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THE THEOREMS OF BETH AND CRAIG IN ABSTRACT MODEL THEORY II. COMPACT LOGICS *

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Abstract

Various compact logics such as stationary logic, positive logic, logics with various cardinality quantifiers and cofinality quantifiers are studied. Counterexamples to the theorems of Beth and Craig are given. Back and forth arguments are studied for the first two logics, transfer theorems presented for positive logic and a new compactness proof for the cofinality quantifiers is given.

0. Introduction

This paper is a companion to [21, 22]. We discuss Beth's and Craig's Definability theorem for several compact logics known from the literature. For some of them new proofs of compactness are given[1].

In Section 1 we discuss extensions of $L(Q_1)$ (cf. [15]) culminating in Shelah's $L(aa)$ (cf. [3]). We present in Section 2 various Ehrenfeucht games mostly due to the first author, which give criteria for elementary equivalence for some of the logics discussed in the paper.

In Section 3 we then prove

Theorem A. *Not* BETH $(L(Q_1), L_{\infty\omega}(aa))$.

In Section 4 we study the same for $L(Q_\alpha)$ culminating in the first author's L^p_α using Chang's transfer principle (cf. [18]).

In Section 5 we study extensions of Malitz and Magidor's $L(Q^{<\omega})$ (cf. [17]). In particular we prove a result of Stavi, that L^r (the analogue of L^p) is not compact. Our main result here is

Theorem B. *Not* BETH $(L(Q_1), L^r)$.

In Section 6 we finally discuss a family of quantifiers due to the second author [26] and give a new proof of their compactness, which is due to the second author

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alone. Again we have counterexamples to Beth's theorem. For all three counterexamples we use Shelah's construction described in [21, Section 6]. We assume the reader is familiar with [21]).

1. Stationary Logic and its Sublogics

Let L be a first order language $x_i (i<\omega)$ countably many individual variables, and $s_i (i<\omega)$ countably many relation variables, not in L.

$L(aa)$ and L^p are the least sets of formulae closed under the formation rules (i), (ii) and $(\mathrm{iii})_{aa}$ $(\mathrm{iii})_p$ respectively:

(i) L-atomic formulae and $s_i(x_j)$ are in $L(aa)$ (L^p).

(ii) $L(aa)$ (L^p respectively) is closed under the finitary operations $\wedge$, $\vee$, $\Rightarrow$, $\sim$, $\exists x_i$, $\forall x_j$.

$(\mathrm{iii})_{aa}$ $L(aa)$ is closed under $aa\, s_i$ and stat s_i i.e. if $\varphi(s_i)$ is in $L(aa)$ so are $aa\, s_i \varphi$ and $\mathrm{stat}\, s_i \varphi$.

$(\mathrm{iii})_p$ L^p is closed under $\exists s_i \varphi(s_i)$ provided s_i does not occur negatively in φ.

$L_{\kappa\omega}(aa)$, $L_{\infty\omega}(aa)$ etc. are defined in the obvious way. Satisfaction (in $\aleph_1$-interpretation) for L^p is defined as usually with the additional clause:
$\mathfrak{A} \models \exists s_i \varphi(s_i)$ if there is a countable $R \subset A$ such that $\langle \mathfrak{A}, R\rangle \models \varphi(R)$.
The $\aleph_\alpha$-interpretation is given by replacing "countable" by "of cardinality less than $\aleph_\alpha$".
Satisfaction for $L(aa)$ is explained via the cub-filter on $P_{<\aleph_1}(A)$:
$\mathfrak{A} \models aa\, s\varphi(s)$ if $\{R \in P_{<\aleph_1}(A) \mid \langle \mathfrak{A}, R\rangle \models \varphi(R)\}$ contains a closed and unbounded family of countable sets

$$\mathfrak{A} \models \mathrm{stat}\, s\, \varphi(s) \quad \text{if} \quad \mathfrak{A} \models \neg\, aa\, s \neg\, \varphi(s).$$

The infinitary cases are defined in the obvious way.
$L(aa)$ was invented in [26] and extensively studied in [3]. L^p was invented by the first author and introduced in [23]. It is extensively studied in [18].
The following resumes what is known and needed here on $L(aa)$ and L^p. Note that L^p is a sublogic of $L(aa)$ due to the equivalence of $\exists s\varphi(s)$ and $aa\, s\varphi(s)$ for $\exists s\varphi(s) \in L^p$.
There are axioms Γ_p (due to Stavi) and Γ_{aa} (due to Barwise and Makkai) such that

Theorem 1.1. *Let Σ be a countable set of sentences of $L(aa)$ (L^p respectively). Then the following are equivalent*

(i) *Σ has a model;*

(ii) *Σ has a model of cardinality $\leqq \aleph_1$;*

(iii) *every finite subset of Σ has a model ($L(aa)$ is (ω,ω)-compact);*

(iv) *$\Sigma \cup \Gamma_{aa}(\Sigma \cup \Gamma_p)$ is consistent.*

Similar theorems [but (iii)] are true for the infinitary case $L_{\omega_1\omega}(aa)$ and $L^p_{\omega_1\omega}$.
For later use we state here without proofs (cf. [24]).

Theorem 1.2. *Every formula of $L(aa)$ is equivalent to a formula of $L(aa)$ which is in prenex normal form. (Similarly for L^P).*

A weak structure $\mathfrak{A}$ is quadruple $\langle A, P, E, F\rangle$ where A is an L-structure, P is an additional sort, E is a binary relation on AxP and F is a family of subsets of P. Without loss of generality P can be identified with a family of subsets of A via E and then it makes sense to speak of F as a filter on P. Letting the variables s_i range over P and identifying $s_i(x_j)$ with $x_j E s_i$ we have an obvious interpretation of $L(aa)$ and L^P formulas on weak structures, taking as "countable sets" the elements of P and interpreting $aas\varphi(s)$ by

$$\mathfrak{A} \models aas\varphi(s) \quad \text{iff} \quad \{X \in P : \langle \mathfrak{A}, X\rangle \models \varphi(X)\} \in F .$$

$L(aa)$ is the culmination of the search of extensions of $L(Q_1)$ which hopefully satisfy the interpolation theorem.

Our main result is:

Theorem 1.3.
(i) *Not* CRAIG $(L(Q_1), L_{\infty\omega}(aa)$.
(ii) *Assuming* $MA_{\aleph_1}$ *and* $2^{\aleph_0} > \aleph_1$.
Not Δ Int$(L(Q_1), L_{\infty\omega}(aa))$.
(iii) *Not* BETH $(L(Q_1), L_{\infty\omega}(aa))$.

This improves many theorems for sublogics of $L(aa)$ and solves a problem left open in [3]. The proof will be given in Section 3. A presentation of 1.3(i) can also be found in [14].

Among the sublogics are:

1. $L(Q_1)$
Here not Δ-Int(LQ_1, LQ_1) is easy and was observed by many people. Not BETH(LQ_1, LQ_1) was proved by Friedmann [10].

2. In [23] a quantifier,

$$Q^{B(n,m)} x_1, \ldots, x_n, y_1, \ldots, y_m (x_1, \ldots, x_n, y_1, \ldots, y_m)$$

binding $n+m$ variables, was introduced.
It's semantics is defined by $\models Q^{B(n,m)}\bar{x}, \bar{y}\,\varphi(\bar{x}, \bar{y})$ iff there exists a countable set $C \subsetneqq A$ such that for every $a_1, \ldots, a_n \in A$ there are $c_1, \ldots, c_m \in C$ such that if $\mathfrak{A} \models \exists \bar{y}\,\varphi(\bar{a}, \bar{y})$ then $\mathfrak{A} \models \varphi(\bar{a}, \bar{c})$. Obviously this quantifier is expressible in L^P.
Ebbinghaus [8] proved not CRAIG$(L(Q_1), L[Q^{B(n,m)}, n, m \in \omega])$. His example is our starting point.

3. L^P
Though Theorem 1.3 applies to L^P there are natural weakened Definability theorems which derive from Lindström's alternative proof of Craig's theorem via

his characterization of $L_{\omega\omega}$ (cf. [19]). The same applies for $L(Q_1)$ which was first proved by Stavi and for $L(aa)$ in [19]. Similar results were also obtained by Caicedo [6].

2. Some Ehrenfeucht Games

To prove Theorem 1.3 we shall use a Back and Forth criterion for elementary equivalence in $L_{\infty\omega}(aa)$ which is taken from [20], and was independently found by Weese and Seese as well as Caicedo [6]. A slightly different game was introduced by Kaufmann [14]. The Back and Forth arguments described here have several ancestors: the Ehrenfeucht-Fraisse game for predicate calculus, its extension to $L(Q)$ due to several authors among which are Vinner and Slomson, the observation that this can be extended to arbitrary monotone quantifiers as described in [24]. This will be put together to yield our criterion for $L(aa)$ and $L_{\infty\omega}(aa)$.

Ziegler [33] designed a Back and Forth argument for topological logic and its dualized version yields our criterion for L^p.

Let $\mathfrak{A}_i = \langle A_i, P_i, E_i, F_i \rangle$ $(i=0,1)$ be two weak structures. Then $\bar{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ $(n \in \omega+1)$ is the following game:

There are two players, I and II. The length of the game is n. In the k^{th} move I chooses $i \in \{0,1\}$ and either $a_k^i \in A_i$ or $f_k^i \in F_i$.

II replies with the choice of $a_k^{1-i} \in A_{1-i}$ or $f_k^{1-i} \in F_{1-i}$ respectively. In the first case the move is completed and the outcome is a_k^0, a_k^1. In the other case I continues with the choice of $s_k^{1-i} \in f_k^{1-i}$ and II replies with a choice of $s_k^i \in f_k^i$ and the outcome is s_k^0, s_k^1.

After n moves we have an outcome sequence

$$\begin{matrix} x_0^0, x_1^0, \ldots, x_{n-1}^0 \\ x_0^1, x_1^1, \ldots, x_{n-1}^1 \end{matrix} \quad \text{with} \quad x_j^i = \begin{cases} a_j^i & \text{according to the} \\ s_j^i & \text{type of the move.} \end{cases}$$

II has won if the map

$$a_k^0 \mapsto a_k^1 \quad (\text{for } x_k^i = a_k^i) \text{ is an } L\text{-isomorphism}$$

from $\{a_0^0, \ldots, a_n^0\}$ onto $\{a_0^1, \ldots, a_n^1\}$ and

$$\mathfrak{A}_0 \models a_k^0 E s_j^0 \quad \text{iff} \quad \mathfrak{A}_1 \models a_k^1 E s_j^1$$

(for $x_k^i = a_k^i$ and $x_j^i = s_j^i$ and $k, j < n$).

The game $\overset{+}{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ is similar but instead of f_k^i I chooses directly $s_k^i \in P_i$ and II replies with $s_k^{1-i} \in P_{1-i}$ and the outcome is s_k^{i+}, s_k^{i-} in the second type of the move. Here the $+(-)$marks which set has been choosen by player I(II). In this game II has won if the map $a_k^0 \mapsto a_k^1$ is an L-isomorphism and $\mathfrak{A}_i \models a_k^i E s_j^i$ implies $\mathfrak{A}_{i-1} \models a_k^{1-i} E s_j^{1-i}$ provided that $x_j^i = s_j^{i+}$.

If II has a winning strategy for $\bar{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ or $\overset{+}{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ we write $\mathfrak{A}_0 \underset{n}{\cong} \mathfrak{A}_1(\bar{o})$ or $\mathfrak{A}_0 \underset{n}{\cong} \mathfrak{A}_1(\overset{+}{o})$ respectively.

In the following we prove our main theorems for $L(aa)$ and $\bar{o}_n$, the proofs for L^P and $\overset{+}{o}_n$ are exactly the same as for topological logic and are verified in [12]. Throughout the rest of this paper we assume L to be finite and relational (including individual constants).

Theorem 2.1.
(i) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L(aa))$ *iff for all* $n \in \omega\, \mathfrak{A}_0 \underset{n}{\simeq} \mathfrak{A}_1(\bar{o})$.
(ii) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L_{\infty\omega}(aa))$ *iff* $\mathfrak{A}_0 \underset{\omega}{\simeq} \mathfrak{A}_1(\bar{o})$.
(iii) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^P)$ *iff for all* $n \in \omega\, \mathfrak{A}_0 \underset{n}{\simeq} \mathfrak{A}_1(\overset{+}{o})$.
(iv) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^P_{\infty\omega})$ *iff* $\mathfrak{A}_0 \underset{\omega}{\simeq} \mathfrak{A}_1(\overset{+}{o})$.

To prove this we need a few lemmata.

Let $\sum\limits_{n,k_1,k_2}$ be the set of $L(aa)$-formulae with exactly $x_1, \ldots, x_{k_1}, s_1, \ldots, s_{k_2}$ as its free variables and $n-(k_1+k_2)$ bound variables. For $x: k_1 \to k_2$, let $\prod\limits_{n,k_1,\chi}$ be the set of L^P-formulas with exactly $x_1, \ldots, x_{k_1}, s_1, \ldots, s_{k_2}$ as its free variables, and $n-(k_1+k_2)$ bound variables; s_i occurs only positively (negatively) if $\chi(i)=1$ ($\chi(i)=0$).

Lemma 2.2. $\sum\limits_{n,k_1,k_2}$ *and* $\prod\limits_{n,k_1,\chi}$ *are, up to logical equivalence, finite.*

Proof. For $n-(k_1+k_2)=0$ the formulae are quantifierfree, so the lemma is true as for predicate logic. For the induction step we use Theorem 1.2. □

Let us denote by $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$ a possible outcome of $\bar{o}_k(\mathfrak{A}_0, \mathfrak{A}_1)$ ($\overset{+}{o}_k(\mathfrak{A}_0, \mathfrak{A}_1)$) where $k=k_1+k_2$ and $\bar{a}^i = a^i_1, \ldots, a^i_{k_1}$, $\bar{s}^i = s^i_1, \ldots, s^i_{k_2}$. We write $\mathfrak{A}_0 \underset{n}{\simeq} \mathfrak{A}_1(o)$ over $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$ if $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$ is a winning position for $\bar{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ and similarily for $\overset{+}{o}_n$.

Lemma 2.3. *If* $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$ *is a winning position for* $\bar{o}_n(\mathfrak{A}_0, \mathfrak{A}_1)$ *then for every* φ *in* $\sum\limits_{n,k_1,k_2}$ *we have*

$$\mathfrak{A}_0 \models \varphi(\bar{a}^0, \bar{s}^0) \quad \textit{iff} \quad \mathfrak{A}_1 \models \varphi(\bar{a}^1, \bar{s}^1).$$

Proof. The lemma is proved by induction on $n-(k_1+k_2)$ and follows closely the proof in [9] or [24]. The case $n-(k_1+k_2)=0$ and all the cases for the usual connectives are left for the reader. Now assume the lemma proved for $m=n-(k_1+k_2)$ and $\mathfrak{A}_0 \models aas\varphi(s, \bar{a}^0, \bar{s}^0)$. By the definition of satisfaction for $L(aa)$ there is $f \in F_0$ such that for all $s \in f$ we have $\mathfrak{A}_0 \models \varphi(s, \bar{a}^0, \bar{s}^0)$. Let I choose f and let II reply with $g \in F_1$ according to the winning strategy for $\bar{o}_{n+1}(_0, _1)$. If I now choose $t \in g$ and II replies with $s \in f$ then, by induction hypothesis and the winning strategy, we have $\mathfrak{A}_0 \models \varphi(s, \bar{a}^0, \bar{s}^0)$ iff $\mathfrak{A}_1 \models \varphi(t, \bar{a}^1, \bar{s}^1)$. Hence for all $t \in g$ we have $\mathfrak{A}_1 \models \varphi(t, \bar{a}^1, \bar{s}^1)$ and $g \subset \{s : \mathfrak{A}_1 \models \varphi(s, \bar{a}^1, \bar{s}^1)\} \in F_1$, therefore $\mathfrak{A}_1 \models aas\varphi(s, \bar{a}^1, \bar{s}^1)$. The symmetric case is similar. □

Lemma 2.4. *If $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$ are such that for $\varphi \in \sum_{n,k_1,k_2}$ we have $\mathfrak{A}_0 \models \varphi(\bar{a}^0, \bar{s}^0)$ iff $\mathfrak{A}_1 \models \varphi(\bar{a}^1, \bar{s}^1)$ then $\mathfrak{A}_0 \simeq \mathfrak{A}_1(\bar{o})$ over $(\bar{a}^0, \bar{a}^1, \bar{s}^0, \bar{s}^1)_{k_1,k_2}$.*

Proof. Again we restrict ourselves to the only nontrivial step and proceed by induction on $n-(k_1+k_2)$. Assume I chooses $f \in F_0$. Put φ_s

$$\varphi_s = \bigwedge\!\!\!\bigwedge \Big\{\varphi \in \sum_{n,k_1,k_0+1} : \mathfrak{A}_0 \models \varphi(s, \bar{a}^0, \bar{s}^0)\Big\} \quad \text{and} \quad \psi = \bigvee\!\!\!\bigvee \{\varphi_s : s \in f\}.$$

By Lemma 2.2 we can assume ψ to be finite. Put $f' = \{s \in P_0 : \mathfrak{A}_0 \models \psi(s, \bar{a}^0, \bar{s}^0)\}$. $f' \in F_0$ since $f \subsetneqq f'$. Therefore we have $\mathfrak{A}_0 \models \mathrm{aas}\psi(s, \bar{a}^0, \bar{s}^0)$ and, since $aas\psi \in \sum_{n,k_1,k_2}$, $\mathfrak{A}_1 \models aas\psi(s, \bar{a}^1, \bar{s}^1)$. We now let II choose $g = \{s \in P_1 : \mathfrak{A}_1 \models \psi(s, a^1, s^1)\} \in F_1$. If I now chooses $t \in g$ there is an $s' \in f$ such that $\mathfrak{A}_1 \models \varphi_{s'}(t, \bar{a}^1, \bar{s}^1)$ and we let II choose any $s \in f'$ such that $\mathfrak{A}_0 \models \varphi_{s'}(s, \bar{a}^0, \bar{s}^0)$. $\square$

With these and similar lemmata for L^p we can prove (i) and (iii) of Theorem 2.1 (ii) and (iv) are proved from this in a similar way as Karp's theorem in [2].
Note that we did not use more of the properties of F, but that it is a monotone family over P.
Note also that the game for L^p functions also for interpretations in other cardinals. This will be used in Section 4.

Let us denote by $L^{p,\alpha}$ the logic which syntactically looks like L^p, but in the semantics the sets range over the sets of cardinality $<\aleph_\alpha$. $L^{p,\alpha}_{\infty\omega}$ is defined similarly. So $L^p = L^{p,1}$ and the game $\overset{+}{o}_\alpha$ is the obvious modification of $\overset{+}{o} = \overset{+}{o}_1$.

Theorem 2.5.
(i) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^{p,\alpha})$ *iff for all* $n \in \omega$ $\mathfrak{A}_0 \underset{n}{\simeq} \mathfrak{A}_1(\overset{+}{o}_\alpha)$.
(ii) $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^{p,\alpha}_{\infty\omega})$ *iff* $\mathfrak{A}_0 \underset{\omega}{\equiv} \mathfrak{A}_1(\overset{+}{o}_\alpha)$.

The proof is like the proof of 2.1. As an application of this we get:

Theorem 2.6.
(i) *If* $\mathfrak{A}_i \equiv \mathfrak{B}_i(L^{p,\alpha})$ $(i=0,1)$
then $[\mathfrak{A}_0, \mathfrak{A}_1] \equiv [\mathfrak{B}_0, \mathfrak{B}_1]$ $(L^{p,\alpha})$.
(ii) *If* $\mathfrak{A}_i \equiv \mathfrak{B}_i(L^{p,\alpha}_{\infty\omega})$ $(i=0,1)$
then $[\mathfrak{A}_0, \mathfrak{A}_1] \equiv [\mathfrak{B}_0, \mathfrak{B}_1]$ $(L^{p,\alpha}_{\infty\omega})$.
(iii) *If* $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^{p,\alpha})$ *then*

$$P(\mathfrak{A}_0, \mathfrak{A}_1) \equiv \bar{P}(\mathfrak{A}_0, \mathfrak{A}_1)\ (L^{p,\alpha})$$

Here $P(\mathfrak{A}_0, \mathfrak{A}_1)$ *and* $\bar{P}(\mathfrak{A}_0, \mathfrak{A}_1)$ *is the construction described in Section 3.*

In the last section we shall show that the corresponding theorem for $L(aa)$ fails, unless one changes the definition of the pair $[\mathfrak{A}, \mathfrak{B}]$.

3. Proof of Theorem 1.3[1]

In this section we give a proof that both Beth's and Craig's definability theorem fail for $L(aa)$. The construction of the counterexamples is based on Shelah's simplification of [8] and Section 5 of [21].
For the rest of this section trees are partially ordered sets with a root. They need not necessarily be wellfounded, but they satisfy

$$\forall xyz(x \leqq z \wedge y \leqq z \Rightarrow x \leqq y \vee y \leqq x).$$

We now define two classes of trees of cardinality $\geqq \omega_1$. K_1 are the trees with at least one ω_1-like branch and K_2 are the trees where there is an order preserving map from the tree into the rationals.

Proposition 3.1. *Let L consist of $\leqq$ only. K_1 and K_2 are both in $PC(L(Q_1))$ and disjoint.*

Proof. Let R, $\prec$ be new binary relation symbols. For K_1 we say: (1.1) For every x the set $R(x, -)$ is totally ordered by $\leqq$. (1.2) There is an x such that $R(x, -)$ is an ω_1-like ordering. Clearly (1.1) is expressible in $L_{\omega\omega}$ and (1.2) in $L(Q_1)$.

For K_2 we say:
(2.1) The range of R is totally ordered by $\leqq$.
(2.2) R is a total function and order preserving.
(2.3) The domain of R is totally ordered by $\prec$ and is isomorphic to the rationals.
(2.4) The tree is uncountable.
Clearly (2.1) and (2.2) are expressible in $L_{\omega\omega}$ and (2.3), (2.4) in $L(Q_1)$. For (2.3) we add a branch to the tree, i.e., we use relativized PC-classes.
Clearly also, $K_1 \cap K_2 = \emptyset$. $\square$
Now let us construct two trees T_1, T_2 with $T_i \in K_i (i=1,2)$ and $T_1 \equiv T_2(L_{\infty\omega}(aa))$.
Recall that a tree T is *normal* if
(i) every branch is well ordered.
For $x \in T$ put $\hat{x} = \{y \in T : y < x\}$ and $o(x)$ be the order-type of $\hat{x}$. We set $o(T) = \sup\{o(x)+1 : x \in T\}$.
(ii) $o(T) = \omega_1$
Let α, β, δ be ordinals, $U_\alpha = \{x \in T : o(x) = \alpha\}$.
(iii) For each $\alpha < \omega_1$, U_α is at most countable, U_0 has exactly one element.
Put $T_x = \{y \in T : y \geqq x\}$, $T_\alpha = \{y \in T : o(y) < \alpha\}$.
(iv) For each $x \in T$, $o(T_x) = \omega_1$.
(v) Each $x \in T$ has exactly $\aleph_0$ immediate successors.
(vi) if $o(x) = o(y) = \delta$ is limit and $\hat{x} = \hat{y}$ then $x = y$.
Normal trees were introduced by Kurepa. A reference for what we need is [11].

[1] The first author is endebted to the referee of [20] and D. Giorgetta, for very valuable remarks. [20] has been incorporated in this paper.

Lemma 3.2. *Let T, T' be two normal trees.*
(i) *For all $\alpha<\omega_1$ there is an isomorphism $g:T_\alpha\to T'_\alpha$.*
(ii) *If $\alpha<\beta<\omega_1$ and $g:T_{\alpha^+}\to T'_{\alpha^+}$ is an isomorphism then there is an isomorphism $h:T_\beta\to T'_\beta$ with $h\restriction_{T_\alpha}=g$.*

A proof may be found in [11], but it is straightforeward using a Cantor type argument.

Theorem 3.3. *If T^1 and T^2 are two normal trees then $T^1\equiv T^2(L_{\infty\omega}(aa))$.*

Proof. We use Theorem 2.1(ii) and prove $T^1 \underset{\omega}{\cong} T^2(\bar{o})$. Let $(\bar{a}^1,\bar{s}^1,\bar{a}^2,\bar{s}^2)_{k_1,k_2}$ be an outcome already played according to a winning strategy. We describe the winning strategy inductively over $n=k_1+k_2$. Put $X_i=\bar{s}^i\cup\{\bar{a}^i\}$ $(i=1,2)$. W.l.o.g. we can assume that $X_i\subseteq T^i_\alpha$ for some $\alpha\in\omega_1$ and $g_\alpha:T^1_\alpha\to T^2_\alpha$ is an isomorphism such that $g_\alpha(\bar{a}^1)=\bar{a}^2$ and $g_\alpha(\bar{s}^1)=g_\alpha(\bar{s}^2)$ (abusing vectorial notation). If player I now chooses $a\in T^1_\alpha$ II chooses $g_\alpha(a)$. If player I chooses $a\in T^1_\beta(\beta>\alpha)$ we can find g_α extending g_α as in 3.2(ii) and II chooses $g_\beta(a)$. If player I chooses in $a\in T^2$ we use g_α^{-1} or g_β^{-1} respectively.
If player I chooses $f_1\in F_1$, for each $s\in f_1$ there is $\beta(s)\in\omega_1$ such that $X_1\cup s\in T^1_\beta$. Let $g_\beta:T^1_\beta\to T^2_\beta$ be an isomorphism extending g_α. Now let II play

$$f_2=\bigcup_{\beta<\omega_1} f_2^\beta \quad\text{with}\quad f_2^\beta=\{g_\beta(s)\subseteq T^2_\beta : s\in f\wedge s\subseteq T^1_\beta\}.$$

Clearly f_2 is in F_2.
If player II now chooses $t\in f_2$, $t\subset T^2_\beta$, II chooses $g_\beta^{-1}(t)$. If player I chooses $f_2\in F_2$ the argument is similar. $\square$

To prove 1.3(i) we only have to construct normal trees $T^i\in K^i(i=1,2)$.
For T^2 take any normal Aronszajn-tree which is special.
For their existence cf. [13].
For T^1 we start with ω_1 many copies of T^2, take their disjoint union, and if $a_\alpha(\alpha<\omega_1)$ is the root of the copy no. α we put $a_\alpha<a_\beta$ for $\alpha<\beta<\omega_1$. Clearly T^1 is normal and in K^1. This proves 1.3(i). To prove 1.3(ii) we use a result of Baumgartner [4] to the effect that $MA_{\aleph_1}$ and $2^{\aleph_0}>\aleph_1$ imply that K_1 and K_2 are complementary if restricted to ranked trees of length ω_1 with countable levels. But the latter is clearly $PC(L(Q_1))$.
To prove 1.3(iii) we have to work more. In [21] a very general theorem was proved which says that subject to a certain hypothesis Beth's theorem implies Craig's theorem for compact logics. The hypothesis was a Feferman-Vaught type theorem for a rather complicated sumlike operation. The general theorem fails for $L(aa)$, but for our special structures $\langle T^1,\leqq\rangle$ and $\langle T^2,\leqq\rangle$ it holds. Therefore we repeat the construction of Section 6 of [21] here for the special case. Let T_1^* and T_2^* be the expansions of the trees T^1, T^2 to structures as described in Proposition 3.1.

Let us redefine $\mathcal{N}_1 = P(T^1, T^2)$ and $\mathcal{N}_2 = \bar{P}(T^1, T^2)$ for this special case:
$\mathcal{N}_i = \langle N_i, \leqq, \prec, R, s, c\rangle$ is defined by:
$\leqq$, $\prec$ are partial orders of N_i.
R is a binary relation on N_i.

Here $\leqq$ is the partial order of the trees T_1^*, T_2^* and $\prec$ is an order between elements of T_2^* which will serve as the domain of the function whose graph is R. R together with $\prec$ make the tree into a ranked tree (cf. the proof of 3.1).
s is a function from N_i into $N_i \times N_i$ [as ternary relation, but we write for $s(x, y, z)$ $s(x) = (y, z)$] such that

(i) $s(c) = (c, c)$, $s(x) = (x, x)$ iff $x = c$.
(ii) $\exists x\, s(x) = (y, z)$ iff $s(y) = s(z)$
Denote by $N_i(a, b)$ the set $\{a' \in N_i : s(a') = (a, b)\}$.
(iii) The structure $\langle N_i(a, b), \leqq, \prec, R\rangle$ is isomorphic to T_j^* for $s(a) = s(b)$ and
$$j = \begin{cases} 1 & \text{if } (a, b) \in R \text{ or } a \prec b \\ 2 & \text{if } (a, b) \notin R \text{ and not } a \prec b. \end{cases}$$
(iv) $\leqq$, $\prec$, R on N_i are defined as the unions of the corresponding relations induced by the isomorphisms described in (iii).
(v) $(c, c) \in R$ in N_i iff $i = 1$.
(vi) For every $a \in N_i$ there is a natural number n such that $s_1^n(a) = (c, c)$ with
$s_1(a) = b$ iff $s(a) = (b, b')$
$s_1^1(a) = s_1(a)$
$s_1^{k+1}(a) = s_1(s_1^k(a))$.

To prove 1.3(iii) it is sufficient to prove the following Proposition 3.4 since by a result of [21] both R and $\prec$ can be defined implicitly. Now if R was definable explicitly with a formula $\theta(x, y)$ then we had $\mathcal{N}_1 \models \theta(c, c)$ and $\mathcal{N}_2 \models \neg\theta(c, c)$ contradicting 3.4.

Proposition 3.4. *$\mathcal{N}_1 \equiv \mathcal{N}_2(L_{\infty\omega}(aa))$ for $L = \{\leqq, s, c\}$.*

Proof. Call a set X in N_i *large* if (i) X is countable, (ii) X is closed under s, i.e. if $a' \in X$ and $s(a') = (a, b)$ then $a, b \in X$, and (iii) for all a, b $X \cap N_i(a, b)$ is either empty or of the form $(T_j)_\alpha$ for some $\alpha \in \omega_1$ independent of a, b where $T_j \cong N_i(a, b)$ for $L = \{\leqq, s, c\}$. We put $o(X) = \alpha$ in the latter case and $o(\phi) = 0$.

Lemma 3.5. *The collection of large sets in N_i is closed and unbounded.* □

Now the winning strategy is defined similarily as in the proof of Theorem 3.3. Let $T(\mathcal{N}_i)$ be the underlying ω-tree of $\mathcal{N}_i$, i.e. the nodes are the $N_i(a, b)$ for $a, b \in \mathcal{N}_i$ and s^{-1} is the successor function. Clearly $T(\mathcal{N}_1) \cong T(\mathcal{N}_2)$ by an isomorphism, say φ, and there is a natural projection $\pi : \mathcal{N}_i \to T(\mathcal{N}_i)$. Put $\mathcal{N}_i(X) = \langle N_i \cap X, \leqq\restriction_X, \subseteq\restriction_x\rangle$ for any $X \subseteq N_i$. This is possible by (vi).

Lemma 3.6. *Let $X_i \subseteqq \mathcal{N}_i$ be large ($i \in 1, 2$), $\varphi(\pi(X_1)) = \pi(X_2)$ and $o(X_1) = o(X_2)$. Then there is an isomorphism $g: \mathcal{N}_i(X_i) \to \mathcal{N}_2(X_2)$. Furthermore if $X_i \subseteqq Y_i \subseteqq N_i$ and Y_i large $\varphi(\pi(Y_1) = \pi(Y_2)$ and $o(Y_1) = o(Y_2)$ then there is an isomorphism $h: \mathcal{N}_1(Y_1) \to \mathcal{N}_2(Y_2)$ extending g.*

The lemma is easely proved using Lemma 3.2. To prove Proposition 3.4 we proceed as in the proof of 3.3 replacing the T^i_β by large X_i with $o(X_i) = \beta$ and using φ. □

4. L^p in the α-Interpretation

$L^{p,\alpha}$ (L^p in the $\aleph_\alpha$-interpretation) shares many properties of L^p in the $\aleph_1$-interpretation. Similarly as for $L(Q_\alpha)$ (in the $\aleph_\alpha$-interpretation) we have

Theorem 4.1 (GCH). *L^p in the $\aleph_{\alpha+1}$-interpretation is countably compact provided $\aleph_\alpha$ is regular.*

The proof uses Chang's Two cardinal theorem as described in [7, Chapter 7].
In fact the proof gives that the same axioms Γ_p give completeness for any $\aleph_{\alpha+1}$ such that $\aleph_\alpha$ is regular.
This transfer principle gives us immediately for $\aleph_{\alpha+1}$ as above

Theorem 4.2.
(i) *Not* CRAIG $(L(Q_{\alpha+1}), L^{p,\alpha+1}_{\infty})$.
(ii) *Not* BETH $(L(Q_{\alpha+1}), L^{p,\alpha+1}_{\infty\omega})$ *(using* GCH*)*.

The question now is if this can be proved with fewer or no set theoretical assumptions.

Discussion 4.3. Much of the model theory for $L(Q_1)$ (besides compactness) can be carried over to the $L(Q_\alpha)$ for arbitrary $\aleph_\alpha$ without further set theoretic assumptions.
Among them
- The Back and Forth criteria (due to Lipner [16], Vinner [30], and Slomson [29].
- Several forms of Feferman–Vaught type theorems, as URP, FVP, FVT (cf. Wojciechowska [31]).
- The counterexample to Craig's theorem or even Δ-Interpolation. To see this put $K_\alpha = \{\mathfrak{A}/\mathfrak{A} = \langle A, \equiv\rangle$ where $\equiv$ is an equivalence relation with each class of cardinality $\geqq \aleph_\alpha\}$.
  $K_1 = \{\mathfrak{A} \in K_\alpha / \equiv$ has $< \aleph_\alpha$ many equivalence classes$\}$.
  $K_2 = \{\mathfrak{A} \in K_\alpha / \equiv$ has at least $\aleph_\alpha$ many equivalence classes$\}$.
  $K_1 \cup K_2 = K_\alpha$, $K_1 \cap K_2 = \emptyset$ and K_1, K_2 are PC-classes. One easely finds $\mathfrak{A}_1 \in K_1$, $\mathfrak{A}_2 \in K_2$ such that $\mathfrak{A}_1 \equiv \mathfrak{A}_2(\mathrm{L}_{\infty\omega}(\mathrm{Q}_\alpha))$.

With the technique of [21, Section 6] we get

$$\text{not BETH}(L(Q_1), L_{\infty\omega}(Q_1)).$$

Note that Yasuhara [32] proved that $L(Q_\alpha)$ *without* equality satisfies Craig's theorem, provided, $\aleph_\alpha$ is singular. For $\aleph_\alpha$ regular the example above works without equality.
Concerning compactness one observes that 4.1 is equivalent to Chang's two cardinal theorem, hence all the independence results for the latter carry over (cf. [1]).
A similar situation occurs for $L^{P,\alpha}$.
To carry over the argument from the previous section it is sufficient to know of the existence of special κ^+-Aronszajn-trees. But Mitchell [25] has shown that they need not exist.

Theorem 4.4[1]. *Assume there are special κ^+-Aronszajn-trees and $\kappa^+ = \aleph_{\alpha+1}$. Then*
(i) *Not* CRAIG$(LQ_{\alpha+1}, L^{P,\alpha+1}_{\infty\omega})$.
(ii) *Not* BETH$(LQ_{\alpha+1}, L^{P,\alpha+1}_{\infty\omega})$.
So let us resume what we have got: Theorem 4.4(i) *and* (ii) *hold under the hypothesis of* GCH *for $\aleph_\alpha$ regular, and $V=L$ for all $\aleph_\alpha$ (cf.* [7]*).*

Problem 4.5. *Find basically new counterexamples for* CRAIG$(L(Q_\alpha), L^{P,\alpha}_{\infty\omega})$ *which function in* ZFC *alone.*

The rest of this section is devoted to $L^{P,\alpha}$ for a limit cardinal. The results here are due to Makowsky and Stavi. The main result for $L^{P,\alpha}$, $\aleph_\alpha$ singular is

Theorem 4.6. *Let $\aleph_\alpha$ be singular. Then*
(i) $L(Q_{\alpha+1}, Q_\alpha) < \Delta(L^{P,\alpha})$ *i.e. there are PC-classes K_1, K_2 for $L^{P,\alpha}$ with $K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 =$ all sets, K_1 contains all sets of cardinality $\leqq \aleph_\alpha$, K_2 all sets of cardinality $\geqq \aleph$.*
(ii) $\Delta(L^{P,\alpha}) \subsetneqq L^{P,\alpha+1}$.

Corollary 4.7. *Under the assumption of 4.4*
(i) *Not* Δ-Int$(LQ_{\alpha+1}, L^{P,\alpha}_{\infty\omega})$.
(ii) *Not* BETH$(LQ_{\alpha+1}, L^{P,\alpha}_{\infty\omega})$.

Proof of Corollary. (i) The counterexample in Discussion 4.3 for $L(Q_{\alpha+1})$ turns out to function as well using Theorem 2.5(ii) for $\aleph_\alpha$.
(ii) This is done as in Section 3.
To prove the Theorem 4.6 we need a definition and some Lemmata. Here it helps to be familiar with [23].

[1] Stavi noted that the result can be extended to Not CRAIG$(LQ_{\alpha+1}, L_{\infty\alpha+1}(aa))$ using the same counterexamples, by a more refined use of Back and Forth techniques.

Denote by $P_{<\kappa}(A)$ the set $\{X \subsetneqq A / \bar{\bar{X}} < \kappa\}$.
The property $P(\kappa)$ holds if for every set A of cardinality κ there is a $S \subsetneqq P_{<\kappa}(A)$ with $\operatorname{card}(S) = \kappa$ and for every $X \in P_{<\kappa}(A)$ there is $Y \in S$ with $X \subsetneqq Y$ [i.e. S is cofinal in $P_{<\kappa}(A)$].

Lemma 4.8. *κ is singular iff $P(\kappa)$ does not hold.*

Proof. Let $\mathrm{cf}(\kappa) = \aleph_\alpha$, and $S \subset P_{<\kappa}(A)$ be cofinal. Let $\{s_j^m\}$, $m < \alpha$, $j < \kappa$ be an enumeration of all members of S such that S_j^m has cardinality $\leqq \aleph_m$. Now $\bigcup_{j \leqq \aleph_n} s_j^m \neq A$ since its cardinality is $\leqq \aleph_n \cdot \aleph_m$, so there is $\chi_n^m \notin \bigcup_{j \leqq \aleph_n} S_j^m$.
Now put $A_0 = \{\chi_n^m \mid m < \alpha,\ n < \alpha\}$.
Obviously there is no $X \in S$ with $A_0 \subsetneqq X$.
For the other direction well order A of order type κ and take for S the initial segments. □

Lemma 4.9. *There is a sentence φ in $L^{P,\alpha}$ which has always models of cardinality $\aleph_{\alpha+1}$, but has a model of cardinality $\aleph_\alpha$ iff $P(\aleph_\alpha)$ holds.*

Proof. Let R be a binary predicate symbol and let φ_1 be $\forall x \neg Q_1 y R(x, y)$ and φ_2 be

$$\neg \exists s \forall x (\exists y (\neg R(x, y)) \Rightarrow \exists y (y \in S \wedge \neg R(x, y))$$

and φ be $\varphi_1 \wedge \varphi_2$. φ is clearly in $L^{P,\alpha}$. Now let $\mathfrak{A} \models \varphi$ and $|\mathfrak{A}|$ has cardinality $\aleph_\alpha$. Then $S = \{\{y : R(x, y)\} : x \in A\}$ is cofinal and $P(\aleph_\alpha)$ holds. Conversely if $P(\aleph_\alpha)$ holds we easily construct a model of cardinality $\aleph_\alpha$. To construct a model of φ in $\aleph_{\alpha+1}$ is straightforward. □

Note that a somewhat weaker logic than $\underline{L}^{P,\alpha}$ is sufficient.
Now to prove 4.6 we use 4.9 to make $\{S : \bar{\bar{S}} \geqq \aleph_{\alpha+1}\}$ a PC-class. Its complement is PC as well taking $\aleph_\alpha$-like orderings and their initial segments. □

Problem 4.10. *Does $L^{P,\alpha}$ for $\aleph_\alpha$ inaccessible satisfy any interpolation theorem?*

5. The Magidor-Malitz Quantifier and L^n

Magidor and Malitz defined generalized quantifiers Q^n $(n < \omega)$ with the satisfaction clause $\mathfrak{A} \models Q_\alpha^n x, \ldots, x_n \varphi(x, \ldots, x_n)$ iff there is a set $X \subsetneqq |\mathfrak{A}|$ of cardinality $\geqq \aleph_\alpha$ such that $\forall a, \ldots, a_n \in X$ we have $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$. Parallel to L^P it is natural to look at the following logic L^n which is defined as L^P but the clause $(\mathrm{iii})_+$ is replaced by
$(\mathrm{III})_-$ if s occurs only negatively in $\varphi(s) \in L^n$ then $\exists s \varphi(s^-) \in L^n$.
Satisfaction is defined by
$(\mathrm{III})_-$ $\mathfrak{A} \models_\alpha \exists s \varphi(s)$ iff there is an $X \subsetneqq A$ of cardinality $\geqq \aleph_\alpha$ such that $\mathfrak{A} \models \varphi(X)$.
Magidor-Malitz and Shelah have proved [17, 27] that

Theorem 5.1.
(i) [17] ($\diamondsuit_{\aleph_1}$)
$L^{<\omega} = L_{\omega\omega}[Q^n]_{n<\omega}$ *is countably compact in the* $\aleph_1$ *interpretation.*
(ii) [27] *(*$\diamondsuit_{\aleph_\alpha}$ *and* $\aleph_\alpha$ *regular)* $L^{<\omega}$ *is countably compact in the* $\aleph_\alpha$*-interpretation.*
(iii) [27] *(*$\aleph_\alpha$ *inaccessible)* $L^{<\omega}$ *is countably compact in the* $\aleph_\alpha$*-interpretation.*

Note that $L^{\prime\prime}$ is an extension of $L^{<\omega}$ by

$$Q^n x, \ldots, x_n \varphi(x, \ldots, x_n) \Leftrightarrow \exists s \forall x, \ldots, x_n$$
$$\left(\bigwedge_{i=1,\ldots,n} x_i \in S \Rightarrow \varphi(x, \ldots, x_n)\right).$$

The authors conjectured that using $\diamondsuit_{\aleph_1}$ one could prove countable compactness of $L^{\prime\prime}$. But Stavi has constructed the following counterexample, which we include here with his kindest permission.

Theorem 5.2[2]**.** (Stavi). *There is an* $L^{\prime\prime}$*-sentence* φ *with* $L = (<, P_1 \ldots P_k)$ *such that whenever* $\mathfrak{A} \models \varphi$ *then*

$$\mathfrak{A} \restriction_{L_0} \cong \langle \omega_1, < \rangle \ (L_0 = \{<\}),$$

hence $L^{\prime\prime}$ *is not compact.*

Proof. Let φ be $\varphi_0 \wedge \varphi_1 \wedge \varphi_2$ with φ_0 the conjunction of a finite number of axioms or theorems of ZFC comprising what we shall need to know about ω_1 and its subsets. (φ_0 is thus first order.)
φ_1 says that the countable ordinals of the model are ω_1-like ordered (in the real universe). This can be expressed by an $L(Q_1)$-sentence, hence in $L^{\prime\prime}$.
φ_2 says that every (external) uncountable subset of the countable ordinals of the model has an internal uncountable subset, i.e.
$\forall s \exists x[(\forall y \in s)\ (y < \omega_1) \rightarrow (x$ is a uncountable subset of ω_1 and $(\forall z \in x)\, z \in s))]$.
Since s occurs only positively in the above and s is quantified by $\forall$ this formula is in $L^{\prime\prime}$.

Claim. Let $\mathfrak{M} = \langle M, \in, \omega_1, < \rangle \models \varphi$ and $\psi(x)$ say that x is a countable ordinal, and $M_0 = \{x \in M : \mathfrak{M} \models \psi(x)\}$ then $\langle M_0, < \rangle \cong \langle \omega_1, > \rangle$.

Proof. By φ_1 $\langle M_0, < \rangle$ is an ω_1-like ordering.

Hence there is a strictly increasing sequence $A = \langle a_\alpha | \alpha < \omega_1 \rangle$ in $\langle M_0, < \rangle$. By φ_2 there exists a set $b \in M$ such that $\mathfrak{M} \models$"b is an uncountable subset of ω_1" and $b^m = \{x \in M | M \models x \in b\} \subsetneqq A$.
It is a theorem of ZFC (which we include in φ_0) that every uncountable subset of ω_1 is isomorphic to ω_1 as an ordered set. Now A is well-ordered (since it is an

[2] Malitz has found another counterexample independently.

external set) so b^m is well-ordered as well. But b^m is a subset of M_0 and both M_0 and b^m are internal, so $\langle M_0, <\rangle \cong \langle b^m, <\rangle$ inside the model and $\langle M_0, <\rangle$ is well-ordered and isomorphic to $\langle \omega_1, <\rangle$ (externally). □

That $L^{<\omega}$ did not satisfy even Δ-interpolation was proved by Makowsky and Magidor without set theoretic assumptions. A counterexample to Beth's theorem was constructed by Badger [1]. Whether the weak Beth property holds is still open. The rest of this section is devoted to a counterexample for

$$\mathrm{BETH}(L(Q_1), L^r).$$

Let $\mathfrak{A} \overset{\mathrm{neg}}{\underset{n}{\sim}} \mathfrak{B}$ be the dual equivalence relation of $\mathfrak{A} \overset{\alpha}{\underset{n}{\sim}} \mathfrak{B}$, $(n \in \omega + 1)$. *in* [20] it is proved that

Theorem 5.3.
(i) $\mathfrak{A} \equiv \mathfrak{B}(L^r)$ *for L finite and relational iff* $\mathfrak{A} \overset{\mathrm{neg}}{\underset{n}{\sim}} \mathfrak{B}$ *for all* $n < \omega$.
(ii) $\mathfrak{A} \equiv \mathfrak{B} L^r_{\infty\omega}$ *iff* $\mathfrak{A} \overset{\mathrm{neg}}{\underset{\omega}{\sim}} \mathfrak{B}$.

Let $K_1 = \{\langle A, <\rangle \mid A$ is a dense ordering without first nor last element and $\exists P$ (P is a countable dense subset of A)$\}$. Let $K_2 = \{\langle A, <\rangle \mid A$ is a dense ordering without first nor last element and $\exists P$ (P is an uncountable set of quadrupels coding disjoint rectangles on A^2)$\}$.

Proposition 5.4. (Kurepa).
K_1 *and* K_2 *are complementary disjoint PC-classes of* $L(Q_1)$.

Proof. The only non trivial step is to prove that they are complementary. Let Q be an uncountable set of disjoint rectangles and assume for contradiction that P is a countable dense subset of A.[3] We define a map $\sigma: Q \to P$ by $\sigma(a_1 a_2 b_1 b_2) =$ any $p \in (a_1 a_2)$ so there is some $p_0 \in P$ with $\sigma^{-1}(p)$ uncountable. But then the

$$\{(b_1 b_2)/\sigma(a_1 a_2 b_1, b_2) = p\}$$

form an uncountable set of disjoint intervals on A, since all the $(a_1 a_2 b_1 b_2) \in Q$ are disjoint.
Otherwise let Q be a countable maximal set of disjoint rectangles then either $\pi_1(Q)$ (first projection) or $\pi_2(Q)$ (second projection) is dense in A. For suppose not then there are (a_1, a_2), (b_1, b_2) such that $(a_1 a_2) \cap \pi_1(Q)$ and $(b_1 b_2) \cap \pi_2(Q)$ are empty. But then $(a_1 a_2 b_1 b_2)$ is a rectangle disjoint from all the rectangles in Q, which contradicts the maximality of Q. □

Now put $\mathfrak{A}_1 = \langle I, <\rangle$ to be the irrational numbers with their natural ordering and form also $\mathfrak{A}_2 = \langle I \cdot \omega_1, <\rangle$.

Proposition 5.5.
(i) $\mathfrak{A}_1 \in K_1, \mathfrak{A}_2 \in K_2$.
(ii) $\mathfrak{A}_1 \equiv \mathfrak{A}_2(L^r)$.

[3] W.l.o.g. all rectangles in Q are disjoint from the diagonal.

Proof. (i) is obvious. To prove (ii) one uses Theorem 3.3(i). The description of the winning strategy is rather involved. The critical case occurs when the first player chooses an uncountable set in $\mathfrak{A}_1$ which intersects any interval with only countably many points, i.e. an uncountable set $S' \subsetneqq A_1$ such that for every $x, y \in S'$, if $x<y$ then there are only countably many $z \in s'$ with $x<z<y$. The second player then replies with an uncountable set S $s \cdot t$:
1) All previously chosen points are smaller than any point in S, 2) if $x, y \in S$ then there we infinitely many $z \notin S$ with $x<z<y$, and 3) there is a z' such that for all $x>x'>z'$ there is a $y \in S$ with $x>y>x'$. The further details are tedious but straight foreward. □

To prove our main theorem of this section we need a Feferman–Vaught type theorem for L^n:

Theorem 5.6. *Let I be a set*

$$\mathfrak{A}_i \equiv \mathfrak{B}_i(L^n) \text{ for all } i \in I$$

then $[\mathfrak{A}_i, i \in I] \equiv [\mathfrak{B}_i, i \in I]$.

Proof. Again we use Theorem 3.3(i). We have to prove that $[\mathfrak{A}_i, i\in I]\overset{\mathrm{neg}}{\underset{n}{\sim}}[\mathfrak{B}_i, i\in I]$ for all $n \in \omega$. Fix n. Let g_i^n be a winning strategy for $\mathfrak{A}_i \overset{\mathrm{neg}}{\underset{n}{\sim}} \mathfrak{B}_i$. We construct now g^n for $\mathfrak{A} = [\mathfrak{A}_i, i\in I]\overset{\mathrm{neg}}{\underset{n}{\sim}}[\mathfrak{B}_i, i\in I] = \mathfrak{B}$.
For choices of points we use the g_i^n's. If player I chooses an uncountable set $S \subset A$ and for some $iS \cap A_i$ is uncountable, we use g_i^n again. If $S_i = S \cap A_i$ is countable for all $i \in I$ then for an uncountable $I_0 \subsetneqq I$ and for all $i \in I_0 S_i \neq \emptyset$. Let C be a choice set of the S_i for $i \in I_0$. Put $S' = \{x \in B \mid x = g_i^n(c) \text{ for some } c \in C\}$ S' is uncountable. One easily verifies that $g^n(S) = S'$ is a winning strategy.

Corollary (to the Proof:) 5.7. *In the notation of* [21, *Section 6*], *cf. also p. 21 in this paper.*

$$P(\mathfrak{A}_1, \mathfrak{A}_2) \equiv \bar{P}(\mathfrak{A}_1 \mathfrak{A}_2)(L^n). \quad \square$$

Theorem 5.8. *Not* BETH($L(Q_1), L^n$)

Proof. By theorem [21, Section 6] it suffices to prove:
1) There are two disjoint PC-classes in $L(Q_1)$, K_1, K_2 and structures $\mathfrak{A}_1 \in K_1$, $\mathfrak{A}_2 \in K_2$ such that $\mathfrak{A}_1 \equiv \mathfrak{A}_2(L^n)$.
Here we take those from Proposition 5.5.
2) $P(\mathfrak{A}_1 \mathfrak{A}_2) \equiv \bar{P}(\mathfrak{A}_1 \mathfrak{A}_2)(L^n)$ which we have from Corollary 5.7. □

6. Cofinality Quantifiers

In this section we give a simplified proof of the compactness of the cofinality quantifiers, introduced in [26]. The presentation is based on lectures of S. Shelah, held in Berlin in July 1977.

Let C be a class of regular cardinals. Let $Q^C xy\varphi(x, y)$ be a binary quantifier (as in [23]) with the following additional satisfaction clause:

$$\mathfrak{A} \models Q^C xy\varphi(x, y)[\bar{a}] \text{ iff } \varphi(\ulcorner, \urcorner)[\bar{a}] \text{ is a linear order}$$
$$\text{of its domain which has cofinality } \alpha \in C.$$

More formally put

$$D_\varphi(\bar{a}) := \{a \in |\mathfrak{A}| : \mathfrak{A} \models \exists y \varphi(a, y)[\bar{a}]\} \text{ and}$$
$$O_\varphi(\bar{a}) := \{(a, b) \in |\mathfrak{A}|^2 : \mathfrak{A} \models \varphi(a, b)[\bar{a}]\}.$$

Then $O_\varphi(\bar{a})$ linearily orders $D_\varphi(\bar{a})$ and cf $(D_\varphi(\bar{a}), O_\varphi(\bar{a})) \in C$.

Let $C_1, \ldots, C_n (n \in \omega)$ be convex classes of cardinals and $L^{**} = L_{\omega\omega}(Q^{C_1}, \ldots, Q^{C_n})$ be the logic obtained from $L_{\omega\omega}$ adding the formation rules for the $Q^{C_i}(i \leqq n)$ and the above satisfaction rules for each C_i. We call this the $(C_1, \ldots, C_n)$-interpretation for L^{**}.

Theorem 6.1. *L^{**} is (κ, ω)-compact for each $\kappa \in$ Card in every $(C_1, \ldots, C_n)$-interpretation.*

Let $\mathrm{Val}(C_1^i, \ldots, C_n^i)$ be the set of valid sentences of L^{**} in the $(C_1^i, \ldots, C_n^i)$-interpretation. Let B_n^i be the Boolean Algebra generated by $C_1^i, \ldots, C_n^i (i = 0, 1)$. Let

$$\varphi_0 : \{C_i^0, \ldots, C_n^0\} \to \{C_1^1, \ldots, C_n^1\}$$

be given by $\varphi_0(C_j^0) = C_j^1 (j = 1, \ldots, n)$ and $\varphi : B_n^0 \to B_n^1$ its natural extension.

Theorem 6.2. $\mathrm{Val}(C_1^0, \ldots, C_n^0) = \mathrm{Val}(C_1^1, \ldots, C_n^1)$ *iff φ is an isomorphism.*

Theorem 6.3. *For no $(C_1, \ldots, C_n)$-interpretation such that L^{**} properly extends $L_{\omega\omega}$ does* BETH(L^{**}, L^{**}) *hold.*

Remarks. 1. W.l.o.g. the $C_i (i \leqq n)$ can be assumed to be disjoint.
2. L^{**} is equivalent to $L_{\omega\omega}$ if we restrict ourselfs to countable structures. (For then the Q^{C_i}'s can be eliminated.)
The proofs of 6.1 and 6.2 given here depend on a theorem of Shelah [28].
Let $\mathfrak{A} = \langle A, P_1, \ldots, P_n, <_1, \ldots, <_n, \ldots \rangle$ be an L structure where P_i are unary predicates and $<_i$ is a linear order of P_i. We denote by $\mathrm{cf}(\mathfrak{A}) = (\lambda_1, \ldots, \lambda_n)$ the n-tuple of the $\lambda_i = \mathrm{cf}(P_i, <_i)$.
By $(\lambda_1, \ldots, \lambda_n) \to (\mu_1, \ldots, \mu_n)$ we abreviate the statement: For every $\mathfrak{A}$ with $\mathrm{cf}(\mathfrak{A}) = (\lambda_1, \ldots, \lambda_n)$ there is $\mathfrak{B} \equiv \mathfrak{A}(L_{\omega\omega})$ with $\mathrm{cf}(\mathfrak{B}) = (\mu_1, \mu_n)$.

Theorem 6.4. *If $\lambda_i (i \leqq n)$ are regular and distinct and $\mu_i (i \leqq n)$ regular then $(\lambda_1, \ldots, \lambda_n) \to (\mu_1, \ldots, \mu_n)$.*

Proof of 6.1. Let $C_1, \ldots, C_n$ be given. W.l.o.g. $C_l = [\lambda_{l-1}, \lambda_l)$ where $\lambda_i (\mathrm{i} = 0, \ldots, n)$ is either a regular cardinal or ∞. Let $\mathfrak{A}$ be a L-structure of cardinality $> \lambda_{n-1}$. We define an expansion $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ which is an analogue to Skolem expansions for classical logic.

(i) $<^A$ is a binary relation not from L which is a well-ordering of A of ordertype $\mathrm{card}(\mathfrak{A})+1$.

This makes $\mathfrak{A}$ into an ordinal and $<^A$ can be thought of as a membership relation inducing some set theory on $\mathfrak{A}$.

(ii) For every $\varphi \in L^{**}$, $\varphi = \varphi(x_1, \ldots, x_n)$, let $R_{\varphi(x_1,\ldots,x_n)}$ be a new n-ary relation not from $L \cup \{<\}$ and put
$R_{\varphi(x_1,\ldots,x_n)} = \{\bar{a} \in A / \mathfrak{A} \models \varphi[\bar{a}]\}$.

(iii) Let $\underline{\mathrm{Reg}}$ be a unary predicate not in $L \cup \{<\} \cup \{R_\varphi : \varphi \in L^{**}\}$ and put $\underline{\mathrm{Reg}}^A = \{\alpha \in \mathrm{card}(\mathfrak{A})+1 : \alpha \text{ regular}\}$.

(iv) $\underline{\mathrm{Let}}$ $\lambda_i (i=0, \ldots, n-1)$ constants not from L and $\lambda_i^A = \lambda_i$ and $\lambda_n = \mathrm{card}(\mathfrak{A})$. Here we use that $\mathfrak{A}$ is of cardinality $> \lambda_{n-1}$.

(v) Let F_φ, $\varphi = Q^{C_i}\psi(x, y)\bar{a}^n)$, be n-ary functions not from L and put $F_\varphi(\bar{a}^n) = \mathrm{cf}(Q_\psi(\bar{a}^n)$ if $\mathfrak{A} \models Q^{C_i} xy\psi(x, y, \bar{a}^n)$ and $F_\varphi(\bar{a}^n) = 2$ otherwise.

(vi) Let G_φ, $\varphi = Q^{C_i} xy\psi(x, y, \bar{a}^n)$ be unary functions not from $L \cup \{F_\varphi\}$ with n parameters and put
$G_\varphi : \mathrm{cf}(O_{\psi(x,y,\bar{a}^n)}) \to D_{\psi(x,y,\bar{a}^n)}$, $1-1$, orderpreserving and cofinal.

We denote by $\tilde{\mathfrak{A}}$ the structure $\mathfrak{A}$ expanded by (i)–(vi) and by $\tilde{L}$ its language.

Lemma 6.5. *Let $\tilde{\mathfrak{A}}$ be a given expanded structure and $\mu_1, \ldots, \mu_n$ be regular cardinals. Then there is $\mathfrak{B} \equiv \tilde{\mathfrak{A}}(L_{\omega\omega})$ such that whenever*

$$\mathfrak{B} \models \underline{\mathrm{Reg}}(a) \wedge \underline{\lambda}_{l-1} \leqq a \leqq \underline{\lambda}_l (l \leqq n)$$

then

$$\mathrm{cf}(\langle \{b \in B : \mathfrak{B} \models b < a\} < \rangle) = \mathrm{cf}(a)^{\mathfrak{B}} = \mu_l .$$

(Here we assume $\tilde{L}$ to be countable.)

Proof of 6.1. Let T be a theory in L^{**} of arbitrary cardinality, and $\{T_\alpha : \alpha < |T|\}$ an enumeration of the finite subsets of T. By assumption each T_α has a model. W.l.o.g. each T_α has a model of cardinality $\geqq \lambda_{n-1}$. For, let $\mathfrak{A}_\alpha \models T_\alpha$ and κ be any cardinal $> \mathrm{card}(\mathfrak{A}_\alpha)$. Using relativization to a new unary predicate P we can form T_α^P and embed $\mathfrak{A}_\alpha$ in $\langle \kappa, P, \ldots \rangle$ with $P = A_\alpha$. Clearly T_α^P has a model iff T_α has a model and T^P has a model iff T has a model.

Let now $\tilde{\mathfrak{A}}_\alpha$ be the above described expansion of $\mathfrak{A}_\alpha$. $\tilde{\mathfrak{A}}_\alpha$ can be assumed to be a $\tilde{L}_\alpha$-structure with $\tilde{L}_\alpha$ countable for T_α is finite. So all the $\underline{\lambda}_i (i=0, \ldots, n)$ in $\tilde{\mathfrak{A}}_\alpha$ are distinct. Take $\mu_i (i=1, \ldots, n)$ to be distinct regular cardinals such that $\mu_i^{|T|} = \mu_i (i=1, \ldots, n)$. Applying Lemma 6.5 we get $\mathfrak{B}_\alpha \equiv \tilde{\mathfrak{A}}_\alpha(\tilde{L}_{\omega\omega})$ with the property that for each $\varphi \in L^{**} \mathfrak{B}_\alpha \models Q^{C_i} xy\varphi$ iff $\mathrm{cf}(O_\varphi^{\mathfrak{B}_\alpha}) = \mu_i$ [using (i)–(vi) of the definition of $\tilde{\mathfrak{A}}_\alpha$][4]. Now let $\mathscr{D}$ be a non-principal ultrafilter on $\beta = |P_{<\omega}(T)| = |T| \geqq \aleph_0$ and let $\mathfrak{B} = \prod \mathfrak{B}_\alpha / \mathscr{D}$.

[4] We omit the parameters in the notation to facilitate reading.

Sh:101

Claim. If $\mathfrak{B} \models \underline{\mathrm{Reg}}(a) \wedge \underline{\lambda}_{l-1} \leqq a < \underline{\lambda}_l$ then $\mathrm{cf}(a) = \mu_l$. Let $a = (a_\alpha : \alpha < \beta) \in B$. For almost all α there is a sequence $(c_\alpha^j : j < \mu_l)$ cofinal in $\{b \in B_\alpha : \mathfrak{B}_\alpha \models b < a_\alpha\}, <\}$ and $\mathfrak{B}_\alpha \models c_\alpha^j < c_\alpha^{j+1}$ for $j < \mu_l$. Now let $d \in B$, $\mathfrak{B} \models d < a$ and $d = (d_\alpha : \alpha < \beta)$. Then for almost all α there is $j(\alpha)$ such that $\mathfrak{B}_\alpha \models d_\alpha < c_\alpha^{j(\alpha)}$, so for $c = (c_\alpha^{j(\alpha)} : \alpha < \beta)$ we have $\mathfrak{B} \models d < c$. Now since $\mu_l^\beta = \mu_l$ this proves the claim. Now we apply Theorem 6.4 in the form $(\mu_1, \ldots, \mu_n) \to (\lambda_0, \ldots, \lambda_{n-1})$ and get a model $\mathscr{C} \equiv \mathfrak{B}(\tilde{L}_{\omega\omega})$.

Claim. $\mathscr{C} \models T$.
From the above we have for each $\varphi \in L^{**} \mathscr{C} \models Q^{C_i} xy\varphi$ iff $\mathrm{cf}(O_\varphi^{\mathscr{C}}) = \lambda_{i-1}$. The rest is by induction over the formulas of L^{**} and using (i)–(vi) of the definition of $\tilde{\mathfrak{A}}_\alpha$. This ends the proof of 6.1. □

Theorem 6.2 is obvious from the proof above. One proves that $\varphi \in L^{**}$ has a model in the $(C_1^0, \ldots, C_n^0)$-interpretation iff it has a model in the $(C_1^1, \ldots, C_n^1)$-interpretation. For this let $\mathfrak{A}_0 \models \varphi$ in the $(C_1^0, \ldots, C_n^0)$-interpretation. Expand $\mathfrak{A}_0$ to $\tilde{\mathfrak{A}}_0$ and find $\tilde{\mathfrak{A}}_1$ via Theorem 6.4. So $\mathfrak{A}_1 \models \varphi$ in the $(C_1^1, \ldots, C_n^1)$-interpretation. □

Before we prove Theorem 6.3 let us prove first Lemma 6.5. The proof is by induction on n. (The number of fixed cofinalities.) Let $\tilde{\mathfrak{A}}$ be given. For $n=1$, $\mu_1 = \aleph_0$ take $\mathfrak{B}$ to be a countable elementary submodel of $\tilde{\mathfrak{A}}$, for $\mu_1 > \aleph_0$ take a proper elementary chain of submodels in $\tilde{\mathfrak{A}}$ of length μ_1. W.l.o.g. we can assume $\tilde{\mathfrak{A}}$ to be bigger than any of the μ_i's.

Induction Step. Assume the lemma has been proved for $n-1$. By Theorem 6.4 it suffices to prove the lemma for some distinct $\mu_1, \ldots, \mu_n$, so w.l.o.g. we can assume $\mu_n = \aleph_0$.

We define by induction on $l < \omega$ submodels $\mathfrak{B}_l < \tilde{\mathfrak{A}}$ such that
(i) $\mathrm{card}(\mathfrak{B}_l) < \lambda_{n-1}$.
(ii) $\mathfrak{B}_l < \mathfrak{B}_{l+1} < \tilde{\mathfrak{A}}$.
(iii) If $a \in \underline{\mathrm{Reg}}^{\mathfrak{B}_l}$, $a < \underline{\lambda}_{n-1}^{\mathfrak{B}_l}$ then for all $b \in |\tilde{\mathfrak{A}}|$ such that $\tilde{\mathfrak{A}} \models b < a$ we have $b \in |\mathfrak{B}_l|$.
(iv) For every $a \in \underline{\mathrm{Reg}}^{\mathfrak{B}_l}$ with $a \geqq \underline{\lambda}_{n-1}^{\mathfrak{B}_l}$ there is $b_{a,l}$, $b_{a,l} \in |\mathfrak{B}_{l+1}|$ with $\mathfrak{B}_{l+1} \models b_{a,l} < a$ (to prevent wrong cofinality) but for all $c \in |\mathfrak{B}_l|$ if $\mathfrak{B}_{l+1} \models c < a$ then $\mathfrak{B}_{l+1} \models c < b_{a,l}$.

It is easy to find $\mathfrak{B}_0$ satisfying (i)–(iii). Now we take $b_{a,0} \in |\tilde{\mathfrak{A}}|$ for each $a \in \underline{\mathrm{Reg}}^{\mathfrak{B}_0}$ $a \geqq \underline{\lambda}_{n-1}$ as in (iv). There are less than $\mathrm{card}(\mathfrak{B}_0)$ such $b_{a,0}$, hence less than λ_{n-1} by (i). So we choose $\mathfrak{B}_1$ satisfying (i)–(iii) such that $B_0 \cup \{b_{a,0} : a \in \underline{\mathrm{Reg}}^{\mathfrak{B}_0}, a \geqq \underline{\lambda}_{n-1}\} \subseteqq B_1$. Now we put $\tilde{\mathfrak{A}}^* = \langle \tilde{\mathfrak{A}}, P_i \rangle_{i \in \omega}$ where $P_i = B_i$ and apply the induction hypothesis, i.e. the lemma for $n-1$ to $\tilde{\mathfrak{A}}^*$. We have

$$\lambda_0 = \lambda_0' < \lambda_1 = \lambda_1' < \ldots < \lambda_{n-1} = \lambda_{n-1}' < \lambda_{n-1}' = \infty$$

and $\mu_1, \ldots, \mu_{n-1}$. So there is $\mathscr{M}^* = \langle \mathscr{M}, P_i \rangle_{i \in \omega} \equiv \tilde{\mathfrak{A}}^*$ such that for each $a \in \underline{\mathrm{Reg}}^{\mathscr{M}^*}$ and $\underline{\lambda}_{l-1}^{\mathscr{M}^*} \leqq a < \underline{\lambda}_l^{\mathscr{M}^*}$ $\mathrm{cf}^{\mathscr{M}^*}(a) = \mu_l$ $(l = 1, \ldots, n-1)$. So in $\mathscr{M}^*$ if $a \in \underline{\mathrm{Reg}}^{\mathscr{M}^*}$ and $a \geqq \underline{\lambda}_{n-1}^{\mathscr{M}^*}$ $\mathrm{cf}^{\mathscr{M}^*}(a) = \mu_{n-1}$, but we want it to be ω.

Therefore we define $\mathcal{M}_l = \langle P_l^*, \ldots \rangle < \mathcal{M}$, the built in elementary submodels. We have

1) $\mathcal{M}_l < \mathcal{M}_{l+1} < \mathcal{M}$,
2) if $a \in \underline{\mathrm{Reg}}^{\mathcal{M}} \cap \left(\bigcup_{l<\omega}\right) \mathcal{M}_l$ and $a < \underline{\lambda}_{n-1}$ then for some $l \in \omega$, $a \in |\mathcal{M}_l|$.

Hence $\{b \in M : b < a\} \subseteqq M_l \subseteqq \bigcup_{l<\omega} M_l$ by (iii).

Put $\mathcal{N} = \bigcup_{l<\omega} \mathcal{M}_l$ and $\mathcal{N}$ is the required model. □

Proof of 6.3. First we show

Proposition 6.6. *For every $(C_1, \ldots, C_n)$-interpretation of L^{**} there are $K_i \in PC(L^{**})$ and $\mathfrak{A}_i \in K_i (i=1,2)$ with $K_1 \cap K_2 = \emptyset$ and $\mathfrak{A}_1 \equiv \mathfrak{A}_2 (L^{**})$.*

To prove 6.3 we take the construction $P(\mathfrak{A}_0, \mathfrak{A}_1)$ and $\bar{P}(\mathfrak{A}_0, \mathfrak{A}_1)$ from Section 3 and observe that $P(\mathfrak{A}_0, \mathfrak{A}_1) \equiv \bar{P}(\mathfrak{A}_0, \mathfrak{A}_1)$ (L^{**}).
The latter will be obvious from Lemma 6.7 below.
To prove 6.6 put K to be the class of partially ordered structures $\langle A, < \rangle$ such that

(i) Every two elements have a least upper and a greatest lower bound.
(ii)_k Between two elements, which are comparable, there are at least k mutually incomparable elements $(k \in \omega)$.
(iii) There are no extremal elements.
(iv) There are no elements comparable to every element.

A set $X \subseteqq A$ is cofinal if for every $a \in A$ there is $b \in X$ with $a < b$.
If $\mathfrak{A} \in K$ has a cofinal chain we can speak of its cofinality and hence of the cofinality of $\mathfrak{A}$ $(\mathrm{cf}\,(\mathfrak{A}) = \kappa)$.
Now put

$$K_1 = \{\mathfrak{A} \in K : \mathrm{cf}(\mathfrak{A}) \in C_0\}$$

$$K_2 = \left\{\mathfrak{A} \in K : \mathrm{cf}(\mathfrak{A}) \in \bigcup_{i=1}^{n} C_i\right\},$$

where we can assume w.l.o.g. that the C_i's are disjoint and $\bigcup_{i=0}^{n} C_i$ are the regular cardinals (by 6.2).
Clearly $K_i \in PC(L^{**})$ $(i=1,2)$, $K_1 \cap K_2 = \emptyset$.
Let Γ be the axioms which one obtains from (i), (ii)_k, and (iii). $\Gamma \subseteqq L_{\omega\omega}$.

Lemma 6.7. *Γ admits elimination of quantifiers.*

Proof. For $\exists$ and $\forall$ this is easily checked. For Q^{C_i} we observe that for each quantifierfree $\varphi(x_1, \ldots, x_n)$ and $\mathfrak{A} \in K$ we have
$\mathfrak{A} \models \forall x_3 \ldots x_n (\neg Q^{C_i} x_1 x_2 \varphi)$, since no definable infinite set is a total order [here we need (iv)]. □

To end the proof of 6.6 we take $\mathfrak{A}_i \in K_i (i=1,2)$. □

Sh:101

In [26] another quantifier was introduced. Let C be again a class of regular cardinals. We define a binary quantifier $Q_C^{dc}xy$ by
$\mathfrak{A} \models Q_C^{dc}xy\varphi(x,y)\ [\bar{a}]$ iff $O\varphi(\bar{a})$ linearily orders $D\varphi(\bar{a})$ and there is a Dedekind cut (A_1, A_2) of $D\varphi(\bar{a})$ such that $\mathrm{cf}(A_1, O\varphi(\bar{a}))$ and $\mathrm{cf}^*(A_2, O\varphi(\bar{a}))$ are both in C. (The C-interpretation of Q_C^{dc}.) Let $L^* = L_{\omega\omega}[Q^{C_i}, Q_{C_i}^{dc}]_{i\in n}$ for some $(C_0, \ldots, C_{n-1})$-interpretation.

Theorem 6.8. *For no $(C_0, \ldots, C_{n-1})$-interpretation does* BETH(L^*, L^*) *hold, provided L^* properly extends $L_{\omega\omega}$.*

The proof uses the same counterexamples as described in Proposition 6.6 and the remark immediately after it. For, with the argument in Lemma 6.7 we can even eliminate the quantifier $Q_{C_i}^{dc}$. □

7. The Feferman-Vaught Theorem for Pairs

7.1. The Counterexamples

Let κ be an ordinal (cardinal) and $S \subsetneqq \kappa$. We put for $i \in \kappa$

$$\theta_i^S = \begin{cases} 1+\eta & \text{if} \quad i \in S \quad \text{or} \quad i = 0 \\ \eta & \text{if} \quad i \notin S \quad \text{and} \quad i \neq 0 \end{cases}$$

and

$$\xi_i^S = \begin{cases} 1+\omega^* & i \in S \\ \omega^* & i \notin S. \end{cases}$$

Now we define

$$M(\kappa, S_j) = \left\langle \sum_{i<\kappa} \xi_i^S, < \right\rangle = M_j$$

and

$$N(\kappa, S_j) = \left\langle \sum_{i\in\kappa} \Theta_i^S, < \right\rangle = N_j$$

with their natural orderings.
By a simple cardinality argument we get

Proposition 7.1. *Let L^* be a logic such that $2^{|L^*|} < 2^{\kappa^+}$ for all finite L. Then there are stationary sets S_1, S_2 of κ^+ with $S_1 \Delta S_2$ stationary and $M(\kappa^+, S_1) \equiv M(\kappa^+, S_2)$ (L^*).* □

Proof. There are at most $2^{|L^*|} < 2^{\kappa^+}$ many L^*-theories but there are 2^{κ^+} many such stationary set. □

Let $[M_1, M_2]$ be the two-sorted structure with universes $|M_1|$ and $|M_2|$, two binary relations (linear orders) $<_1$, $<_2$ and where the set variables range over countable subsets of $|M_1| \cup |M_2|$ (the disjoint union). Let the language for such structures be denoted by L_0. Let ψ be the following sentence:

$$\psi = aas(\exists x\varphi(x,s) \Leftrightarrow \exists\varphi(y,s)),$$

where x is a variable of the first, y of the second sort and $\varphi(x,s)$ says that x is a first element for $<_1$ of $|M_1| - S$, formally

$$\forall x_0(x_0 < x \Rightarrow x_0 \in s)$$

and similarly for $\varphi(y,s)$

$$\forall y_0(y_0 < y \Rightarrow y_0 \in s).$$

Proposition 7.2 (cf. also [14]).
(i) $[M_j, M_j] \models \psi$ *but*
(ii) $[M_1, M_2] \models \neg\psi$.

The proof will follow in the next section. From this we conclude

Theorem 7.3. *Let L^* be a logic such that L^* is a set for each finite L and such that $L(aa) < L^*$, then* FVP$(L(aa), L^*)$ *fails.*

Proof. We have to find structures $\mathfrak{A}_1 \equiv \mathfrak{A}_2 \equiv \mathfrak{B}_1 \equiv \mathfrak{B}_2(L^*)$ such that $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv [\mathfrak{B}_1, \mathfrak{B}_2]$ $(L(aa))$. So we put $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{B}_1 = \mathrm{M}(\kappa^+, S_1)$ and $\mathfrak{B}_2 = M(\kappa^+, S_2)$ for suitable κ, S_1, S_2 from 7.1 and 7.2. □

7.3 leaves two questions open: What about $L_{\infty\omega}(aa)$ and can κ be chosen to be ω_1? Let $S \subsetneq \omega_1$ be stationary and costationary and put $N_1 = N(\omega_1, S)$, $N_2 = N(\omega_1, \omega_1 - S)$.

Proposition 7.4. $N_1 \equiv N_2(L_{\infty\omega}(aa))$.

Proposition 7.5.
(i) $[N_j N_j] \models \psi$ *but*
(ii) $[N_1, N_2] \models \neg\psi$.

The proofs will follow in the next section.
Again we conclude

Theorem 7.6. *Not* WFVP$(L(aa), L_{\infty\omega}(aa))$.

7.2. Proofs of the Propositions

Here we prove Proposition 7.5 leaving the proof of Proposition 7.2 to the reader. Both proofs are entirely analogous.

Sh:101

Claim 1. $[N_j, N_j] \models \psi$ $(j=1,2)$.
Put $\psi_0(s) = (\exists x\varphi(x,s) \Leftrightarrow \exists y\varphi(y,s))$. We have to show that the set C

$$C = \{s=(u,v): u \in P_{\leqq\omega}(N_j), v \in P_{\leqq\omega}(N_j) \quad \text{and}$$
$$\langle N_j, u\rangle \models Ex\varphi(x,u) \quad \text{iff} \quad \langle N_j, v\rangle \models Ey\varphi(y,v)\}$$

contains a c.u.b. Clearly C is unbounded by the definition of N_j. Let $C_0 = \{s=(u,v): s \in C \text{ and } u=v\}$. Clearly also C_0 is c.u.b. and $C_0 \subset C$.

Claim 2. $[N_1, N_2] = \neg\psi$.

Consider the set $D = \left\{(u,v): u=v \text{ and } v = \sum_{\alpha<\beta} \eta_\alpha \text{ for some } \beta \in S\right\}$. Clearly D is stationary so $D \cap C \neq \emptyset$. □

Finally to prove 7.4 we apply the back and forth criteria from Section 2.

7.3. An Alternative Definition of the Pair $[\mathfrak{A}, \mathfrak{B}]$

Let $[[\mathfrak{A}, \mathfrak{B}]]$ denote the structure of the same type as $[\mathfrak{A}, \mathfrak{B}]$, but where the set variables are *two sorted* as well and range over subsets of $\mathfrak{A}$ or $\mathfrak{B}$ respectively, but we have no set variables for $|\mathfrak{A}| \cup |\mathfrak{B}|$. Then we get easily with the methods of Section 2

Theorem 7.7. *If* $\mathfrak{A} \equiv \mathfrak{A}'$, $\mathfrak{B} \equiv \mathfrak{B}'$ *in* $L(aa)$ *then* $[[\mathfrak{A}, \mathfrak{B}]]$ $[[\mathfrak{A}', \mathfrak{B}']]$ $(L(aa))$ *and similarly for* $L_{\infty\omega}(aa)$.

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