangle$ is a P-name,*

(ii) *$P * P(\boldsymbol{D}) \lessdot Q$,*

(iii) $V^{P*P(\boldsymbol{D})} \models$ "$Q/(P*P(\boldsymbol{D}))$ *has the Laver property*".
Then in V^Q *the Borel Conjecture holds.*

Proof. Use Lemma 2.0 working in V^P. □

2.2. Theorem (Laver). *If* $V \models \mathrm{CH}$, *then there exists P a forcing notion such that*

$$V^P \models \text{"Borel Conjecture"}.$$

Proof. Let $Q = \langle P_\alpha: \boldsymbol{Q}_\alpha; \alpha < \omega_2 \rangle$ be such that:

(i) For every $\alpha < \omega_2$, there exists a P_α-name $\boldsymbol{D}$ for a Ramsey ultrafilter such that

$$\Vdash_{P_\alpha} \text{"}\boldsymbol{Q}_\alpha = P(\boldsymbol{D})\text{"}.$$

(ii) P_α is the mixed support limit of $Q \restriction \alpha$.

Let P be the mixed support limit of Q. Then P satisfies the assumption given in Theorem 2.1. Therefore in V^P the Borel Conjecture holds.

It remains to show that there exists in V a Q satisfying (i) and (ii). The problem is (i) but for every α,

$$\Vdash_{P_\alpha} \text{"CH"}$$

and this implies that in V^{P_α} there are Ramsey ultrafilters. □

2.3. Corollary. *Let V be a model of* ZFC *and let D be an ultrafilter in V. Then for every sequence* $\langle f_\alpha: \alpha < \omega_1 \rangle \in V$ *of distinct real numbers we have*

$$V^{P(D)} \models \text{"}\langle f_\alpha: \alpha < \omega_1 \rangle \text{ does not have strong measure zero"}.$$

Remark. The problem is that if D is not a Ramsey ultrafilter, an iteration of $P(D)$ may add Cohen reals, therefore we need to use Ramsey ultrafilters in order to obtain the Borel Conjecture in a generic extension.

3. Adding random reals to the usual models for the Borel Conjecture

Let R be a measure algebra (i.e., the positive sets of a measure product of $\{0, 1\}$ endowed with the equidistributive probability). This forcing notion has an absolute definition and satisfies the following fact:

3.1. Fact. *If f is an R-name for a real in* $V \subseteq V^1$, *then f is an R-name for a real in* V^1.

(Note that $R^V \subseteq R^{V^1}$ but not $R^V \lessdot R^{V^1}$, this holds only for maximal antichains which are in V.)

3.2. Definition. Let η be an R-name for a member of ${}^n2$ and let $h: {}^n2 \to [0, 1]$. We define

$$\mathrm{Ex}(h(\eta)) = \sum_{\varepsilon \in {}^n2} h(\varepsilon) \cdot \mu([\![\eta = \varepsilon]\!])$$

where $[\![\ ]\!]$ is the Boolean value with respect to the measure algebra, and μ is the Lebesgue measure.

From now on we fix a ground model V, a sequence $\bar{f} = \langle f_\alpha : \alpha < \omega_1 \rangle \in V$, of distinct reals where each f_α is an R-name for a function from ω to 2. Let V^1 be a generic extension of V by forcing the Laver real "Lv" (you can use $P(D)$, or Mathias). Let $\langle n_i : i < \omega \rangle = \bar{n}$ be the Laver real over V, clearly $\bar{n} \in V^1$. Also we fix $k: \omega \to \omega$, $k \in V$. Before we establish the main fact for the proof of the theorem we need the following definition:

3.3. Definition. Let $V^2 \supseteq V^1$ be a model of ZFC, then we say $V^2 \models (*)(\bar{f}, \bar{n}, k)$ iff for every family of functions $\langle h_i ; i < \omega \rangle$ such that

$$h_i : {}^{n_i}2 \to [0, 1], \qquad \sum_{\eta \in {}^{n_i}2} h_i(\eta) \leq k(i),$$

there exists $\alpha < \omega_1$ such that

$$\sum_{i < \omega} \mathrm{Ex}(h_i(f_\alpha \restriction n_i)) < \infty.$$

3.4. Fact. *For every $k \in V$,*

$$V^1 \models (*)(\bar{f}, \bar{n}, k).$$

Proof. Let $N \prec (H(2^{2^{\aleph_0}}), \in, \leq)$ be countable and $\{\langle \mathbf{h}_i : i < \omega \rangle \bar{f}, \bar{\mathbf{n}}, k\} \subseteq N$; remember that $\mathbf{h}_i$ are Lv-names and $\bar{\mathbf{n}}$ is the canonical name for the Laver sequence. We will show that the following holds:

(1) For every $\alpha \in \omega_1 - N$, for every $p \in Lv \cap N$, for every $\mathrm{stem}(p) \subseteq \rho \in p$ and for every $\varepsilon > 0$, there exists $A \in [\omega]^\omega$ such that for all $m \in A$, there exists $q_m \in Lv \cap N$, $p^{[\rho^\wedge \langle m \rangle]} \leq_{\mathrm{pr}} q_m$ and $q_m \Vdash_{Lv}$ "$\mathrm{Ex}(h_{|\rho|+1}(f_\alpha \restriction m)) \leq \varepsilon$". Also $A \subseteq \{m : \rho^\wedge \langle m \rangle \in p\}$. ($p^{[\eta]} = \{\theta \in p : \eta \subseteq \theta$ or $\theta \subseteq \eta\}$ for $\eta \in \omega^{<\omega}$, $q \leq_{\mathrm{pr}} r$ iff $\mathrm{stem}(q) = \mathrm{stem}(r)$).

Clearly by proving (1), and using induction on the levels of the tree we can give q, $p \leq_{\mathrm{pr}} q$ and $q \Vdash (*)(\bar{f}, \bar{n}, k)$.

In order to show (1) let $p \in Lv \cap N$, $\rho \in p$ be given. Let $A = \{m : \rho^\wedge \langle m \rangle \in p\}$. Then $A \in N$.

We define the following function

$$T_1(n) = 2^n, \qquad T_{r+1}(n) = 2^{T_r(n)} \quad \text{for } r \geq 1.$$

Now for every $m \in A$, let $q_m \in Lv \cap N$ and $h^m \in N$ be such that

(a) $p^{[\rho^\wedge\langle n\rangle]} \leqslant_{\mathrm{pr}} q_m$,

(b) $h^m: {}^m2 \to \left\{\frac{l}{T_3(m+k(m))} : 0 \leqslant l \leqslant T_3(m+k(m))\right\}$,

(c) $q_m \Vdash_{Lv}$ "$\left(\forall \eta \in {}^m2)(\boldsymbol{h}_{|\rho|+1}(\eta) \leqslant h^m(\eta) \leqslant \boldsymbol{h}_{|\rho|+1}(\eta) + \frac{1}{T_3(m+k(m))}\right)$".

(Therefore $\sum_{\eta\in{}^m2} h^m(\eta) \leqslant k(|\rho|+1)+1$. These q_m's are found by using the Laver condition and rational approximations to $\boldsymbol{h}_{|\rho|+1}(\eta)$.)

Set $A_0 = \{m: \mathrm{Ex}(h^m(f_\alpha \restriction m)) \leqslant \varepsilon/2\}$. If $A_0 \in [A]^\omega$, then for all $m \in A_0 - m_0$, for a m_0 fixed,

$$q_m \Vdash \mathrm{Ex}(\boldsymbol{h}_{|\rho|+1}(f_\alpha \restriction m)) \leqslant \mathrm{Ex}(h^m(f_\alpha \restriction m)) + \frac{1}{T_3(m+k(m))} < \varepsilon$$

and this implies (1).

Hence the problem is when A_0 is finite. In this case we define

$$H = \{\alpha: (\exists m \in \omega)(\forall n \in A - m)(\mathrm{Ex}(h^n(f_\alpha \restriction n)) > \varepsilon/2)\}.$$

Clearly the parameters in the definition of H belong to N, therefore $H \in N$.

Claim. *H is finite.*

Proof. Let $l = 4(k(|\rho|+1)+1)/\varepsilon + 1$ and fix $\langle \alpha_i: 0 \leqslant i \leqslant l\rangle \subseteq H$ distinct elements of H, and m_i witnessing $\alpha_i \in H$. Let $m > \sup\{m_i: i \leqslant l\} + m_0$, $m \in A$ and $B \in R$ such that

$$\mu(B) \geqslant 1 - \frac{\varepsilon}{4(k(|\rho|+1)+1)}$$

and

$$\mu([\![f_{\alpha_i} \restriction m = f_{\alpha_j} \restriction m]\!] \cap B) = 0$$

for every $i \neq j$ less than $l+1$. (Remember that R forces that f_{α_i} and f_{α_j} are distinct.)

Then

$$\begin{aligned}\sum_{i=0}^{l} \mathrm{Ex}(h^m(f_\alpha \restriction m)) &= \sum_{i=0}^{l} \sum_{\eta\in{}^m2} h^m(\eta)\cdot \mu([\![f_{\alpha_i} \restriction m = \eta]\!]) \\ &= \sum_{\eta\in{}^m2} h^m(\eta) \cdot \sum_{i=0}^{l} \mu([\![f_{\alpha_i} \restriction m = \eta]\!]) \\ &\leqslant \sum_{\eta\in{}^m2} h^m(\eta)\left(1 + \frac{\varepsilon}{4(k(|\rho|+1)+1)}\cdot l\right)\end{aligned}$$

(remember that, inside B, $\mu([\![f_{\alpha_i} \restriction m]\!] \cap [\![f_{\alpha_j} \restriction m]\!]) = 0$)

$$\leqslant k(|\rho|+1) + 1 + \frac{\varepsilon}{4}\cdot l.$$

Therefore there exists $i \leq l$ such that

$$\mathrm{Ex}(h^m(f_{\alpha_i} \restriction m)) < \frac{k(|\rho|+1)+1}{l} + \frac{\varepsilon}{4}$$

but, by the choice of l, we get

$$\mathrm{Ex}(h^m(f_{\alpha_i} \restriction m)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

a contradiction to the fact that $m \in A - m_i$. □ (Claim)

Therefore $H \subseteq N$ and this implies that $\alpha \notin H$, a contradiction. □

3.5. Lemma. *Suppose that $P \in V^1$ is a forcing notion satisfying the Laver property and suppose that*

(i) $V^1 \models (*)(\bar{f}, \bar{n}, k^0)$,

(ii) $k^1: \omega \to \omega$ *is such that* $\lim_{k\to\infty} k^0(k)/k^1(k) = \infty$.

Then $(V^1)^P \models (*)(\bar{f}, \bar{n}, k^1)$.

Proof. Let $\langle \boldsymbol{h}_i : i < \omega \rangle$ be P-names for functions h_i in V^P from ${}^{n_i}2$ to $[0, 1]$, for each $i < \omega$, such that

$$\sum_{\eta \in {}^{n_i}2} h_i(\eta) \leq k^1(i).$$

By using the Laver property we get

$$\langle h_i^* : i < \omega \rangle \in V^1$$

satisfying:

(a) for every $i \in \omega$, for every $\eta \in {}^{n_i}2$, $h_i(\eta) \leq h_i^*(\eta)$;

(b) $\sum_{\eta \in {}^{n_i}2} h_i^*(\eta) \leq k^0(i)$ (by the Laver property we can get, for each i, $k^0(i)/2k^1(i)$ approximations for $\langle h_i(\eta): \eta \in {}^{n_i}2 \rangle$ with error less than $1/T_3(n_i + k^1(i) + k^0(i))$, using rational numbers). Now, as $V^1 \models (*)(\bar{f}, \bar{n}, k^0)$, we obtain $\alpha < \omega_1$ such that

$$\sum_i \mathrm{Ex}(h_i^*(f_\alpha \restriction n_i)) < \infty.$$

This α works for showing

$$\sum_i \mathrm{Ex}(h_i(f_\alpha \restriction n_i)) < \infty. \qquad \square$$

3.6. Corollary. *If $\langle k^l : l < \omega \rangle \in V$, each $k^l: \omega \to \omega$ and for every l*

$$\lim_{i\to\infty} k^l(i)/k^{l+1}(i) = \infty$$

and

$$V^1 \models \text{“}(\forall l < \omega)((*)(\bar{f}, \bar{n}, k^l))\text{”}$$

and $P \in V^1$ satisfies the Laver property, then

$$(V^1)^P \vDash (\forall l < \omega)((*)(\bar{f}, \bar{n}, k^l)).$$

3.7. Corollary. *For every $k \in V$, if $P \in V^1$ has the Laver property, then*

$$(V^1)^P \vDash (*)(\bar{f}, \bar{n}, k).$$

(Use Corollary 3.6 and Fact 3.4.)

3.8. Lemma. *Let $P \in V^1$ satisfy the Laver condition. Then*

$$(V^1)^{P*R} \vDash \text{“}\langle f_\alpha : \alpha < \omega_1 \rangle \textit{ does not have strong measure zero”}.$$

Proof. If the condition of the theorem does not hold, then there exist $\langle \rho_i : i < \omega \rangle$ such that for every $i < \omega$, ρ_i is an R-name for a member of ${}^{n_i}2$ and $\langle \rho_i : i < \omega \rangle \in (V^1)^P$, and $B \in R$,

$$(V^1)^P \vDash \text{“}B \Vdash_R \text{“}(\forall \alpha \in \omega_1)(\exists^\infty i)(\rho_i = f_\alpha \restriction n_i)\text{” ”}.$$

In $(V^1)^P$ we define the following functions

$$h_i : {}^{n_i}2 \to [0, 1], \qquad h_i(\rho) = \mu([\![\rho_i = \rho]\!]).$$

Then clearly $\langle h_i : i < \omega \rangle \in (V^1)^P$ and for every i,

$$\sum_{\eta \in {}^{n_i}2} h_i(\eta) \leq 1$$

and for every i, for every $\alpha < \omega$,

$$\mathrm{Ex}(h_i(f_\alpha \restriction n_i)) = \mu([\![\rho_i = f_\alpha \restriction n_i]\!]).$$

Therefore, there exists α such that

$$\sum_i \mu([\![\rho_i = f_\alpha \restriction n_i]\!]) < \infty.$$

Therefore there exists $i_0 < \omega$ such that

$$\sum_{i \geq i_0} \mu([\![\rho_i = f_\alpha \restriction n_i]\!]) < \mu(B)/2.$$

Therefore there exists $C \in R$ such that $\mu(B \cap C) > 0$ and

$$\mu(C \cap [\![\rho_i = f_\alpha \restriction n_i]\!]) = 0$$

for all $i > i_0$, but then

$$C \Vdash_R \text{“}(\forall i > i_0)(\rho_i \neq f_\alpha \restriction n_i)\text{”}$$

a contradiction. $\square$

3.9. Theorem. *Let V be a model of* ZFC. *Let Q be a forcing notion satisfying: For every Q-name $\langle f_\alpha : \alpha < \omega_1 \rangle$ for a sequence of real number there exists P such that*

(i) $\langle f_\alpha : \alpha < \omega_1 \rangle$ *is a P-name.*
(ii) $P * Lv \lessdot Q$.
(iii) $V^{P*Lv} \models$ "*$Q/P * Lv$ has the Laver property*".

And suppose that R is the measure algebra. Then

$$V^{Q*R} \models \text{"Borel Conjecture"}.$$

Proof. Use Lemma 3.8. The only remark is that an R-name for a real is essentially a real number. □

Clearly the iteration of ω_2-Laver reals is like the Q of the above theorem. Many other forcing notions satisfy this fact. We will finish with the following conjecture:

Conjecture. $V \models$ "Borel Conjecture" iff $V^R \models$ "Borel Conjecture" when R is random forcing.

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