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Was Sierpiński right? III
Can continuum-c.c. times c.c.c. be continuum-c.c.?

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Abstract

We prove the consistency of: if B_1, B_2 are Boolean algebras satisfying the c.c.c. and the $2^{\aleph_0}$-c.c. respectively then $B_1 \times B_2$ satisfies the $2^{\aleph_0}$-c.c. We start with a universe with a Ramsey cardinal (less suffice).

0. Introduction

We heard the problem from Velickovic who got it from Todorcevic, it says "are there P, a c.c.c. forcing notion, and Q, a $2^{\aleph_0}$-c.c. forcing, such that $P \times Q$ is not $2^{\aleph_0}$-c.c.?" We can phrase it as a problem of cellularity of Boolean algebras or topological spaces.

We give a negative answer even for $2^{\aleph_0}$ regular, this by proving the consistency of the negation. The proof is close to [2, §3] which continues [1, §2] and is close to [3]. A recent use is [4].

We start with $V \models$ "λ is a Ramsey cardinal", then use c.c.c. forcing blowing the continuum to λ. Originally the paper contained the consistency of e.g. $2^{\aleph_0} \to [\aleph_2]^2_3$, $2^{\aleph_0}$ the first k^2_2-Mahlo (weakly inaccessible; remember $k^2_2 < \omega$), but the theorem presented here is (for me) satisfactory. See more in [5]. I thank Mariusz Rabus for corrections.

What problems do [1–4] and this paper raise? The most important are (we state the simplest uncovered case for each point):

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A. Question. (1) Can we get e.g. $Con(2^{\aleph_0} \to [\aleph_2]^2_3)$? More generally, raise μ^+ to higher cardinals. (See [5].)

(2) Can we get $Con(\aleph_\omega > 2^{\aleph_0} \to [\aleph_1]^2_3)$? Generally lower 2^μ; the exact $\aleph_n$ seems to me less exciting.

(3) Can we get e.g. $Con(2^\mu > \lambda \to [\mu^+]^2_3)$?

Also concerning [4]:

B. Question. (1) Can we get the continuity on a nonmeager set for functions $f: {}^\kappa 2 \to {}^\kappa 2$?

(2) What can we say about the continuity of 2-place functions? (See [7].)

(3) What about n-place functions (after [2])?

C. Question. (1) Can we get e.g. for $\mu = \mu^{<\mu} > \aleph_0$, Con(if P is 2^μ-c.c., Q is μ^+-c.c. then $P \times Q$ is 2^μ-c.c.)?

(2) Can we get e.g. Con(if P is $2^{\aleph_0}$-c.c., Q is $\aleph_2$-c.c. then $P \times Q$ is $2^{\aleph_0}$-c.c.)?

(3) Can we get e.g. $Con(2^{\aleph_0} > \lambda > \aleph_0$, and if P is λ-c.c., Q is $\aleph_2$-c.c. then $P \times Q$ is λ-c.c.)?

Preliminaries

0.A. Let $<^*_\chi$ be a well ordering of $H(\chi) = \{x$: the transitive closure of x has cardinality $< \chi\}$ agreeing with the usual well ordering of the ordinals.

P (and Q, R) will denote forcing notion(s), i.e., partial order with a minimal element $\emptyset = \emptyset_P$. A forcing notion P is λ-closed if every increasing sequence of members of P, of length less than λ, has an upper bound.

0.B. For sets of ordinals, A and B, define $H^{\mathrm{OP}}_{B,A}$ as the maximal order preserving bijection between initial segments of A and B, i.e., it is the function with domain $\{\alpha \in A\colon \mathrm{otp}(\alpha \cap A) < \mathrm{otp}(B)\}$ and $H^{\mathrm{OP}}_{A,B}(\alpha) = \beta$ if and only if $\alpha \in A$, $\beta \in B$ and $\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B)$.

Definition 0.1. $\lambda \to^+ (\alpha)^{<\omega}_\mu$ holds provided that: whenever F is a function from $[\lambda]^{<\omega}$ to λ, $F(w) < \min(w)$, $C \subseteq \lambda$ is a club then there is $A \subseteq C$ of order type α such that $[w_1, w_2 \in [A]^{<\omega}, |w_1| = |w_2| \Rightarrow F(w_1) = F(w_2)]$. (See [6, XVII, 4.x].)

0.1A. Remark. (1) If λ is a Ramsey cardinal then $\lambda \to^+ (\lambda)^{<\omega}_\mu$.

(2) If $\lambda = \min\{\lambda\colon \lambda \to (\alpha)^{<\omega}_\mu\}$ then λ is regular and $\lambda \to^+ (\alpha)^{<\omega}_\mu$.

Definition 0.2. $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{F(w)\colon w \in [A]^n\}$ has power $\leqslant \theta$.

Definition 0.3. A forcing notion P satisfies the Knaster condition (has property K) if for any $\{p_i: i < \omega_1\} \subseteq P$ there is an uncountable $A \subseteq \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

1. Consistency of "c.c.c. $\times\ 2^{\aleph_0}$-c.c. $= 2^{\aleph_0}$-c.c."

The a_i's are not really necessary but (hopefully) clarify.

1.1. Definition. (1) $\mathscr{K}_{\mu,\kappa}$ is the family of $\bar{Q} = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta: \gamma \leqslant \alpha, \beta < \alpha\rangle$, where

(a) $\langle P_\gamma, \underset{\sim}{Q}_\beta: \gamma \leqslant \alpha, \beta < \alpha\rangle$ is a finite support iteration;

(b) every P_γ, $\underset{\sim}{Q}_\gamma$ satisfies the c.c.c.;

(c) $\underset{\sim}{Q}_\beta$ is a P_β-name which depends just on $G_{P_\beta} \cap P^*_{a_\beta}$ (see below; hence it is in $V[G_{P^*_{a_\beta}}]$), and $|\underset{\sim}{Q}_\beta| \leqslant \kappa$ and its set of members $\subseteq V$ (for simplicity);

(d) $a_\beta \subseteq \beta$, $|a_\beta| \leqslant \mu$ and $\gamma \in a_\beta \Rightarrow a_\gamma \subseteq a_\beta$.

(2) For such $\bar{Q}$ we call $a \subseteq \ell g(\bar{Q})$, $\bar{Q}$-closed if $[\beta \in a \Rightarrow a_\beta \subseteq a]$ and let

$$P^*_a = P^{\bar{Q}}_a =: \{p \in P_a: \operatorname{dom}(p) \subseteq a \text{ and for all } \beta \in \operatorname{dom}(p): p(\beta) \in V$$

$$\text{(not a name) and } p \restriction a_\beta \Vdash \text{“}p(\beta) \in \underset{\sim}{Q}_\beta\text{”}\}$$

(so we are defining P^*_a by induction on $\sup(a)$) ordered by the order of $P_{\sup(a)}$.

(3) $\mathscr{K}^k_{\mu,\kappa}$ is the class of $\bar{Q} \in \mathscr{K}_{\mu,\kappa}$ such that if $\beta < \gamma \leqslant \ell g(\bar{Q})$, $cf(\beta) \neq \aleph_1$ then P_γ/P_β satisfies the Knaster condition (actually we can use somewhat less). Let $\mathscr{K}^n_{\mu,\kappa} = \mathscr{K}_{\mu,\kappa}$.

(4) If defining $\bar{Q}$, we omit P_α to mean $\bigcup_{\beta<\alpha} P_\beta$ if α is limit, $P_\beta * \underset{\sim}{Q}_\beta$ if $\alpha = \beta + 1$.

(5) We do not lose generality, if we assume $\underset{\sim}{Q}_\beta \subseteq [\kappa]^{<\aleph_0}$ and the order is $\subseteq$ (then 1.2(1)(g) becomes trivial as for closed a and $p, q \in P^*_j$, we have $p \leqslant q \Rightarrow p \restriction a \leqslant q \restriction a$).

1.2. Claim. (1) *Assume $x \in \{n, k\}$ and $\bar{Q} = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta: \beta < \alpha, \gamma \leqslant \alpha\rangle \in \mathscr{K}^x_{\mu,\kappa}$. Then*

(a) *for $\alpha^* < \alpha$, $\bar{Q} \restriction \alpha^* =: \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta: \beta < \alpha^*, \gamma \leqslant \alpha^*\rangle$ belongs to $\mathscr{K}^x_{\mu,\kappa}$;*

(b) *P^*_α is a dense subset of P_α;*

(c) *for any $\bar{Q}$-closed $a \subseteq \alpha$, $P^*_a \lessdot P_\alpha$ (in particular, P^*_α is a dense subset of P_α); moreover, if $p \in P^*_\alpha$ then $p \restriction a \in P^*_a$ and $[p \restriction a \leqslant q \in P^*_a \Rightarrow r =: q \cup p \restriction (\alpha \backslash a) \in P^*_\alpha$ & $p \leqslant r$ & $q \leqslant r]$;*

(d) *for a $\bar{Q}$-closed $a \subseteq \alpha$, $\langle P^*_{a\cap\gamma}, \underset{\sim}{Q}_\beta, a_\beta: \beta \in a, \gamma \in a\rangle$ belongs to $\mathscr{K}^x_{\mu,\lambda}$ (except renaming; not used);*

(e) *if $\underset{\sim}{Q}_\alpha$ is a P^*_a-name of a c.c.c. forcing notion of cardinality $\leqslant \kappa$, each member of $\underset{\sim}{Q}_\alpha$ is from V, $a \subseteq \alpha$ is $\bar{Q}$-closed, $|a| \leqslant \mu$ and $P_{\alpha+1} = P_\alpha * \underset{\sim}{Q}_\alpha$ and when $x = k$, $\underset{\sim}{Q}_\alpha$ satisfies the Knaster condition or at least $cf(\alpha) = \aleph_1$ & $(\beta < \alpha \Rightarrow P_\alpha * \underset{\sim}{Q}_\alpha/P_{\beta+1}$ satisfies the Knaster condition) then $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta: \beta < \alpha + 1, \gamma \leqslant \alpha + 1\rangle \in \mathscr{K}^x_{\mu,\lambda}$;*

(f) *if $n < \omega$, $p_1, \ldots, p_n \in P_{\alpha^*}$ and*

(∗) *for every $\beta \in \bigcup_{\ell=1}^n dom(p_\ell)$ for some $m = m_{\beta,\ell} \in \{1, \ldots, n\}$ we have $p_m \restriction \beta \Vdash$ “$p_\ell(\beta) \leqslant_{\underset{\sim}{Q}_\beta} p_m(\beta)$ for $\ell \in \{1, \ldots, n\}$”*

then $p_1, \ldots, p_n$ *has a least common upper bound* p *which is defined by:* $\mathrm{dom}(p) = \bigcup_{\ell=1}^{n} \mathrm{dom}(f)$, $p_\ell(\beta) = p_{m_{\beta,\ell}}(\beta)$, *so in particular* $p \in P_{\alpha^*}$ *and* $\bigwedge_{\ell=1}^{n} p_\ell \in P^*_{\alpha^*} \Rightarrow p \in P^*_{\alpha^*}$;

(g) *if* $p_\ell \leqslant p$ *and* $p_\ell \in P^*_\gamma$ *for* $\ell < n$, *and* a_k *is* $\bar{Q}$-*closed for* $k < m$ *then there is* $p' \in P^*_\gamma$, *such that* $p \leqslant p'$ *and* $P^*_{a_k} \models p_\ell \restriction a_k \leqslant p' \restriction a_k$ *for* $\ell < n$, $k < m$.

(2) *If* $x \in \{n, k\}$ *and* $\delta < \lambda$ *is a limit ordinal, and for* $\alpha < \delta$ *we have* $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta \colon \beta < \alpha, \gamma \leqslant \alpha \rangle \in \mathscr{K}^x_{\mu,\lambda}$ *and* $P_\delta = \bigcup_{\gamma<\delta} P_\gamma$ *then* $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta \colon \beta < \delta, \gamma \leqslant \delta \rangle$ *belongs to* $\mathscr{K}^x_{\mu,\lambda}$.

Proof. Straightforward.

Essentially by [3, 2.4(2), p. 176] (which is slightly weaker and its proof is left to the reader, so we give details here).

1.3. Claim. *Assume* $\lambda \to^+ (\omega\alpha^*)^{<\omega}_\mu$ (*e.g.* λ *a Ramsey cardinal*, $\alpha^* = \lambda$), $\chi > \lambda$, $x \in H(\chi)$.

(1) *There is a strong* $(\chi, \lambda, \alpha^*, \mu, \aleph_0, \omega)$-*system for* x (*see Definition* 1.3A).

(2) *There is an end extension strong* $(\chi, \lambda, \alpha, \mu, \aleph_0, \omega)$-*system for* x *if* λ *is Ramsey or* $\lambda = \min\{\lambda \colon \lambda \to (\omega\alpha^*)^{<\omega}_\mu\}$ (*also then the condition holds for every* $\mu' < \mu$).

1.3A. Definition. (1) We say $\bar{N} = \langle N_s \colon s \in [B]^{<1+n} \rangle$ is a $(\chi, \lambda, \alpha, \theta, \sigma, n)$-system if:

(a) $N_s \prec (H(\chi), \in)$ (or of some expansion), $\theta + 1 \subseteq N_s$, $\|N_s\| = \theta$, ${}^{\sigma>}(N_s) \subseteq (N_s)$;

(b) $B \subseteq \lambda$, $\mathrm{otp}(B) = \alpha$;

(c) $n \leqslant \omega$ (equality is allowed but $1 + \omega = \omega$ so s is always finite);

(d) $N_s \cap N_t \subseteq N_{s\cap t}$;

(e) $N_s \cap B = s$;

(f) if $|s| = |t|$ then $N_s \cong N_t$ say $H_{s,t}$ is an isomorphism from N_t onto N_s (necessarily $H_{s,t}$ is unique);

(g) if $s' \subseteq s$, $t' = \{\alpha \in t \colon (\exists \beta \in s')\,[|\beta \cap s'| = |\alpha \cap t|]\}$ then $H_{s',t'}$, $H_{s,t}$ are compatible functions; $H_{s,s} = id$, $H_{s,t} \supseteq H^{\mathrm{OP}}_{s,t}$, $H_{s_0,s_1} \circ H_{s_1,s_2} = H_{s_0,s_2}$, $H_{t,s} = (H_{s,t})^{-1}$;

(h) $\sup(N_s \cap \lambda) < \min\{\alpha \in B \colon \bigwedge_{\gamma \in s} \gamma < \alpha\}$.

(2) We add the adjective "strong" if we strengthen clause (d) by

(d)$^+$ $N_s \cap N_t = N_{s\cap t}$ (so in clause (g), $H_{s',t'} \subseteq H_{s,t}$).

(3) We add the adjective "end extension" if

(i) $s \lhd t \Rightarrow N_s \cap \lambda \lhd N_t \cap \lambda$ (where $A \lhd B$ means $A = B \cap \min(B \setminus A)$).

(4) We add "for x" if $x \in N_s$ for every $s \in [B]^{<1+n}$, and $H_{s,t}(x) = x$.

1.3B. Remark. If λ is a Ramsey cardinal (or much less, see [6, XVII, 4.x] and [3, §4]) then we have if $\gamma \in s \cap t$, $s \cap \gamma = t \cap \gamma$ and $y \in N_s$ then in $(H(\chi), \in, <^*_\chi)$ the elements y and $H_{t,s}(y)$ realize the same type over $\{i \colon i < \gamma\}$.

Proof of Claim 1.3. (1) Let $C = \{\delta < \lambda \colon$ for every $\alpha < \delta$ there is $N \prec (H(\chi), \in, <^*_\chi)$ such that $\mu + 1 + \alpha \subseteq N$ and $\sup(N \cap \lambda) < \delta\}$. Clearly C is a club of λ.

Let $B_0 = \{\alpha_i: i < \omega\alpha^*\} \subseteq C$ (α_i strictly increasing) be indiscernible in $(H(\chi), \in, <^*_\chi, x)$ (see Definition 0.1). Let $B = \{\alpha_i: i < \omega\alpha^* \text{ limit}\}$. For $s \in [B_0]^{<\aleph_0}$ let N^0_s = the Skolem hull of $s \cup \{i: i \leqslant \mu\} \cup \{x, \lambda\}$ under the definable functions of $(H(\chi), \in, <^*_\chi)$ and

$$N_s = \bigcup \{N^0_{t_1} \cap N^0_{t_2}: t_1, t_2 \in [\{\alpha_i: i < \omega\alpha^*\}]^{<\aleph_0} \quad \text{and} \quad s = t_1 \cap t_2\}.$$

Clearly

(*) $\quad \|N_s\| \leqslant \mu$ and $\{x, \lambda\} \subseteq N_s$.

Now we shall show

$(*)_1$ if $s \in [B]^{<\aleph_0}$, $y \in N_s$ then for every finite $t \subseteq B_0$ there is $u \in [B_0]^{<\aleph_0}$ such that $s \subseteq u, u \cap t \subseteq s$ and $y \in N^0_u$.

As $y \in N_s$ there are $s_1, s_2 \in [B_0]^{<\aleph_0}$ such that $y \in N^0_{s_1} \cap N^0_{s_2}$ and $s = s_1 \cap s_2$. Let $s_1 \cup s_2 = \{\alpha_{i_0}, \ldots, \alpha_{i_{m-1}}\}$ (increasing), and let $n^* = \sup\{n: \text{for some } \beta,\ \beta + n \in N_1 \cup N_2\} + 1$, and define for $\ell \leqslant m$ a function f_ℓ with domain $s_1 \cup s_2$, such that

$$f_\ell(\alpha_{i_k}) = \begin{cases} \alpha_{i_k + n^*} & \text{if } k \geqslant m - \ell \text{ and } i_k \notin s, \\ \alpha_{i_k} & \text{otherwise.} \end{cases}$$

Note that

$\otimes_1$ for $\ell < m, f_\ell \restriction s_1 = f_{\ell+1} \restriction s_1$ or $f_\ell \restriction s_2 = f_{\ell+1} \restriction s_2$ (or both).

[Why? As $i_\ell \in s_2 \backslash s_1 \backslash s$ or $i_\ell \in s_2 \backslash s_1 \backslash s$ or $i_\ell \in s = s_1 \cap s_2$.]

$\otimes_2$ f_ℓ is order preserving with domain $s_0 \cup s_1, f_\ell \restriction s$ = the identity.

As $y \in N^0_{s_1} \cap N^0_{s_2}$ there are terms τ_1, τ_2 such that

$$y = \tau_1(\ldots, \alpha_{i_k}, \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, \alpha_{i_k}, \ldots)_{\alpha_{i_k} \in s_2}.$$

Using the indiscernibility of B_0 we can prove by induction on $\ell \leqslant m$ that

$\otimes_{3,\ell}$ $\quad y = \tau_1(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}.$

[Why? For $\ell = 0$ this is given by the choice of τ_1, τ_2. For $\ell + 1$ note that by $\otimes_2$, $f_{\ell+1} \circ f_\ell^{-1}$ is an order-preserving function from $\operatorname{ran}(f_\ell)$ onto $\operatorname{ran}(f_{\ell+1})$. By $\otimes_{3,\ell}$ and "B_0 is indiscernible" we know $\tau_1(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}$. By the last two sentences and the indiscernibility of B_0

$$\tau_1(\ldots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \ldots)_{\alpha_{i_k} \in s_2}.$$

But $(f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})) = f_{\ell+1}(\alpha_{i_k})$ so

$$\tau_1(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}.$$

But by $\otimes_1$ for some $e \in \{1,2\}$ we have $f_\ell \restriction s_e = f_{\ell+1} \restriction s_e$, so $\tau_e(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_e} = \tau_e(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_e}$ but the latter is equal to y (by the induction hypothesis), hence the former is so by the last sentence

$$y = \tau_1(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}.$$

So we have carried out the induction on $\ell \leqslant m$, and for $\ell = m$ we get $y \in N^0_{f_m(s_1)}$, but by the choice of n^* and f_m clearly $f_m(s_1) \cap t \subseteq s$, and we have proved $(*)_1$.]

Now we can note

$(*)_2$ if $s \in [B]^{<\aleph_0}$ and $y_1, \ldots, y_n \in N_s$ then for some $s_1, s_2 \in [B_0]^{<\aleph_0}$ we have: $s = s_1 \cap s_2$ and $y_1, \ldots, y_n \in N^0_{s_1} \cap N^0_{s_2}$.

[Why? We can find $u_1, \ldots, u_n \in [B_0]^{<\aleph_0}$ such that $s \subseteq u_\ell$, $y_\ell \in N^0_{u_\ell}$ (as $y_\ell \in N_s$). Now by $(*)_1$ for each $\ell = 1, 2, \ldots, n$ we can find $v_\ell \in [B_0]^{<\aleph_0}$ such that $s \subseteq v_\ell$, $s = v_\ell \cap (\bigcup_{m=1}^n u_m)$ and $y_\ell \in N^0_{v_\ell}$. Let $u = \bigcup_{i=1}^n u_\ell$, $v = \bigcup_{\ell=1}^n u_\ell$, clearly $y_1, \ldots, y_n \in N^0_u \cap N^0_v$ and $u \cap v = s$, as required.]

Now, as we have Skolem functions, $(*)_2$ implies

$(*)_3$ $N_s \prec (H(\chi), \in, <^*_\chi)$.

Also trivially

$(*)_4$ $N^0_s \prec N_s$ hence $\mu + 1 \subseteq N_s$,

$(*)_5$ $s \subseteq t \Rightarrow N_s \prec N_t$.

(For $(*)_5$, use $(*)_1$.) Also

$(*)_6$ $N_{s_1} \cap N_{s_2} = N_{s_1 \cap s_2}$ for $s_1, s_2 \in [B]^{<\aleph_0}$.

[Why? The inclusions $N_{s_1 \cap s_2} \subseteq N_{s_1} \cap N_{s_2}$ follow from $(*)_5$; for the other direction let $y \in N_{s_1} \cap N_{s_2}$. By $(*)_1$ as $y \in N_{s_1}$ there is t_1 such that $s_1 \subseteq t_1 \in [B_0]^{<\aleph_0}$, $t_1 \cap (s_1 \cup s_2) = s_1$ and $y \in N^0_{t_1}$. By $(*)_1$, as $y \in N_{s_2}$ there is t_2 such that $s_2 \subseteq t_2 \in [B_0]^{<\aleph_0}$, $t_2 \cap (s_1 \cup s_2 \cup t_1) = s_2$ and $y \in N^0_{t_2}$. So $y \in N^0_{t_1} \cap N^0_{t_2}$, but easily $t_1 \cap t_2 = s_1 \cap s_2$.]

$(*)_7$ $\sup(N_s \cap \lambda) < \min\{\alpha \in B: \bigwedge_{\gamma \in s} \gamma < \alpha\}$.

[Why? As $B_0 \subseteq C$ and see the definition of C.]

Now check that (a)–(h) of Definition 1.3A hold. Now $\langle N_s : s \in [B]^{<\aleph_0} \rangle$ is as required.

(2) If λ is Ramsey, without loss of generality $\mathrm{otp}(B_0) = \lambda$ and it is easy to check 1.3A(i). The other case is like [3, §4]. □

1.4. Theorem. *Assume $\aleph_0 < \mu \leqslant \kappa < \lambda = cf(\lambda)$, λ strongly inaccessible, λ a Ramsey cardinal, and $\diamondsuit_{\{\delta < \lambda:\ cf(\delta) = \aleph_1\}}$ (can be added by a preliminary forcing). Then we have P such that*

(a) *P is a c.c.c. forcing of cardinality λ adding λ reals (so the cardinals and cardinal arithmetic in V^P should be clear), in particular in V^P we have $2^{\aleph_0} = \lambda$.*

(b) $\Vdash_P$ "*MA holds for c.c.c. forcing notions of cardinality* $\leqslant \mu$ *and* $< \lambda$ *dense sets (and even for c.c.c. forcing notions of cardinality* $\leqslant \kappa$ *which are from* $V[A]$ *for some* $A \subseteq \mu$)".

(c) $\Vdash_P$ "*if* B *is a* λ-*c.c. Boolean algebra,* $x_i \in B \backslash \{0\}$ *for* $i < \lambda$ *then for some* $Z \subseteq \lambda$, $|Z| = \aleph_1$ *and* $\{x_i: i \in Z\}$ *generates a proper filter of* B *(i.e., no finite intersection is* 0_B)".

(d) $\Vdash_P$ "*if* B_1 *is a c.c.c. Boolean algebra,* B_2 *is a* λ-*c.c. Boolean algebra then* $B_1 \times B_2$ *is a* λ-*c.c. Boolean algebra*".

Proof. Let $\langle A_\delta: \delta < \lambda, cf(\delta) = \aleph_1 \rangle$ exemplify the diamond. We choose by induction on $\alpha < \lambda$, $\bar{Q}^\alpha = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta: \gamma \leqslant \alpha, \beta < \alpha \rangle \in \mathscr{K}^n_{\mu,\kappa}$ such that $\alpha^1 < \alpha \Rightarrow \bar{Q}^{\alpha^1} = \bar{Q}^\alpha \restriction \alpha^1$. In limits α use 1.2(2). For $\alpha = \beta + 1$, $cf(\beta) \neq \aleph_1$ take care of (b) by suitable bookkeeping using 1.2(1)(e). If $\alpha = \beta + 1$, $cf(\beta) = \aleph_1$ and A_β codes $p \in P_\beta$ and P_β-names of a Boolean algebra $\underset{\sim}{B}_\beta$ and sequence $\langle \underset{\sim}{x}^\beta_i: i < \beta \rangle$ of non-zero members of $\underset{\sim}{B}_\beta$, and p forces ($\Vdash_{P_\beta}$) that there is in $V[\underset{\sim}{G}_{P_\beta}]$ some c.c.c. forcing notion Q of cardinality $\leqslant \mu$ adding some $Z \subseteq \beta$, $|Z| = \aleph_1$ with $\{x^\beta_i: i \in Z\}$ generating a proper filter of $\underset{\sim}{B}_\beta$ then we choose $\underset{\sim}{Q}_\beta$, if $p \in \underset{\sim}{G}_{P_\beta}$, as such Q. If $p \notin \underset{\sim}{G}_{P_\beta}$ or there is no such $\underset{\sim}{Q}$ in $V[\underset{\sim}{G}_{P_\beta}]$, then $\underset{\sim}{Q}_\beta$ is e.g. Cohen forcing.

So every $\bar{Q}^\alpha$ is defined, let $P = \bigcup_{\gamma < \lambda} P_\gamma$. Clearly (a) and (b) hold, and (d) follows by (c). So the rest of the proof is dedicated to proving (c).

So let $p \in P$, $p \Vdash$ "$\underset{\sim}{B}$ a λ-c.c. Boolean algebra, $\underset{\sim}{x}_i \in B \backslash \{0_B\}$ for $i < \lambda$". Without loss of generality the set of members of $\underset{\sim}{B}$ is λ.

Let $x = \langle P, p, \underset{\sim}{B}, \langle \underset{\sim}{x}_i: i < \lambda \rangle \rangle$, $\chi = \lambda^+$. By Claim 1.3 there are $A \in [\lambda]^\lambda$ and $\langle N_s: s \in [A]^{<\aleph_0} \rangle$ as there (for $\kappa = \mu + \kappa$ here standing for μ there). Let

$$
\begin{aligned}
C = \{\delta < \lambda:\ & \delta \text{ a strong limit cardinal } > \kappa + \mu, [\alpha < \delta \Rightarrow \bar{Q} \restriction \alpha \in H(\delta)], \\
& \delta = \sup(A \cap \delta), s \in [A \cap \delta]^{<\aleph_0} \Rightarrow \sup(\lambda \cap N_s) < \delta, \\
& \underset{\sim}{B} \restriction \delta \text{ a } P_\delta\text{-name, and for } i < \delta \text{ we have } \underset{\sim}{x}_i \text{ a } P_\delta\text{-name}\}.
\end{aligned}
$$

For some accumulation point δ of C, $cf(\delta) = \aleph_1$ and A_δ codes $\langle p, \underset{\sim}{B} \restriction \delta, \langle \underset{\sim}{x}_i: i < \delta \rangle \rangle$. We shall show that for some q, $p \leqslant q \in P_\delta$ and $q \Vdash_{P_\delta}$ "there is Q as required above". By the inductive choice of $\underset{\sim}{Q}_\delta$ this suffices.

Let $A^* \subseteq A \cap \delta$, $\text{otp}(A^*) = \omega_1$, $\delta = \sup(A^*)$ and $\langle \delta_i: i < \omega_1 \rangle$ increasing continuous, $\delta = \bigcup_{i < \omega_1} \delta_i$, $\delta_i \in C$, $A^* \cap \delta_0 = \emptyset$, $|A^* \cap [\delta_i, \delta_{i+1})| = 1$.

In V^{P_δ} we define:

$$\underset{\sim}{Q} = \left\{ u: u \in [A^*]^{<\aleph_0}, \text{ and } B \models \text{“}\bigcap_{i \in u} \underset{\sim}{x}_i \neq 0_B\text{”} \right\}$$

ordered by inclusion. It suffices to prove that some $q, p \leqslant q \in P_\delta$, q forces that: $\underset{\sim}{Q}$ is c.c.c. with $\bigcup \underset{\sim}{G}_Q$ an uncountable set; now clearly q forces that $\{\underset{\sim}{x}_i : i \in \bigcup \underset{\sim}{G}_Q\}$ generates a proper filter of $\underset{\sim}{B}$.

If not, we can find q_i, u_i such that

$$p \leqslant q_i \in P^*_\delta \quad \text{and} \quad q_i \Vdash_{P_\delta} \text{“} u_i \in \underset{\sim}{Q}\text{”} \quad (\text{where } u_i \in [A^*]^{<\aleph_0})$$

and $\langle (q_i, u_i) : i < \omega_1 \rangle$ are pairwise incompatible in $P_\delta * \underset{\sim}{Q}$.

Let v_i be a finite subset of A^* such that: $u_i \subseteq v_i$, and

$$(*) \qquad [v \subseteq A^* \ \& \ v \text{ finite} \ \& \ \gamma \in (\operatorname{dom} q_i) \cap N_v \Rightarrow \gamma \in (\operatorname{dom} q_i) \cap N_{v \cap v_i}].$$

By Fodor's Lemma for some stationary $S \subseteq \omega_1$, u^*, v^*, n^* and $i(*)$ we have: for $i < j$ in S,

$$v_i \cap \delta_i = v^* \subseteq \delta_{i(*)}, \qquad v_i \subseteq \delta_j, \qquad |v_i| = n^*,$$

$$u_i \cap \delta_i = u^*, \qquad i(*) = \min(S),$$

$$\{|\gamma \cap v_i)| : \gamma \in u_i\} \text{ does not depend on } i,$$

$$q_i \restriction \delta_i \in P^*_{\delta_{i(*)}}, \qquad q_i \in P^*_{\delta_j}.$$

Let $b_i =: N_{v_i} \cap \lambda$, so b_i is necessarily $\bar{Q}^\delta$-closed and $|b_i| = \kappa$. Let $q^1_i = q_i \restriction b_i$, so necessarily $q^1_i \in P^*_{b_i}$ (see 2.2(1)(c)). Easily $P^*_{b_i} \subseteq N_{v_i}$ (though do not belong to it) so $q^1_i \in N_{v_i}$.

Let $q^2_i =: H_{v_{i(*)}, v_i}(q^1_i)$, so $q^2_i \in P^*_{b_{i(*)}}$. Let

$$q^3_i =: (q_i \restriction \delta_{i(*)}) \cup [q^2_1 \restriction (N_{v_{i(*)}} \cap \lambda \backslash \delta_{i(*)})],$$

by 1.2(1)(c) we know $q^3_i \in P^*_{\sup(b_{i(*)})+1}$ and $q^2_i \leqslant q^3_i$, even without loss of generality $q^2_i \leqslant q^3_i \restriction b_{i(*)}$. As $P^*_{\sup(b_{i(*)})+1} \lessdot P_\delta$ and P_δ satisfies the c.c.c. clearly for some $i < j$ from S, q^3_i, q^3_j are compatible in $P^*_{\sup(b_{i(*)})+1}$, so let $r \in P^*_{\sup(b_{i(*)})+1}$ be a common upper bound. So $q^3_i \restriction (\delta_{i(*)} \cap b_{i(*)}) \leqslant r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and $q^3_j \restriction (\delta_{i(*)} \cap b_{i(*)}) \leqslant r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and $q^3_i \restriction b_{i(*)} \leqslant r \restriction b_{i(*)}$ and $q^3_j \restriction b_{i(*)} \leqslant r \restriction b_{i(*)}$.

Without loss of generality $\operatorname{dom}(r) \subseteq b_{i(*)} \cup \delta_{i(*)}$ (allowed as $b_{i(*)}$ and $\delta_{i(*)}$ are closed, see 1.2(1)(c)); let $r_i = H_{v_i, v_{i(*)}}(r \restriction b_{i(*)})$ and similarly $r_j = H_{v_j, v_{i(*)}}(r \restriction b_{i(*)})$.

Note that $r_i \in P^*_{\delta_j}$, $r_j \in P^*_\delta$, $r_j \restriction \delta_j = r_i \restriction \delta_i = r \restriction \delta_{i(*)}$. Hence $r_i \cup r_j \in P^*_\delta$.

Case 1: $r_i \cup r_j$ do not force (i.e. $\Vdash_{P_\delta}$) that

$$\underset{\sim}{B} \Vdash \text{“} \bigcap_{\alpha \in u_i \cup u_j} \underset{\sim}{x}_\alpha = 0_{\underset{\sim}{B}}\text{”}.$$

Then there is $r' \in P_\delta$, $r_i \leqslant r'$, $r_j \leqslant r'$ forcing the negation. So without loss of generality $r' \in P^*_\delta$, and (as all parameters appearing in the requirements on r' are in $N_{v_i \cup v_j}$ also) $r' \in P^*_{\lambda \cap (N_{v_i \cup v_j})}$. Now r', r, q_i, q_j have an upper bound $r'' \in P_\delta$. [Why? By 1.2(1)(f), we have to check the condition $(*)$ there, so let

$$\beta \in \operatorname{dom}(r') \cup \operatorname{dom}(r) \cup \operatorname{dom}(q_i) \cup \operatorname{dom}(q_j).$$

Subcase 1a: $\beta \in \delta_{i(*)} \setminus N_{v_i \cup v_j}$. Note that $N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$ (see choice of the N_u's and definition of the b_v's) but $\mathrm{dom}(r') \subseteq N_{v_i \cup v_j} \cap \lambda$, so $\beta \notin \mathrm{dom}(r')$. Now

$$q_i \restriction \delta_i = q_i \restriction \delta_{i(*)} = q_i^3 \restriction \delta_{i(*)} \leqslant r,$$

$$q_j \restriction \delta_j = q_j \restriction \delta_{i(*)} = q_j^3 \restriction \delta_{i(*)} \leqslant r.$$

So $r \restriction \beta \Vdash_{P_\beta}$ "$q_i(\beta) \leqslant r(\beta), q_j(\beta) \leqslant r(\beta)$" and $\beta \notin \mathrm{dom}(r')$. So we have confirmed $(*)$ from 1.2(1)(f) for this subcase.

Subcase 1b: $\beta \in \delta_{i(*)} \cap N_{v_i \cup v_j}$. Exactly as above: $N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$, so $\beta \in N_{v^*}$, $\beta \in \delta_{i(*)} \cap b_{i(*)}$. Also

$$q_i \restriction b_{i(*)} = q_i^1 \restriction \delta_{i(*)} = q_i^2 \restriction \delta_{i(*)} = q_i^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \leqslant r \restriction (\delta_{i(*)} \cap b_{i(*)})$$

and

$$q_j \restriction b_{i(*)} = q_j^1 \restriction \delta_{i(*)} = q_j^2 \restriction \delta_{i(*)} = q_j^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \leqslant r \restriction (\delta_{i(*)} \cap b_{i(*)})$$

and

$$r \restriction (\delta_{i(*)} \cap b_{i(*)}) \leqslant r'$$

(as $H_{v_i, v_{i(*)}}$ is the identity on $\delta_{i(*)} \cap b_{i(*)}$). The last three inequalities confirm the requirement in 1.2(1)(f) (as $\beta \in \delta_{i(*)} \cap b_{i(*)}$, see above).

Subcase 1c: $\beta \in (\delta \setminus \delta_{i(*)}) \setminus N_{v_i \cup v_j}$. In this case $\beta \notin \mathrm{dom}(r')$ (as $r' \in N_{v_i \cup v_j}$). Also $\delta_{i(*)} < \delta_i < \delta_j < \delta$ and:

$$\mathrm{dom}(r) \setminus \delta_{i(*)} \subseteq (b_{i(*)} \cup \delta_{i(*)}) \setminus \delta_{i(*)} \subseteq [\delta_{i(*)}, \delta_i),$$

$$\mathrm{dom}(q_i) \setminus \delta_{i(*)} \subseteq [\delta_i, \delta_j), \qquad \mathrm{dom}(q_j) \setminus \delta_{i(*)} \subseteq [\delta_j, \delta).$$

So β belongs to at most one of $\mathrm{dom}(r')$, $\mathrm{dom}(r)$, $\mathrm{dom}(q_i)$, $\mathrm{dom}(q_j)$ so the requirement $(*)$ from 1.2(1)(f) holds trivially.

Subcase 1d: $\beta \in (\delta \setminus \delta_{i(*)}) \cap N_{v_i \cup v_j}$. Clearly $\beta \notin \mathrm{dom}(r)$. We know $q_i \restriction b_i = q_i^1$, $r_i \leqslant r'$, $H_{v_{i(*)}, v_i}(q_i^1) = q_i^2 \leqslant q_i^3 \restriction b_{i(*)} \leqslant r \restriction b_{i(*)}$ hence

$$q_i^1 \leqslant H^{-1}_{v_{i(*)}, v_i}(r \restriction b_{i(*)}) = H_{v_i, v_{i(*)}}(r \restriction b_{i(*)}) = r_i$$

but $r_i \leqslant r'$, so together $q_i^1 \leqslant r'$, and similarly $q_j^1 \leqslant r'$. As we have noted $\beta \notin \mathrm{dom}(r)$ we have finished confirming condition $(*)$ from 1.2(1)(f)].

So really r, r', q_i, q_j have a least common upper bound, say r''s hence $(r'', u_i \cup u_j) \in P_\delta * \underset{\sim}{Q}$; exemplified (q_i, u_i), (q_j, u_j) are compatible, as required.

Case 2: not Case 1. Let $\langle s_\beta \colon \beta < \lambda \rangle$ be such that:

$$s_\beta \in [A]^{<\aleph_0}, \qquad v^* \subseteq s_\beta, \qquad |s_\beta \setminus v^*| = |v_i \setminus v^*|,$$

$$\sup(v^*) < \delta_{i(*)} < \min(s_\beta \setminus v^*),$$

$$\delta < \min(s_\beta \setminus v^*) \quad \text{(for simplicity)},$$

$$\beta < \gamma \Rightarrow \max(s_\beta) < \min(s_\gamma \setminus v^*).$$

As the truth value of $\bigcap_{\alpha\in u_i} x_\alpha$ is a P^*_a-name for some closed $a \in N_{v_i}$ of cardinality $\leqslant \mu$, and $q_i \Vdash [\underset{\sim}{B} \models \text{“}\bigcap_{\alpha\in u_i} x_\alpha \neq 0_{\underset{\sim}{B}}\text{”}]$ clearly,

$$q^1_i \Vdash [\underset{\sim}{B} \models \text{“}\bigcap_{\alpha\in u_i} x_\alpha \neq 0_{\underset{\sim}{B}}\text{”}].$$

For $\beta < \lambda$ let $r^\beta = H_{s_\beta, v_{i(*)}}(r \restriction b_{i(*)})$, and $u'_\beta = H_{s_\beta, v_{i(*)}}(u_{i(*)})$. Let

$$\underset{\sim}{Y} = \{\beta < \lambda \colon r^\beta \in \underset{\sim}{G}_P\}.$$

Clearly,

$$r^\beta \Vdash_{P_\lambda} [\underset{\sim}{B} \Vdash \text{“}\bigcap_{i\in u'_\beta} \underset{\sim}{x}_i \neq 0_{\underset{\sim}{B}}\text{”}].$$

Clearly $p \leqslant r^\beta$ and for some β we have $r^\beta \Vdash \text{“}\underset{\sim}{Y} \in [\lambda]^\lambda$ (and $p \in \underset{\sim}{G}_P$)” and by the assumption of the case:

$$p \Vdash \text{“}\left\{\bigcap_{i\in u'_\beta} x_i \colon \beta \in Y\right\} \text{ is a set of non-zero members of } \underset{\sim}{B}$$

any two having zero intersection in $\underset{\sim}{B}$”.

This contradicts an assumption on B. □

We can phrase the consistency result as one on colouring.

1.5. Lemma. *In* 1.4 *we can add*:

(e) *If c is a symmetric function from $[2^{\aleph_0}]^{<\omega}$ to $\{0, 1\}$ then at least one of the following holds*:

(α) *we can find pairwise disjoint $w_i \subseteq 2^{\aleph_0}$ for $i < 2^{\aleph_0}$ such that $c \restriction [w_i]^{<\aleph_0}$ is constantly zero but*

$$\bigwedge_{i<j} (\exists u \subseteq w_i, \exists v \subseteq w_j)[c[u \cup v] = 1];$$

(β) *we can find an unbounded $B \subseteq 2^{\aleph_0}$ such that $c \restriction [B]^{<\omega}$ is constantly* 0.

It is natural to ask:

1.6. Question. Can we replace $2^{\aleph_0}$ by $\lambda < 2^{\aleph_0}$? $\aleph_1$ by $\mu < \lambda$? What is the consistency strength of the statements we prove consistent? (see later). Does λ strongly inaccessible k^2_2-Mahlo (see [3]) suffice?

1.7. Discussion. Of course, 1.5(e) ⇒ 1.4(c) ⇒ 1.4(d). Starting with λ weakly compact, seemingly we can get a c.c.c. forcing notion P of cardinality λ, such that in V^P, $2^{\aleph_0} = \lambda$ and (e) of 1.5 holds for $c\colon [2^{\aleph_0}]^2 \to \{0, 1\}$ (so $c(u) = 0$ if $|u| \neq 2$) and this suffices for the result. Also we can generalize to higher cardinals. We shall discuss this elsewhere.

1.8. Theorem. *Concerning the consistency strength, in* 1.4 *it suffices to assume*

(∗) λ *is strongly inaccessible and for every* F: $[\lambda]^{<\aleph_0} \to \mu$ *and club* C *we can find* $B \subseteq C$ (*or just* $B \subseteq \lambda$), otp $(B) = \omega_1$ *such that*

(a) B *is* F-*indiscernible, i.e., if* $n < \omega$, $u, v \in [B]^n$ *then* $F(u) = F(v)$;

(b) *for every* $n < \omega$ *there is* $B' \in [C]^\lambda$ *such that*

$$\text{if } u \in [B']^n \text{ and } v \in [B]^n \text{ then } F(u) = F(v).$$

Proof. Let $R = \{\bar{Q}: \bar{Q} \in H(\lambda), \bar{Q} \in \mathscr{K}^n_{\mu,\kappa}\}$ ordered by $\bar{Q}^1 < \bar{Q}^2$ if $\bar{Q}^1 = \bar{Q}^2 \restriction \ell g(\bar{Q}^1)$. Clause (b) takes care also of "the end extension" clause and for 1.3A(4), clause (b) the proof is the same.

A somewhat less natural property though suffices. (Note: Clause (b) also helps to get rid of the club C.)

1.9. Claim. *In* 1.4 *it suffices to assume*

(∗)′ *if* F: $[\lambda]^{<\aleph_0} \to \mu$ *then there is* $B \subseteq \lambda$, otp $(B) = \omega_1$ *such that*

(a) $F \restriction [B]^n$ *is constant for* $n < \omega$;

(b) *if* $u \lhd v^\ell \in [B]^{<\aleph_0}$ *for* $\ell = 1,2$ *then we can find* $v_i \in [\lambda]^n$ *for* $i < \lambda$, $u \subseteq v_i$, $\min(v_i \backslash u) \geqslant i$, *and* $i < j \Rightarrow F(v^1 \cup v^2) = F(v_i \cup v_j)$.

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