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ON THE NUMBER OF STRONGLY $\aleph_\varepsilon$-SATURATED MODELS OF POWER λ

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We prove that for superstable T the number is small and for unsuperstable T the number is large.

0. Introduction

We deal with models of a fixed complete first-order theory T. We rely on [1].

0.1. Definition. (1) A model M is strongly κ-saturated if:

(i) M is strongly κ-homogeneous, i.e., if $\bar{a}$, $\bar{b}$ are sequences of elements of M, of the same length which is $<\kappa$, and realizes the same type, then for some automorphism f of M, $f(\bar{a})=\bar{b}$; and

(ii) M is $\aleph_\varepsilon$-saturated ($=\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-saturated, see VI Definition 1.1(4), 2.1), i.e., every type which is almost over a finite subset of M is realized in M.

(2) Let $I^{\mathrm{sa}}(\lambda, T)$ be the number of strongly $\aleph_0$-saturated models of T of power λ up to isomorphism.

We shall compute $I^{\mathrm{sa}}(\lambda, T)$ for $\lambda \geq 2^{|T|}$. We do not need the new methods needed for classifying theories (see [2]). Moreover the main dividing line is simply superstability. So we have gotten a direct characterization of "T is superstable" in terms of some spectrum function.

More explicitly our results are:

0.2. Theorem. (1) *If T is superstable, then for some cardinal* $\mathrm{nde}(T) \leq 2^{|T|}$ *for every* $\aleph_\alpha \geq 2^{|T|}$, $I^{\mathrm{sa}}(\aleph_\alpha, T) \leq |\alpha|^{\mathrm{nde}(T)}$.

(2) *If T is not superstable, then for every* $\lambda \geq 2^{|T|}$, $I^{\mathrm{sa}}(\lambda, T) = 2^\lambda$.

If we change 0.1(1)(i) by demanding $\bar{a}$, $\bar{b}$ realize the same strong type (over $\emptyset$)

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the change is immaterial. E.g. expand T by a name for every equivalence class of each $E \in \mathrm{FE}(\emptyset)$

Notation. References like V 1.1 are to [1].

1. On superstable T

Hypothesis. *T is superstable.*

1.1. Claim. *Suppose M^* is a model of T. If $A \subseteq M^*$, $\bar{a}, \bar{b} \in M^*$, $p = \mathrm{tp}(\bar{b}, \bar{a})$ is stationary and orthogonal to A, q its stationarization over $A \cup \bar{a}$, then* $\dim(p, M^*)$ *and* $\dim(q, M^*)$ *are equal or both finite.*

Remark. Remember that if $B \subseteq M^*$, $r \in S^m(B)$, then $\dim(r, M) = \mathrm{Min}\{|\boldsymbol{I}| : \boldsymbol{I}$ is a family of sequences of length m of members of M, realizing p, which is independent over B, and maximal under the restrictions listed so far$\}$.

If r is regular we can omit the 'Min', but even generally for any two such $\boldsymbol{I}_1, \boldsymbol{I}_2$: $|\boldsymbol{I}_1| \leq |\boldsymbol{I}_2|\, w(r)$ where $w(r)$ is a natural number (see V 3.13(2)), so if at least one is infinite they are equal.

Proof. Clearly $\dim(q, M^*) \leq \dim(p, M^*)$, more exactly, by the remark, $\dim(q, M^*) < \dim(p, M^*)^+ + \aleph_0$. We can find a maximal $\boldsymbol{I} \subseteq p(M^*)$ independent over $\bar{a}$ such that $|\boldsymbol{I}| = \dim(p, M^*)$. We can also find $\bar{c} \in A$ such that $\mathrm{tp}(\bar{a}, A)$ does not fork over $\bar{c}$. Now (as $\kappa(T) = \aleph_0$) by III 3.5(2) for some finite $\boldsymbol{J} \subseteq \boldsymbol{I}$, $\boldsymbol{I} - \boldsymbol{J}$ is independent over $(\bar{c} \cup \bar{a}, \bar{a})$. As p is orthogonal to A, it is orthogonal to $\mathrm{stp}_*(A, \bar{c})$ (see Definition V 1.1). By V 1.5, $\mathrm{tp}_*(\bigcup(\boldsymbol{I} - \boldsymbol{J}), \bar{a})$ is orthogonal to $\mathrm{tp}_*(A, \bar{c})$. Remember that $\mathrm{stp}_*(A, \bar{a} \cup \bar{c})$, $\mathrm{stp}_*(\bigcup(\boldsymbol{I} - \boldsymbol{J}), \bar{a} \cup \bar{c})$ does not fork over $\bar{c}, \bar{a}$ resp. Hence (by V 1.2(4)) $\mathrm{stp}_*(\bigcup(\boldsymbol{I} - \boldsymbol{J}), \bar{a} \cup \bar{c})$, $\mathrm{stp}_*(A, \bar{a} \cup \bar{c})$ are orthogonal, hence (by V 1.2(1)) $\mathrm{tp}(\bigcup(\boldsymbol{I} - \boldsymbol{J}), \bar{a} \cup \bar{c} \cup A)$ does not fork over $\bar{c}$. So

$$\begin{aligned} \dim(q, M^*) &\geq (|\boldsymbol{I} - \boldsymbol{J}|)/w(p) = (|\boldsymbol{I}| - |\boldsymbol{J}|)/w(p) \\ &= (\dim(p, M^*) - |\boldsymbol{J}|)/w(p), \qquad \boldsymbol{J} \text{ finite.} \end{aligned}$$

Together with the first sentence of the proof, we finish. □

1.2. Claim. *Let A be a set, $\boldsymbol{I}$ independent over A, and N is $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime ove $A \cup \bigcup \boldsymbol{I}$.*

Then for every $\bar{a}, \bar{b} \in N$, if $p = \mathrm{tp}(\bar{b}, \bar{a})$ is stationary and orthogonal to A, the $\dim(p, N) = \aleph_0$.

Proof. By 1.1, $\dim(p, N)$ is $\leq \aleph_0 + \dim(p_1, N)$ where p_1 is the stationarization c p over $A \cup \bar{a}$. Also for some finite $\boldsymbol{J} \subseteq \boldsymbol{I}$, $\mathrm{tp}(\bar{a}, A \cup \bigcup \boldsymbol{I})$ does not fork ove

$A \cup \bigcup J$, and by III 3.5, $\dim(p_1, N) + \aleph_0 = \dim(p_2, N) + \aleph_0$ where p_2 is the stationarization of p over $A \cup \bar{a} \cup \bigcup J$. As p is orthogonal to $\operatorname{tp}(\bar{c}, A)$ and even to $\operatorname{tp}(\bar{c}, A \cup \bigcup J)$ for $\bar{c} \in I - J$, by V 1.4(1), it is orthogonal to $\operatorname{tp}_*(\bigcup (I - J), A \cup \bigcup J)$. Hence every $J' \subseteq p_2(N)$ independent over $A \cup \bar{a} \cup \bigcup J$, is independent over $A \cup \bar{a} \cup J \cup (I - J) = A \cup \bar{a} \cup I$, hence $\dim(p_2, N) = \dim(p_3, N)$ where p_3 is the stationarization of p over $A \cup \bar{a} \cup \bigcup I$. But N is $F^a_{\aleph_0}$-prime over $A \cup \bigcup I$, hence (by IV 2.12(3)) over $A \cup \bar{a} \cup \bigcup I$, hence (by VI 4.9(2)) $\dim(p_3, N) \leq \aleph_0$. So $\dim(p, N) \leq \dim(p_3, N) + \aleph_0 \leq \aleph_0$, but $\bar{a}$ is finite hence equality holds as N is $F^a_{\aleph_0}$-saturated. □

1.3. Proposition. *Suppose N, M^* are $F^a_{\aleph_0}$-saturated, $N \subseteq M^*$, $I \subseteq M^*$ is independent over N, and $\operatorname{stp}(\bar{c}, N)$ is regular for every $\bar{c} \in I$. Then we can find $N_0 \subseteq M^*$ $F^a_{\aleph_0}$-prime over $N \cup \bigcup I$ such that:*

(∗) *if $\bar{a}, \bar{b} \in N_0$, $p = \operatorname{tp}(\bar{b}, \bar{a})$ is stationary, regular orthogonal to N, and q is the stationarization over M^* of p, then $\dim(q \restriction (N \cup \bar{a}), M^*) = \dim(q \restriction N_0, M^*)$.*

Proof. Let N_0 be an $F^a_{\aleph_0}$-primary model over $N \cup \bigcup I$. Let $\{p_i : i < \alpha\}$ be a maximal family of complete over N_0, regular, orthogonal to N, pairwise orthogonal types. Let $\{\bar{a}^i_n : n < \omega\}$ be independent over N_0, $\bar{a}^i_n$ realizing p_i (so $\{a^i_n : i < \alpha, n < \omega\}$ is independent over N_0, see V 1.4(2)), and N_1 be $F^a_{\aleph_0}$-primary over $N_0 \cup \bigcup \{\bar{a}^i_n : i < \alpha, n < \omega\}$. Now $\operatorname{tp}(\bar{a}^i_n, N_0)$ is orthogonal to N.

1.3A. Fact. *N_1 is $F^a_{\aleph_0}$-atomic over $N \cup \bigcup I$.*

Proof. Suppose $\bar{c} \in N_1$. Then by IV 3.12(2), $\operatorname{tp}(\bar{c}, N_0 \cup \{\bar{a}^i_n : i < \alpha, n < \omega\}$ is $F^a_{\aleph_0}$-isolated, so (see IV 2.1) there are finite $B \subseteq N_0$, $u \subseteq \alpha$ and $k < \omega$ such that

$$\operatorname{stp}(\bar{c}, B \cup \{\bar{a}^i_n : i \in u, n < k\}) \vdash \operatorname{stp}(\bar{c}, N_0 \cup \{\bar{a}^i_n : i < \alpha, n < \omega\}).$$

Clearly it suffices to find $\bar{b}^i_n$ $(i \in u, n < k)$ in N_0 such that $\langle \bar{b}^i_n : i \in u, n < k \rangle$ realize $\operatorname{stp}(\langle \bar{a}^i_n : i \in u, n < k \rangle, N \cup I \cup B)$ [as then we can find in N_0 $\bar{c}'$ such that $\bar{c}' {}^\wedge \langle \bar{b}^i_n : i \in u, n < k \rangle$ realizes $\operatorname{stp}(\bar{c} {}^\wedge \langle \bar{a}^i_n : i \in u, n < k \rangle, I \cup B)$; so $\operatorname{tp}(\bar{c}, N \cup I) = \operatorname{tp}(\bar{c}', N \cup I)$ hence they are $F^a_{\aleph_0}$-isolated, remembering that N_0 is $F^a_{\aleph_0}$-atomic over $N \cup I$]. As $\{\bar{a}^i_n : i \in u, n < k\}$ is independent over N_0 (and $N \cup I \subseteq N_0$) it suffices to prove:

(∗)$_1$ for every finite $A \subseteq N_0$ and $i < \alpha$, $n < \omega$, some $\bar{b} \in N_0$ realizes $\operatorname{stp}(\bar{a}^i_n, N \cup I \cup A)$

(as then we define $\bar{b}^i_n$ by induction on $i \in \omega$ and $n < \omega$).

Proof of (∗)$_1$. W.l.o.g. $\operatorname{tp}(\bar{a}^i_n, n_0)$ does not fork over A. As N_0 is $F^a_{\aleph_0}$-saturated w.l.o.g. $\operatorname{tp}_*(\bar{a}^i_n, A)$ is stationary. Also w.l.o.g. $\operatorname{tp}(A, N)$ does not fork over $A \cap N$. Hence $\operatorname{tp}_*(N, A)$ does not fork over $A \cap N$.

Now $\operatorname{tp}(\bar{a}^i_n, N_0)$ is orthogonal to N, hence to $A \cap N$ hence to $\operatorname{tp}_*(N, A \cap N)$.

But $\mathrm{stp}(\bar a^i_n, A)$, $\mathrm{stp}_*(N, A \cap N)$ are parallel to $\mathrm{tp}(\bar a^i_n, N_0)$ and $\mathrm{stp}_*(N, A)$ resp., hence the latter are orthogonal so $\mathrm{stp}(\bar a^i_n, A) \vdash \mathrm{stp}(\bar a^i_n, N \cup A)$.

For some finite $\boldsymbol{J} \subseteq \boldsymbol{I}$, $\boldsymbol{I} - \boldsymbol{J}$ is independent over $(N, N \cup A \cup \boldsymbol{J})$; so w.l.o.g. $(\forall \bar d \in \boldsymbol{J})(\bar d \subseteq A)$ so similarly $\mathrm{stp}(\bar a^i_n, N \cup A) \vdash \mathrm{stp}(\bar a^i_n, N \cup A \cup (\boldsymbol{I} - \boldsymbol{J}))$. So $\mathrm{stp}(\bar a^i_n, A) \vdash \mathrm{stp}(\bar a^i_n, N \cup \boldsymbol{I})$, but the former is realized in N_0, as A is finite. So we have proved $(*)_1$, hence Fact 1.3A. □

Continuation of the proof of 1.3. We want to show that N_1 is $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime over $M \cup \bigcup \boldsymbol{I}$. By IV 4.18 (and see Definition IV 4, p. 192) it suffices to show that:

1.3B. Fact. *For every regular stationary $p \in S^m(N \cup \bigcup \boldsymbol{I} \cup \bar b)$ (for some $\bar b \in N_1$), $\dim(p, N_1) \leq \aleph_0$.*

Proof. If p is orthogonal to N_0, this follows by Claim 1.2. Suppose p is not orthogonal to N_0.

Let $\bar c$ realize p. W.l.o.g. p does not fork over $\bar b$, $p \restriction \bar b$ stationary and $\mathrm{tp}(\bar b, N_0)$ does not fork over some finite $A \subseteq N_0$, and p is not orthogonal to A. Choose $\bar b' {}^\wedge \bar c' \in N_0$ realizing $\mathrm{stp}(\bar b {}^\wedge \bar c, A)$. By V 3.4, $\mathrm{tp}(\bar c', \bar b')$ is not orthogonal to p, and clearly it is regular and stationary and let $q \in S^m(N \cup \boldsymbol{I} \cup \bar b')$ be the stationarization of $\mathrm{tp}(\bar b', \bar c')$. Let $p' \in S^m(N \cup \boldsymbol{I} \cup \bar b \cup \bar b')$, $q' \in S^m((N \cup \boldsymbol{I} \cup \bar b \cup \bar b')$ be stationarizations of p, q resp. By III 3.5,

$$\dim(p, N_1) + \aleph_0 = \dim(p', N_1) + \aleph_0 \qquad \text{and} \quad \dim(q, N_1) + \aleph_0 = \dim(q', N_1) + \aleph_0.$$

By V 1.14 and V 2.7, $\dim(p', N_1) + \aleph_0 = \dim(q', N_1) + \aleph_0$. So it suffices to prove $\dim(q', N_1) \leq \aleph_0$, i.e., w.l.o.g. $\bar b \in N_0$. By IV 4.9, $\dim(p, N_0) \leq \aleph_0$. Let $p' \in S^m(N_0)$ be the stationarization of p over N_0. By V 1.16(3),

$$\dim(p, N_1) = \dim(p, N_0) + \dim(p', N_1).$$

Let U be the set of $i < \alpha$ such that p_i is orthogonal to p. By V 1.13(1), $|\alpha - U| \leq 1$. Now easily $\mathrm{tp}_*(\bigcup \{\bar a^i_n : i \in U\}, N_0)$ is orthogonal to p. We also know that there is N' $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime over $N_0 \cup \bigcup \{\bar a^i_n : i \in U, n < \omega\}$, and if $j \in \alpha - U$, $\mathrm{tp}(\bigcup \{\bar a^j_n : n < \omega\}, N_0) \vdash \mathrm{tp}(\bigcup \{\bar a^j_n : n < \omega\}, N')$, so w.l.o.g. N_1 is $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime over $N' \cup \bigcup \{\bar a^i_n : i \in \alpha - U, n < \omega\}$. Now by V 1.16(3),

$$\dim(p, N_1) = \dim(p, N_0) + \dim(p', N') + \dim(p'', N_1)$$

where p'' is the stationarization of p over N'. By IV 4.9 and III 3.5, $\dim(p'', N_1) \leq \aleph_0$. Lastly note that $\dim(p', N') = 0$ because $\mathrm{tp}_*\{\bar a^i_n : i \in v, n < \omega\}, N_0)$ is orthogonal to p (by V 1.4) using V 3.2. Together $\dim(p'', N_1) \leq \aleph_0$.

So we have proved Fact 1.3B. □

Continuation of the proof of 3.1. So N_1 is really $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime over $N \cup \bigcup \boldsymbol{I}$. By the definition of $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime w.l.o.g. $N_1 \subseteq M^*$, replacing, of course, our N_0 by another choice.

Now we shall prove that N_0 is as required. Let $\bar{a}, \bar{b}, p, q$ be as assumed in $(*)$. By Claim 1.2 (as p is orthogonal to N), $\dim(q\restriction(N \cup \bar{a}), N_0)$ is $\leqslant\aleph_0$. As by V 1.16(3),

$$\dim(q\restriction(N \cup \bar{a}), M^*) = \dim(q\restriction(N \cup \bar{a}), N_0) + \dim(q\restriction N_0, M^*),$$

we are almost finished.

The only non-immediate case is $\dim(q\restriction N_0, M^*) \leqslant \aleph_0$. But N_1 witness $\dim(q\restriction N_0, N_1) \geqslant \aleph_0$ hence

$$\aleph_0 \geqslant \dim(q\restriction(N \cup \bar{a}), M^*) \geqslant \dim(q\restriction N_0, M^*) \geqslant \dim(q\restriction N_0, N_1) \geqslant \aleph_0,$$

thus finishing. □

1.4. Claim. *Suppose M^* is $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-saturated. There is $N_0 \subseteq M^*$ $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-prime over $\emptyset$ such that*:

$(*)$ *if $\bar{a}, \bar{b} \in N_0$, $p = \mathrm{tp}(\bar{b}, \bar{a})$ is stationary regular, q the stationarization of p over M^*, then* $\dim(q\restriction\bar{a}, M^*) = \dim(q\restriction N_0, M^*)$.

Proof. Similar to that of 1.3. □

1.5. Theorem. *Suppose M^*_l $(l = 1, 2)$ are $\boldsymbol{F}^{\mathrm{a}}_{\aleph_0}$-saturated, and for every $\bar{a}_l, \bar{b}_l \in M^*_l$, $p_l = \mathrm{tp}(\bar{a}_l, \bar{a}_l)$ regular and stationary (for $l = 1, 2$)*:

$$\mathrm{tp}(\bar{a}^\wedge\bar{b}_1, \emptyset) = \mathrm{tp}(\bar{a}_2{}^\wedge\bar{b}_2, \emptyset) \quad \textit{implies} \quad \dim(p_1, M^*_1) = \dim(p_2, M^*_2).$$

Then M^*_1, M^*_2 are isomorphic.

Remark. Another variant is when we demand $\mathrm{stp}(\bar{a}^\wedge\bar{b}_1, \emptyset) = \mathrm{stp}(\bar{a}_2{}^\wedge\bar{b}_2, \emptyset)$. We can deduce it from 1.5 by expanding $\mathfrak{C}^{\mathrm{eq}}$ by the individual constants c for all $c \in \mathrm{acl}\,\emptyset$.

Proof. We define by induction on $k < \omega$

$$N^l_k, \quad \{p^{l,k}_\alpha : \alpha < \alpha_k\}, \quad \langle \boldsymbol{J}^{l,k}_{\alpha,m} : \alpha < \alpha_k,\ k \leqslant m < \omega\rangle \quad (\text{for } l = 1, 2) \quad \text{and} \quad F_k$$

such that:

(1) $N^l_k \subseteq M^*_l$, N^l_k is $F^{\mathrm{a}}_{\aleph_0}$-saturated.
(2) F_k is an isomorphism from N^1_k onto N^2_k.
(3) $N^l_k \subseteq N^l_{k+1}$, $F_k \subseteq F_{k+1}$.
(4) If $p \in S^m(N^l_k)$ is regular, does not fork over $\bar{a}$, $p\restriction\bar{a}$ stationary, $\bar{a} \in N^l_k$, then $\dim(p\restriction\bar{a}, M^*_l) = \dim(p, M^*_l)$.
(5) $\{p^{l,k}_\alpha : \alpha < \alpha_k\}$ is a maximal family of regular, complete over N^l_k, orthogonal to N^l_{k-1} (when $k > 0$) pairwise orthogonal types.

(6) F_k maps $p_\alpha^{1,k}$ to $p_\alpha^{2,k}$.

(7) $\bigcup_{m\geq k} J_{\alpha,m}^{l,k} \subseteq M_l^*$ is a maximal family of sequences realizing $p_\alpha^{l,k}$, independent over N_k^l.

(8) $\langle J_{\alpha,m}^{l,k} : k \leq m < \omega \rangle$ are pairwise disjoint. $|J_{\alpha,m}^{2,k}| = |J_{\alpha,m}^{1,k}| = |J_{\alpha,k}^{l,k}| \geq \aleph_0$.

(9) $J_{\alpha,m}^{l,k} \subseteq N_{m+1}^l$, F_{m+1} maps $J_{\alpha,m}^{1,k}$ onto $J_{\alpha,m}^{2,k}$.

(10) $\bigcup_{m\geq m(0)} J_{\alpha,m}^{l,k}$ is independent over $(N_k^l, N_{m(0)}^l)$ (when $k \leq m(0)$).

First Case: $k = 0$.

For $l = 1, 2$, let $N_0^l \subseteq M_l^*$ be $F_{\aleph_0}^a$-prime over $\emptyset$, such that for $\bar{a}, \bar{b} \in N_0^l$ if $p = \text{tp}(\bar{b}, \bar{a})$ is stationary and regular, q the stationarization of p over N_0^l, then $\dim(p, M_l^*) = \dim(q, M_l^*)$; this is possible by 1.4. Easily (4) holds. As N_0^1, N_0^2 are $F_{\aleph_0}^a$-prime over $\emptyset$, by IV 4.18 there is an isomorphism F_0 from N_0^0 onto N_0^1.

Let $\{p_\alpha^{1,0} : \alpha < \alpha_0\}$ be a maximal family of types in $\bigcup_m S^m(N_0^1)$, regular and orthogonal in pairs and $p_\alpha^{2,0} = F_0(p_\alpha^{1,0})$. Condition (6) holds and easily also condition (5). Let $J_\alpha^{l,0} \subseteq M_l^*$ be a maximal family of sequences, independent over N_0^l, realizing $p_\alpha^{l,k}$. As (4) holds, $J_\alpha^{l,0}$ is infinite, so we can partition $J_\alpha^{l,0}$ to $J_{\alpha,m}^{l,0} (m < \omega)$ such that $|J_{\alpha,m}^{l,0}| = |J_\alpha^{l,0}|$. Now $|J_\alpha^{1,0}| = |J_\alpha^{2,0}|$ by the hypothesis of 1.5 and the choice of N_k^0.

We can check the other conditions.

Second Case: $k+1$.

So we have defined already for $l = 1, 2$ and $m \leq k$: N_k^l, $N_\alpha^{l,m}$ for $\alpha < \alpha_m$ and $J_{\alpha,n}^1$. Let N_{k+1}^l be $F_{\aleph_0}^a$-prime over $N_k^l \cup \bigcup \{J_{\alpha,k}^{1,k(*)} : k(*) \leq k,\ \alpha \leq \alpha_{k(*)}\}$. Then $N_{k+1}^l \subseteq M_l^*$ and w.l.o.g. condition (4) holds; this is possible by 1.3.

By V 3.2 condition (10) holds. By conditions (10) and (6) for $\bar{c}^1 \in J_{\alpha,k}^{1,k(*)}$, $\bar{c}^2 \in J_{\alpha,k}^{2,k(*)}$, $k(*) \leq k$

$$F_k(\text{tp}(\bar{c}^1, N_k^1)) = \text{tp}(\bar{c}^2, N_k^2),$$

hence by (10), for any $n < \omega$, distinct $\bar{a}_0, \ldots, \bar{a}_{n-1} \in J_{\alpha,k}^{1,k(*)}$ and distinct $\bar{b}_0, \ldots, \bar{b}_{n-1} \in J_{\alpha,k}^{2,k(*)}$

$$F_k(\text{tp}_*(\bar{a}_0{}^\wedge \cdots {}^\wedge \bar{a}_{n-1}, N_k^l)) = \text{tp}_*(\bar{b}_0{}^\wedge \cdots {}^\wedge \bar{b}_{n-1}, N_k^2).$$

As by (8) $J_{\alpha,k}^{1,k(*)}$, $J_{\alpha,k}^{2,k(*)}$ have the same cardinality, we can extend F_k to an elementary mapping $F_{k,k(*),\alpha}^*$ from $N_k^1 \cup \bigcup J_{\alpha,k}^{1,k(*)}$ onto $N_k^2 \cup \bigcup J_{\alpha,k}^{2,k(*)}$.

By condition (5), the types $p_\alpha^{l,k(*)}$ $(k(*) \leq k,\ \alpha < \alpha_{k(*)})$ are pairwise orthogonal, hence by V (1.4(1) (and (10)) also the types

$$\text{tp}_*(\bigcup \{J_{\beta,k}^{l,k(1)} : k(1) \leq k,\ \beta < \alpha_{k(1)},\ (k(1), \beta) \neq (k(*), \alpha)\}, N_k^l)$$
$$\text{and}\quad \text{tp}_*(J_{\alpha,k}^{l,k(*)}, N_k^l)$$

are orthogonal $(k(*) \leq k,\ \alpha < \alpha_{k(1)})$, hence by V 1.2 weakly orthogonal.

Hence $F_k^* = \bigcup \{F_{k,k(*),\alpha} : k(*) \leq k,\ \alpha < \alpha_{k(*)}\}$ is an elementary mapping.

As N_{k+1}^l is $F_{\aleph_0}^a$-prime over $N_k^l \cup \bigcup \{J_{\alpha,k}^{l,k(*)} : k(*) \leq k,\ \alpha < \alpha_{k(*)}\}$, by IV 4.18 we can extend F_k^* to an isomorphism from N_{k+1}^1 onto N_{k+1}^2 and call it F_{k+1}.

Let $\{p_\alpha^{1,k+1}: \alpha < \alpha_{k+1}\}$ be a maximal family of pairwise orthogonal, complete over N_{k+1}^1, orthogonal to N_k^l types. Let $p_\alpha^{2,k+1} = F_{k+1}(p_\alpha^{1,k+1})$.

Now there is no problem to find $J_{\alpha,m}^{l,k+1}$ $(k < m < \omega)$ to satisfy condition (7), for each $\alpha < \alpha_{k+1}$. There is no problem to check the conditions (1)–(10). So we have carried out the induction.

To finish the proof it suffices to prove that (for $l = 1, 2$) $M_l^* = \bigcup_{k<\omega} N_k^l$ (as then $\bigcup_{k<\omega} F_k$ is an isomorphism from M_1^* onto M_2^*). Suppose $M_l^* \neq \bigcup_{k<\omega} N_k^l$. As both models are $\boldsymbol{F}_{\aleph_0}^{\mathrm{a}}$-saturated and $\bigcup_{k<\omega} N_k^l \subseteq M_l^*$, by V 3.5 for some $c \in M_l^*$, $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$ is regular. As T is superstable for some $m < \omega$, $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$ does not fork over N_m^l, hence $\operatorname{tp}(c, N_m^l)$ is parallel to $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$ and so is regular. Let $n \leq m$ be minimal such that $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$ is not orthogonal to N_n^l. Using V 3.4 we can find a regular type $r \in S(N_n^l)$ not orthogonal to $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$, so by V 1.13(2), r is orthogonal to N_{n-1}^l (if $n > 0$) hence for some $\alpha < \alpha_n$, r is not orthogonal to $p_\alpha^{l,n}$. By V 1.13, $\operatorname{tp}(c, \bigcup_{k<\omega} N_k^l)$ is not orthogonal to $p_\alpha^{l,n}$, so by V 1.12 some $\bar{d} \in M_l^*$ realizes the stationarization of $p_\alpha^{l,n}$ over $\bigcup_{k<\omega} N_k^l$. But $\operatorname{tp}(\bar{d}, N_n^l \cup \bigcup_{m \geq n} J_{\alpha,m}^{l,n}) \subseteq \operatorname{tp}(\bar{d}, \bigcup_{k<\omega} N_k^l)$ does not fork over N_n^l, contradicting condition (7) in the induction hypothesis. $\square$

2. The non-structure theorem

2.1. Theorem. *Suppose T is not superstable.*

$$K = \{M : M \text{ a strongly } \boldsymbol{F}_{\aleph_0}^{\mathrm{a}}\text{-saturated model of } T\}.$$

Then for every $\lambda > \lambda(T)$ there are 2^λ pairwise non-isomorphic models from K of power λ.

Remark. (1) $\lambda(T)$ is the first cardinality such that for M a model of T and finite $A \subseteq M$, the number of non-equivalent $\operatorname{stp}(\bar{a}, A)$ (in M), $\bar{a} \in M$, is $\leq \lambda(T)$. It is known that $\lambda(T) \leq 2^{|T|}$ and w.l.o.g. $|T| \leq \lambda(T)$ (as $|D(T)| \leq \lambda(T)$, and if $|D(T)| < |T|$, then T is a definitional extension of some $T' \subseteq T$, $|T'| \leq |D(T)|$).

(2) Note that the proof shows that if $T \subseteq T_1$ (T_1 a first-order theory), then for $\lambda > |T_1| + \lambda(T)$ there are 2^λ non-isomorphic models from K of power λ which are reducts of models of T_1.

Notation. We identify ${}^{\omega\geq}\lambda$ with the model $({}^{\omega\geq}\lambda, \lessdot, <_{\mathrm{lx}}, \ldots, P_\alpha, \ldots)_{\alpha\leq\omega}$ where $<_{\mathrm{lx}}$ is the lexicographic order, $\lessdot$ is being initial segment and $P_\alpha = {}^\alpha\lambda$. The slight variation from VII Definition 2.1 is inessential.

Proof. We know (II 3.14, 9) that there are formulas $\varphi_n(\bar{x}, \bar{y}_n)$ (of $L(T)$) and $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in {}^{\omega\geq}\omega$ ($\mathfrak{C}$ a quite saturated model of T, see [1, p. 7]) such that if $\eta \in {}^\omega\omega$, $\nu \in {}^n\omega$, then

$$(*) \qquad \models \varphi_n[\bar{a}_\eta, \bar{a}_\nu] \quad \text{iff} \quad \nu \lessdot \eta.$$

We now define by induction on n, $\bar{a}^n_\nu$ $(\nu \in {}^{\omega\geq}\omega)$ and L_n such that:

(1) M_n is an L_n-model, $|L_n| \leq \lambda(T)$.

(2) $\bar{a}^n_\nu \in M_n$ and $\langle a^n_\nu : \nu \in {}^{\omega\geq}\omega\rangle$ is indiscernible in M_n (for indiscernability with respect to $({}^{\omega\geq}\omega, <, <_{\text{lx}}, \ldots, P_\alpha, \ldots)$ see VII Definitions 2.2, 2.3, 2.4).

(3) $L_n \subseteq L_{n+1}$, and $(M_{n+1} \restriction L_0)$ is an elementary extension of $M_n \restriction L_0$.

(4) For any $\nu_1, \ldots, \nu_k \in {}^{\omega\geq}\omega$, the quantifier free L_n-type of $\bar{a}^{n+1}_{\nu_1}{}^\wedge \cdots {}^\wedge \bar{a}^{n+1}_{\nu_k}$ in M_{n+1} is equal to the quantifier free L_n-type of $\bar{a}^n_{\nu_1}{}^\wedge \cdots {}^\wedge \bar{a}^n_{\nu_k}$ in M_n: moreover $M_n < (M_{n+1} \restriction L_n)$.

(5) $M_0 = \mathfrak{C}$, $\bar{a}^0_\nu = \bar{a}_\nu$.

(6) For $n > 1$, $\text{Th}(M_n)$ has Skolem functions.

(7) For $n > 1$, and $\nu_1, \ldots, \nu_k \in {}^{\omega\geq}\omega$, every (L_0, m)-type in $M^n_{\nu_1,\ldots,\nu_k}$ (= the Skolem Hull of $\{\bar{a}^n_{\nu_1}, \ldots, \bar{a}^n_{\nu_k}\}$ in M_n) which is almost over a finite set, is realized in $M^{n+1}_{\nu_1,\ldots,\nu_k}$.

(8) For any $m < \omega$, F_m is a $(2m+1)$-place function symbol from L_1, such that if $\bar{a}, \bar{b} \in M^n_{\nu_1,\ldots,\nu_k}$, $n \geq 2$, $\bar{a}, \bar{b}$ realize the same L_0-type in M_n, then $x \to F(x, \bar{a}, \bar{b})$ is an automorphism of $M_n \restriction L_0$ taking $\bar{a}$ to $\bar{b}$.

For the case $n = 0$ there is nothing to do.

Case $n+1$: Choose λ large enough (e.g., $\beth((2^{2^{|T|}})^+)$). We can find an L_n-model N_n, of $\text{Th}(M_n)$, and $b_\eta \in N_n$ $(\eta \in {}^{\omega\geq}\lambda)$ such that if $\eta_1, \ldots, \eta_k \in {}^{\omega\geq}\lambda$, $\nu_1, \ldots, \nu_k \in {}^{\omega\geq}\omega$, and $\langle \eta_1, \ldots, \eta_k\rangle$, $\langle \nu_1, \ldots, \nu_k\rangle$ realize the same atomic type in ${}^{\omega\geq}\lambda$, ${}^{\omega\geq}\omega$ resp., then $\bar{b}_{\eta_1}{}^\wedge \cdots {}^\wedge \bar{b}_{\eta_k}$, $\bar{a}_{\nu_1}{}^\wedge \cdots {}^\wedge \bar{a}_{\nu_k}$ realize the same L_n-type in N_n, M_n resp.

For $I \subseteq {}^{\omega\geq}\lambda$ let N^n_I be the Skolem Hull of $\{a_\eta : \eta \in I\}$ in N_n.

Let N^1_n be an elementary extension of N_n, such that $(N^1_n \restriction L_0)$ is strongly $\|N_n\|^+$-saturated. For any $\bar{a}, \bar{b} \in N^n_{({}^{\omega\geq}\lambda)}$ (of length m) realizing the same L_0-type, the function $x \to F_m(x, \bar{a}, \bar{b})$ is an automorphism of $N^n_{({}^{\omega\geq}\lambda)} \restriction L_0$. We can extend this automorphism of $N^n_{({}^{\omega\geq}\lambda)} \restriction L_0$ to an automorphism of $N^1_n \restriction L_0$. Also if $\bar{a}, \bar{b} \in N^1_n$ realize the same L_0-type $\bar{a}{}^\wedge\bar{b} \nsubseteq N^n_{({}^{\omega\geq}\lambda)}$, we find an automorphism of $N^n_{({}^{\omega\geq}\lambda)} \restriction L_0$ taking $\bar{a}$ to $\bar{b}$. So we can define a model N^2_n, which is like N^1_n, but we change the interpretation of F_m so that:

(a) for $x, \bar{a}, \bar{b} \in N^n_{({}^{\omega\geq}\lambda)}$: $F^{N^2_n}_m(x, \bar{a}, \bar{b}) = F^{N^1_n}_m(x, \bar{a}, \bar{b})$;

(b) if $\bar{a}, \bar{b} \in N^2_n$ realize the same L_0-type, then $x \to F(x, \bar{a}, \bar{b})$ is an automorphism of $N^2_n \restriction L_0$.

Next for every complete atomic $L({}^{\omega\geq}\omega)$-type r realized in ${}^{\omega\geq}\omega$, we choose a tuple say $\bar{\nu}^r = \langle \nu^r_1, \ldots, \nu^r_{k(r)}\rangle$ realizing it in ${}^{\omega\geq}\omega$, and we choose $\lambda(T)$ complete (L_0, m)-types over M^n, $\{p^{n,r}_i : i < \lambda(T)\}$, such that any L_0-type in $N^n_{\{\nu^r_1,\ldots,\bar{\nu}^r_k\}}$ (i.e. its parameters are from it and it is finitely satisfiable there) almost over a finite set is included in some $p^{n,r}_i$. Now we define in N^2_n k-placed functions $G^{n,k}_i$ (for each m) such that:

(**) if $\nu_1, \ldots, \nu_k \in {}^{\omega\geq}\lambda$, $\langle \nu_1, \ldots, \nu_k\rangle$ realizes in ${}^{\omega\geq}\lambda$ the complete atomic type

r, then for every L_n-term $\tau_1, \ldots, \tau_m$ and L_0-formulas ψ:

$$\psi(x, \tau_1(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}), \ldots, \tau_m(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k})) \in p_i^{n,r}$$
$$\text{iff} \quad N_n^2 \vDash \psi[G_i^{n,k}(\bar a_{\nu_k}, \ldots, \bar a_{\nu_k}), \tau_1(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}), \ldots, \tau_m(\bar a_{\nu_1}, \ldots, \bar a_{\nu_k})].$$

Let $N_n^3 = (N_n^2, \ldots, G_i^{n,k}, \ldots)_{n,k<\omega,\, i<\lambda(T)}$. Lastly let N_n^4 be an expansion of N_n^3 by Skolem functions but still $|L(N_n^4)| \leqslant \lambda(T)$. Let $L_{n+1} = L(N_n^4)$. Now by [1, AP 2.6] (just as in the proof of VII 3.6(1)) there is an L_{n+1}-model N_n^5 of $\mathrm{Th}(N_n^4)$ and $\bar c_\eta \in N_n^5$ ($\eta \in {}^{\omega\geqslant}\omega$) such that:

(i) $\langle \bar c_\eta : \eta \in {}^{\omega\geqslant}\omega\rangle$ is indiscernible in N_n^5.

(ii) If $\eta_1, \ldots, \eta_k \in {}^{\omega\geqslant}\omega$, $\varphi \in L_{n+1}$, $N_n^5 \vDash \varphi[\bar a_{\eta_1}, \ldots, \bar a_{\eta_k}]$, then there are $\nu_1, \ldots, \nu_k \in {}^{\omega\geqslant}\lambda$ such that $\langle \eta_1, \ldots, \eta_k\rangle$, $\langle \nu_1, \ldots, \nu_k\rangle$ realize the same atomic type in ${}^{\omega\geqslant}\omega$, ${}^{\omega\geqslant}\lambda$ resp. and $N_n^4 \vDash \varphi[\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}]$.

(iii) N_n^5 is the Skolem Hull of $\{\bar c_\eta : \eta \in {}^{\omega\geqslant}\omega\}$.

Now by renaming we can assume $\bar c_\eta = \bar a_\eta$, $M_n \subseteq N_n^5 \restriction L_n$. So we can let $M_{n+1} \stackrel{\text{def}}{=} N_n^5$. Let M^* be the limit of the M_n (i.e., $L(M^*) = \bigcup_n L_n$, $M^* \restriction L_n = \bigcup\{(M_m \restriction L_n) : n \leqslant m < \omega\}$). So M^*, $\bar a_\eta$ ($\eta \in {}^{\omega\geqslant}\omega$) define Φ proper for ${}^{\omega\geqslant}\omega$ (see VII Definition 2.6) such that for any atomic $\varphi \in \bigcup_n L_n$:

$$\mathrm{EM}^1({}^{\omega\geqslant}\omega, \Phi) \vDash \varphi[\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}] \quad \text{iff} \quad M^* \vDash \varphi[\bar a_{\nu_1}, \ldots, \bar a_{\nu_k}].$$

As $M_n \restriction L_0 (M_{n+1} \restriction L_0)$, and M_n has a Skolem function clearly for every $I \in K({}^{\omega\geqslant}\omega)$, (see VII 3.1 and the Notation above), $\mathrm{EM}(I, \Phi)$ ($= \mathrm{EM}^1(I, \Phi) \restriction L_0$) is a model of T and for $\eta \in P_\omega^I$, $\nu \in P_n^I$:

$$\mathrm{EM}(I, \Phi) \vDash \varphi_n[\bar a_\eta, \bar a_\nu] \quad \text{iff} \quad \nu < \eta.$$

By VIII §2 we get the desired conclusion. □

Remark. Though the theorems there (mainly VIII 2.1 for our purpose) speak about $\mathrm{PC}(T_1, T)$, T unsuperstable, they give that for Φ, L, φ_n as above, $\mathrm{card}\{\mathrm{EM}(I, \Phi)/\cong : |I| = \lambda\}$ is 2^λ, if

$$\lambda > |L(\mathrm{EM}^1({}^{\omega\geqslant}\omega, \Phi))| \quad \text{or e.g.} \quad \lambda = \lambda^{\aleph_0} \geqslant |L(\mathrm{EM}({}^{\omega\geqslant}\omega, \Phi))|.$$

Remark. (a) Really "$\lambda \geqslant \lambda(T) + \aleph_1$" is enough in 2.1.

(b) If we assume a Ramsey cardinal exists above $|T|$, then by (4) we can simplify the proof using one stage instead of ω.

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