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KILLING LUZIN AND SIERPINSKI SETS

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ABSTRACT. We will kill the old Luzin and Sierpinski sets in order to build a model where $U(\mathscr{M}) = U(\mathscr{N}) = \aleph_1$ and there are neither Luzin nor Sierpinski sets. Thus we answer a question of J. Steprans, communicated by S. Todorcevic on route from Evans to MSRI.

In this note we will build a model where there are nonmeasurable sets and nonmeager sets of size $\aleph_1$ and there are neither Luzin nor Sierpinski sets. All our notation is standard and can be found in [Ku, BJ1]. Let us start with the basic concept underlying this work.

Let $U(\mathscr{M})$ be the minimal cardinal of a nonmeager set.

Let $U(\mathscr{N})$ be the minimal cardinal of a nonnull set.

We say that a set of reals X is a Luzin set if X is uncountable and $X \cap M$ is countable for every meager set M. We say that a set X is a Sierpinski set if X is uncountable and $X \cap N$ is countable for every null set N.

Fact. (a) If there is a Luzin set, then $U(\mathscr{M}) = \aleph_1$.
(b) If there is a Sierpinski set, then $U(\mathscr{N}) = \aleph_1$.

In [Sh] it was proved that if ZF is consistent, then there is a model where there are no Luzin sets and $U(\mathscr{M}) = \aleph_1$. In [BGJS] it was proved that if there is a Sierpinski set, then there is a nonmeasurable meager filter on ω. It was natural to ask if from $U(\mathscr{N}) = \aleph_1$ we can get such a filter. Clearly it will be enough to answer positively the following question.

(Steprans) Does $U(\mathscr{N}) = \aleph_1$ imply the existence of a Sierpinski set?

We give a negative answer to this question by proving the following

Theorem. $\mathrm{Cons}(ZF) \to \mathrm{Cons}(ZFC + U(\mathscr{M}) = U(\mathscr{N}) = \aleph_1 +$ *there are neither Luzin nor Sierpinski sets*).

We will prove this theorem by iterating with countable support iteration Miller reals (rational perfect forcing). We will use the machinery produced by "preservation theorems" to show that the old reals are a nonmeager, nonmeasurable set. We will show that Miller reals kill Luzin and Sierpinski sets from the ground model.

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The reader can find a complete analysis of Luzin and Sierpsinki sets in [BJ2].

1. **Definition.** Let $P = \{T : T \subseteq \omega^{<\omega}$ & T is a tree & $(\forall s \in T)(s$ is increasing$)$ & $(\forall s \in T)(\exists t \in T)(\exists^\infty n)(s \subset t^\wedge\langle n\rangle \in T)\}$.
Let $\leq$ be defined by $T \leq S$ if and only if $S \subseteq T$.

$\langle P, \leq\rangle$ is called rational perfect forcing [Mi], and if $G \subseteq P$ is generic, then $\mathbf{m} = \bigcap G \in \omega^\omega$ is called a Miller real. From our assumption we have that $\mathbf{m}$ is increasing.

2. **Definition.** Let $r \in \omega^\omega$ be increasing. We define the set

$$B(r) = \bigcup_{j<\omega} B_j(r)$$

where

$$B_j(r) = \{\eta \in 2^\omega : (\forall i > j)\ (\eta \restriction [r(i), r(i) + 10(i+1)) \text{ is not identically zero})\}.$$

3. **Fact.** $\mu(B_j(r)) \geq 1 - \frac{1}{j+1}$ and $B_j(r)$ is closed.

Therefore, $\mu(B(r)) = 1$.

4. **Lemma.** *Let A be a set of reals such that $\mu^*(A) > 0$. Let $\underset{\sim}{\mathbf{m}}$ be the canonical name for the Miller real. Then*

$$\Vdash_P \text{“}A - B(\underset{\sim}{\mathbf{m}}) \textit{ is uncountable”}.$$

Proof. Let $p \in P$, $p \Vdash_P$ “$\underset{\sim}{A'} = A \backslash B(\underset{\sim}{\mathbf{m}})$ is countable”. As P is proper, without loss of generality for some countable set $A^* \subseteq A$ and $q \geq p$ we have $q \Vdash_P$ “$\underset{\sim}{A'} \subseteq A^*$”. Let $N \prec \langle H((2^{\aleph_0})^+), \in\rangle$ be countable, $q \in N$, $A \in N$, $A^* \in N$. As $\mu^*(A) > 0$ there is $\eta \in A$, η random over N. Therefore, $\eta \notin A^*$. Let $t \in q$ be such that $mc_q(t) = \{n : t^\wedge\langle n\rangle \in q\}$ is infinite. Let us write this set as $mc_q(t) = \{k^t_\ell : \ell < \omega\}$, where $k^t_\ell < k^t_{\ell+1}$. Let $i_t = |t|$. For $n < \omega$, we define

$$E^n_t = \{x \in 2^\omega : (\forall \ell \geq n)(x \restriction [k^t_\ell, k^t_\ell + 10(i_t+1)) \text{ is not identically zero})\}.$$

5. **Fact.** $\mu(E^n_t) = 0$.

Therefore, $E_t = \bigcup_n E^n_t$ is null and $E_t \in N$; thus, $\eta \notin E_t$. Hence,

$$D_t = \{k^t_\ell : \eta \restriction [k^t_\ell, k^t_\ell + 10(i_t+1)) \text{ is identically zero}\}$$

is infinite.

Now using this we can define, inductively, $q' \geq q$ satisfying

$$\text{if } t \in q' \text{ and } mc_q(t) \text{ is infinite, then } mc_{q'}(t) = D_t.$$

Therefore, $q' \Vdash_P$ “$\eta \notin B(\underset{\sim}{\mathbf{m}})$”, a contradiction. □

6. **Corollary.** *If $Y \in V$ is a Sierpinski set, then Y is not a Sierpinski set in any extension of V containing a Miller real over V.*

7. *Remark.* The same result can be obtained if you replace Miller real by Laver real.

8. **Definition.** Let $r \in \omega^\omega$ be increasing. We define the set

$$T(r) = \bigcup_{j<\omega} [T_j(r)]$$

where $T_j(r)$ is the tree defined by

$$\eta \in [T_j(r)] \quad \text{if and only if} \quad \eta \in 2^\omega \ \& \ (\forall i > j)(\eta(r(i)) = 0).$$

We say that for a tree T, $[T]$ is the set of ω-branches of T.

9. **Fact.** $[T_j(r)]$ is a closed nowhere dense set.

Therefore, $T(r)$ is a meager set.

10. **Lemma.** *Let A be a nonmeager set of reals. Let $\underset{\sim}{\mathbf{m}}$ be the canonical name for the Miller real. Then*

$$\Vdash_P \text{“}A \cap T(\underset{\sim}{\mathbf{m}}) \textit{ is uncountable”}.$$

Proof. Let $p \in P$ and let $N \prec (H((2^{\aleph_0})^+), \in)$ be countable such that $p \in N$. Then there is $\eta \in A$ such that η is Cohen over N. We will find q such that

$$p \leq q \in P \text{ and } q \Vdash_P \text{“}\eta \in T(\underset{\sim}{\mathbf{m}})\text{”}.$$

Clearly this is enough. Let $\langle \nu_\rho : \rho \in {}^{\omega>}\omega \rangle$ be the list of splitting nodes of p such that $\rho_1 \subsetneq \rho_2$ implies $\nu_{\rho_1} \subsetneq \nu_{\rho_2}$. Thus $\langle \nu_{\rho^\wedge\langle n\rangle}(|\nu_\rho|) : n < \omega \rangle$ are distinct and without loss of generality are strictly increasing, so

$$(*) \qquad \nu_{\rho^\wedge\langle n\rangle}(|\nu_\rho|) \geq n.$$

For each $\rho \in {}^{\omega>}\omega$ let

$$A_\rho = \{n < \omega : \eta \restriction (\text{Range } \nu_{\rho^\wedge\langle n\rangle} \setminus \text{Range } \nu_\rho) \text{ is identically zero}\}.$$

11. **Fact.** For $\rho \in {}^{\omega>}\omega$, A_ρ is infinite.

Proof. $p \in N$ and let $s \in 2^{<\omega}$ be a condition in Cohen forcing. Then there is n, by $(*)$, such that

$$\text{dom}(s) \cap (\text{Range } \nu_{\rho^\wedge\langle n\rangle} \setminus \text{Range } \nu_\rho) = \varnothing.$$

Thus we can extend s to $t \in 2^{<\omega}$ such that $t \restriction (\text{Range } \nu_{\rho^\wedge\langle n\rangle} \setminus \text{Range } \nu_\rho)$ is identically zero. Thus, because η is Cohen over N, we have that A_ρ is infinite. □

Now we define q by

$$q = \{\nu \in p : (\forall \ell \leq |\nu|)(\nu \restriction \ell = \nu_{\rho^\wedge\langle n\rangle} \rightarrow n \in A_\rho)\}$$

and $q \Vdash_P$ “$\eta \in T(\underset{\sim}{\mathbf{m}})$”. □

12. **Corollary.** *If $X \in V$ is a Luzin set, then X is not a Luzin set in any extension of V containing a Miller real over V.*

Now we are ready to show the main Theorem.

13. **Theorem.** Cons(ZF) *implies* Cons($ZFC + U(\mathscr{M}) = U(\mathscr{N}) = \aleph_1 +$ *there are neither Luzin nor Sierpinski sets*).

Proof. Let us start with $V = L$. Let P_{ω_2} be the countable support iteration of P, of length ω_2. Then the following hold in $V^{P_{\omega_2}}$.

(i) $U(\mathscr{M}) = \aleph_1$: In [Go] it is proved that the property of being nonmeager is preserved by a countable support iteration. It is easy to see that P satisfies the covering properties established in [Go, §6.20]. Therefore, $V \cap 2^\omega$ is a nonmeager set in $V^{P_{\omega_2}}$.

(ii) $U(\mathscr{N}) = \aleph_1$: In [Go] it is proved that the property of being nonnull is preserved by countable support iteration. In [BJS] it is proved that P satisfies the covering properties established in [Go, §6.8]. Therefore, $V \cap 2^\omega$ is a nonnull set in $V^{P_{\omega_2}}$.

(iii) There are no Luzin sets in $V^{P_{\omega_2}}$: by Corollary 6.

(iv) There are no Sierpinski sets in $V^{P_{\omega_2}}$: by Corollary 12. □

14. *Remark.* In the ω_2-iteration of Laver reals we have that $U(\mathscr{N}) = \aleph_1$ and there are no Sierpinski sets. We do not know if in this model there are uncountable strongly meager sets. We know that Miller reals do not kill strong measure zero sets. This is a consequence of a Rothberger theorem. See [BJ2].

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