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CHARACTERIZING AN $\aleph_\epsilon$-SATURATED MODEL OF SUPERSTABLE NDOP THEORIES BY ITS $\mathbb{L}_{\infty,\aleph_\epsilon}$-THEORY

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ABSTRACT

Assume a complete countable first order theory is superstable with NDOP. We know that any $\aleph_\epsilon$-saturated model of the theory is $\aleph_\epsilon$-prime over a non-forking tree of "small" models and its isomorphism type can be characterized by its $\mathbb{L}_{\infty,\kappa}$ (dimension qualifiers)-theory, or, if you prefer, appropriate cardinal invariants. We go one step further by providing cardinal invariants which are as finitary as seem reasonable.

0. Introduction

After the main gap theorem was proved (see [Sh:c]), in a discussion, Harrington expressed a desire for a finer structure — of finitary character (when we have a structure theorem at all). I point out that the logic $\mathbb{L}_{\infty,\aleph_0}$ (d.q.) (where d.q. stands for dimension quantifier) does not suffice: suppose, e.g., for $T = \mathrm{Th}(\lambda \times {}^\omega 2, E_n)_{n<\omega}$ where $(\alpha,\eta)E_n(\beta,\nu) =: \eta \restriction n = \nu \restriction n$ and for $S \subseteq {}^\omega 2$ we define $M_S = M \restriction \{(\alpha,\eta) : [\eta \in S \Rightarrow \alpha < \omega_1]$ and $[\eta \in {}^\omega 2 \backslash S \Rightarrow \alpha < \omega]\}$. Hence,

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it seems to me we should try $\mathbb{L}_{\infty,\aleph_\epsilon}$ (d.q.) (essentially, in $\mathfrak{C}$ we can quantify over sets which are included in the algebraic closure of finite sets, see below 1.1, 1.3), and Harrington accepts this interpretation. Here the conjecture is proved for $\aleph_\epsilon$-saturated models.

I.e., the main theorem is $M \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})} N \Leftrightarrow M \cong N$ for $\aleph_\epsilon$-saturated models of a superstable countable (first order) theory T without dop. For this we analyze further regular types, define a kind of infinitary logic (more exactly, a kind of type of $\bar{a}$ in M), "looking only up" in the definition (when thinking of the decomposition theorem). Recall that for a $\aleph_\varepsilon$-saturated model M of a superstable DNOP theory a $\aleph_\varepsilon$-decomposition is $\langle M_\eta, a_\eta : \eta \in \mathcal{T}\rangle$, where

(a) $I \subseteq {}^{\omega>}\text{ord}$ is nonempty closed under initial segments,
(b) $M_\eta \prec M$ is $\aleph_\varepsilon$-saturated,
(c) $\nu \triangleleft \eta \in I \Rightarrow M_\nu \prec M_\eta$,
(d) if $\nu = \eta\hat{}\langle\alpha\rangle \in I$ then M_ν is $\aleph_\varepsilon$-prime over $M_\eta \cup \{a_\nu\}$ and $\text{tp}(a_\eta, M_\eta)$ is orthogonal to M_ρ for $\rho \triangleleft \nu$, and (the last is not essential but clarifies)
(e) $\langle M_\eta : \eta \in I\rangle$ is nonforking enough: for every $\nu \in I$ the set $\{a_\eta : \eta \in \text{Suc}_I(\nu)\} \subseteq M$ is independent over M_ν.

The point is that if $\eta = \nu\hat{}\langle\alpha\rangle, M_{\eta_\nu}, a_\eta$ are chosen, then to a large extent $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$ is determined. But the amount of "to a large extent" which suffices in [Sh:c] is not sufficient here; we need to find a finer understanding. In particular, we certainly do not like to "know" (M_ν, a_η). So we consider a pair (A, B) where $A \subseteq M_\nu, A \cup \{a_\eta\} \subseteq B \subseteq M_\eta$, $\text{stp}_*(B, A) \vdash \text{stp}_*(B, M_\nu)$ and we try to define the type of such pairs in a way satisfying:

(a) it can be impressed in our logic $\mathbb{L}_{\infty,\aleph_\varepsilon}$,
(b) it expresses the essential information in $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$.

To carry out the isomorphism proof we need: (1.27) the type of the sum is the sum of types (infinitary types) assuming first order independence. The main point of the proof is to construct an isomorphism between M_1 and M_2 when $M_1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)} M_2, Th(M_\ell) = T$ where T and $\equiv_{\mathbb{L}_{\infty,\aleph_\varepsilon}(\text{q.d.})}$ are as above. So by [Sh:c, X] it is enough to construct isomorphic decompositions. The construction of isomorphic decompositions is by ω approximations; in stage n, $\sim n$ levels of the decomposition tree are approximated, i.e. we have $I_n^\ell \subseteq {}^{n\geq}\text{Ord}$ and $\bar{a}_\eta^{n,\ell} \in M_\ell$ for $\eta \in I_n, \ell = 1,2$ such that $\text{tp}(\bar{a}_{\eta\restriction 0}^{n,1}\hat{}\bar{a}_{\eta\restriction 1}^{n,1}\hat{}\cdots\hat{}\bar{a}_\eta^{n,1}, \emptyset, M) = \text{tp}(\bar{a}_{\eta\restriction 0}^{n,2}\hat{}\bar{a}_{\eta\restriction 1}^{n,2}\hat{}\cdots\hat{}\bar{a}_\eta^{n,2}, \emptyset, M_2)$ with $\bar{a}_\eta^{\eta,\ell}$ being ε-finite, so in stage $n+1$, choosing $\bar{a}_{\langle\rangle}^{n+1,\ell}$ we cannot take care of all types $\bar{a}_{\langle\rangle}^{n+1,\ell}\hat{}\bar{a}_{\langle\alpha\rangle}^{\eta,\ell}$ so the addition theorem takes care. So though we are thinking on $\aleph_\varepsilon$-decomposition (i.e. the M_η's are $\aleph_\varepsilon$-saturated), we get just a decomposition.

In the end of section 1 (in 1.37) we point out that the addition theorem holds in fuller generalization. In the second section we deal with a finer type needed for shallow T; in the appendix we discuss how absolute is the isomorphism type.

Of course, we may consider replacing "$\aleph_\varepsilon$-saturated models of an NDOP superstable countable T" by "models of an NDOP $\aleph_0$-stable countable T". But the use of ε-finite sets seems considerably less justifiable in this context; it seems more reasonable to use finite sets, i.e., $\mathbb{L}_{\infty,\aleph_0}$ (d.q.). But subsequently Hrushovski and Bouscaren proved that even if T is $\aleph_0$-stable, $\mathbb{L}_{\infty,\aleph_0}$ (d.q.) is not sufficient to characterize models of T up to isomorphism. This is not sufficient even if one considers the class of all $\aleph_\varepsilon$-saturated models rather than all models. The first example is $\aleph_0$-stable shallow of depth 3, and the second one is superstable (non-$\aleph_0$-stable), NOTOP, non-multidimensional.

If we deal with $\aleph_\epsilon$-saturated models of shallow (superstable NDOP) theories T, we can bound the depth of the quantification $\gamma = DP(T)$; i.e., $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ suffice.

We assume the reader has a reasonable knowledge of [Sh:c, V,§1,§2] and mainly [Sh:c, V,§3] and [Sh:c, X].

Here is a slightly more detailed guide to the paper. In 1.1 we define the logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ and in 1.3 we give a back and forth characterization of equivalence in this logic which is the operative definition for this paper.

The major tools are defined in 1.7, 1.11. In particular, the notion of tp_α defined in 1.5 is a kind of a depth α look-ahead type which is actually used in the final construction. In 1.28 we point out that equivalence in the logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ implies equivalence with respect to tp_α for all α. Proposition 1.14 contains a number of important concrete assertions which are established by means of Facts 1.16–1.23. In general, these explain the properties of decompositions over a pair $\binom{B}{A}$. Claim 1.27 (which follows from 1.26) is a key step in the final induction. Definition 1.30 establishes the framework for the proof that two $\aleph_\epsilon$-saturated structures that have the same tp_∞ are isomorphic. The induction step is carried out in 1.35.

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0.1 Notation: The notation is from [Sh:c], with the following additions (or reminders).

If $\eta = \nu\hat{\ }\langle\alpha\rangle$ then we let $\eta^- = \nu$; for I a set of sequences ordinals we let $\mathrm{Suc}_T(\eta) = \{\nu :$ for some $\alpha, \nu = \eta\hat{\ }\langle\alpha\rangle \in I\}$.

We work in $\mathfrak{C}^{\mathrm{eq}}$ and for simplicity every first order formula is equivalent to a relation.

(1) $\perp$ means orthogonal (so q is $\perp$ p means q is orthogonal to p), remember $p \perp A$ means p orthogonal to A; i.e., $p \perp q$ for every $q \in S(ac\ell(A))$ (in $\mathfrak{C}^{eq}$).

(2) $\perp_a$ means almost orthogonal.

(3) $\perp_w$ means weakly orthogonal.

(4) $\frac{\bar{a}}{B}$ and $\bar{a}/B$ means $\mathrm{tp}(\bar{a}, B)$.

(5) $\frac{A}{B}$ or A/B means $\mathrm{tp}_*(A, B)$.

(6) $A + B$ means $A \cup B$.

(7) $\bigcup_{A}\{B_i : i < \alpha\}$ means $\{B_i : i < \alpha\}$ is independent over A.

(8) $A \bigcup_{B} C$ means $\{A, C\}$ is independent over B.

(9) $\{C_i : i < \alpha\}$ is independent over (B, A) means that[1]

$$j < \alpha \Rightarrow \mathrm{tp}_*\left(C_j, \bigcup_{i \neq j} C_i \cup B\right) \text{ does not fork over } A.$$

(10) Regular type means stationary regular type $p \in S(A)$ for some A.

(11) For $p \in S(A)$ regular and C a set of elements realizing p, $\dim(C, p)$ is

$$\mathrm{Max}\{|\mathbf{I}| : \mathbf{I} \subseteq C \text{ is independent over } A\}.$$

(12) $ac\ell(A) = \{c : \mathrm{tp}(c, A) \text{ is algebraic}\}$.

(13) $dc\ell(A) = \{c : \mathrm{tp}(c, A) \text{ is realized by one and only one element}\}$.

(14) $\mathrm{Dp}(p)$ is depth (of a stationary type); see [Sh:c, X, Definition 4.3, p. 528, Definition 4.4, p. 529].

(15) $\mathrm{Cb}(p)$ is the canonical base of a stationary type p (see [Sh:c, III, 6.10, p. 134]).

(16) B is $\aleph_\varepsilon$-atomic over A <u>if</u> for every finite sequence $\bar{b}$ from A, for some finite $A_0 \subseteq A$ we have $\mathrm{stp}(\bar{b}, A_0) \vdash \mathrm{stp}(\bar{b}, A)$, equivalently for some ε-finite $A_0 \subseteq ac\ell(A)$ we have $\mathrm{tp}(\bar{b}, A_0) \vdash \mathrm{tp}(\bar{b}, ac\ell(A))$.

1. $\aleph_\epsilon$-saturated models

We first define our logic, but, as noted in section 0, we shall only use the condition from 1.4. T is always superstable complete first order theory.

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1 Actually, by the nonforking calculus this is equivalent to: $\{C_i : i \leq \alpha\}$ is independent over A, where we let $C_\alpha = B$.

1.1 Definition: (1) The logic $\mathbb{L}_{\infty,\aleph_\epsilon}$ is slightly stronger than $\mathbb{L}_{\infty,\aleph_0}$; it consists of the set of formulas in $\mathbb{L}_{\infty,|T|^+}$ such that any subformula of ψ of the form $(\exists \bar{x})\varphi$ is actually the form

$$(\exists \bar{x}^0, \bar{x}^1)\left[\varphi_1(\bar{x}^1, \bar{y}) \& \bigwedge_{i<\ell g \bar{x}^1} (\theta_i(x^1_i, \bar{x}^0) \& (\exists^{<\aleph_0} z)\theta_i(z, \bar{x}^0))\right],$$

with $\bar{x}^0$ finite, $\bar{x}^1$ not necessarily finite but of length $< |T|^+$; so φ "says" $\bar{x}^1 \subseteq ac\ell(\bar{x}^0)$; note that our final proof of the theorem always uses $|T| \geq \aleph_0$.
(2) $\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)$ is like $\mathbb{L}_{\infty,\aleph_\epsilon}$ but we have cardinality quantifiers and, moreover, dimensional quantifiers (as in [Sh:c, XIII, 1.2, p. 624]); see below.
(3) The logic $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ consist of the formulas of $\mathbb{L}_{\infty,\aleph_\epsilon}$ such that φ has quantifier depth $< \gamma$ (but we start the inductive definition by defining the quantifier depth of all first order as zero).
(4) $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}(d.q.)$ is like $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$ but we have cardinality quantifiers and, moreover, dimensional quantifiers.

1.2 Remark: (1) In fact the dimension quantifier is used in a very restricted way (see Definition 1.10 and Claim 1.28 + Claim 1.30).
(2) The reader may ignore this logic altogether and use just the characterization of equivalence in Claim 1.4.

1.3 CONVENTION: (1) T is a fixed first order complete theory, $\mathfrak{C}$ is the "monster" model, as in [Sh:c], so is $\bar{\kappa}$-saturated; $\mathfrak{C}^{\rm eq}$ is as in [Sh:c, III, 6.2, p. 131]. We work in $\mathfrak{C}^{\rm eq}$ so M, N vary on elementary submodels of $\mathfrak{C}^{\rm eq}$ of cardinality $< \bar{\kappa}$. We assume T is superstable with NDOP (countability is used only in the Proof of 1.5 for bookkeeping, i.e. in the proof of 1.30 (and 1.29)).

Remember a, b, c, d denote members of $\mathfrak{C}^{\rm eq}$; $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ denote finite sequences of members of $\mathfrak{C}^{\rm eq}$; A, B, C denote subsets of $\mathfrak{C}^{\rm eq}$ of cardinality $< \bar{\kappa}$.

Remember $ac\ell(A)$ is the algebraic closure of A, i.e.,

$$\{b : \text{for some first order and } n < \omega, \varphi(x, \bar{y}) \text{ and } \bar{a} \subseteq A \text{ we have } \mathfrak{C}^{\rm eq} \models \varphi[b, \bar{a}] \& (\exists^{\leq n} y)\varphi(y, \bar{a})\}$$

and $\bar{a}$ denotes $\mathrm{Rang}(\bar{a})$ in places where it stands for a set (as in $ac\ell(\bar{a})$). We write $\bar{a} \in A$ instead of $\bar{a} \in {}^{\omega>}(A)$.
(2) A is ϵ-finite, if for some $\bar{a} \in {}^{\omega>}A, A = ac\ell(\bar{a})$. (So for stable theories a subset of an ϵ-finite set is not necessarily ϵ-finite but, as T is superstable, a subset of an ϵ-finite set is ϵ-finite as if $B \subseteq ac\ell(\bar{a})$; $\bar{b} \in B$ is such that $\mathrm{tp}(\bar{a}, B)$ does not fork over $\bar{b}$; then trivially $ac\ell(\bar{b}) \subseteq A$ and, if $ac\ell(\bar{b}) \neq B, \mathrm{tp}_*(B, \bar{a}{}^\frown\bar{b})$

forks over B, hence ([Sh:c, III, 0.1]) $\mathrm{tp}(\bar{a}, B)$ forks over $\bar{b}$, a contradiction. So if $ac\ell(A) = ac\ell(B)$, then A is ϵ-finite iff B is ϵ-finite.)
(3) When T is superstable by [Sh:c, IV, Table 1, p. 169] for $\mathbf{F} = \mathbf{F}^a_{\aleph_0}$, all the axioms there hold and we write $\aleph_\varepsilon$ instead of $\mathbf{F}$ and may use implicitly the consequences in [Sh:c, IV, §3].

Instead of Definition 1.1 we may use directly the standard characterization from 1.4; as actually less is used we state the condition we shall actually use:

1.4 CLAIM: *For models M_1, M_2 of T we have $M_1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)} M_2$ if*

$\bigotimes$ *there is a non-empty family $\mathcal{F}$ such that:*

- (a) *each $f \in \mathcal{F}$ is an (M_1, M_2)-elementary mapping (so $\mathrm{Dom}(f) \subseteq M_1$, $\mathrm{Rang}(f) \subseteq M_2$),*
- (b) *for $f \in \mathcal{F}, \mathrm{Dom}(f)$ is ϵ-finite (see 1.3(2) above),*
- (c) *if $f \in \mathcal{F}, \bar{a}_\ell \in M_\ell (\ell = 1, 2)$ then for some $g \in \mathcal{F}$ we have: $f \subseteq g$ and $ac\ell(\bar{a}_1) \subseteq \mathrm{Dom}(f)$ and $ac\ell(\bar{a}_2) \subseteq \mathrm{Rang}(f)$,*
- (d) *if $f \cup \{\langle a_1, a_2\rangle\} \in \mathcal{F}$ and $\mathrm{tp}(a_1, \mathrm{Dom}(f))$ is stationary and regular then $\dim(\{a^1_1 \in M_1 : f \cup \{\langle a^1_1, a_2\rangle\} \in \mathcal{F}\}, M_1)$
  $= \dim(\{a^1_2 \in M_2 : f \cup \{\langle a_1, a^1_2\rangle\} \in \mathcal{F}\}, M_2)$.*

Our main theorem is

1.5 THEOREM: *Suppose T is countable (superstable complete first order theory) with NDOP. Then:*

1. *The $\mathbb{L}_{\infty,\aleph_\epsilon}$(d.q.) theory of an $\aleph_\epsilon$-saturated model characterizes it up to isomorphism.*
2. *Moreover, if M_1, M_2 are $\aleph_\epsilon$-saturated models of T (so $M_\ell \prec \mathfrak{C}^{\mathrm{eq}}$) and $\bigotimes_{M_0,M_1}$ of 1.4 holds, then M_1, M_2 are isomorphic.*

* * *

By 1.4, it suffices to prove part (2).
The proof is broken into a series of claims (some of them do not use NDOP, almost all do not use countability; but we assume T is superstable complete all the time (1.3(1))).

1.6 DISCUSSION: Let us motivate the notation and Definition below.

Recall from the introduction that we are thinking of a triple (M, N, a) which may appear in $\aleph_\varepsilon$-decomposition $\langle M_\eta, a_\eta : \eta \in I\rangle$ of N, in the sense that for some $\eta \in I\backslash\{<>\}$ we have $(M, M', a) = (M_{\eta^-}, M_\eta, a_\eta)$ so M, M' are $\aleph_\varepsilon$-saturated, $a_\eta \in M'\backslash M', M'$ is $\aleph_\varepsilon$-prime over $M + a$ and $\mathrm{tp}(a, M)$ is regular. But this is "too large for us", hence we consider an approximation (A, B) where $A \subseteq M$

$(= M_{\eta^-}), A \subseteq B \subseteq M'(= M_\eta)), a = a_\eta \in B$ and $B/M(= B/M_{\eta^-})$ does not fork over A. We would like to define the α-type of (A, B) in N, which tries to say something on the decomposition above $(M, M', a) = (M_{\eta^-}, M_\eta, a_\eta)$, i.e., on $\langle M_\rho, a_\rho : \eta \triangleleft \rho \in I\rangle$. There are two natural "successors" of (A, B) we may choose in this context: the first, 1.7 below, replaces (A, B) by (A', B') such that $A \subseteq A' \subseteq M(= M_{\eta^-}), B \subseteq B' \subseteq M'(= M_\eta)$ and (as M' is $\aleph_\varepsilon$-prime over $M + a$) we have $\mathrm{stp}_*(B', A' \cup B) \vdash \mathrm{stp}(B', M)$, so $\mathrm{tp}(B', A' \cup B)$ is almost orthogonal to A'; we can think of this as "advancing in the same model"; in other words, as A, B are ε-finite, we have to increase them in order to capture even (M, M'). This is formalized by $\leq_{\mathrm{a}}$ in Definition 1.7 below.

The second is to pass from (M_{η^-}, M_η, a) to (M_η, M_ν, a_ν) for some ν an immediate successor (in I) of $\eta \in I$. So the old B is included in the new A' and $B' = A' \cup \{a\}$ where $\mathrm{tp}(a, A')$ is regular and is orthogonal to A (as in the decomposition we require $\mathrm{tp}(a_\eta, M_{\eta^-})(M_\nu$ when $\nu \triangleleft \eta^-))$. This is formalized by $\leq_{\mathrm{b}}$ in Definition 1.7 below.

1.7 Definition: (1) $\Gamma = \{(A, B) : A \subseteq B$ are ϵ-finite$\}$. Let

$$\Gamma(M) = \{(A, B) \in \Gamma : A \subseteq B \subseteq M\}.$$

(2) For members (A, B) of Γ we may also write $\binom{B}{A}$; <u>if</u> $A \not\subseteq B$ we mean $\binom{B \cup A}{A}$.
(3) $\binom{B_1}{A_1} \leq_{\mathrm{a}} \binom{B_2}{A_2}$ (usually we omit a) <u>if</u> (both are in Γ and) $A_1 \subseteq A_2$, $B_1 \subseteq B_2, B_1 \bigcup\limits_{A_1} A_2$ and $\frac{B_2}{B_1 + A_2} \perp_{\mathrm{a}} A_2$.
(4) $\binom{B_1}{A_1} \leq_b \binom{B_2}{A_2}$ <u>if</u> $A_2 = B_1, B_2 \backslash A_2 = \bar{b}$ and $\frac{\bar{b}}{A_2}$ is regular orthogonal to A_1.
(5) $\leq^*$ is the transitive closure of $\leq_{\mathrm{a}} \cup \leq_{\mathrm{b}}$. (So it is a partial order, whereas in general $\leq_{\mathrm{a}} \cup \leq_{\mathrm{b}}$ and $\leq_{\mathrm{b}}$ are not.)
(6) We can replace A, B by sequences listing them (we do not always strictly distinguish).

Remark: The following observation may clarify.

1.8 Observation: If $\binom{B_1}{A_1} \leq^* \binom{B_2}{A_2}$ <u>then</u> we can find $\langle B'_\ell : \ell \leq n\rangle$ and $\langle c_\ell : 1 \leq \ell < n\rangle$ for some $n \geq 1$, satisfying $\binom{B_1}{A_1} \leq_{\mathrm{b}} \binom{B'_1}{B'_0}, c_\ell \in B'_{\ell+1}, \frac{c_\ell}{B'_\ell}$ regular, $\frac{B'_{\ell+1}}{c_\ell + B'_\ell} \perp_{\mathrm{a}} B'_\ell, A_2 = B'_{n-1}, B_2 = B'_n$.

Remark: (1) Note that actually $\leq_{\mathrm{a}}$ is transitive. This means that in a sense $\leq_{\mathrm{b}}$ is enough, $\leq_{\mathrm{a}}$ inessential. (2) We may in 1.7(4) use $\bar{b} = \langle c\rangle$; it does not matter.

Proof: By the definition of $\leq^*$ there are $k < \omega$ and $\binom{B^\ell}{A^\ell}$ for $\ell \leq k$ such that: $\binom{B^\ell}{A^\ell} \leq_{x(\ell)} \binom{B^{\ell+1}}{A^{\ell+1}}$ for $\ell \leq k$ and $x(\ell) \in \{a, b\}$ and $\binom{B^0}{A^0} = \binom{B_1}{A_1}, \binom{B^k}{A^k} = \binom{B_2}{A_2}$ and without loss of generality, $x(2\ell) = a$, $x(2\ell+1) = b$. Let $N_0 \prec \mathfrak{C}$ be $\aleph_\epsilon$-prime over $\emptyset$ such that $A^0 \subseteq N_0, B_0 \bigcup\limits_{A^0} N_0$ and $f_0 = \mathrm{id}_{A_0}$. We choose by induction on $\ell \leq k, N_{\ell+1}, f_{\ell+1}$ such that:

(a) $\mathrm{Dom}(f_{\ell+1}) = B^\ell$,

(b) $N_\ell \prec N_{\ell+1}$,

(c) if $x(\ell) = b$, then $f_{\ell+1}$ is an extension of f_ℓ (which necessarily has domain A_ℓ, check) with domain B^ℓ such that $f_\ell(B^\ell) \bigcup\limits_{f_\ell(A^\ell)} N_\ell$ and $N_{\ell+1}$ is $\aleph_\epsilon$-prime over $N_\ell \cup f_\ell(B^\ell)$,

(d) if $x(\ell) = a$, then $f_{\ell+1}$ maps A^ℓ into $N_{\ell-1}, B^\ell$ into N_ℓ and $N_{\ell+1} = N_\ell$.

This is straightforward. Now on $\langle N_\ell : \ell \leq k+1\rangle$ we repeat the argument (of choosing $\langle B_\ell : \ell \leq n\rangle$) in the proof of 1.14(6) above, i.e., choose $B^\ell \subseteq N_\ell$ by downward induction on ℓ large enough as required. $\blacksquare_{1.8}$

1.9 Definition: (1) We define $\mathrm{tp}_\alpha[\binom{B}{A}, M]$ (for $A \subseteq B \subseteq M, A$ and B are ϵ-finite and α is an ordinal) and $\mathcal{S}_\alpha(\binom{B}{A}, M), \mathcal{S}_\alpha(A, M)$ and $\mathcal{S}^r_\alpha(\binom{B}{A}, M), \mathcal{S}^r_\alpha(A, M)$ by induction on α (we mean simultaneously; of course, we use appropriate variables):

(a) $\mathrm{tp}_0[\binom{B}{A}, M]$ is the first order type of $A \cup B$,

(b) $\mathrm{tp}_{\alpha+1}[\binom{B}{A}, M] =$ the triple $\langle Y^{1,\alpha}_{A,B,M}, Y^{2,\alpha}_{A,B,M}, \mathrm{tp}_\alpha(\binom{B}{A}, M)\rangle$ where: $Y^{1,\alpha}_{A,B,M}$ $=: \{\mathrm{tp}_\alpha[\binom{B'}{A'}, M]$: for some A', B' we have $\binom{B}{A} \leq_{\mathrm{a}} \binom{B'}{A'} \in \Gamma(M)\}$, and $Y^{2,\alpha}_{A,B,M} =: \{\langle \Upsilon, \lambda^\Upsilon_{M,B}\rangle : \Upsilon \in \mathcal{S}^r_\alpha(B, M)\}$ where

$$\lambda^\Upsilon_{M,B} = \dim[\{d : \mathrm{tp}_\alpha[\binom{B+d}{B}, M] = \Upsilon\}, B],$$

(c) for δ a limit ordinal, $\mathrm{tp}_\delta[\binom{B}{A}, M] = \langle \mathrm{tp}_\alpha[\binom{B}{A}, M] : \alpha < \delta\rangle$ (this includes $\delta = \infty$, really $\|M\|^+$ suffice),

(d) $\mathcal{S}_\alpha(A, M) = \{\mathrm{tp}_\alpha[\binom{B}{A}, M]$: for some B such that $B \subseteq M$, and $\binom{B}{A} \in \Gamma(M)\}$,

(e) $\mathcal{S}^r_\alpha(\binom{B}{A}, M) = \{\mathrm{tp}_\alpha[\binom{B+c}{B}, M]$: for some $c \in M$ we have $\frac{c}{B} \perp A$ and $\frac{c}{B}$ is regular$\}$,

(f) $\mathcal{S}^r_\alpha(A, M) = \{\mathrm{tp}_\alpha[\binom{A+c}{A}, M] : c \in M$ and $\frac{c}{A}$ regular$\}$.

(2) We define also $\mathrm{tp}_\alpha[A, M]$, for A an ϵ-finite subset of M:

(a) $\mathrm{tp}_0[A, M] =$ first order type of A,

(b) $\mathrm{tp}_{\alpha+1}[A, M]$ is the triple $\langle Y^{1,\alpha}_{A,M}, Y^{2,\alpha}_{A,M}, \mathrm{tp}_\alpha[A, M]\rangle$ where $Y^{1,\alpha}_{A,M} =: \mathcal{S}_\alpha(A; M)$ and $Y^{2,\alpha}_{A,M} =: \{\langle \Upsilon, \dim\{d \in M : \mathrm{tp}_\alpha[\binom{A+d}{A}, M] = \Upsilon\}\rangle : \Upsilon \in \mathcal{S}^r_\alpha(A, M)\}$,

(c) $\mathrm{tp}_\delta[A, M] = \langle \mathrm{tp}_\alpha(A, M) : \alpha < \delta\rangle$.

(3) $\mathrm{tp}_\alpha[M] = \mathrm{tp}_\alpha[\emptyset, M]$.

1.10 Discussion: Clearly $\mathrm{tp}[\binom{B}{A}, M]$ is intended, on the one hand, to be expressible by our logic and, on the other hand, to express the isomorphism type of M "in the direction of $\binom{B}{A}$". To really say it we need to go back to the $\aleph_\varepsilon$-decompositions of M, a central notion of [Sh:c, Ch. X].

For the reader's benefit, at the referee's request, let us review informally the proof in [Sh:c, Ch. X]. Let M be an $\aleph_\varepsilon$-saturated model, and we choose $\langle M_\eta : \eta \in I \cap {}^n\mathrm{Ord}\rangle, \langle a_\eta : \eta \in I \cap {}^{n+1}\lambda\rangle$ by induction on n. For $n = 0$, of course, $I \cap {}^0\mathrm{Ord} = \{<>\}$, we let $N_{<>} \prec M$ be $\aleph_\varepsilon$-prime over $\emptyset$ and let $\mathbf{I}_{<>}$ be a maximal subset of $\{c \in M : \mathrm{tp}(c, N_{<>}) \text{ regular}\}$ which is independent over $N_{<>}$; let $\langle a_{<\alpha>} : \alpha < |\mathbf{I}_{<>}|\rangle$ list $\mathbf{I}_{<>}$. Similarly for $n+1, \eta \in I \cap {}^{n+1}\mathrm{Ord}$, let $N_\eta \prec M$ be $\aleph_\varepsilon$-prime over $M_{\eta^-} + a_\eta$, let $\mathbf{I}_\eta$ be a maximal subset of $\{c \in M : \mathrm{tp}(c, M_\eta)$ is regular orthogonal to $M_{\eta^-}\}$ independent over N_η. Lastly, let $\langle c_{\eta\hat{}<\alpha>} : \alpha < |\mathbf{I}_\eta|\rangle$ list $\mathbf{I}_\eta$ and let $I \cap {}^{n+1}\mathrm{Ord} = \{\eta\hat{} < \alpha >: \eta \in I \cap {}^n\mathrm{Ord} \text{ and } \alpha < |\mathbf{I}_\eta|\}$.

To carry this we use the existence of $\aleph_\varepsilon$-prime models (and the local character of indpendent). Also, looking at the set $\cup\{M_\eta : \eta \in I\}$, its first order type is determined by the nonforking calculus. In fact, for any $\eta \in I\backslash\{<>\}$, the sets $\cup\{N_\nu : \eta \triangleleft \nu \in I\}, \cup\{N_\eta : \neg(\eta \leq \nu) \text{ and } \nu \in I\}$ are independent over N_η. Let $N \prec M$ be $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in I\}$. Now if $M = N$, we are done decomposing M; if not, some $c \in M\backslash N$ realize a regular type (we use density of regular types). By NDOP, the $\mathrm{tp}(c, N)$ is not orthogonal to some N_η. Choose η of minimal length, hence $\nu \triangleleft \eta \Rightarrow \mathrm{tp}(c, M_\eta) \perp N_\nu$. By properties of regular types, without loss of generality $\mathrm{tp}(c, N)$ does not fork over N_η, so we get a contradiction to the maximality of $\{a_\nu : \nu \in \mathrm{Suc}_I(\eta)\}$ (this explains the role of $\mathcal{P}$ in Definition 1.11(5) below).

We are interested in the possible trees $\langle N_\nu : \eta \triangleleft \nu \in I\rangle$.

Now the tree determines M up to isomorphism, but there are "incidental" choices, so two trees may give isomorphic models (for investigating the number of non-isomorphic models it is enough to find sufficiently pairwise far trees I).

Here we like to get exact information and in as finitary a way as we can. So we replace $(M_{\eta^-}, M_\eta, a_\eta)$ by $\binom{B}{A}$, where $A \subseteq M_{\eta^-}, A + a_\eta \subseteq B \subseteq M_\eta$, $\mathrm{tp}(B, M_{\eta^-})$ does not fork over A.

Now for $\eta \in I\backslash\{<>\}$ we are interested in the possible trees $\langle N_\nu : \eta \triangleleft \nu \in I\rangle$, over $(N_{\eta_*^-}, N_\eta, a_\eta)$. But not only different trees may be equivalent (giving isomorphic $\aleph_\varepsilon$-prime models) <u>but</u> the other part of the tree, $\langle N_\nu : \nu \in I$ but $\neg(\eta \triangleleft \nu)\rangle$, may apriori cause non-equivalent trees to contribute the same toward understanding M. This is done in [Sh:c, Ch. XII], <u>but</u> here we have to deal with ε-finite A, B.

The following claim 1.11 really does not add to [Sh:c, Ch. X], it just collects the relevant information which is proved there, or which follows immediately (particularly using the parameter (A, B)). We allow here a_η / M_η to be not regular, but this is not serious: we can here deal exclusively with this case and we can omit this requirement in [Sh:c, Ch. X]; however, this does not eliminate the use of regular types (in the proof that M is $\aleph_\varepsilon$-prime over every $\aleph_\varepsilon$-decomposition of it).

1.11 Definition: (1) $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\epsilon$-decomposition inside M above (or over) the pair $\binom{B}{A}$ (but we may omit the "$\aleph_\epsilon -$ ") if:

(a) I is a set of finite sequences of ordinals closed under initial segments,

(b) $\langle\rangle, \langle 0 \rangle \in I, \eta \in I \backslash \{\langle\rangle\} \Rightarrow \langle 0 \rangle \trianglelefteq \eta$, let $I^- = I \backslash \{\langle\rangle\}$, really $a_{\langle\rangle}$, is meaningless,

(c) $A \subseteq N_{\langle\rangle}, B \subseteq N_{\langle 0 \rangle}, N_{\langle\rangle} \bigcup\limits_A B$ and $dc\ell(a_{\langle 0 \rangle}) \subseteq dc\ell(B)$,

(d) if $\nu = \eta\hat{\ }\langle \alpha \rangle \in I$ <u>then</u> N_ν is $\aleph_\epsilon$-primary over $N_\eta \cup \bar{a}_\nu, N_{\langle\rangle}$ is $\aleph_\epsilon$-prime over A,

(e) for $\eta \in I$ such that $k = \ell g(\eta) > 1$ the type $a_\eta / N_{\eta \restriction (k-1)}$ is orthogonal to $N_{\eta \restriction (k-2)}$,

(f) $\eta \triangleleft \nu \Rightarrow N_\eta \prec N_\nu$,

(g) M is $\aleph_\epsilon$-saturated and $N_\eta \prec M$ for $\eta \in I$,

(h) if $\eta \in I \backslash \{\langle\rangle\}$, then $\{a_\nu : \nu \in \mathrm{Suc}_I(\eta)\}$ is (a set of elements realizing over N_η types orthogonal to N_{η^-} and is) an independent set over N_η.

(2) We replace "inside M" by "of M" if, in addition,

(i) in clause (h) the set is maximal.

(3) $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\epsilon$-decomposition inside M <u>if</u> (a), (d), (e), (f), (g), (h) of part (1) holds and in clause (h) we allow $\eta = \langle\rangle$ (call this $(h)^+$). We add "over A" if $A \subseteq M_{<>}$.

(4) $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\epsilon$-decomposition of M <u>if</u> in addition to 1.11(3) we have the stronger version of clause (i) of 1.11(2) by including $\eta = \langle\rangle$, i.e., we have:

(i)$^+$ for $\nu \in I$, the set $\{a_\eta : \eta \in \mathrm{Suc}_I(\nu)\}$ is a maximal subset of M independent over N_ν.

We may add "over A" <u>if</u> $A \subseteq M$.

(5) If $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\aleph_\epsilon$-decomposition inside M we let

$$\mathcal{P}(\langle N_\eta, a_\eta : \eta \in I \rangle, M) = \{p \in S(M) : p \text{ regular and for some } \eta \in I \backslash \{\langle\rangle\} \text{ we have } p \text{ is orthogonal to } N_{\eta^-} \text{ but not to } N_\eta\}.$$

As noted earlier, it is natural to use regular types.

1.12 Definition: (1) We say that $\langle N_\eta, a_\eta : \eta \in I\rangle$, an $\aleph_\epsilon$-decomposition inside M, is J-regular <u>if</u> $J \subseteq I$ and:

$$(*)\qquad \begin{array}{l}\text{for each } \eta \in I\backslash J \text{ there}^{\dagger} \text{ is } c_\eta \text{ such that } a_\eta \in ac\ell(N_\eta^- + c_\eta),\\ \dfrac{c_\eta}{N_\eta} \text{ is regular and if } \eta \neq \langle\rangle \text{ then } \dfrac{a_\eta}{N_\eta + c_\eta} \perp_{\mathrm{a}} N_{(\eta^-)}.\end{array}$$

(2) We say "$\langle N_\eta, a_\eta : \eta \in I\rangle$ is a regular $\aleph_\epsilon$-decomposition inside M [of M]" <u>if</u> it is an $\aleph_\epsilon$-decomposition inside M [of M] which is $\emptyset$-regular.
(3) We say "$\langle N_\eta, a_\eta : \eta \in I\rangle$ is a regular $\aleph_\epsilon$-decomposition inside M [of M] over $\binom{B}{A}$" <u>if</u> it is an $\aleph_\epsilon$-decomposition inside M [of M] over $\binom{B}{A}$ which is $\{\langle\rangle\}$-regular.

1.13 CLAIM: (1) *Every $\aleph_\epsilon$-saturated model has an $\aleph_\epsilon$-decomposition (i.e., of it).*
(2) If M is $\aleph_\epsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M, <u>then</u> for some J, and N_η, a_η for $\eta \in J\backslash I$ we have: $I \subseteq J$ and $\langle N_\eta, a_\eta : \eta \in J\rangle$ is an $\aleph_\varepsilon$-decomposition of M (even a $(J\backslash I)$-regular one).
(3) If M is $\aleph_\epsilon$-saturated, $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M, <u>then</u> M is $\aleph_\epsilon$-prime and $\aleph_\epsilon$-minimal‡ *over $\bigcup_{\eta\in I} N_\eta$; if in addition $\langle N_\eta, a_\eta : \eta \in \{\langle\rangle, \langle 0\rangle\}\rangle$ is an $\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$, <u>then</u> $\langle N_\eta, a_\eta : \eta \in I \& (\eta \neq \langle\rangle \to \langle 0\rangle \trianglelefteq \eta)\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$.*
(4) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$, <u>then</u> it is an $\aleph_\epsilon$-decomposition inside M.
(5) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M [above $\binom{B}{A}$], $\eta \in I$, [$\eta \in I\backslash\{\langle\rangle\}$], $\alpha = \mathrm{Min}\{\beta : \eta\hat{\ }\langle\beta\rangle \notin I\}, \nu =: \eta\hat{\ }\langle\alpha\rangle, a_\nu \in M\backslash N_\eta$, $\frac{a_\nu}{N_\eta}$ is orthogonal to M_{η^-} if η^- if $\neq \langle\rangle, N_\nu \prec M$ is $\aleph_\epsilon$-primary over $N_\eta + a_\nu$ and $a_\nu \underset{N_\eta}{\bigcup\!\!\!\!\bigcup} (\bigcup_{\rho\in I} N_\rho)$ (enough to demand $\{a_\rho : \rho^- = \eta \text{ and } \rho \in I\}$ is independent over a_ν/N_η), <u>then</u> $\langle N_\rho, a_\rho : \rho \in I \cup \{\nu\}\rangle$ is an $\aleph_\epsilon$-decomposition inside M [over $\binom{B}{A}$].
(6) Assume $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M, <u>if</u> p is regular (stationary) and is not orthogonal to M (e.g., $p \in S(M)$), <u>then</u> for one and only one $\eta \in I$, there is a regular (stationary) $q \in S(N_\eta)$ not orthogonal to p such that: if η^- is well defined (i.e., $\eta \neq \langle\rangle$), then $p \perp N_{\eta^-}$.
(7) Assume $I = \bigcup_{\alpha<\alpha(*)} I_\alpha$, for each α we have $\langle N_\eta, a_\eta : \eta \in I_\alpha\rangle$ is an $\aleph_\epsilon$-decomposition inside M [above $\binom{B}{A}$] and for each $\eta \in I$ for every $n < \omega$ and $\nu_\ell = \eta\hat{\ }\langle\beta_\ell\rangle \in I$ for $\ell < n$, for some α we have: $\{\nu_\ell : \ell < n\} \subseteq I_\alpha$ (e.g., I_α increasing). <u>Then</u> $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M [above $\binom{B}{A}$].
(8) In (7), if $\eta \neq \langle\rangle$ and some ν_ℓ is not $\triangleleft$-maximal in I and $\frac{a_{\nu_\ell}}{N_\eta}$ is regular, it is

† Wlog $c_\eta = a_\eta$.
‡ Here we use NDOP.

enough:

$$\ell_1 < \ell_2 < n \Rightarrow \bigvee_{\alpha<\alpha(*)} [\{\nu_{\ell_1}, \nu_{\ell_2}\} \subseteq I_\alpha].$$

(9) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside $M, I_1, I_2 \subseteq I$ are closed under initial segments and $I_0 = I_1 \cap I_2$, then $(\bigcup_{\eta\in I_1} N_\eta) \underset{\bigcup_{\eta\in I_0} N_\eta}{\Cup} (\bigcup_{\eta\in I_2} N_\eta)$.

(10) Assume that for $\ell = 1,2$ that $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M_ℓ, and for $\eta \in I$ the function f_η is an isomorphism from N^1_η onto N^2_η and $\eta \triangleleft \nu \Rightarrow f_\eta \subseteq f_\nu$. Then $\bigcup_{\eta\in I} f_\eta$ is an elementary mapping; if in addition $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M_ℓ (for $\ell = 1,2$), then $\bigcup_{\eta\in I} f_\eta$ can be extended to an isomorphism from M_1 onto M_2.
(11) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M (above $\binom{B}{A}$) and $M^- \prec M$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$, then $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M (above $\binom{B}{A}$).
(12) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ in an $\aleph_\varepsilon$-decomposition inside M/of M (above $\binom{B}{A}$) and $a'_\eta \in N_\eta$ and N_η is $\aleph_\varepsilon$-prime over $N_{\eta^-} + a'_\eta$ for $\eta \in I \setminus \{\langle\,\rangle\}$ (and $a'_{\langle 0\rangle} = a_{\langle 0\rangle}$ or at least $\mathrm{dcl}(a'_{\langle 0\rangle}) \subseteq \mathrm{dcl}(B)$), then $\langle N_\eta, a'_\eta : \eta \in I\rangle$ in an $\aleph_\varepsilon$-decomposition inside M/of M (above $\binom{B}{A}$).

Proof: (1), (2), (3), (5), (6), (9), (10). Repeat the proofs of [Sh:c, X]. (Note that here a_η/N_η is not necessarily regular, a minor change.)
(4), (7). Check.
(8) As $\mathrm{Dp}(p) > 0 \Rightarrow p$ is trivial, by [Sh:c, Ch. X, 7.2, p. 551] and [Sh:c, Ch. X, 7.3]. $\blacksquare_{1.13}$

We shall prove:

1.14 Claim: *(1) If M is $\aleph_\epsilon$-saturated, $\binom{B}{A} \in \Gamma(M)$, then there is $\langle N_\eta, a_\eta : \eta \in I\rangle$, an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$.*
(2) Moreover if $\langle N_\eta, a_\eta : \eta \in I\rangle$ satisfies clauses (a) – (h) of Definition 1.11(1), we can extend it to satisfy clause (i) of 1.11(2), too.
(3) If $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$, $M^- \prec M$ is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$, then:
- (a) *$\langle N_\eta : \eta \in I\rangle$ is a $\aleph_\epsilon$-decomposition of M^-,*
- (b) *we can find an $\aleph_\epsilon$-decomposition $\langle N_\eta, a_\eta : \eta \in J\rangle$ of M such that $J \supseteq I$ and $[\eta \in J\backslash I \Leftrightarrow (\eta \neq \langle\rangle$ and $\neg\langle 0\rangle \triangleleft \eta)]$; moreover, the last phrase follows from the previous ones.*

(4) If in (3)(b) the set $J\backslash I$ is countable (finite is enough for our applications), then necessarily M, M^- are isomorphic, even adding all members of an ϵ-finite subset of M^- as individual constants.

(5) *If* $\langle N_\eta, a_\eta : \eta \in I\rangle$ *is an* $\aleph_\epsilon$*-decomposition of* M *above* $\binom{B}{A}$, $I \subseteq J$ *and* $\langle N_\eta, a_\eta : \eta \in J\rangle$ *is an* $\aleph_\epsilon$*-decomposition of* M, $M^- \prec M$ *is* $\aleph_\epsilon$*-prime over* $\bigcup_{\eta\in I} N_\eta$ *and* $\binom{B}{A} \leq^* \binom{B_1}{A_1}$ *and* $B_1 \subseteq M$ *and* $c \in M$ *and* $\frac{c}{B_1} \perp A_1$ *and* $\frac{c}{B_1}$ *is (stationary and) regular,* <u>then</u>

(α) $\frac{c}{B_1} \perp \frac{\cup\{N_\eta:\eta\in J\setminus I\}}{N_{\langle\rangle}}$,

(β) $\frac{c}{B_1}$ *is not orthogonal to some* $p \in \mathcal{P}(\langle N_\eta, a_\eta : \eta \in I\rangle, M^-)$.

(6) *If* $\langle N_\eta, a_\eta : \eta \in I\rangle$ *is an* $\aleph_\epsilon$*-decomposition of* M *above* $\binom{B}{A}$ *and* M^- *is* $\aleph_\epsilon$*-prime over* $\bigcup_{\eta\in I} N_\eta$, <u>then</u> *the set* $\mathcal{P} = \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M)$ *depends on* $\binom{B}{A}$ *and* M *only (and not on* $\langle N_\eta : \eta \in I\rangle$ *or* M^-*), recalling:*

$$\mathcal{P} = \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M) = \{p \in S(M) : p \text{ regular and for some } \eta \in I\setminus\{<>\}, \text{ we have: } p \text{ is orthogonal to } N_{\eta^-} \text{ but not to } N_\eta\}.$$

So let $\mathcal{P}(\binom{B}{A}, M) =: \mathcal{P}(\langle N_\eta : \eta \in I\rangle, M)$.

(7) *If* $\frac{B}{A}$ *is regular of depth zero or just* $\frac{b}{A} \leq_{\mathrm{a}} \frac{B}{A}$, $\frac{b}{A}$ *regular of depth zero and* M *is* $\aleph_\epsilon$*-saturated and* $B \subseteq M$, <u>then</u>

(a) *for any* α, *we have* $\mathrm{tp}_\alpha(\binom{B}{A}, M)$ *depends just on* $\mathrm{tp}_0(\binom{B}{A}, M)$,

(b) *if* $\binom{B}{A} \leq^* \binom{B'}{A} \in \Gamma(M)$ <u>then</u> $\mathrm{tp}_\alpha(\binom{B'}{A}, M)$ *depends just on* $\mathrm{tp}_0(\binom{B}{A}, M)$ *(and* (A, B, A', B) *but not on* M*).*

(8) *For* $\alpha < \beta$, *from* $\mathrm{tp}_\beta(\binom{B}{A}, M)$ *we can compute* $\mathrm{tp}_\alpha(\binom{B}{A}, M)$.

(9) *If* f *is an isomorphism from* M_1 *onto* M_2, $A_1 \subseteq B_1$ *are* ε*-finite subsets of* M_1 *and* $f(A_1) = A_2, f(B_1) = B_2$, <u>then</u>

$$\mathrm{tp}_\alpha(\binom{B_1}{A_1}, M_1) = \mathrm{tp}_\alpha(\binom{B_2}{A_2}, M_2)$$

(more pedantically $\mathrm{tp}_\alpha(\binom{B_2}{A_2}, M_2) = f[\mathrm{tp}_\alpha(\binom{B_1}{A_1}, M_1)]$ *or consider the* A_ℓ, B_ℓ *as indexed sets).*

We delay the proof (parts (1), (2), (3) are proved after 1.22, part (4), (6) after 1.23, and after it parts (5), (7), (8)). Part (9) is obvious.

1.15 Definition: (1) If $\binom{B}{A} \in \Gamma(M)$, M is $\aleph_\epsilon$-saturated, let $\mathcal{P}^M_{\binom{B}{A}}$ be the set $\mathcal{P}$ from Claim 1.14(6) above (by 1.14(6) this is well defined as we shall prove below).
(2) Let $\mathcal{P}_{\binom{B}{A}} = \{p : p$ is (stationary regular and) parallel to some $p' \in \mathcal{P}^{\mathfrak{C}^{\mathrm{eq}}}_{\binom{B}{A}}\}$.

1.16 Definition: If $\langle N^\ell_\eta, a_\eta : \eta \in J\rangle$ is a decomposition inside $\mathfrak{C}$ for $\ell = 1, 2$ we say that $\langle N^1_\eta, a_\eta : \eta \in J\rangle \leq^*_{\mathrm{direct}} \langle N^2_\eta, a_\eta : \eta \in J\rangle$ <u>if</u>:

(a) $N^1_{\langle\rangle} \prec N^2_{\langle\rangle}$,

(b) $N^2_{\langle\rangle} \underset{N^1_{\langle\rangle}}{\bigcup\!\!\!\!\bigcup} \{a_{\langle\alpha\rangle} : \langle\alpha\rangle \in J\}$,

(c) for $\eta \in J\backslash\{\langle\rangle\}, N^2_\eta$ is $\aleph_\epsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$.

1.17 CLAIM: (1) *M is $\aleph_\epsilon$-prime over A <u>iff</u> M is $\aleph_\epsilon$-primary over A <u>iff</u> M is $\aleph_\epsilon$-saturated, $A \subseteq M$, M is $\aleph_\varepsilon$-atomic over A (see 0.1(16)) for every $\mathbf{I} \subseteq M$ indiscernible over A we have: $\dim(\mathbf{I}, M) \leq \aleph_0$ <u>iff</u> M is $\aleph_\epsilon$-saturated, $A \subseteq M$, M is $\aleph_\varepsilon$-atomic over A and for every finite $B \subseteq M$ and regular (stationary) $p \in S(A \cup B)$, we have $\dim(p, M) \leq \aleph_0$.*

(2) *If N_1, N_2 are $\aleph_\epsilon$-prime over A, <u>then</u> they are isomorphic over A.*

Proof: By [Sh:c, IV, 4.18] (see Definition [Sh:c, IV, 4.16], noting that we replace $\mathbf{F}^a_{\aleph_0}$ by $\aleph_\varepsilon$ and that part (4) there disappears when we are speaking on $\mathbf{F}^a_{\aleph_0}$). $\blacksquare_{1.16}$

However, we need more specific information saying that "minor changes" preserve being $\aleph_\varepsilon$-prime. This is done in 1.18 below; parts of it are essentially done in [Sh 225] but we give a full proof.

1.18 FACT: (0) If A is countable, N is $\aleph_\epsilon$-primary over A <u>then</u> N is $\aleph_\epsilon$-primary over $\emptyset$.

(1) If N is $\aleph_\epsilon$-prime over $\emptyset$, A countable, N^+ is $\aleph_\epsilon$-prime over $N \cup A$, <u>then</u> N^+ is $\aleph_\epsilon$-prime over $\emptyset$.

(2) If $\langle N_n : n < \omega\rangle$ is increasing, each N_n is $\aleph_\epsilon$-prime over $\emptyset$ or just $\aleph_\varepsilon$-constructible over $\emptyset$ and N_ω is $\aleph_\epsilon$-prime over $\bigcup_{n<\omega} N_n$, <u>then</u> N_ω is $\aleph_\epsilon$-prime over $\emptyset$ (note that if each N_n is $\aleph_\varepsilon$-saturated then $N_\omega = \bigcup_{n<\omega} N_n$).

(2A) If N is $\aleph_\varepsilon$-prime over C, $\bar{a}\hat{\ }\bar{b} \subseteq N$, $\mathrm{tp}(\bar{b}, \bar{a})$ is regular (stationary) and orthogonal to C, <u>then</u> $\dim(\mathrm{tp}(\bar{b}, \bar{a}), N) \leq \aleph_0$; also, if $q \in S(C \cup \bar{a})$ is a nonforking extension of $\mathrm{tp}(\bar{b}, \bar{a})$ then $\dim(q, C \cup \bar{a}) = \dim(\mathrm{tp}(\bar{b}, \bar{a}), N) = \aleph_0$.

(2B) If $C \cup \bar{a}\hat{\ }\bar{b} \subseteq N$ and $\bar{a}/\bar{b}$ is a regular type orthogonal to C and $q \in S^{\ell g(\bar{a})}(N)$ is a nonforking extension of $\bar{a}/\bar{b}$, <u>then</u> $\dim(p \restriction (C + \bar{b}), N) \leq \dim(\bar{a}/\bar{b}, N) \leq \dim(p \restriction (C + \bar{b}), N) + \aleph_0$; moreover, $\dim(p \restriction (C + \bar{b}), N) \leq \dim(\bar{a}/\bar{b}, N) < \dim(p \restriction (C + \bar{b}), N)^+ + \aleph_0$.

(3) If $N_2 \underset{N_0}{\bigcup\!\!\!\!\bigcup} N_1$, each N_ℓ is $\aleph_\epsilon$-saturated, N_2 is $\aleph_\epsilon$-prime over $N_0 \cup \bar{a}$, and N_3 is $\aleph_\epsilon$-prime over $N_2 \cup N_1$, <u>then</u> N_3 is $\aleph_\epsilon$-prime over $N_1 \cup \bar{a}$.

(4) If $N_1 \prec N_2$ are $\aleph_\epsilon$-primary over $\emptyset$, <u>then</u> for some $\aleph_\epsilon$-saturated $N_0 \prec N_1$ (necessarily $\aleph_\epsilon$-primary over $\emptyset$) we have: N_1, N_2 are isomorphic over N_0.

(5) In part (4), if $A \subseteq N_1$ is ϵ-finite <u>then</u> we can demand $A \subseteq N_0$.

(6) If M_0 is $\aleph_\epsilon$-saturated, $A \underset{M_0}{\bigcup\!\!\!\!\bigcup} B, M_1$ is $\aleph_\epsilon$-primary over $M_0 \cup A$, <u>then</u> $M_1 \underset{M_0}{\bigcup\!\!\!\!\bigcup} B$.

(7) Assume $N_0 \prec N_1 \prec N_2$ are $\aleph_\epsilon$-saturated, N_2 is $\aleph_\epsilon$-primary over $N_1 + a$ and $\frac{a}{N_1} \perp N_0$ (and $a \notin N_1$). If $N'_0 \prec N_0, N'_0 \prec N'_1 \prec N_1, N'_1 \underset{N'_0}{\bigcup\!\!\!\!\bigcup} N_0$ and N_1 is $\aleph_\epsilon$-primary over $N_0 \cup N'_1, A^*_1 \subseteq N'_1, A^*_2 \subseteq N_2$ are ε-finite and $\mathrm{tp}_*(A^*_2, N_1)$ does not fork over A^*_1, then we can find a', N'_2 such that: N'_2 is $\aleph_\epsilon$-saturated, $\aleph_\epsilon$-primary over $N'_1 + a', N'_1 \prec N'_2 \prec N_2, N_1 \underset{N'_1}{\bigcup\!\!\!\!\bigcup} N'_2$ and N_2 is $\aleph_\epsilon$-primary over $N_1 \cup N'_2$ and $A^*_2 \subseteq N'_2$.
(8) Assume $N'_0 \prec N_0 \prec N_1$ and $a \in N_1$ and N_1 is $\aleph_\epsilon$-prime over $N_0 + a$ and $\frac{a}{N_0} \pm N'_0$ and $A^*_0 \subseteq N'_0, A^*_1 \subseteq N_1$ are ε-finite and $\mathrm{tp}_*(A^*_1, N_0)$ does not fork over A^*_0; then we can find a', N'_1 such that $a' \in N', N'_0 \prec N'_1 \prec N_1, N'_1 \underset{N'_0}{\bigcup\!\!\!\!\bigcup} N_0, N'_1$ is $\aleph_\epsilon$-prime over $N'_0 + a$ and N_1 is $\aleph_\epsilon$-prime over $N_0 + N'_1$ and $A^*_1 \subseteq N'_1$.
(9) If N_1 is $\aleph_\epsilon$-prime over $\emptyset$ and $A \subseteq B \subseteq N_1$ and A, B are ϵ-finite, then we can find N_0 such that: $A \subseteq N_0 \prec N_1, N_0$ is $\aleph_\epsilon$-prime over $\emptyset, A \subseteq N_0, B \underset{A}{\bigcup\!\!\!\!\bigcup} N_0$, and N_1 is $\aleph_\epsilon$-prime over $N_0 \cup B$.
(10) If N_0 is $\aleph_\epsilon$-prime over A and $B \subseteq N_0$ is ϵ-finite, then N_0 is $\aleph_\epsilon$-prime over $A \cup B$ (and also over A' if $A \subseteq A' \subseteq \mathrm{acl}(A)$).

1.19 Remark: In the proof of 1.18(1)–(6),(10) we do not use "T has NDOP".

Proof: (0) There is $\{a_\alpha : \alpha < \alpha^*\}$, a list of members of N in which every member of $N \setminus A$ appears such that for $\alpha < \alpha(*)$ we have: $\mathrm{tp}(a_\alpha, A \cup \{a_\beta : \beta < \alpha\})$ is $\aleph_\varepsilon$-isolated (which means just $\mathbf{F}^a_{\aleph_0}$-isolated).
[Why? By the definition of "N is $\aleph_\epsilon$-primary over A".] Let $\{b_n : n < \omega\}$ list A (if $A = \emptyset$ the conclusion is trivial, so without loss of generality $A \neq \emptyset$, hence we can find such a sequence $\langle b_n : n < \omega \rangle$). Now define $\beta^* = \omega + \beta$ and $b_{\omega+\alpha} = a_\alpha$ for $\alpha < \alpha^*$. So $\{b_\beta : \beta < \beta^*\}$ lists the elements of N (possibly with repetition, remember $A \subseteq N$ and check). We claim that $\mathrm{tp}(b_\beta, \{b_\gamma : \gamma < \beta\})$ is $\mathbf{F}^a_{\aleph_0}$-isolated for $\beta < \beta^*$.
[Why? If $\beta \geq \omega$, let $\beta' = \beta - \omega$ (so $\beta < \alpha^*$); now the statement above means $\mathrm{tp}(a_{\beta'}, A \cup \{a_\gamma : \gamma < \beta'\})$ is $\mathbf{F}^a_{\aleph_0}$-isolated, which we know. If $\beta < \omega$ this statement is trivial.] By the definition of "$\mathbf{F}^a_{\aleph_0}$-primary", clearly $\langle b_\beta : \beta < \omega + \alpha \rangle$ exemplifies that N is $\mathbf{F}^a_{\aleph_0}$-primary over $\emptyset$.
(1) Note

- $(*)_1$ if N is $\aleph_\epsilon$-primary over $\emptyset$ and $A \subseteq N$ is finite, then N is $\aleph_\epsilon$-primary over A [why? see [Sh:c, IV, 3.12(3), p. 180] (of course, using [Sh:c, IV, Table 1, p. 169] for $\mathbf{F}^a_{\aleph_0}$];
- $(*)_2$ if N is $\aleph_\epsilon$-primary over $\emptyset, A \subseteq N$ is finite and $p \in S^m(N)$ does not fork over A and $p \restriction A$ is stationary, then for some $\{\bar{a}_\ell : \ell < \omega\}$ we have: $\bar{a}_\ell \in N$

realize p, $\{\bar{a}_\ell : \ell < \omega\}$ is independent over A and $p \restriction (A \cup \bigcup_{\ell<\omega} \bar{a}_\ell) \vdash p$ [why? [Sh:c, IV, proof of 4.18] (i.e., by it and [Sh:c, 4.9(3), 4.11]) or let N' be $\aleph_\epsilon$-primary over $A \cup \bigcup_{\ell<\omega} \bar{a}_\ell$ and note: N' is $\aleph_\epsilon$-primary over A (proof like the one of 1.18(0)) but also N is $\aleph_\epsilon$-primary over A, so by uniqueness of the $\aleph_\epsilon$-primary model N' is isomorphic to N over A, so without loss of generality $N' = N$; and easily N' is as required].

Now we can prove 1.18(1), for any $\bar{c} \in {}^{\omega>}A$, we can find a finite $B^1_{\bar{a}} \subseteq N$ such that $\mathrm{tp}(\bar{c}, N)$ does not fork over $B^1_{\bar{c}}$, let $\bar{b}_{\bar{c}} \in {}^{\omega>}N$ realize $\mathrm{stp}(\bar{a}, B^1_{\bar{a}})$ and let $B_{\bar{c}} = B^1_{\bar{c}} \cup \bar{b}_{\bar{c}}$, so $\mathrm{tp}(\bar{c}, N)$ does not fork over $B_{\bar{c}}$ and $\mathrm{tp}(\bar{c}, B_{\bar{c}})$ is stationary, hence we can find $\langle \bar{a}^{\bar{c}}_\ell : \ell < \omega\}$ as in $(*)_2$ (for $\mathrm{tp}(\bar{c}, B_{\bar{c}})$). Let $A' = \cup\{B_{\bar{c}} : \bar{c} \in {}^{\omega>}A\} \cup \{\bar{a}^{\bar{c}}_\ell : \bar{c} \in {}^{\omega>}A$ and $\ell < \omega\}$, so A' is a countable subset of N and $\mathrm{tp}_*(A, A') \vdash \mathrm{tp}(A, N) = \mathrm{stp}(A, N)$. As N is $\aleph_\epsilon$-primary over $\emptyset$ we can find a sequence $\langle d_\alpha : \alpha < \alpha^*\rangle$ and $\langle w_\alpha : \alpha < \alpha^*\rangle$ such that $N = \{d_\alpha : \alpha < \alpha^*\}$ and $w_\alpha \subseteq \alpha$ is finite and $\mathrm{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha\}) \vdash \mathrm{stp}(d_\alpha, \{d_\beta : \beta < \alpha\})$ and $\beta < \alpha \Rightarrow d_\beta \neq d_\alpha$.

We can find a countable set $W \subseteq \alpha^*$ such that $A' \subseteq \{d_\alpha : \alpha \in W\}$ and $\alpha \in W \Rightarrow w_\alpha \subseteq W$. Let $A'' = \{a_\alpha : \alpha \in W\}$. By [Sh:c, IV, §2, §3] without loss of generality W is an initial segment of α^*. Easily

$$\alpha < \alpha^* \& \alpha \notin W \Rightarrow \mathrm{stp}(d_\alpha, \{d_\beta : \beta \in w_\alpha) \vdash \mathrm{stp}(d_\alpha, A \cup \{d_\beta : \beta < \alpha\}).$$

As N^+ is $\aleph_\epsilon$-primary over $N \cup A$ we can find a list $\{d_\alpha : \alpha \in [\alpha^*, \alpha^{**})\}$ of $N^+ \backslash (N \cup A)$ such that $\mathrm{tp}(d_\alpha, N \cup A \cup \{d_\beta : \beta \in [\alpha^*, \alpha^{**})\})$ is $\aleph_\epsilon$-isolated. So $\langle d_\alpha : \alpha \notin W, \alpha < \alpha^{**}\rangle$ exemplifies that N^+ is $\aleph_\epsilon$-primary over $A \cup A''$, hence by 1.18(0) we know that N^+ is $\aleph_\epsilon$-primary over $\emptyset$.

(2) We shall use the characterization of "N is $\mathbf{F}^a_{\aleph_0}$-prime over A" in 1.17; more exactly we use the last condition in 1.17(1) for $A = \emptyset$, $M = N_\omega$. Clearly N_ω is $\aleph_\epsilon$-saturated (as it is $\aleph_\epsilon$-prime over $\bigcup_{n<\omega} N_n$). Suppose $B \subseteq N_\omega$ is finite and $p \in S(B)$ is (stationary and) regular.

CASE 1: p not orthogonal to $\bigcup_{n<\omega} N_n$.

So for some $n < \omega$, p is not orthogonal to N_n, hence there is a regular $p_1 \in S(N_n)$ such that p, p_1 are not orthogonal. Let $A_1 \subseteq N_n$ be finite such that p_1 does not fork over A and $p_1 \restriction A_1$ is stationary. So by [Sh:c, V, §2] we know $\dim(p, N_\omega) = \dim(p_1 \restriction A_1, N_\omega)$, hence it suffices to prove that the latter is $\aleph_0$. Now this holds by [Sh:c, V, 1.16(3), p. 237] or imitate the proof of $(*)_2$ above.

CASE 2: p is orthogonal to $\bigcup_{n<\omega} N_n$.

Note that if each N_n is $\aleph_\varepsilon$-prime, then $\bigcup_{n<\omega} N_n$ is $\aleph_\varepsilon$-saturated, hence $N = \bigcup_{n<\omega} N_n$ hence this case does not arise. Let $A = \bigcup_{n<\omega} N_n$, so $\dim(p, N) \leq \aleph_0$

follows from (2A) below.

Alternatively (and work even if we replace N_n by a set A_n, $\mathbf{F}^a_{\aleph_0}$-constructible over $\emptyset$), see below.

(2A) By (2B).

(2B) The first inequality is immediate (as T is superstable and $\bar{a}, \bar{b}$ are finite), so let us concentrate on the second. Let $B \subseteq C$ be a finite set such that $\mathrm{tp}_*(\bar{a}\hat{\ }\bar{b}, C)$ does not fork over B and $\mathrm{stp}_*(\bar{a}\hat{\ }\bar{b}, B) \vdash \mathrm{stp}_*(\bar{a}\hat{\ }\bar{b}, C)$. Recall $q \in S(N)$ extend $\bar{a}/\bar{b}$ and do not fork over $\bar{b}$, let $b^* \in \mathfrak{C}$ realize q and let $q_1 = \mathrm{stp}(\bar{b}^*, B \cup \bar{b})$ and $q_2 = \mathrm{stp}(\bar{b}^*, C \cup \bar{b})$. Now by the assumption of our case q_1 is orthogonal to $\mathrm{tp}_*(C, B)$ hence (see [Sh:c, V, §3]) $q_1 \vdash q_2$ and let $\{a_\alpha : \alpha < \alpha^*\} \subseteq (q_1 \restriction (\bar{b} \cup B))(N)$ be a maximal set independent over $C + \bar{b}$, so $|\alpha^*| \leq \dim(\bar{a}/(C + \bar{b}), N)$ and $q \restriction (C \cup \bar{b} \cup \{a_\alpha : \alpha < \alpha^*\}) \vdash q$. Also clearly $\mathrm{stp}_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup B) \vdash \mathrm{stp}_*(\{a_\alpha : \alpha < \alpha^*\}, \bar{b} \cup C)$. Together $\dim(q_1, N) \leq |\alpha^*|$ and as $|B| < \aleph_0 = \kappa_r(T)$ clearly $\dim(\bar{a}/\bar{b}, N) < \aleph_0 + \dim(q_1, N)^+$, so we are done.

We can use a different proof for part (2), note:

- $\otimes_1$ if $\kappa = \mathrm{cf}(\kappa) \geq \kappa_r(T)$ and B_α is $\mathbf{F}^a_\kappa$-constructible over A for $\alpha < \delta, \delta \leq \kappa$ and $\alpha < \beta < \delta \Rightarrow B_\alpha \subseteq B_\beta$, then $\bigcup_{\alpha<\delta} B_\alpha$ is $\mathbf{F}^a_\kappa$-constructible over A. [Why? See [Sh:c, IV, §3], [Sh:c, IV, 5.6, p. 207] for such arguments; assume $\mathcal{A}_\alpha = \langle A, \langle a^\alpha_i : i < i_\alpha\rangle, \langle B^\alpha_i : i < i_\alpha\rangle\rangle$ is an $\mathbf{F}^a_\kappa$-construction of B_α over A. Without loss of generality $i < j < i_\alpha \Rightarrow a^\alpha_i \neq a^\alpha_i$, and choose by induction on $\zeta, \langle u^\alpha_\zeta : \alpha < \delta\rangle$ such that: $u^\alpha_\zeta \subseteq i_\alpha, u^\alpha_\zeta$ increasing continuous in $i, u^\alpha_0 = \emptyset, |u^\alpha_{\zeta+1} \backslash u^\alpha_\zeta| \leq \kappa, u^\alpha_\zeta$ is $\mathcal{A}_\alpha$-closed and $\alpha < \beta < \delta$ implies $\{a^\alpha_j : j \in u^\alpha_\zeta\} \subseteq \{a^\beta_j : j \in u^\beta_\zeta\}$ and $\mathrm{tp}_*(\{a^\beta_i : i \in u^\beta_\zeta\}, A \cup \{a^\alpha_i : i < i_\alpha\})$ does not fork over $A \cup \{a^\alpha_i : i \in u^\alpha_\zeta\}$. Now find a list $\langle a_j : j < j^*\rangle$ such that for each $\zeta, \{j : a_j \in a^\alpha_i : i \in u^\alpha_\varepsilon$ for some $\alpha < \delta, \varepsilon < \zeta\}$ is an initial segment β_ζ of j^* and $\beta_{\zeta+1} \leq \beta_\zeta + \kappa$.]

We use $\otimes_1$ for $\kappa = \aleph_0$. So each N_n is $\aleph_\epsilon$-constructible over $\emptyset$, hence $\bigcup_{n<\omega} N_n$ is $\aleph_\epsilon$-constructible over $\emptyset$ and also N_ω is $\aleph_\epsilon$-constructible over $\bigcup_{n<\omega} N_n$, hence N_ω is $\aleph_\epsilon$-constructible over $\emptyset$. But N_ω is $\aleph_\epsilon$-saturated, hence N_ω is $\aleph_\epsilon$-primary over $\emptyset$. Alternatively use: if B is $\mathbf{F}^a_\kappa$-constructible over A, $\kappa \geq \kappa_i(\tau)$ and $\mathbf{I}$ is indiscernible over A, $|\mathbf{I}| > \kappa$ <u>then</u> for some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $\leq \kappa$, $\mathbf{I} \setminus \mathbf{J}$ is an indiscernible set over B.

(3) Suppose N'_3 is $\aleph_\epsilon$-saturated and $N_1 + \bar{a} \subseteq N'_3$. As N_2 is $\aleph_\epsilon$-prime over $N_0 + \bar{a}$ and $N_0 + \bar{a} \subseteq N_1 + \bar{a} \subseteq N'_3$ we can find an elementary embedding f_0 of N_2 into N'_3 extending $\mathrm{id}_{N_0+\bar{a}}$. By [Sh:c, V, 3.3], the function $f_1 = f_0 \cup \mathrm{id}_{N_1}$ is an elementary mapping and clearly $\mathrm{Dom}(f_1) = N_1 \cup N_2$. As N_3 is $\aleph_\epsilon$-prime over $N_1 \cup N_2$ and f_1 is an elementary mapping from $N_1 \cup N_2$ into N'_3, which is an $\aleph_\varepsilon$-saturated

model, there is an elementary embedding f_3 of N_3 into N_3' extending f_2. So as for any such N_3' there is such f_3, clearly N_3 is $\aleph_\epsilon$-prime over $N_1+\bar{a}$, as required.
(4) Let N_0 be $\aleph_0$-prime over $\emptyset$ and let $\{p_i : i<\alpha\} \subseteq S(N_0)$ be a maximal family of pairwise orthogonal regular types. Let $\mathbf{I}_i = \{\bar{a}^i_n : n<\omega\} \subseteq \mathfrak{C}$ be a set of elements realizing p_i independent over N_0 and let $\mathbf{I} = \bigcup_{i<\alpha} \mathbf{I}_i$ and N_1' be $\mathbf{F}^a_{\aleph_0}$-prime over $N_0 \cup \mathbf{I}$. Now

(∗) if $\bar{a}, \bar{b} \subseteq N_1'$ and $\bar{a}/\bar{b}$ is regular (hence stationary), then $\dim(\bar{a}/\bar{b}, N_1') \leq \aleph_0$.
[Why? If $\bar{a}/\bar{b} \perp N_0$, then $\dim(\bar{a}/\bar{b}, N_1') \leq \aleph_0$ by part (2A) and the choice of the p_i and $\mathbf{I}_i$ for $i<\alpha$. If $\bar{a}/\bar{b} \pm N_0$, then for some $\bar{b}'\hat{\ }\bar{a}' \subseteq N_0$ realizing $\mathrm{stp}(\bar{b}\hat{\ }\bar{a}, \emptyset)$, we have $\bar{a}'/\bar{b}' \pm \bar{a}/b$ hence $\dim(\bar{a}/b, N_1') = \dim(\bar{a}'/\bar{b}', N_1')$, so without loss of generality $\bar{b}\hat{\ }\bar{a} \subseteq N_0$; similarly, without loss of generality there is $i(*)<\alpha$ such that $\bar{a}/\bar{b} \subseteq p_{i(*)}$ and $p_{i(*)}$ do not fork over $\bar{b}$, now easily $\dim(\bar{a}/\bar{b}, N_1') = \dim(\bar{a}/\bar{b}, N_0) + \dim(p_{i(*)}, N_0) \leq \aleph_0 + \aleph_0 = \aleph_0$ (see [Sh:c, V, 1.6(3)]). So we have proved (∗).]

Now use 1.17(1) to deduce: N_1' is $\mathbf{F}^a_{\aleph_\epsilon}$-prime over $\emptyset$, hence (by uniqueness of $\aleph_\epsilon$-prime model, 1.17(2)) $N_1' \cong N_1$.

By renaming, without loss of generality $N_1' = N_1$. Now

(∗∗)(α) $(N_1, c)_{c\in N_0}, (N_2, c)_{c \in N_0}$ are $\aleph_\epsilon$-saturated and

(β) if $\bar{a} \in \mathfrak{C}, \bar{b} \in N_\ell, \bar{a}/\bar{b}$ a regular type and $\bar{a} \underset{\bar{b}}{\bigcup\!\!\!\!\bigcup} (N_0+\bar{b})$ (for $\ell=1$ or $\ell=2$), then $\dim(\bar{a}/(\bar{b}\cup N_0), N_\ell) = \aleph_0$.

[Why? Remember that we work in $(\mathfrak{C}^{\mathrm{eq}}, c)_{c\in N_0}$. The "$\aleph_\epsilon$-saturated" follows from the second statement.
Note: $\dim(\bar{a}/(\bar{b}\cup N_0), N_\ell) \leq \dim(\bar{a}/\bar{b}, N_\ell) \leq \aleph_0$ (the first inequality by monotonicity, the second inequality by 1.17(1) and the assumption "N_ℓ is $\aleph_\epsilon$-prime over $\emptyset$"). If $\bar{a}/\bar{b}$ is not orthogonal to N_0, then for some $i<\alpha$ we have $p_i \pm (\bar{a}/\bar{b})$, so easily (using "$N_\ell$ is $\aleph_\varepsilon$-saturated") we have $\dim(\bar{a}/(\bar{b}\cup N_0), N_\ell) = \dim(p_i, N_\ell) \geq \|\mathbf{I}_i\| = \aleph_0$; so together with the previous sentence we get equality. Lastly, if $\bar{a}/\bar{b} \perp N_0$ by part (2B) of 1.18, we have $\dim(\bar{a}/(\bar{b}\cup N_0), N_\ell) < \aleph_0 \Rightarrow \dim(\bar{a}/\bar{b}, N_\ell) < \aleph_0$, which contradicts the assumption "N_ℓ is $\aleph_\epsilon$-saturated".] So we have proved (∗∗), hence by 1.17(1) we get "N_1, N_2 are isomorphic over N_0''" as required.
(5) This is proved similarly, because if N is $\aleph_\varepsilon$-prime over A and $B \subseteq N$ is ε-finite, then N is $\aleph_\varepsilon$-prime over $A+B$ and also over A' if $A+B \subseteq A' \subseteq \mathrm{acl}(A+B)$; see part (10).
(6) By [Sh:c, V, 3.2].
(7) First assume that $A_2^* \subseteq N_1$ and a/N_1 is regular. As N_1 is $\aleph_\epsilon$-prime over $N_0 \cup N_1'$ and as T has NDOP (i.e., does not have DOP), we know (by [Sh:c, X, 2.1, 2.2, p. 512]) that N_1 is $\aleph_\epsilon$-minimal over $N_0 \cup N_1'$ and $\frac{a}{N_1}$ is not orthogonal to N_0 or to N_1'. But $a/N_1 \perp N_0$ by an assumption, so a/N_1 is not orthogonal to

N_1', hence there is a regular $p' \in S(N_1')$ not orthogonal to $\frac{a}{N_1}$, hence (by [Sh:c, V, 1.12, p. 236]) p' is realized say by $a' \in N_2$. By [Sh:c, V, 3.3], we know that N_2 is $\aleph_\epsilon$-prime over $N_1 + a'$. We can find N_2' which is $\aleph_\epsilon$-prime over $N_1' + a'$ and N_2'' which is $\aleph_\epsilon$-prime over $N_1 \cup N_2'$, hence by part (3) of 1.18 we know that N_2'' is $\aleph_\varepsilon$-prime over $N_1 + a'$, so by uniqueness, i.e., 1.17(1), without loss of generality $N_2'' = N_2$, hence we are done.

In general, by induction on α choose $N_{2,\alpha}'$ such that $N_{2,0}'$ is $\aleph_\varepsilon$-prime over $N_1' \cup A_2^*$, $N_{2,\alpha}'$ is increasing with α and $N_i \bigcup\limits_{N_1'} N_{2,\alpha}'$. Easy for some α, $N_{2,\alpha}'$ is defined but not $N_{2,\alpha+1}'$. Necessarily N_2 is $\aleph_\varepsilon$-prime over $N_1' \cup N_{2,\alpha}'$. Lastly, let $a' \in N_{2,\alpha}'$ be such that $\mathrm{tp}(a, N_1 \cup N_{2,\alpha}')$ dnf over $N_1 + a'$. Easily $N_{2,\alpha}'$ is $\aleph_\varepsilon$-prime over $N_1' + a'$ (by 1.17(1)).

(8) A similar, easier proof.

(9) Let N_0' be $\aleph_\epsilon$-prime over A such that $B \bigcup\limits_{A} N_0'$, and let N_1' be $\aleph_\epsilon$-prime over $N_0' \cup B$. By 1.18(1), we know that N_1' is $\aleph_\epsilon$-prime over $\emptyset$, and by 1.18(10) below N_1' is $\aleph_\epsilon$-prime over $A \cup B$; hence by 1.17(2) we know that N_1', N_1 are isomorphic over $A \cup B$, hence without loss of generality $N_1' = N_1$ and so $N_0 = N_0'$ is as required.

(10) By [Sh:c, IV, 3.12(3), p. 180]. $\blacksquare_{1.18}$

1.20 FACT: Assume $\langle N_\eta^1, a_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N_\eta^2, a_\eta : \eta \in I\rangle$ (see Definition 1.16) and $A \subseteq B \subseteq N^1_{<0>}$ and $\bigwedge_{\eta \in I} N_\eta^2 \prec M$.

(1) If $\nu = \eta\hat{\ }\langle\alpha\rangle \in I$, then $N_\eta^2 \bigcup\limits_{N_\eta^1} N_\nu^1$ and even $N_\eta^2 \bigcup\limits_{N_\eta^1} (\bigcup_{\substack{\rho \in I \\ \neg\eta\triangleleft\rho}} N_\rho^1)$; and $\eta \triangleleft \nu \in I$ implies $N_\nu^2 \bigcup\limits_{N_\eta^1} (\bigcup_{\substack{\rho \in T \\ \neg\eta\triangleleft\rho}} N_\rho^1)$.

(2) $\langle N_\eta^2, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$ <u>iff</u> $\langle N_\eta^1, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$.

(3) Similarly, replacing "$\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$" by "$\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$".

Proof: (1) We prove the first statement by induction on $\ell g(\eta)$. If $\eta = <>$ this is clause (b) by the Definition 1.16 and clause (d) of Definition 1.11(1) (and [Sh:c, V, 3.2]). If $\eta \neq <>$, then $\frac{a_\nu}{N_\eta} \perp N^1_{(\eta^-)}$ (by condition (e) of Definition 1.11(1)). By the induction hypothesis $N^2_{(\eta^-)} \bigcup\limits_{N^1_{(\eta^-)}} N_\eta^1$ and we know N_η^2 is $\aleph_\epsilon$-primary over $N^2_{(\eta^-)} \cup N_\eta^1$; we know this implies that no $p \in S(N_\eta^1)$ orthogonal to $N^1_{\eta^-}$ is realized in N_η^2, hence $\frac{a_\nu}{N_\eta^1} \perp \frac{N_\eta^2}{N_\eta^1}$, so $\frac{a_\nu}{N_\eta^1} \vdash \frac{a_\nu}{N_\eta^2}$, hence $\frac{N_\nu^1}{N_\eta^1} \perp \frac{N_\eta^2}{N_\eta^1}$, hence $N_\nu^1 \bigcup\limits_{N_\eta^1} N_\eta^2$ as required. The other statements hold by the non-forking calculus (remember, if $\eta = \nu\char`\^\langle\alpha\rangle \in I$ then use $\mathrm{tp}(\cup\{N^1_\rho : \eta \trianglelefteq \rho \in I\}, N^1_\eta)$ is orthogonal to N^1_ν or see details in the proof of 1.21(1)(α)).

(2) By Definition 1.16, for $\ell = 1,2$ we have: $\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is a decomposition inside $\mathfrak{C}$ and by assumption $\bigwedge_{\eta\in I} N^1_\eta \prec N^2_\eta \prec M$. So for $\ell = 1,2$ we have to prove "$\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M for $\binom{B}{A}$" assuming this holds for $1-\ell$. We have to check Definition 1.11(1).

Clauses 1.5(1)(a),(b) for ℓ hold because they hold for $1-\ell$.

Clause 1.5(1)(c) holds, as by the assumptions $A \subseteq B \subseteq N^1_{<0>} \prec N^2_{<0>}, A \subseteq N^1_{<>}$ and $N^1_{<0>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} N^2_{<>}$.

Clauses 1.5(1)(d),(e),(f),(h) hold as $\langle N^\ell_\eta, a_\eta : \eta \in I\rangle$ is a decomposition inside $\mathfrak{C}$ (for $\ell = 1$ given, for $\ell = 2$ easily checked).

Clause 1.5(1)(g) holds as $\bigwedge_\eta N^1_\eta \prec N^2_\eta \prec M$ is given and M is $\aleph_\epsilon$-saturated.

(3) First we do the "only if" direction; i.e., prove the maximality of $\langle N^1_\eta, a_\eta : \eta \in I\rangle$ as an $\aleph_\epsilon$-decomposition inside M for $\binom{B}{A}$ (i.e., condition (i) from 1.11(2)), assuming it holds for $\langle N^2_\eta, a_\eta : \eta \in I\rangle$. If this fails, then for some $\eta \in I\backslash\{<>\}$ and $a \in M, \{a_{\eta\char`\^<\alpha>} : \eta\char`\^ <\alpha> \in I\} \cup \{a\}$ is independent over N^1_η and $a \notin \{a_{\eta\char`\^\langle\alpha\rangle} : \eta\char`\^\langle\alpha\rangle \in I\}$ and $\frac{a}{N^1_\eta} \perp N^1_{\eta^-}$. Hence, if $\eta\char`\^\langle\alpha_\ell\rangle \in I$ for $\ell < k$ then $\bar{a} = \langle a\rangle\char`\^\langle a_{\eta\char`\^<\alpha_\ell>} : \ell < k\rangle$ realizes over N^1_η a type orthogonal to $N^1_{\eta^-}$, but $N^1_{\eta^-} \prec N^1_\eta, N^1_{\eta^-} \prec N^2_{\eta^-}$ and $N^1_\eta \underset{N^1_{\eta^-}}{\bigcup\!\!\!\!\bigcup} N^2_\eta$ (see 1.20(1), hence (by [Sh:c, V, 2.8]) $\mathrm{tp}(\bar{a}, N^2_\eta) \perp N^2_{\eta^-}$, hence $\{a\} \cup \{a_{\eta\char`\^\langle\ell\rangle} : \ell < k\}$ is independent over $N^2_\eta$; but $k, \eta\char`\^\langle\alpha_\ell\rangle I$ for $\ell < k$ were arbitrary, so $\{a\} \cup \{a_{\eta\char`\^\langle\alpha\rangle} : \eta\char`\^\langle\alpha\rangle \in I\}$ is independent over N^2_η, contradicting condition (i) from Definition 1.11(2) for $\langle N^2_\eta, a_\eta : \eta \in I\rangle$.

For the other direction use: if the conclusion fails, then for some $\eta \in I\backslash\{<>\}$ and $a \in M\backslash N^2_\eta \setminus \{a_{\eta^1\langle\alpha\rangle} : \eta^1\langle\alpha\rangle \in I\}$ the set $\{a_{\eta\char`\^\langle\alpha\rangle} : \eta\char`\^\langle\alpha\rangle \in I\} \cup \{a\}$ is independent of N^2_η and $\mathrm{tp}(a, N^2_\eta)$ is orthogonal to $N^2_{\eta^-}$; let $N' \prec M$ be $\aleph_\varepsilon$-prime over $N^2_\eta + a$. But N^2_η is $\aleph_\varepsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$ (by the definition of $\leq_{\mathrm{direct}}$) so by NDOP $\mathrm{tp}(a, N^2_\eta) \not\perp N^1_\eta$, hence there is a regular $q \in S(N^1_\eta)$ such that $q \pm \mathrm{tp}(a, N^2_\eta)$. Hence some $a' \in N'$ realizes q; clearly $\{a_{\eta\char`\^<\alpha>} : \eta\char`\^ <\alpha> \in I\} \cup \{a'\}$ is independent over N^2_η (and $a' \notin \{a^1_{\eta^1\langle\alpha\rangle} : \eta^1\langle\alpha\rangle \in I\}$), hence over (N^2_η, N^1_η) and easily we get a contradiction. $\blacksquare_{1.20}$

1.21 Fact: Assume $\langle N^1_\eta, a^1_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M.

(1) If $N^1_{<>} \prec N^2_{<>} \prec M, N^2_\eta$ is $\aleph_\epsilon$-prime over $\emptyset$ and $N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} \{a^1_{<\alpha>} :< \alpha > \in I\}$, then

(α) $\left[N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\bigcup} \bigcup_{\eta\in I} N^1_\eta\right]$ and

(β) we can find $N^2_\eta(\eta \in I\backslash\{<>\})$ such that $N^2_\eta \prec M$, and

$$\langle N^1_\eta, a^1_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N^2_\eta, a^1_\eta : \eta \in I\rangle.$$

(2) If Cb $\frac{a^1_{<\alpha>}}{N^1_{<>}} \subseteq N^0_{<>} \prec N^1_{<>}$ or at least $N^0_{<>} \prec N^1_{<>}$ and $\frac{a^1_{<\alpha>}}{N^1_{<>}} \pm N^0_{<>}$ whenever $<\alpha>\in I$, then we can find $N^0_\eta \prec M$ and $a^0_\eta \in N_\eta$ (for $\eta \in I\backslash\{<>\}$) such that $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N^1_\eta, a^0_\eta : \eta \in I\rangle$.
(3) In part (2), if in addition we are given $\langle B^*_\eta : \eta \in I\rangle$ such that B^*_η is an ε-finite subset of N_η, $\text{tp}_*(B^*_\eta, N_\eta)$ does not fork over $B^*_{\eta^-}$ and $B^*_{<>} \subseteq N^0_{<>}$, then we can demand in the conclusion that $\eta \in I \Rightarrow B^*_\eta \subseteq N^0_\eta$.

Proof: (1) For proving (α) let $\{\eta_i : i < i^*\}$ list the set I such that $\eta_i \triangleleft \eta_j \Rightarrow i < j$, so $\eta_0 =<>$ and, without loss of generality, for some α^* we have $\eta_i \in \{<\alpha>:<\alpha>\in I\} \Leftrightarrow i \in [1,\alpha^*)$. Now we prove by induction on $\beta \in [1, i^*)$ that $N^2_{<>} \underset{N^1_{<>}}{\bigcup\!\!\!\!\cup} \cup\{N^1_{\eta_i} : i < \beta\}$. For $\beta = 1$ this is assumed. For β limit use the local character of non-forking.

If $\beta = \gamma+1 \in [1,\alpha^*)$, then by repeated use of [Sh:c, V, 3.2] (as $\{a_{\eta_0} : j \in [1,\beta)\}$ is independent over $(N^1_{<>}, N^2_{<>})$ and $N^1_{<>}$ is $\aleph_\epsilon$-saturated and $N^1_{\eta_j}(j \in [1,\gamma))$ is $\aleph_\epsilon$-prime over $N^1_{<>} + a_{\eta_j}$) we know that $\text{tp}(a_{\eta_\gamma}, N^2_{<>} \cup \bigcup_{i<\gamma} N^1_{\eta_i})$ does not fork over $N^1_{<>}$. Again by [Sh:c, V, 3.2], the type $\text{tp}_*(N^1_{\eta_\gamma}, N^2_{<>} \cup \bigcup_{i<\gamma} N^1_{\eta_i})$ does not fork over $N^1_{<>}$, hence $\bigcup_{i<\beta} N^1_{\eta_i} \underset{N^1_{<>}}{\bigcup\!\!\!\!\cup} N^2_{<>}$ and use symmetry.

Lastly, if $\beta \in \gamma + 1 \in [\alpha^*, i^*), \text{tp}(a_{\eta_\gamma^-}, N_{\eta_\gamma})$ is orthogonal to $N^1_{<>}$ and even to $N^1_{(\eta_\gamma^-)^-}$, so again by non-forking and [Sh:c, V, 3.2] we can do it, so clause (α) holds.

For clause (β), we choose $N^2_{\eta_i}$ for $i \in [1, i^*)$ by induction on $i < i^*$ such that $N^2_{\eta_i} \prec M$ is $\aleph_\epsilon$-prime over $N^2_{\eta_i^-} \cup N^1_{\eta_i}$. By the non-forking calculus we can check Definition 1.7.
(2) We let $\{\eta_i : i < i^*\}$ be as above. Now we choose $N^0_{\eta_i}, a^0_{\eta_i}$ by induction on $i \in [1, i^*)$ such that:

($*$) $N^0_{\eta_i} \prec N^1_{\eta_i}$ and $N^1_{\eta_i^-} \underset{N^0_{\eta_i^-}}{\bigcup\!\!\!\!\cup} N^0_{\eta_i}$ and $N^1_{\eta_i}$ is $\aleph_\epsilon$-prime over $N^0_{\eta_i} \cup N^1_{\eta_i^-}$,

($**$) $a^0_{\eta_i} \in N^0_{\eta_i}$ and $N^0_{\eta_i}$ is $\aleph_\epsilon$-prime over $N^0_{\eta_i^-} + a^0_{\eta_i}$.

The induction step has already been done: if $\ell g(\eta_i) > 1$ by 1.18(7) and if $\ell g(\eta_i) = 1$ by 1.18(8).

(3) Similar. $\blacksquare_{1.21}$

1.22 Fact: (1) If $\langle N^1_\eta, a_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N^2_\eta, a_\eta : \eta \in I\rangle$ and both are $\aleph_\epsilon$-

decompositions of M above $\binom{B}{A}$, <u>then</u>

$$\mathcal{P}(\langle N^1_\eta, a^1_\eta : \eta \in I\rangle, M) = \mathcal{P}(\langle N^2_\eta, a^2_\eta : \eta \in I\rangle, M).$$

Proof: By Defintion 1.11(5) it suffices to prove, for each $\eta \in I\backslash\{<>\}$, that

(∗) for regular $p \in S(M)$ we have $p \perp N^1_{\eta^-} \& p \pm N^1_\eta \Leftrightarrow p \perp N^2_{\eta^-} \& p \pm N^2_\eta$.

Now consider any regular $p \in S(M)$: first assume $p \perp N^1_{\eta^-} \& p \pm N^1_\eta$ where $\eta \in I \setminus \{\langle\,\rangle\}$ so $p \pm N^2_\eta$ (as $N^1_\eta \prec N^2_\eta$ and $p \pm N^1_\eta$) and we can find a regular $q \in S(N^1_\eta)$ such that $q \pm p$; so as $p \perp N^1_{\eta^-}$ also $q \perp N^1_{\eta^-}$, now $q \perp N^2_{\eta^-}$ (as $N^1_\eta \bigcup_{N^1_{\eta^-}} N^2_{\eta^-}$ and $q \perp N^1_\eta$ see [Sh:c, V, 2.8]), hence $p \perp N^2_{\eta^-}$.

Second, assume $p \perp N^2_{\eta^-} \& p \pm N^2_\eta$ where $\eta \in I \setminus \{\langle\,\rangle\}$; remember $N^1_{\eta^-}, N^1_\eta, N^2_\eta$, N^3_η are $\aleph_\epsilon$-saturated, $N^1_\eta \bigcup_{N^1_{\eta^-}} N^2_{\eta^-}$ and N^2_η is $\aleph_\epsilon$-prime over $N^1_\eta \cup N^2_{\eta^-}$ and T does not have DOP. Hence N^2_η is $\aleph_\epsilon$-minimal over $N^1_\eta \cup N^2_{\eta^-}$ and every regular $q \in S(N^2_\eta)$ is not orthogonal to N^1_η or to $N^2_{\eta^-}$. Also, as $p \pm N^2_\eta$ there is a regular $q \in S(N^2_\eta)$ not orthogonal to p, so as $p \perp N^2_{\eta^-}$ also $q \perp N^2_{\eta^-}$; hence by the previous sentence $q \pm N^1_\eta$, hence $p \pm N^1_\eta$. Lastly, as $p \perp N^2_{\eta^-}$ and $N^1_{\eta^-} \prec N^2_{\eta^-}$ clearly $p \perp N^1_{\eta^-}$, as required. $\blacksquare_{1.22}$

At last we start proving 1.14.

Proof of 1.14: (1) Let $N^0 \prec \mathfrak{C}$ be $\aleph_\epsilon$-primary over A; without loss of generality $N^0 \bigcup_A B$ (but not necessarily $N^0 \prec M$), and let N^1 be $\aleph_\epsilon$-primary over $N^0 \cup B$. Now by 1.18(0) the model N^0 is $\aleph_\epsilon$-primary over $\emptyset$ and by 1.18(1) the model N^1 is $\aleph_\epsilon$-primary over $\emptyset$, hence (by 1.18(10)) is $\aleph_\epsilon$-primary over B, hence without loss of generality $N^1 \prec M$. Let $N_{<>} =: N^0, N_{<0>} = N^1, I = \{<>, <0>\}$ and $a_{<0>} = B$. More exactly a_η is such that $\mathrm{dcl}(\{a_\eta\}) = \mathrm{dcl}(B)$. Clearly $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M above $\binom{B}{A}$. Now apply part (2) of 1.14 proved below.

(2) By 1.13(4) we know $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition inside M. By 1.18(2) we then find $J \supseteq I$ and N_η, a_η for $\eta \in J\backslash I$ such that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M. By 1.18(3), $\langle N_\eta, a_\eta : \eta \in J'\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$ where $J' =: \{\eta \in J : \eta = <> \text{ or } \langle 0\rangle \trianglelefteq \eta \in J\}$.

(3) Part (a) holds by 1.13(2),(3). As for part (b), by 1.13(2) there is $\langle N_\eta, a_\eta : \eta \in J\rangle$, an $\aleph_\epsilon$-decomposition of M with $I \subseteq J$; easily $[\langle 0\rangle \trianglelefteq \eta \in J \Rightarrow \eta \in I]$. $\blacksquare_{1.14(1),(2),(3)}$

1.23 Fact: If $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell\rangle$ are $\aleph_\epsilon$-decompositions of M above $\binom{B}{A}$, for $\ell = 1, 2$ and $N^1_{<>} = N^2_{<>}$, <u>then</u> $\mathcal{P}(\langle N^1_\eta, a^1_\eta : \eta \in I^1\rangle, M) = \mathcal{P}(\langle N^2_\eta, a^2_\eta : \eta \in I^2\rangle, M)$.

Proof: By 1.14(3)(b) we can find $J^1 \supseteq I^1$ and N^1_η, a^1_η for $\eta \in J^1 \backslash I^1$ such that $\langle N^1_\eta, a^1_\eta : \eta \in J^1 \rangle$ is an $\aleph_\epsilon$-decomposition of M and moreover $\eta \in J^1 \backslash I^1 \Leftrightarrow \eta \neq \langle\rangle \& \neg(\langle 0 \rangle \triangleleft \eta)$. Let $J^2 = I^2 \cup (J^1 \backslash I^1)$ and for $\eta \in J^2 \backslash I^2$ let $a^2_\eta =: a^1_\eta$, $N^2_\eta =: N^1_\eta$. Easily $\langle N^2_\eta, a^2_\eta : \eta \in J^2 \rangle$ is an $\aleph_\epsilon$-decomposition of M. By 1.13(6) we know that for every regular $p \in S(M)$ there is (for $\ell = 1, 2$) a unique $\eta(p,\ell) \in J^\ell$ such that $p \perp N_{\eta(p,\ell)} \& p \perp N_{\eta(p,\ell)^-}$ (note $\langle\rangle^-$ — meaningless). By the uniqueness of $\eta(p,\ell)$, if $\eta(p,1) \in J^1 \backslash I^1$ then as it can serve as $\eta(p,2)$ clearly it is $\eta(p,2)$, so $\eta(p,2) = \eta(p,1) \in J^1 \backslash I^1 = J^2 \backslash I^2$; similarly $\eta(p,2) \in J^2 \backslash I^2 \Rightarrow \eta(p,1) \in J^1 \backslash I^1$ and $\eta(p,1) = \langle\rangle \Leftrightarrow \eta(p,2) = \langle\rangle$. So

(*) $\eta(p,1) \in I^1 \backslash \{\langle\rangle\} \Leftrightarrow \eta(p,2) \in I^2 \backslash \{\langle\rangle\}$.

But

(**) $\eta(p,\ell) \in I^\ell \backslash \{\langle\rangle\} \Leftrightarrow p \in \mathcal{P}(\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell \rangle, M)$.

Together we finish. $\blacksquare_{1.23}$

We continue proving 1.14.

Proof of 1.14(4): Let $A^* \subseteq M^-$ be ε-finite, so we can find an ε-finite $B^* \subseteq \cup\{N_\eta : \eta \in I\}$ such that $\mathrm{stp}(A^*, B^*) \vdash \mathrm{stp}(A^*, \cup\{N_\eta : \eta \in I\})$. Hence, there is a finite non-empty $I^* \subseteq I$ such that $<> \in I^*$, I^* is closed under initial segments and $B^* \subseteq \cup\{N_\eta : \eta \in I^*\}$, so of course

$$\mathrm{stp}_*(A^*, \cup\{N_\eta : \eta \in I^*\}) \vdash \mathrm{stp}(A^*, \cup\{N_\eta : \eta \in I\}).$$

We can also find $\langle B^*_\eta : \eta \in I^* \rangle$ such that B^*_η is an ε-finite subset of N_η, $B^*_\eta = \mathrm{acl}(B^*_\eta)$ and $B^* \subseteq \cup\{B^*_\eta : \eta \in I^*\}$, $\eta \neq <> \Rightarrow a_\eta \in B^*_\eta$, and if $\eta \triangleleft \nu \in I^*$ then $B^*_\eta \subseteq B^*_\nu$ and $\mathrm{tp}_*(B^*_\nu, N_\nu)$ does not fork over B^*_η. W.l.o.g. $B \subseteq B^*_{\langle 0 \rangle}$.

For $\eta \in I \backslash I^*$ let $B^*_\eta = B^*_{\eta \restriction \ell}$ where $\ell < \ell g(\eta)$ is maximal such that $\eta \restriction \ell \in I^*$; such ℓ exists as $\ell g(\eta)$ is finite and $<> \in I^*$.

Let $N^1_\eta = N_\eta$ and $a^1_\eta = a_\eta$ for $\eta \in I$ and, without loss of generality, $J \neq I$ hence $J \backslash I \neq \emptyset$.

Let $N^2_{<>} \prec M$ be $\aleph_\epsilon$-prime over $\bigcup_{\nu \in J \backslash I} N_\nu$; letting $J \backslash I = \{\eta_i : i < i^*\}$ be such that $[\eta_i \triangleleft \eta_j \Rightarrow i < j]$ we can find $N^2_{<>,i}$ (for $i \leq i^*$) increasing continuous, $N^2_{<>,0} = N_{<>}$ and $N^2_{<>,i+1}$ is $\aleph_\epsilon$-prime over $N^2_{<>,i} \cup N_{\eta_i}$, hence over $N^2_{<>,i} + a_{\eta_i}$. Lastly, w.l.o.g. $N^2_{<>,i^*} = N^2_{<>}$.

By 1.18(1),(2) we know $N^2_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$ and (using repeatedly 1.18(6) + finite character of forking) we have $N^2_{<>} \underset{N^1_{<>}}{\bigcup} a_{<0>}$. By 1.18(4) (with $N^1_{<>}$, $N^2_{<>}, B^*_{<>} \supseteq \mathrm{Cb}(a_{<>}/N^1_{<>})$ here standing for N_1, N_2, A there) we can find a model $N^0_{<>}$ such that $a_{<0>} \underset{N^0_{<>}}{\bigcup} N^1_{<>}$ and $\mathrm{Cb}(a_{<>}/N^1_{<>}) \subseteq B^*_{<>} \subseteq N^0_{<>}, N^0_{<>} \prec$

$N^1_{<>}, N^0_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$ and $N^1_{<>}, N^2_{<>}$ are isomorphic over $N^0_{<>}$. By 1.21(1) we can for $\eta \in I$ choose $N^2_\eta \prec M$ with $N^1_\eta \prec N^2_\eta$ and $\langle N^1_\eta, a^1_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N^2_\eta, a^1_\eta : \eta \in I\rangle$. Similarly, by 1.21(2) (here $\mathrm{Suc}_I(<>) = \{\langle 0\rangle\}$) we can choose an $\aleph_\epsilon$-decomposition $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle$ with $\langle N^0_\eta, a^0_\eta : \eta \in I\rangle \leq^*_{\text{direct}} \langle N^1_\eta, a^0_\eta : \eta \in I\rangle$. Moreover, we can demand $\eta \in I^* \Rightarrow B^*_\eta \subseteq N^0_\eta$ using 1.21(3). By 1.13(12)+1.14(3) we know that $\langle N^1_\eta, a^0_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M^- and easily $\langle N^2_\eta, a^0_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M. Now choose by induction on $\eta \in I$ an isomorphism f_η from N^1_η onto N^2_η over N^0_η such that $\nu \lhd \eta \Rightarrow f_\nu \subseteq f_\eta$ and $\eta \in I^* \Rightarrow f_\eta \restriction B^*_\eta = \mathrm{id}_{B^*_\eta}$. For $\eta =<>$ we have chosen N^0_η such that N^1_η, N^2_η are isomorphic over N^0_η. For the induction step note that $f_{(\eta^-)} \cup \mathrm{id}_{N^0_\eta}$ is an elementary mapping as $N^2_{(\eta^-)} \bigcup\limits_{N^0_{(\eta^-)}} N^0_\eta$ and $f_{(\eta^-)} \cup \mathrm{id}_{N^0_\eta}$ can be extended to an isomorphism f_η from N^1_η onto N^2_η as N^ℓ_η is $\aleph_\epsilon$-primary (in fact even $\aleph_\epsilon$-minimal) over $N^\ell_{(\eta^-)} \cup N^0_\eta$ for $\ell = 1,2$ (which holds easily). If $\eta \in I^*$ there is no problem to add $f_\eta \restriction B^*_\eta = \mathrm{id}_{B^*_\eta}$. Now by 1.13(3) the model M^- is $\aleph_\epsilon$-saturated and $\aleph_\epsilon$-primary and $\aleph_\epsilon$-minimal over $\bigcup_{\eta\in J} N_\eta = \bigcup_{\eta \in I} N^1_\eta$; similarly M is $\aleph_\epsilon$ -primary over $\bigcup_{\eta\in I} N^2_\eta$. Now $\bigcup_\eta f_\eta$ is an elementary mapping from $\bigcup_{\eta \in I} N^1_\eta$ onto $\bigcup_{\eta\in I} N^2_\eta$, hence can be extended to an isomorphism f from M^- into M. Moreover, as $\mathrm{stp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\}) \vdash \mathrm{stp}(A^*, \{N^1_\eta : \eta \in I\})$, by [Sh:c, Ch. XII, §4] we have $\mathrm{tp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\} \vdash \mathrm{tp}(A^*, \cup\{N^1_\eta : \eta \in I\}$, hence $\mathrm{tp}_*(A^*, \cup\{B^*_\eta : \eta \in I^*\})$ has a unique extension as a complete type over $\cup\{N^1_\eta : \eta \in I\}$, hence over $\cup\{N^2_\eta : \eta \in I\}$, so without loss of generality $f \restriction A^* = \mathrm{id}_{A^*}$. By the $\aleph_\epsilon$-minimality of M over $\bigcup_{\eta\in I} N_\eta$ (see 1.13(3)), f is onto M, so f is as required. $\blacksquare_{1.14(4)}$

We delay the proof of 1.14(5).

Proof of 1.14(6): Let $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^\ell\rangle$ for $\ell = 1,2$, be $\aleph_\epsilon$-decompositions of M above $\binom{B}{A}$, so $\mathrm{dcl}(a^\ell_{<>}) = \mathrm{dcl}(B)$. Let $p \in S(M)$, and assume that $p \in \mathcal{P}(\langle N^1_\eta, a^1_\eta : \eta \in I^1\rangle, M)$, i.e., for some $\eta \in I^1 \backslash \{<>\}$, $(p_\eta \perp N_{\eta^-})$ and $p_\eta \pm N_\eta$. We shall prove that the situation is similar for $\ell = 2$, i.e., $p \in \mathcal{P}(\langle N^2_\eta, a^2_\eta : \eta \in I^2\rangle, M)$; by symmetry this suffices.

Let $n = \ell g(\eta)$; choose $\langle B_\ell : \ell \leq n\rangle$ and d such that:

(α) $A \subseteq B_0$,

(β) $B \subseteq B_1$,

(γ) $a_{\eta\restriction\ell} \subseteq B_\ell \subseteq N^1_{\eta\restriction\ell}$, for $\ell \leq n$,

(δ) $B_{\ell+1} \bigcup\limits_{B_\ell} N^1_{\eta\restriction\ell}$,

(ϵ) $\frac{B_{\ell+1}}{B_\ell + a^1_{\eta\restriction(\ell+1)}} \vdash \frac{B_{\ell+1}}{N^1_{\eta\restriction\ell} + a^1_{\eta\restriction(\ell+1)}}$,

(ζ) $d \in B_n$, $\frac{d}{B_n \setminus \{d\}}$ is regular $\pm p$ (hence $\perp B_{n-1}$),

(η) B_ℓ is ϵ-finite.

[Why does such $\langle B_\ell : \ell \leq n \rangle$ exist? We prove by induction on n that for any $\eta \in I$ of length n and ϵ-finite $B' \subseteq N_\eta$, there is $\langle B_\ell : \ell \leq n \rangle$ satisfying $(\alpha) - (\epsilon)$, (η) such that $B' \subseteq B_n$. Now there is $p' \in S(N^1_\eta)$ regular, not orthogonal to p; let $B^1 \subseteq N^1_\eta$ be an ϵ-finite set extending $\mathrm{Cb}(p')$. Applying the previous sentence to η, B^1 we get $\langle B_\ell : \ell \leq n \rangle$; let $d \in N_\eta$ realize $p' \restriction B_n$.

Now as $n > 0$, $\mathrm{tp}(d, B_n) \perp N_{\eta^-}$, hence $\mathrm{tp}(d, B_n) \perp B_{n-1}$, hence $\mathrm{tp}(d, B_n) \perp tp_*(N_{\eta^-}, B_n)$, hence as $\mathrm{tp}(d, B_n)$ is stationary, by [Sh:c, V,1.2(2), p. 231], the types $\mathrm{tp}(d, B_n)$,$\mathrm{tp}_*(N_{\eta^-}, B_n)$ are weakly orthogonal, so $\mathrm{tp}(d, B_n) \vdash \mathrm{tp}(d, N_{\eta^-} \cup B_n)$, hence $\frac{B_n + d}{B_{n-1} + a^1_\eta} \vdash \frac{B_n + d}{N^1_{\eta^-} + a^1_\eta}$.

Now replace B_n by $B_n \cup \{d\}$ and we finish.]

Note that necessarily

$(\delta)^+$ $B_n \underset{B_m}{\bigcup\!\!\!\!\bigcup} N^1_{\eta \restriction m}$ for $m \leq n$.

[Why? By the nonforking calculus.]

$(\epsilon)^+$ $\frac{B_n}{B_m + a^1_{\eta \restriction (m+1)}} \perp_a B_m$ for $m < n$.

[Why? As $N^1_{\eta \restriction m}$ is $\aleph_\epsilon$-saturated.]

Choose $D^* \subseteq N^2_{<>}$ finite such that $\frac{B_n}{N^2_{<>} + B}$ does not fork over $D^* + B$.

[Note: We really mean $D^* \subseteq N^2_{<>}$, not $D^* \subseteq N^1_{<>}$.]

We can find $N^3_{<>}$, $\aleph_\varepsilon$-prime over $\emptyset$ such that $A \subseteq N^3_{<>} \prec N^2_{<>}$ and $D^* \underset{A}{\bigcup\!\!\!\!\bigcup} N^3_{<>}$ and $N^2_{<>}$ is $\aleph_\varepsilon$-prime over $N^3_{<>} \cup D^*$ (by 1.18(9)). Hence $B_n \underset{A}{\bigcup\!\!\!\!\bigcup} N^3_{<>}$ and $B_n \underset{B}{\bigcup\!\!\!\!\bigcup} N^3_{<>}$ (by the non-forking calculus). As $\mathrm{tp}_*(B, N^2_{<>})$ does not fork over $A \subseteq N^3_{<>} \subseteq N^2_{<>}$ by 1.21(2) we can find N^3_η, a^3_η (for $\eta \in I^2 \setminus \{<>\}$), such that $\langle N^3_\eta, a^3_\eta : \eta \in I \rangle$ is an $\aleph_\varepsilon$-decomposition inside M above $\binom{B}{A}$ and $\langle N^3_\eta, a^3_\eta : \eta \in I^2 \rangle \leq^*_{\text{direct}} \langle N^2_\eta, a^3_\eta : \eta \in I^2 \rangle$ and $a^3_{<0>} = a^2_{<0>}$ (remember $\mathrm{dcl}(a^2_{<0>}) = \mathrm{dcl}(B)$). By 1.20(2) we know $\langle N^3_\eta, a^3_\eta : \eta \in I^2 \rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$.

By 1.22 it is enough to show $p \in \mathcal{P}(\langle N^3_\eta, a^3_\eta : \eta \in I^2 \rangle, M)$. Let $N^4_{<>} \prec N^2_{<>} \prec M$ be $\aleph_\epsilon$-prime over $N^3_{<>} \cup B_0$. Now by the non-forking calculus $B \underset{A}{\bigcup\!\!\!\!\bigcup} (N^3_{<>} \cup B_0)$.

[Why? Because

(a) as said above $B_n \underset{B}{\bigcup\!\!\!\!\bigcup} N^3_{<>}$ but $B_0 \subseteq B_n$ so $B_0 \underset{B}{\bigcup\!\!\!\!\bigcup} N^3_{<>}$, and

(b) as $B \underset{A}{\bigcup\!\!\!\!\bigcup} N^1_{<>}$ and $B_0 \subseteq N^1_{<>}$ we have $B \underset{A}{\bigcup\!\!\!\!\bigcup} B_0$ so $B_0 \underset{A}{\bigcup\!\!\!\!\bigcup} B$,

hence (by (a)+ (b) as $A \subseteq B$)

(c) $\frac{B_0}{N^3_{<>} + B}$ does not fork over A, also

(d) $B \bigcup_A N^3_{<>}$ (as $A \subseteq N^3_{<>} \subseteq N^2_{<>}$ and $\mathrm{tp}(B, N^2_{<>})$ does not fork over A);

putting (c) and (d) together we get

(e) $\bigcup_A \{B_0, B, N^3_{<>}\}$,

hence the conclusion.]

Hence $B \bigcup_{N^3_{<>}} B_0$, so $B \bigcup_{N^3_{<>}} N^4_{<>}$ (by 1.18(6)) and so (as $N^3_{\langle 0\rangle}$ is $\aleph_\varepsilon$-prime over $N^3_{\langle\,\rangle} + \mathrm{dcl}(a^3_{\langle\,\rangle}) = N^3_{\langle\,\rangle} + \mathrm{dcl}(B)$) we have $N^4_{<>} \bigcup_{N^3_{<>}} N^3_{<0>}$ and by 1.21(1) we can choose $N^4_\eta \prec M$ (for $\eta \in I^2 \backslash \{<>\}$), such that

$$\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle \geq_{\mathrm{direct}} \langle N^3_\eta, a^3_\eta : \eta \in I^2\rangle.$$

So by 1.20(1) $\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$, hence $a^3_{\langle 0\rangle}/N^4_\eta$ does not fork over A but $A \subseteq B_0 \subseteq N^4_{\langle\,\rangle}$, so $a^3_{\langle\,\rangle}/N^4_\eta$ dnf over B_0, and by 1.22 it is enough to prove $p \in \mathcal{P}(\langle N^4_\eta, a^3_\eta : \eta \in I^2\rangle, M)$. Now as said above $B \bigcup_{N^3_{<>}} N^4_{<>}$ and $B \bigcup_A N^3_{<>}$, so together $B \bigcup_A N^4_{<>}$; also we have $A \subseteq B_0 \subseteq N^4_{<>}$, hence $B \bigcup_{B_0} N^4_{<>}$ and $\frac{B_n}{B_0 + B} \equiv \frac{B_n}{B_0 + a^3_{<0>}} \perp_a B_0$ (by $(\epsilon)^+$ above), but $a^3_{<>} \bigcup_{B_0} N^4_{<>}$, hence $\frac{B_n}{N^4_{<>} + a^3_{<0>}}$ is $\aleph_\epsilon$-isolated. Also, letting $B'_n = B_n \backslash \{d\}$ we have $\frac{B'_n}{N^4_{\langle\,\rangle} + a^3_{\langle\,\rangle}}$ is $\aleph_\varepsilon$-isolated and $\frac{d}{B'_n} \perp B_0$ (by clause (ζ)), and clearly $d \bigcup_{B'_n} (N^4_{<>} \cup B'_n)$ so $\frac{d}{B'_n} \perp N^4_{<>}$. Hence we can find $\langle N^5_\eta, a^5_\eta : \eta \in I^5\rangle$, an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$ such that $N^5_{<>} = N^4_{<>}$, $\mathrm{dcl}(B) = \mathrm{dcl}(a^5_{<0>})$, $B_n \backslash \{d\} \subseteq N^5_{<0>}$ and $d = a^5_{<0,0>}$ (on d see clause (ζ) above), so $d \bigcup_{B_n} N^5_{<0>}$.

By 1.23 it is enough to show $p \in \mathcal{P}(\langle N^5_\eta, a^5_\eta : \eta \in I^5\rangle, M)$, which holds trivially as $\mathrm{tp}(d, B_n \backslash \{d\})$ witness. $\blacksquare_{1.14(6)}$

Proof of 1.14(5): By 1.8, with A, B, A_1, B_1 here standing for A_1, B_1, A_2, B_2 there, there are $\langle B'_\ell : \ell \leq n\rangle, \langle c_\ell : 1 \leq \ell < n\rangle$ as there. By 1.18(9) we can choose $N^1_{<>}$ such that $B_0 \subseteq N^1_{<>}, N^1_{<1>} \bigcup_{B_0} B_n, N^1_{<>}$ is $\aleph_\epsilon$-primary over $\emptyset$. Then we choose $\langle N^1_\eta, a^1_\eta : \eta \in \{<>, <0>, <0,0>, \ldots, \underbrace{<0,\ldots,0>}_{n}\}\rangle$, where $N^1_{\underbrace{<0,\ldots,0>}_{n}} \prec M$, $B'_{\lg \eta} \subseteq N^1_\eta$ and $\ell > 0 \Rightarrow a^1_{\underbrace{<0,\ldots,0>}_{\ell}} = c_\ell$ and we choose N^1_η by induction on $\ell g(\eta)$ being $\aleph_\varepsilon$-prime over $N^1_{\eta^-} \cup a^1_\eta$, hence $a^1_\eta / N^1_{\eta^-}$ does not fork over $B'_{\lg(\eta^-)}$, hence N^1_η is $\aleph_\varepsilon$-prime also over $N^1_{\eta_1} + B'_{\lg(\eta)}$. So

$\langle N^1_\eta, a^1_\eta : \eta \in \{\langle\ \rangle, \dots\}\rangle$ is an $\aleph_\epsilon$-decomposition inside M for $\binom{B_1}{A_1}$. Now apply first 1.14(2) and then 1.14(6).

Proof of 1.14(7): Should be easy. Note that

$(*)_1$ for no $\binom{B'}{A'}$ do we have $\binom{B}{A} \le_b \binom{B'}{A'}$;
why? By the definition of depth zero;

$(*)_2$ if $\binom{B}{A} <_a \binom{B'}{A'}$, then also $\binom{B'}{A'}$ satisfies the assumption.

Hence

$(**)$ for no $\binom{B_1}{A_1}, \binom{B_2}{A_2}$ do we have

$$\binom{B}{A} \le_a \binom{B_1}{A_1} <_b \binom{B_2}{A_2}.$$

[Why? As also $\binom{B_1}{A_1}$ satisfies the assumption.]

Now we can prove the statement by induction on α for all pairs $\binom{B}{A}$ satisfying the assumption. For $\alpha = 0$ the statement is a tautology. For α limit ordinal reread clause (c) of Definition 1.10(1). For $\alpha = \beta + 1$, reread clause (b) of Definition 1.10(1): on $\mathrm{tp}_\beta(\binom{B}{A}, M)$ use the induction hypothesis also for computing $Y^{1,B}_{A,B,M}$ (and reread the definition of tp_0, in Definition 1.10(1), clause (a)). Lastly, $Y^{2,\beta}_{A,B,M}$ is empty by $(*)$ above.

Proof of 1.14(8), (9): Read Definition 1.10. $\blacksquare_{1.14(5),(7),(8),(9)}$

DISCUSSION: In particular, the following Claim 1.26 implies that if $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$ and M^- is $\aleph_\epsilon$-prime over $\cup\{N_\eta : \eta \in I\}$, then $\binom{B}{A}$ has the same tp_α in M and M^-.

1.24 CLAIM: (1) *Assume that $M_1 \prec M_2$ are $\aleph_\varepsilon$-saturated, $\binom{B}{A} \in \Gamma(M_1)$. Then the following are equivalent:*

(a) *if $p \in \mathcal{P}(\binom{B}{A}, M_1)$ (see 1.14(6) for definition; so $p \in S(M_1)$ is regular), then p is not realized in M_2;*

(b) *there is an $\aleph_\varepsilon$-decomposition of M_1 above $\binom{B}{A}$, which is also an $\aleph_\varepsilon$-decomposition of M_2 above $\binom{B}{A}$;*

(c) *every $\aleph_\varepsilon$-decomposition of M_1 above $\binom{B}{A}$ is also an $\aleph_\varepsilon$-decomposition of M_2 above $\binom{B}{A}$.*

(2) *If M is $\aleph_\varepsilon$-saturated, $\binom{B_1}{A_1} \le^* \binom{B_2}{A_2}$ are both in $\Gamma(M)$, then $\mathcal{P}(\binom{B_2}{A_2}, M) \subseteq \mathcal{P}(\binom{B_1}{A_1}, M)$.*

(3) *The conditions in 1.24(1) above imply*

(d) $p \in \mathcal{P}(\binom{B}{A}, M_2) \Rightarrow p \pm M_1$.

Proof: (1) <u>(c) ⇒ (b)</u>. By 1.14(1) there is an $\aleph_\varepsilon$-decomposition of M_1 above $\binom{B}{A}$. By clause (c) it is also an $\aleph_\varepsilon$-decomposition of M_2 above $\binom{B}{A}$, just as needed for clause (b).

<u>(b) ⇒ (a)</u>. Let $\langle N_\eta, a_\eta : \eta \in I\rangle$ be as said in clause (b). By 1.14(3)(b) we can find $J_1, I \subseteq J_1$ and N_η, a_η (for $\eta \in J_1\backslash I$) such that $\langle N_\eta, a_\eta : \eta \in J_1\rangle$ is an $\aleph_\varepsilon$-decomposition of M_1 and $\nu \in J_1\backslash I \Rightarrow \nu(0) > 0$. Then we can find $J_2, J_1 \subseteq J_2$ and N_η, a_η (for $\eta \in J_2\backslash J_1$) such that $\langle N_\eta : \eta \in J_2\rangle$ is an $\aleph_\varepsilon$-decomposition of M_2 (by 1.14(2)). By 1.14(3)(b), $\nu \in J_2\backslash I \Rightarrow \nu(0) > 0$. So $\eta \in I\backslash\{\langle\rangle\} \Rightarrow \mathrm{Suc}_{J_2}(\eta) = \mathrm{Suc}_I(\eta)$, hence

(∗) if $\eta \in I\backslash\{\langle\rangle\}$ and $q \in S(N_\eta)$ is regular orthogonal to N_{η^-}, <u>then</u> the stationarization of q in $S(M_1)$ is not realized in M_2.

Now if $p \in \mathcal{P}(\binom{B}{A}, M_1)$, then $p \in S(M_1)$ is regular and (see 1.14(1), 1.11(5)) for some $\eta \in I\backslash\{\langle\rangle\}, p \perp N_{\eta^-}, p \pm N_\eta$, so there is a regular $q \in S(N_\eta)$ not orthogonal to p. Now no $c \in M_2$ realizes the stationarization of q over M_1 (by (∗) above), hence this applies to p, too.

<u>(a) ⇒ (c)</u>. Let $\langle N_\eta, a_\eta : \eta \in I\rangle$ be an $\aleph_\varepsilon$-decomposition of M_1 above $\binom{B}{A}$. We can find $\langle N_\eta, a_\eta : \eta \in J\rangle$, an $\aleph_\varepsilon$-decomposition of M_1 such that $I \subseteq J$ and $\nu \in J\backslash I \Rightarrow \nu(0) > 0$ (by 1.14(3)(b)), so M is $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in J\}$. We should check that $\langle N_\eta : a_\eta : \eta \in I\rangle$ is also an $\aleph_\varepsilon$-decomposition of M_2 above $\binom{B}{A}$, i.e., Definition 1.11(1),(2). Now in 1.11(1), clauses (a)–(h) are immediate, so let us check clause (i) (in 1.11(2)). Let $\eta \in I\backslash\{\langle\rangle\}$; now is $\{a_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in I\}$ really maximal (among independent over N_η sets of elements of M_2 realizing a type from $\mathcal{P}_\eta = \{p \in S(N_\eta) : p \text{ orthogonal to } N_{\eta^-}\}$)? This should be clear from clause (a) (and basic properties of dependencies and regular types).

(2) By 1.14(5).

(3) Left to the reader. $\blacksquare_{1.24}$

1.25 Conclusion: Assume $M_1 \prec M_2$ are $\aleph_\varepsilon$-saturated and $\binom{B_1}{A_1} \leq^* \binom{B_2}{A_2}$ both in $\Gamma(M_1)$. If clause (a) (equivalently (b) or (c)) of 1.24 holds for $\binom{B_1}{A_1}, M_1, M_2$ then they hold for $\binom{B_2}{A_2}, M_1, M_2$.

Proof: By 1.24(2), clause (a) for $\binom{B_1}{A_1}, M_1, M_2$ implies clause (a) for $\binom{B_2}{A_2}, M_1, M_2$. $\blacksquare_{1.25}$

1.26 Claim: *If $\binom{B_1}{A_1} \in \Gamma(M)$ and $\langle N_\eta, a_\eta : \eta \in I\rangle$ is an $\aleph_\epsilon$-decomposition of M above $\binom{B_1}{A_1}$ and $M^- \subseteq M$ is $\aleph_\epsilon$-saturated and $\bigcup_{\eta\in I} N_\eta \subseteq M^-$ and α is an ordinal, <u>then</u>*

$$\mathrm{tp}_\alpha\left[\binom{B_1}{A_1}, M\right] = \mathrm{tp}_\alpha\left[\binom{B_1}{A_1}, M^-\right].$$

Proof: We prove this by induction on α (for all $B, A, \langle N_\eta, a_\eta : \eta \in I\rangle, I, M$ and M^- as above). We can find an $\aleph_\epsilon$-decomposition $\langle N_\eta, a_\eta : \eta \in J\rangle$ of M with $I \subseteq J$ (by 1.13(4)+ 1.13(2)) such that $\eta \in J\backslash I \Leftrightarrow \eta \neq \langle\rangle$ and $\neg\langle 0\rangle \trianglelefteq \eta$ and so M is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in J} N_\eta$ and also over $M^- \cup \{N_\eta : \eta \in J \setminus I\}$.

CASE 0: $\alpha = 0$.

Trivial.

CASE 1: α is a limit ordinal.

Trivial by induction hypothesis (and the definition of tp_α).

CASE 2: $\alpha = \beta + 1$.

We can find $M^* \prec M^-$ which is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$, so as equality is transitive it is enough to prove

$$\text{tp}_\alpha\left(\begin{pmatrix}B_1\\A_1\end{pmatrix}, M^*\right) = \text{tp}_\alpha\left(\begin{pmatrix}B_1\\A_1\end{pmatrix}, M^-\right)$$

and

$$\text{tp}_\alpha\left(\begin{pmatrix}B_1\\A_1\end{pmatrix}, M^*\right) = \text{tp}_\alpha\left(\begin{pmatrix}B_1\\A_1\end{pmatrix}, M\right).$$

By symmetry, this means that it is enough to prove the statement when M^- is $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$.

Looking at the definition of $\text{tp}_{\beta+1}$ and remembering the induction hypothesis our problems are as follows:

<u>First component of tp_α</u>:

Given $\binom{B_1}{A_1} \leq_{\text{a}} \binom{B_2}{A_2}, B_2 \subseteq M$, <u>it suffices to find</u> $\binom{B_3}{A_3}$ such that:

(∗) there is $f \in \text{AUT}(\mathfrak{C})$ such that: $f \restriction B_1 = \text{id}_{B_1}, f(A_2) = A_3$, $f(B_2) = B_3$ and $B_3 \subseteq M^-$ and $\text{tp}_\beta[\binom{B_2}{A_2}, M] = \text{tp}_\beta[\binom{B_3}{A_3}, M^-]$ (pedantically we should replace B_ℓ, A_ℓ by indexed sets).

We can find J', M' such that:

(i) $I \subseteq J' \subseteq J, |J'\backslash I| < \aleph_0, J'$ closed under initial segments,

(ii) $M' \prec M$ is $\aleph_\epsilon$-prime over $M^- \cup \bigcup\{N_\eta : \eta \in J'\backslash I\}$,

(iii) $B_2 \subseteq M'$.

The induction hypothesis for β applies and gives

$$\text{tp}_\beta\left[\begin{pmatrix}B_2\\A_2\end{pmatrix}, M\right] = \text{tp}_\beta\left[\begin{pmatrix}B_2\\A_2\end{pmatrix}, M'\right].$$

By 1.14(4) there is g, an isomorphism from M' onto M^- such that $g \restriction B_1 = \text{id}$. So clearly $g(B_2) \subseteq M^-$, hence

$$\text{tp}_\beta\left[\begin{pmatrix}B_2\\A_2\end{pmatrix}, M'\right] = \text{tp}_\beta\left[\begin{pmatrix}g(B_2)\\g(A_2)\end{pmatrix}, M^-\right].$$

So $\binom{B_3}{A_3} =: g\binom{A_2}{B_2}$ is as required.

<u>Second component of tp_α</u>:

So we are given Υ, a tp_β type (and we assign the lower part as B), and we have to prove that the dimension in M and in M^- are the same, i.e., $\dim(\mathbf{I}, M) = \dim(\mathbf{I}^-, M)$, where

$$\mathbf{I} = \{c \in M : \Upsilon = \mathrm{tp}_\beta(\begin{pmatrix} c \\ B_1 \end{pmatrix}, M)\} \text{ and } \mathbf{I}^- = \{c \in M^- : \Upsilon = \mathrm{tp}_\beta(\begin{pmatrix} c \\ B_1 \end{pmatrix}, M^-)\}.$$

Let p be such that: $\mathrm{tp}_\beta(\binom{c}{B_1}, M) = \Upsilon \Rightarrow p = \frac{c}{B_1}$. Necessarily $p \perp A_1$ and p is regular (and stationary).

Clearly $\mathbf{I}^- \subseteq \mathbf{I}$, so without loss of generality $\mathbf{I} \neq \emptyset$, hence p is really well defined. Now

(*) for every $c \in \mathbf{I}$ for some $k < \omega, c'_\ell \in M^-$ realizing p for $\ell < k$ we have c depends on $\{c'_0, c'_1, \ldots, c'_{k-1}\}$ over B_1.
[Why? Clearly $p \perp N_{<>}$ (as $B_1 \bigcup_{A_1} N_{<>}$ and $p \perp A_1$), hence $\mathrm{tp}_*(\bigcup_{\eta \in J \setminus I} N_\eta, N_{<>}) \perp p$, hence $\mathrm{tp}_*(\bigcup_{\eta \in J\setminus I} N_\eta, M^-) \perp p$, but M is $\aleph_\epsilon$-prime over $M^- \cup \bigcup_{\eta \in J \setminus I} N_\eta$, hence by [Sh:c, V, 3.2, p. 250] for no $c \in M \setminus M^-$ is $\mathrm{tp}(c, M^-)$ a stationarization of p, hence by [Sh:c, V, 1.16(3)] clearly (*) follows.]

If the type p has depth zero, then (by 1.14(7)):

$$\mathbf{I} = \{c \in M : \mathrm{tp}(c, B) = p\} \quad \text{and} \quad \mathbf{I}^- = \{c \in M^- : \mathrm{tp}(c, B) = p\}.$$

Now we have to prove $\dim(\mathbf{I}, A) = \dim(\mathbf{I}^-, A)$, as A is ε-finite and M, M^- are $\aleph_\epsilon$-saturated and $\mathbf{I}^- \subseteq \mathbf{I}$; clearly $\aleph_0 \leq \dim(\mathbf{I}^-, A) \leq \dim(\mathbf{I}, A)$. Now the equality follows by (*) above.

So we can assume "p has depth > zero", hence (by [Sh:c, X, 7.2]) that the type p is trivial; hence, see [Sh:c, X, 7.3], in (*) without loss of generality $k = 1$ and dependency is an equivalence relation, so for "same dimension" it suffices to prove that every equivalence class (in M, i.e., in $\mathbf{I}$) is representable in M^-, i.e., in $\mathbf{I}^-$. By the remark on (*) in the previous sentence $(\forall d_1 \in \mathbf{I})(\exists d_2 \in \mathbf{I}^-)[\neg d_1 \bigcup_{B_1} d_2]$.

So it is enough to prove that:

⊗ if $d_1, d_2 \in M$ realize the same type over B_1, which is (stationary and) regular, and are dependent over B_1 and $d_1 \in M^-$, <u>then</u> there is $d'_2 \in M^-$ such that $\frac{d'_2}{B_1 + d_1} = \frac{d_2}{B_1 + d_1}$ and $\mathrm{tp}_\beta[(\binom{B_1 + d_2}{B_1}), M] = \mathrm{tp}_\beta[(\binom{B_1 + d'_2}{B_1}), M^-]$.

Let $M_0 = N_{\langle\rangle}$. There are J', M_1, M_1^+ such that

$(*)_1$(i) $J' \subseteq J$ is finite (and, of course, closed under initial segments),

(ii) $\langle\rangle \in J', \langle 0\rangle \notin J'$,
(iii) $M_1 \prec M$ is $\aleph_\varepsilon$-prime over $\cup\{N_\eta : \eta \in J'\}$,
(iv) $M_1^+ \prec M$ is $\aleph_\varepsilon$-prime over $M_1 \cup M^-$ (and $M_1 \bigcup_{M_0} M^-$),
(v) $d_2 \in M_1^+$.

Now the triple $\binom{B_1+d_2}{B_1}, M_1, M$ satisfies the demand on $\binom{B_1}{A_1}, M^-, M$ (because $\binom{B_1}{A_1} \leq^* \binom{B_1+d_2}{B_1}$, by 1.25). Hence by the induction hypothesis we know that

$$\mathrm{tp}_\beta\left[\binom{B_1+d_2}{B_1}, M\right] = \mathrm{tp}_\beta\left[\binom{B_1+d_2}{B_1}, M_1^+\right].$$

By 1.14(4) there is an isomorphism f from M_1^+ onto M^- which is the identity on $B_1 + d_1$; let $d_2' = f(d_2)$, so

$$\mathrm{tp}_\beta\left[\binom{B_1+d_2}{B_1}, M_1^+\right] = \mathrm{tp}_\beta\left[\binom{B_1+d_2'}{B_1}, M^-\right].$$

Together

$$\mathrm{tp}_\beta\left[\binom{B_1+d_2}{B_1}, M\right] = \mathrm{tp}_\beta\left[\binom{B_1+d_2'}{B_1}, M^-\right].$$

As $\{d_1, d_2\}$ is not independent over B_1, also $\{f(d_1), f(d_2)\} = \{d_1, f(d_2)\}$ is not independent over B_1, hence, as p is regular,

(*) $\{d_2, f(d_2)\}$ is not independent over B_2.

Together we have proved $\bigoplus$, hence finishing to prove the equality of the second component.

Third component: Trivial.

So we have finished the induction step, hence the proof. $\blacksquare_{1.26}$

1.27 CLAIM: (1) *Suppose M is $\aleph_\epsilon$-saturated, $A \subsetneq B \subsetneq M, \binom{B}{A} \in \Gamma$, $\bigwedge_{\ell=1}^2[A \subseteq A_\ell \subseteq M]$, $A = ac\ell(A), A_\ell$ are ϵ-finite, $\frac{A_1}{A} = \frac{A_2}{A}, B \bigcup_A A_1$ and $B \bigcup_A A_2$.*

Then $\mathrm{tp}_\alpha[\binom{A_1\cup B}{A_1}, M] = \mathrm{tp}_\alpha[\binom{A_2\cup B}{A_2}, M]$ for any ordinal α.

(2) *Suppose M is $\aleph_\epsilon$-saturated, $B \subsetneq M$, $\binom{B}{A} \in \Gamma, \bigwedge_{\ell=1}^2[A \subsetneq A_\ell \subsetneq M]$, $A = ac\ell(A)$, $B = ac\ell(B), A_\ell = ac\ell(A_\ell), A_\ell$ is ϵ-finite, $\frac{A_1}{A} = \frac{A_2}{A}, B \bigcup_A A_1, B \bigcup_A A_2$, $f : A_1 \xrightarrow{\text{onto}} A_2$ an elementary mapping, $f \restriction A = \mathrm{id}_A, g \supseteq f \cup \mathrm{id}_B, g$ elementary mapping from $B_1 = ac\ell(B \cup A_1)$ onto $B_2 = ac\ell(B \cup A_2)$.*

Then $g(\mathrm{tp}_\alpha[\binom{B_1}{A_1}, M]) = \mathrm{tp}_\alpha[\binom{B_2}{A_2}, M]$ for any ordinal α.

(3) *Assume that*

(a) *$A_\ell = acl(A_\ell) \subseteq B_\ell = acl(B_\ell) \subseteq M^\ell$ for $\ell = 1, 2$,*
(b) *$A_\ell \subseteq A_\ell^+ \subseteq acl(A_\ell^+) \subseteq M^\ell$ for $\ell = 1, 2$,*

(c) $B_\ell \bigcup_{A_\ell} A_\ell^+$ for $\ell = 1, 2$,

(d) f is an elementary mapping from A_1 onto A_2,

(e) g is an elementary mapping from A_1^+ onto A_2^+,

(f) $f \upharpoonright A_1 = g \upharpoonright A_1$,

(g) h is an elementary mapping from $B_1^+ = acl(B_1 \cup A_1^+)$ onto $B_2^+ = acl(B_2 \cup A_2^+)$ extending f and g,

(h) $f(\mathrm{tp}_\alpha[(\frac{B_1}{A_1}), M_1]) = \mathrm{tp}_\alpha[(\frac{B_2}{A_2}), M_2]$.

<u>Then</u> $h(\mathrm{tp}_\alpha[(\frac{B_1^+}{A_1^+}), M_1]) = \mathrm{tp}_\alpha[(\frac{B_2^+}{A_2^+}), M_2]$.

Proof: (1) Follows from part (2).

(2) We can find $A_3 \subseteq M$ such that:

(i) $\frac{A_3}{A} = \frac{A_1}{A}$,

(ii) $A_3 \bigcup_A (B \cup A_1 \cup A_2)$.

Hence without loss of generality $A_1 \bigcup_B A_2$ and even $\bigcup_A \{B, A_1, A_2\}$. Now we can find $N_{<>}$, an $\aleph_\epsilon$-prime model over $\emptyset, N_{<>} \prec M, A \subseteq N_{<>}$ and $(B \cup A_1 \cup A_2) \bigcup_A N_{<>}$ (e.g., choose $\{A_1^\alpha \cup A_i^\alpha \cup B^\alpha : \alpha \le \omega\} \subseteq M$ indiscernible over $A, A_1^\omega = A_1, A_2^\omega = A_2, B^\omega = B$ and let $N_{<>} \prec M$ be $\aleph_\epsilon$-primary over $\bigcup_{n<\omega}(A_1^n \cup A_2^n \cup B^n \cup A)$).

Now find $\langle N_\eta, a_\eta : \eta \in J \rangle$, an $\aleph_\epsilon$-decomposition of M with

$$\mathrm{dcl}(a_{<0>}) = \mathrm{dcl}(B), \mathrm{dcl}(a_{<1>}) = \mathrm{dcl}(A_1), \mathrm{dcl}(a_{<2>}) = \mathrm{dcl}(A_2).$$

Let $I = \{\eta \in J : \eta = <> \text{ or } <0> \trianglelefteq \eta\}$ and $J' = I \cup \{<1>, <2>\}$. Let $N^2_{<>} \prec M^*$ be $\aleph_\epsilon$-prime over $N_{<1>} \cup N_{<2>}$. By 1.21 there is $\langle N_\eta^2, a_\eta : \eta \in I \rangle$, an $\aleph_\epsilon$-decomposition of M above $\binom{B}{A}$ such that $\langle N_\eta, a_\eta : \eta \in I \rangle \le_{\mathrm{direct}} \langle N_\eta^2, a_\eta : \eta \in I \rangle$. Let $M' \prec M$ be $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I} N_\eta^2$ and $M^- \prec M'$ be $\aleph_\epsilon$-prime over $\bigcup_{\eta\in I} N_\eta$. So $M^- \prec M' \prec M$ and M' is $\aleph_\epsilon$-prime over $M^- \cup N_{<1>} \cup N_{<2>}$.

Now by 1.26 we have $\mathrm{tp}_\alpha[\binom{B}{A_\ell}, M] = \mathrm{tp}_\alpha[\binom{B}{A_\ell}, M']$ for $\ell = 1, 2$, hence it suffices to find an automorphism of M' extending g. Let $B^+ = ac\ell(N_{<>} \cup B), A_\ell^* = ac\ell(B \cup A_\ell)$; let $\bar{\mathbf{a}}_\ell$ list A_ℓ^* be such that $\bar{\mathbf{a}}_2 = g(\bar{\mathbf{a}}_1)$. Clearly $\mathrm{tp}(\bar{\mathbf{a}}_\ell, B^+)$ does not fork over $A \subseteq B$ and $ac\ell(B) = B$, and so $\mathrm{stp}(\bar{\mathbf{a}}_1, B^+) = \mathrm{stp}(\bar{\mathbf{a}}_2, B^+)$. Also $\mathrm{tp}_*(A_2, B^+ \cup A_1)$ does not fork over A, hence $\mathrm{tp}(\bar{\mathbf{a}}_2, B^+ \cup \bar{\mathbf{a}}_1)$ does not fork over $A \subseteq B^+$, hence $\{\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2\}$ is independent over B^+, hence there is an elementary mapping g^+ from $ac\ell(B^+ \cup \bar{\mathbf{a}}_1)$ onto $ac\ell(B^+ \cup \bar{\mathbf{a}}_2), g^+ \supseteq \mathrm{id}_{B^+} \cup g$ and even $g' = g^+ \cup (g^+)^{-1}$ is an elementary embedding.

Let $\bar{\mathbf{a}}'_1$ lists $ac\ell(N_{<>} \cup A_1)$, so clearly $\bar{\mathbf{a}}'_2 =: g^+(\bar{\mathbf{a}}'_1)$ list $ac\ell(N_{<>} \cup A_2)$. Clearly $g' \restriction (\bar{\mathbf{a}}'_1 \cup \mathbf{a}'_2)$ is an elementary mapping from $\bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2$ onto itself. Now $N^2_{<>}$ is $\aleph_\epsilon$-primary over $N_{<>} \cup A_1 \cup A_2$ and $N_{<>} \cup A_1 \cup A_2 \subseteq \bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2 \subseteq ac\ell(N_{<>} \cup A_1 \cup A_2)$, so by 1.18(10) $N^2_{<>}$ is $\aleph_\epsilon$-primary over $N_{\langle\,\rangle}\bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2$, hence we can extend $g' \restriction (\bar{\mathbf{a}}'_1 \cup \bar{\mathbf{a}}'_2)$ to an automorphism $h_{<>}$ of $N^2_{<>}$, so clearly $h_{<>} \restriction N_{<>} = \mathrm{id}_{N_{<>}}$. Let $\bar{\mathbf{a}}^+_1$ list $ac\ell(B^+ \cup A_1)$ and $\bar{\mathbf{a}}^+_2 = g^+(\bar{\mathbf{a}}^+_1)$. So $\mathrm{tp}(\bar{\mathbf{a}}^+_\ell, N^2_{<>})$ does not fork over $\bar{\mathbf{a}}'_1 (\subseteq N^2_{<>})$ and $ac\ell(\bar{\mathbf{a}}'_1) = \mathrm{Rang}(\bar{\mathbf{a}}'_1)$ $(= ac\ell(N_{<>} \cup A_1))$ and $h_{<>} \restriction \bar{\mathbf{a}}'_1 = g^+ \restriction \bar{\mathbf{a}}'_1$, hence $h_{<>} \cup g^+$ is an elementary embedding. Remember g^+ is the identity on $B^+ = ac\ell(N_{<>} \cup B)$, and $\mathrm{tp}_*(N_{<0>}, N^2_{<>})$ does not fork over $N_{<>}$, hence $\mathrm{tp}_*(N_{<0>}, B^+ \cup N^2_{<>})$ does not fork over B^+, so as $ac\ell(B^+) = B^+$ necessarily $(h_{<>} \cup g^+) \cup \mathrm{id}_{N_{<0>}}$ is an elementary embedding. But this mapping has domain and range including $N_{<0>} \cup N^2_{<>}$ and included in $N^2_{<0>}$, but the latter is $\aleph_\epsilon$-primary and $\aleph_\epsilon$-minimal over the former. Hence $(h_{<>} \cup g^+) \cup \mathrm{id}_{N_{<>}}$ can be extended to an automorphism of $N^2_{<0>}$ which we call $h_{<0>}$.

Now we define by induction on $n \in [2, \omega)$, for every $\eta \in I$ of length n, an automorphism h_η of N^2_η extending $h_{\eta^-} \cup \mathrm{id}_{N_\eta}$, which exists as N^2_η is $\aleph_\epsilon$-primary over $N^2_{\eta^-} \cup N_\eta$ (and $N^2_{\eta^-} \underset{N_{\eta^-}}{\bigcup\!\!\!\!\bigcup} N_\eta$). Now $\bigcup_{\eta \in I} h_\eta$ is an elementary mapping (as $\langle N^2_\eta : \eta \in I\rangle$ is a non-forking tree; i.e., 1.13(10)), with domain and range $\bigcup_{\eta \in I} N^2_\eta$, hence can be extended to an automorphism h^* of M' (we can demand $h^* \restriction M^- = \mathrm{id}_{M^-}$ but not necessarily). So as h^* extends g, the conclusion follows. (3) Similarly to (2). $\blacksquare_{1.27}$

1.28 Claim: (1) *For every* $\Upsilon = \mathrm{tp}_\delta[\binom{B}{A}, M]$, *and* $\bar{\mathbf{a}}, \bar{\mathbf{b}}$ *listing* A, B *respectively, there is* $\psi = \psi(\bar{x}_A, \bar{x}_B) \in \mathbb{L}_{\infty,\aleph_\epsilon}$ (q.d.) *of depth* δ *such that:*

$$\mathrm{tp}_\delta\left[\binom{B}{A}, M\right] = \Upsilon \Leftrightarrow M \models \psi[\bar{\mathbf{a}}, \bar{\mathbf{b}}].$$

(2) *Assume* $\bigotimes_{M_1,M_2}$ *of 1.4 holds as exemplified by the family* $\mathcal{F}$ *and* $\binom{B}{A} \in \Gamma(M_1)$ *and* $g \in \mathcal{F}, \mathrm{Dom}(g) = B$*; and* α *an ordinal.* <u>*Then*</u>

$$\mathrm{tp}_\alpha\left(\binom{B}{A}, M\right) = \mathrm{tp}_\alpha\left(\binom{g(B)}{g(A)}, M_2\right).$$

(3) *Similarly for* $\mathrm{tp}_\alpha([A], M), \mathrm{tp}_\alpha[M]$.

Proof: Straightforward (remember we assume that every first order formula is equivalent to a predicate). $\blacksquare_{1.28}$

1.29 Proof of Theorem 1.2: [The proof does not require that the M^ℓ are $\aleph_\epsilon$-saturated, but only that 1.27, 1.28 hold except in constructing $g_{\alpha(*)}$ (see $\otimes_{14}, \otimes_{15}$ in 1.30(E)); we could instead use NOTOP.]

So suppose

$(*)_0$ $M^1 \equiv_{\mathbb{L}_{\infty,\aleph_\epsilon}(\text{d.q.})} M^2$ or (at least) $\bigotimes_{M^1,M^2}$ from 1.4 holds.

We shall prove $M^1 \cong M^2$. By 1.28 (i.e., by 1.28(1) if the first possibility in $(*)_0$ holds and by 1.28(2) if the second possibility in $(*)_0$ holds)

$(*)_1$ $\text{tp}_\infty[M^1] = \text{tp}_\infty[M^2]$.

So it suffices to prove:

1.30 CLAIM: *Assume that T is countable. If M^1, M^2 are $\aleph_\epsilon$-saturated models (of T, T as in 1.5), <u>then</u>:*

$(*)_1 \Rightarrow M^1 \cong M^2$.

Proof: Let $\langle W_k, W'_k : k < \omega \rangle$ be a partition of ω to infinite sets (so pairwise disjoint).

1.31 EXPLANATION: (If it seems opaque, the reader may return to it after reading parts of the proof.)

We shall now define an approximation to a decomposition. That is, we are approximating a non-forking tree $\langle N^\ell_\eta, a^\ell_\eta : \eta \in I^* \rangle$ of countable elementary submodels of M^ℓ for $\ell = 1, 2$ and $\langle f^*_\eta : \eta \in I^* \rangle$ such that f^*_η is an isomorphism from N^1_η onto N^2_η increasing with η such that M^ℓ is $\aleph_\epsilon$-prime over $\bigcup_{\eta \in I^*} N^\ell_\eta$.

In the approximation Y we have:

(α) I approximating I^*
[it will not be $I^* \cap {}^{n\geq}\text{Ord}$ but we may "discover" more immediate successors to each $\eta \in I$; as the approximation to N_η improves we have more regular types, but some member of I will later be dropped],

(β) A^ℓ_η approximates N^ℓ_η and is ϵ-finite,

(γ) a^ℓ_η is the a^ℓ_η (if η survives, i.e., will not be dropped),

(δ) $B^\ell_\eta, b^\ell_{\eta,m}$ expresses commitments on constructing A^ℓ_η: we "promise" $B^\ell_\eta \subseteq N^\ell_\eta$ and B^ℓ_η is countable; $b^\ell_{\eta,m}$ for $m < \omega$ list B^ℓ_η (so in the choice $B^\ell_\eta \subseteq M^\ell$ there is some arbitrariness),

(ε) f_η approximate f^*_η,

(ζ) $p^\ell_{\eta,m}$ also expresses commitments on the construction.

Since there are infinitely many commitments that we must meet in a construction of length ω and we would like many chances to meet each of them, the sets W_k, W'_k are introduced as a further bookkeeping device. At stage n in the construction we will deal, e.g., with the $b^\ell_{\eta,m}$ for η that are appropriate and for $m \in W_k$ for some $k < n$ and analogously for $p^\ell_{\eta,m}$ and the W'_k.

Note that while the A^ℓ_η satisfy the independence properties of a decomposition, the B^ℓ_η do not and may well intersect non-trivially. Nevertheless, a conflict arises if an $a^\ell_{\eta\hat{}<i>}$ falls into B^ℓ_η, since the $a^\ell_{\eta\hat{}<i>}$ are supposed to represent independent elements realizing regular types over the model approximated by A^ℓ_η but now $a^\ell_{\eta\hat{}<i>}$ is in that model. This problem is addressed by pruning $\eta\hat{}<i>$ from the tree I.

1.32 Definition: An approximation Y to an isomorphism consists of:

(a) <u>natural numbers n, k^* and index set</u>: $I \subseteq {}^{n\geq}\mathrm{Ord}$ (and n minimal),

(b) $\langle A^\ell_\eta, B^\ell_\eta, a^\ell_\eta, b^\ell_{\eta,m} : \eta \in I$ and $m \in \bigcup_{k<k^*} W_k\rangle$ for $\ell = 1,2$ (this is an approximated decomposition),

(c) $\langle f_\eta : \eta \in I\rangle$,

(d) $\langle p^\ell_{\eta,m} : \eta \in I$ and $m \in \bigcup_{k<k^*} W'_k\rangle$,

such that:

(1) I is closed under initial segments,

(2) $<> \in I$,

(3) $A^\ell_\eta \subseteq B^\ell_\eta \subseteq M^\ell, A^\ell_\eta$ is ϵ-finite, $ac\ell(A^\ell_\eta) = A^\ell_\eta, B^\ell_\eta$ is countable, $B^\ell_\eta = \{b^\ell_{\eta,m} : m \in \bigcup_{k<k^*} W_k\}$,

(4) $A^\ell_\nu \subseteq A^\ell_\eta$ if $\nu \triangleleft \eta \in I$,

(5) if $\eta \in I\backslash\{<>\}$, then $\frac{a^\ell_\eta}{A^\ell_{(\eta^-)}}$ is a (stationary) regular type and $a^\ell_\eta \in A^\ell_\eta$; if, in addition, $\ell g(\eta) > 1$, then $\frac{a^\ell_\eta}{A^\ell_{(\eta^-)}} \perp A^\ell_{(\eta^{--})}$ (note that we may decide $a^\ell_{<>}$ be not defined or $\in A^\ell_{<>}$),

(6) $\frac{A^\ell_\eta}{A^\ell_{\eta^-}+a_\eta} \perp_a A^\ell_{\eta^-}$ if $\eta \in I, \ell g(\eta) > 0$,

(7) if $\eta \in I$, not $\triangleleft$-maximal in I, <u>then</u> the set $\{a^\ell_\nu : \nu \in I$ and $\nu^- = \eta\}$ is a maximal family of elements realizing over A^ℓ_η regular types $\perp A^\ell_{(\eta^-)}$ (when η^- is defined), independent over $(A^\ell_\eta, B^\ell_\eta)$ (and we can add: if $\nu_1^- = \nu_2^- = \eta$ and $\frac{a^\ell_{\nu_1}}{A_\eta} \pm \frac{a^\ell_{\nu_2}}{A_\eta}$ then $a^\ell_{\nu_1}/A_\eta = a^\ell_{\nu_2}/A_\eta$),

(8) f_η is an elementary map from A^1_η onto A^2_η,

(9) $f_{(\eta^-)} \subseteq f_\eta$ when $\eta \in I, \ell g(\eta) > 0$,

(10) $f_\eta(a^1_\eta) = a^2_\eta$,

(11) (α) $f_\eta(\mathrm{tp}_\infty[(\frac{A^1_\eta}{A^1_{(\eta^-)}}), M^1]) = \mathrm{tp}_\infty[(\frac{A^2_\eta}{A^2_{(\eta^-)}}), M^2]$ when $\eta \in I\backslash\{<>\}$,

(β) $f_{<>}(\mathrm{tp}_\infty[A^1_{<>}, M^1]) = \mathrm{tp}_\infty[A^2_{<>}, M^2]$,

(12) $B^\ell_\eta \prec M^\ell$; moreover, $B^\ell_\eta \subseteq_{\mathrm{na}} M^\ell$, i.e., if $\bar a \subseteq N^\ell_\eta, b \in M^\ell\backslash B^\ell_\eta$ and $M^\ell \models \varphi(b, \bar a)$, then for some $b' \in B^\ell_\eta, \models \varphi(b', \bar a)$ and $b \notin acl(\bar a) \Rightarrow b' \notin acl(A)$,

(13) $\langle p^\ell_{\eta,m} : m \in \bigcup_{k<k^*} W'_k\rangle$ is a sequence of types over A^ℓ_η (so $\mathrm{Dom}(p^\ell_{\eta,m})$ may be a proper subset of A^ℓ_η).

1.33 Notation: We write $n = n_Y = n[Y], I = I_Y = I[Y], A^\ell_\eta = A^\ell_\eta[Y]$, $B^\ell_\eta = B^\ell_\eta[Y], f_\eta = f^Y_\eta = f_\eta[Y], a^\ell_\eta = a^\ell_\eta[Y], b^\ell_\eta = b^\ell_\eta[Y], k^* = k^*_Y = k^*[Y]$ and $p^\ell_{\eta,m} = p^\ell_{\eta,k}[Y]$.

Remark: We may decide to demand: each $\frac{a^\ell_{\eta\char`^<i>}}{A_\eta}$ is strongly regular; also: if two such types are not orthogonal then they are equal (or at least have the same witness $\varphi$ for $(\varphi, \frac{a_{\eta\char`^<i>}}{A_\eta})$ regular). This is easy here as the models are $\aleph_\epsilon$-saturated (so take $p' \pm p$, $\mathrm{rk}(p')$ minimal).

1.34 OBSERVATION: $(*)_1$ implies that there is an approximation (see 1.29).

Proof: Let $I = \{<>\}, A^\ell_{<>} = ac\ell(\emptyset), k^* = 1$, and then choose countable $B^\ell_{<>}$ to satisfy condition (12) and then choose $f_\eta, p^\ell_k, b^\ell_{\eta,m}$ (for $k \in W'_0$ and $m \in W_0$) as required.

1.35 MAIN FACT: For any approximation $Y, i \in \bigcup_{k<k^*_Y}(W_k \cup W'_k)$ and $m \leq n_Y$ and $\ell(*) \in \{1,2\}$, we can find an approximation Z such that:

$(\bigotimes)(\alpha)$ $n_Z = \mathrm{Max}\{m+1, n_Y\}, I_Z \cap {}^{m\geq}\mathrm{Ord} = I_Y \cap {}^{m\geq}\mathrm{Ord}$ (we mean m not n_Y) and $k^*_Z = k^*_Y + 1$;

(β) (a) if $\eta \in I_Y, \ell g(\eta) < m$, then

$$A^\ell_\eta[Z] = A^\ell_\eta[Z],$$
$$a^\ell_\eta[Z] = a^\ell_\eta[Z],$$
$$B^\ell_\eta[Z] = B^\ell_\eta[Y],$$

(b) if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W'_k$, then $p^\ell_{\eta,j}[Z] = p^\ell_{\eta,j}[Y]$,

(c) if $\eta \in I_Y \cap I_Z, k < k^*_Y$ and $j \in W_k$, then $b^\ell_{\eta,j}[Z] = b^\ell_{\eta,j}[Y]$;

(γ)[1] <u>if</u> $\eta \in I_Y, \ell g(\eta) = m, k < k^*_Y$ and[2] $i \in W_k$ and the element $b \in M^{\ell(*)}$ satisfies clauses (a), (b) below, <u>then</u> for some such b we have: $A^{\ell(*)}_\eta[Z] = ac\ell(A^{\ell(*)}_\eta[Y] \cup \{b\})$, where

(a) $b^{\ell(*)}_{\eta,i}[Y] \notin A^{\ell(*)}_\eta[Y]$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$,

(b) one of the conditions (i), (ii) listed below holds for b:

(i) $b = b^{\ell(*)}_{\eta,i}[Y]$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$ <u>or</u>

(ii) for no b is (i) satisfied (so $\ell g(\eta) > 0$) and $b \in M^{\ell(*)}$, $b^\ell_{\eta,i} \biguplus_{A^{\ell(*)}_\eta[Y]} b$ and $\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]$;

---

2 Recall that i is part of the information given in the main fact, and, of course, k is uniquely determined by i.

$(\gamma)^2$ if we assume $\eta \in I_Y, \ell g(\eta) = m, k < k^*_Y$ and $i \in W'_k$, then we have:

(a) if $p^{\ell(*)}_{\eta,i}$ is realized by some $b \in M^{\ell(*)}$ such that $\mathrm{Rk}(\frac{b}{A^{\ell(*)}_\eta[Y]}, L, \infty) = \mathrm{R}(p^{\ell(*)}_{\eta,i}, L, \infty)$ and $[\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]]$, then for some such b we have $A^{\ell(*)}_\eta[Z] = ac\ell(A^{\ell(*)}_\eta[Y] \cup \{b\})$,

(b) if the assumption of clause (a) fails but $p^{\ell(*)}_{\eta,i}$ is realized by some $b \in M^{\ell(*)} \backslash A^{\ell(*)}_\eta$ such that $[\ell g(\eta) > 0 \Rightarrow \frac{b}{A^{\ell(*)}_\eta[Y]} \perp_a A^{\ell(*)}_{\eta^-}[Y]]$, then for some such b we have $A^{\ell(*)}_\eta[Z] = ac\ell[A^{\ell(*)}_\eta[Y] \cup \{b\}]$;

(δ) if $\eta \in I_Y$ and $\ell g(\eta) = m$, then $B^\ell_\eta[Z] = \{b^\ell_{\eta,j}[Y] : j \in \cup\{W_k : k < k^*_Z\}\}$ is a countable subset of M^ℓ containing $\{B^\ell_\nu[Z] : \nu \trianglelefteq \eta$ and $\nu \in Y\} \cup B^\ell_\eta[Y]$, with $B^\ell_\eta[Z] \prec M^\ell$; moreover, $B^\ell_\eta[Z] \subseteq_{na} M^\ell$, i.e., if $\bar a \subseteq B^\ell_\eta[Z], \varphi(x, \bar y)$ is first order and $(\exists x \in M^\ell \backslash ac\ell(\bar a))\varphi(x, \bar a)$ then $(\exists x \in B^\ell_\eta[Z] \backslash ac\ell(\bar a))\varphi(\bar x, \bar a))$ and $\{a^\ell_{\eta\hat{}<\alpha>}[Y] : \eta\hat{}\langle\alpha\rangle \in I_Y$ and $a^\ell_{\eta\hat{}<\alpha>}[Y] \notin B^\ell_\eta[Z]\}$ is independent over $(B^\ell_\eta[Z], A^\ell_\eta[Y])$;

(ϵ) if $\eta \in I_Y, \ell g(\eta) > m$, then $\eta \in I_Z \Leftrightarrow a^\ell_{\eta\restriction(m+1)}[Y] \notin B^\ell_{\eta\restriction m}[Z]$;

(ζ) if $\eta \in I_Y \cap I_Z, \ell g(\eta) > m$, then $A^\ell_\eta[Z] = ac\ell(A^\ell_\eta[Y] \cup A^\ell_{\eta\restriction m}[Z])$ and $B^\ell_\eta[Z] = B^\ell_\eta[Y]$;

(η) if $\eta \in I_Z \backslash I_Y$ then $\eta^- \in I_Y$ and $\ell g(\eta) = m + 1$;

(θ) $\{p^\ell_{\eta,i}[Z] : i \in W'_{k^*_Z - 1}\}$ is "rich enough", e.g., includes all finite types over A^ℓ_η;

(ι) $\{b^\ell_{\eta,i} : i \in W_{k^*_z - 1}\}$ list $B^\ell_\eta[Z]$, each appearing infinitely often.

Proof: First we choose $A^{\ell(*)}_\eta[Z]$ for $\eta \in I$ of length m according to condition $(\gamma) = (\gamma)^1 + (\gamma)^2$. (Note: One of the clauses $(\gamma)^1, (\gamma)^2$ necessarily holds trivially as $\bigcup_k W_k \cap \bigcup_k W'_k = \emptyset$.)

Second, we choose (for such η) an elementary mapping f^Z_η extending f^Y_η and a set $A^{3-\ell(*)}_\eta[Z] \subseteq M^{3-\ell(*)}$ satisfying "f^Z_η is from $A^1_\eta[Z]$ onto $A^{3-\ell(*)}_\eta[Z]$" such that

$(*)_2$ if $m > 0$, then $f^Z_\eta(\mathrm{tp}_\infty((\frac{A^1_\eta[Z]}{A^1_{\eta^-}[Y]}), M_1)) = \mathrm{tp}_\infty((\frac{A^2_\eta[Z]}{A^2_{\eta^-}[Y]}), M_2)$,

$(*)_3$ if $m = 0$, then $f^Z_\eta(\mathrm{tp}_\infty(A^1_\eta[Z], M_1)) = \mathrm{tp}_\infty(A^2_\eta[Z], M_2)$.

[Why is it possible? If we ask just the equality of tp_α for an ordinal α, this follows by the first component of $\mathrm{tp}_{\alpha+1}$. But (overshooting) for $\alpha \geq [(\|M_1\| + \|M_2\|)^{|T|}]^+$, equality of tp_α implies equality of tp_∞.]

Third, we choose $B^\ell_\eta[Z]$ for $\eta \in I_Y, \ell g(\eta) = m$ according to condition (δ) (here we use the countability of the language; you can do it by extending it ω times) on both sides, i.e., for $\ell = 1, 2$.

Fourth, let $I' = \{\eta \in I : \text{if } \ell g(\eta) > m \text{ then } a^\ell_{\eta\restriction(m+1)}[Y] \notin B^\ell_{\eta\restriction m}[Z]\}$ (this will be $I_Y \cap I_Z$).

Fifth, we choose $A^\ell_\eta[Z]$ for $\eta \in I'$: if $\ell g(\eta) < m$, let $A^\ell_\eta[Z] = A^\ell_\eta[Y]$; if $\ell g(\eta) = m$, this was done; lastly, if $\ell g(\eta) > m$, let $A^\ell_\eta[Z] = ac\ell(A^\ell_\eta[Y] \cup A^\ell_{\eta\restriction m}[Z])$.

Sixth, by induction on $k \leq n_Y$ we choose f^Z_η for $\eta \in I'$ of length k: if $\ell g(\eta) < m$, let $f^Z_\eta = f^Y_\eta$; if $\ell g(\eta) = m$, this was done; lastly, if $\ell g(\eta) > m$, choose an elementary mapping from A^1_η onto A^2_η extending $f^Y_\eta \cup f^Z_{\eta^-}$ (possible as $f^Y_\eta \cup f^2_{\eta^-}$ is an elementary mapping and $\mathrm{Dom}(f^Y_\eta) \cap \mathrm{Dom}(f^Z_{\eta^-}) = A^{\ell(*)}_{\eta^-}, \mathrm{Dom}(f^Y_\eta) \underset{A^{\ell(*)}_{\eta^-}}{\bigcup\!\!\!\!\!\!\bigcup} \mathrm{Dom}(f^Z_{\eta^-})$ and $A^{\ell(*)}_{\eta^-} = ac\ell(A^{\ell(*)}_{\eta^-})$). Now f^Z_η satisfies clause (11) of Definition 1.32 when $\ell g(\eta) > m$ by applying 1.27(3).

Seventh, for $\eta \in I'$, of length $m < n_Z$, let $v_\eta =: \{\alpha : \eta\hat{\ }\langle\alpha\rangle \in I\}$, and we choose $\{a^1_{\eta\hat{\ }<\alpha>}[Z] : \alpha \in u_\eta\}, [\alpha \in u_\eta \Rightarrow \eta\hat{\ }\langle\alpha\rangle \notin I]$, a set of elements of M^1 realizing (stationary) regular types over $A^1_\eta[Z]$, orthogonal to $A_{\eta^-}[Y]$ when $\ell g(\eta) > 0$, such that it is independent over $(\cup\{a^1_{\eta\hat{\ }<\alpha>}[Y] : \eta\hat{\ }\langle\alpha\rangle \in I'\} \cup B^1_\eta[Z], A^1_\eta[Z])$ and maximal under those restrictions. Without loss of generality, $\sup(v_\eta) < \min(u_\eta)$ and, for $\alpha_1 \in v_\eta \cup u_\eta$ and $\alpha_2 \in u_\eta$, we have:

$(*)_1$ if (for the given α_2 and η) α_1 is minimal such that $\frac{a^1_{\eta\hat{\ }<\alpha_1>}[Z]}{A^1_\eta[Z]} \pm \frac{a^1_{\eta\hat{\ }<\alpha_2>}[Z]}{A^1_\eta[Z]}$, then $\frac{a^1_{\eta\hat{\ }<\alpha_1>}[Z]}{A^1_\eta[Z]} = \frac{a^1_{\eta\hat{\ }<\alpha_2>}[Z]}{A^1_\eta[Z]}$;

$(*)_2$ if $\alpha_1 < \alpha_2$, and $a^1_{\eta\hat{\ }\langle\alpha_1\rangle}[Z]/A^1_\eta[Z] = a^1_{\eta\hat{\ }\langle\alpha_2\rangle}[Z]/A^1_\eta[Z]$ and, for some $b \in M^1$ realizing $\frac{a^1_{\eta\hat{\ }<\alpha_2>}[Z]}{A^1_\eta[Z]}$, we have $b \bigcup\!\!\!\!\!\!\bigcup_{A^1_{\eta,\ell}[Z]} a^1_{\eta\hat{\ }<\alpha_2>}$ and $\mathrm{tp}_\infty[(\frac{b}{A^1_{\eta\hat{\ }<\alpha_2>}}), M] = \mathrm{tp}_\infty[(\frac{a^1_{\eta\hat{\ }<\alpha_1>}}{A^1_{\eta}\hat{\ }<\alpha_2>}), M]$ and α_1 is minimal (for the given α_2 and η), then $\mathrm{tp}_\infty[(\frac{a^1_{\eta\hat{\ }<\alpha_2>}}{A^1_{\eta\hat{\ }<\alpha_2>}}), M] = \mathrm{tp}_\infty[(\frac{a^1_{\eta\hat{\ }<\alpha_1>}}{A^1_{\eta\hat{\ }<\alpha_1>}}), M]$.

Easily (as in [Sh:c, X]), if $\alpha \in u_\eta$ and $\eta\hat{\ }\langle\beta\rangle \in I'$ then $\frac{a^1_{\eta\hat{\ }<\alpha>}[Z]}{A^1_\eta[Z]} \perp \frac{a^1_{\eta\hat{\ }<\beta>}[Y]}{A^1_\eta[Y]}$.

For $\alpha \in u_\eta$ let $A^1_{\eta\hat{\ }<\alpha>}[Z] = ac\ell(A^1_\eta[Y] \cup \{a^1_{\eta\hat{\ }<\alpha>}[Z]\})$.

Eighth, by the second component in the definition of $\mathrm{tp}_{\alpha+1}$ (see Definition 1.10) we can choose (for $\alpha \in u_\eta$)$a^2_{\eta\hat{\ }<\alpha>}[Z], A^2_{\eta\hat{\ }<\alpha>}[Z]$ and then $f^Z_{\eta\hat{\ }<\alpha>}$ as required (see (7) of Definition 1.32).

Ninth, and last, we let

$$I_Z = I' \cup \{\eta\hat{\ } < \alpha >: \eta \in I', \ell g(\eta) = m < n_Z \text{ and } \alpha \in u_\eta\}$$

and we choose B^ℓ_η for $\eta \in I_Z \backslash I_Y$ and the $p^\ell_{\eta,i}, b^\ell_{\eta,j}$ as required (also in the remaining case).

$\blacksquare_{1.35}$

1.36 Finishing the Proof of 1.11: We define by induction on $n < \omega$ an approximation $Y_n = Y(n)$. Let Y_0 be the trivial one (as in observation 1.30(C)).

Y_{n+1} is obtained from Y_n as in 1.35 for $m_n, i_n \leq n, \ell_n(*) \in \{1,2\}$ defined by reasonable bookkeeping (so $i_n \in \bigcup_{k<k^*_{Y(n)}}(W_k \cup W'_k)$) such that any triples appear infinitely often; without loss of generality: if $n_1 < n_2 \& \eta \in I^\ell_{n_1} \cap I^\ell_{n_2}$ then $\eta \in \bigcap_{n=n_1}^{n_2} I_n$.

Let $I^* = I[*] = \lim(I_\ell^{Y(n)}) =: \{\eta : \text{for every large enough } n, \eta \in I_n\}$; for $\eta \in I^*$ let $A^\ell_\eta[*] = \bigcup_{n<\omega} A^\ell_\eta[Y_n], f^\ell_\eta[*] = \bigcup_{n<\omega} f_\eta^{Y(n)}$ and $B^\ell_\eta[*] = \bigcup_{n<\omega} B^\ell_\eta[Y_n]$. Easily

$\bigoplus_0$ $<>\in I^*$ and $I^* \subseteq {}^{\omega>}\mathrm{Ord}$ is closed under initial segments,

$\bigoplus_1$ for $\eta \in I^*, \langle B^\ell_\eta[Y_n] : n < \omega \text{ and } \eta \in I[Y_n]\rangle$ is an increasing sequence of $\subseteq_{\mathrm{na}}$-elementary submodels of M^ℓ.

[Why? By clause (12) of Definition 1.32, Main Fact 1.35, clauses $(\beta)(a), (\delta), (\zeta)$.]

Hence

$\bigoplus_2$ for $\eta \in I^*, B^\ell_\eta[*] \subseteq_{na} M^\ell$.

Also

$\bigoplus_3$ $\nu \triangleleft \eta \in I^* \Rightarrow B^\ell_\nu[*] \subseteq B^\ell_\eta[*]$.

[Why? Because for infinitely many $n, m_n = \ell g(\eta)$ and clause (δ) of Main Fact 1.35.]

$\bigoplus_4$ If $\eta \in I[Y_{n_1}] \cap I^*, \eta^- = \nu$ and $n_1 \leq n_2$, <u>then</u>

$$A^\ell_\eta[Y_{n_1}] \underset{A^\ell_\nu[Y_{n_1}]}{\biguplus} A^\ell_\nu[Y_{n_2}].$$

[Why? Prove by induction on n_2 (using the non-forking calculus); for $n_2 = n_1$ this is trivial, so assume $n_2 > n_1$. If $m_{(n_2-1)} > \ell g(\nu)$ we have $A^\ell_\nu[Y_{n_2}] = A^\ell_\nu[Y_{n_2-1}]$ (see 1.35, clause (β)(a) and we have nothing to prove). If $m_{(n_2-1)} < \ell g(\nu)$, then we note that $A^\ell_\nu[Y_{n_2}] = \mathrm{acl}(A^\ell_\nu[Y_{n_2-1}] \cup A^\ell_{\nu\restriction m_{(n_2-1)}}[Y_{n_2}])$ and $A^\ell_\nu[Y_{n_2-1}] \underset{A^\ell_{\nu\restriction m_{(n_2-1)}}}{\biguplus} A^\ell_{\nu\restriction m_{(n_2-1)}}[Y_{n_2}]$ (as $\nu \in I[Y_{n_2}]$, by 1.35 clause (δ) last phrase) and now use clauses (5), (6) of Definition 1.35. Lastly, if $m_{(n_2-1)} = \ell g(\nu)$ again use $\nu \in I[Y_{n_2}]$ by 1.35, clause (δ), last phrase.]

$\bigoplus_5$ If $\eta \in I[Y_{n_1}] \cap I^*, \eta^- = \nu$ and $n_1 \leq n_2$, <u>then</u>

$$\frac{A^\ell_\eta[*]}{A^\ell_\nu[*] + a^\ell_\eta[*]} \perp_a A^\ell_\nu.$$

[Why? By clause (6) of Definition 1.32, and orthogonality calculus.]

$\bigoplus_6$ If $\eta \in I^*$, then $A^\ell_\eta[*] \subseteq B^\ell_\eta[*] \prec M^\ell$; moreover,

$\bigotimes_7$ $A^\ell_{\eta(*)}[*] \subseteq_{\mathrm{na}} B^\ell_\eta[*] \subseteq_{\mathrm{na}} M^\ell$.

[Why? The second relation holds by $\bigotimes_2$. The first relation we prove by induction on $\ell g(\eta)$; clearly $A^\ell_\eta[*] = ac\ell(A^\ell_\eta[*])$ because $A^\ell_\eta[Y_n]$ increases with n by 1.35 and

$A^\ell_\eta[Y_n] = acl(A^\ell_\eta[Y_n])$ by clause (3) of Definition 1.32. We prove "$A^\ell_{\eta(*)}[*] \subseteq_{\rm na} B^\ell_\eta[*]$" by induction on $m = \ell g(\eta)$, so suppose this is true for every $m' < m, m = \ell g(\eta), \eta \in I^*$, let $\varphi(x)$ be a formula with parameters in $A^\ell_\eta[*]$ realized in M^ℓ as above, say, by $b \in M^\ell$. As $\langle A^\ell_\eta[Y_n] : n < \omega, \eta \in Y_n\rangle$ is increasing with union $A^\ell_\eta[*]$, clearly for some n we have $b \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\bigcup} A^\ell_\eta[*]$.

So $\{\varphi(x)\} = p^\ell_{\eta,i}$ for some i and for some $n' > n$ defining $Y_{n'+1}$ we have used 1.35 with $(\ell(*), i, m)$, there being $(\ell, i, \ell g(\eta))$ here, hence we consider clause $(\gamma)^2$ of 1.35. So the case left is when the assumption of both clauses (a) and (b) of $(\gamma)^2$ fail, in which case we have $\ell g(\eta) > 0$ and

$$b' \notin A^\ell_\eta[Y_{n'}], b' \in M^\ell \models \varphi[b'] \Rightarrow \frac{b'}{A^\ell_\eta[Y_{n'}]} \pm A^\ell_{\eta^-}[Y_{n'}].$$

We can now use the induction hypothesis (and [BeSh 307, 5.3, p. 292]).]

$\bigoplus_8$ If $\eta \in I^*$ and $\ell = 1, 2$, <u>then</u> $\{a^\ell_{\eta\hat{}<\alpha>}[*] : \eta\hat{}\langle\alpha\rangle \in I^*\}$ is a maximal subset of

$$\{c \in M_\ell : \frac{c}{A^\ell_\eta[*]} \text{ regular}, c \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\bigcup} B^\ell_\eta[*] \text{ and } \ell g(\eta) > 0 \Rightarrow \frac{c}{A^\ell_\eta[*]} \perp A^\ell_{\eta^-}[*]\}$$

independent over $(A^\ell_\eta[*], B^\ell_\eta[*])$.

[Why? Note clause (7) of Definition 1.32 and clause (δ) of Main Fact 1.35.]

$\bigotimes_9$ $A^\ell_{<>}[*] = B^\ell_{<>}[*]$.

[Why? By the bookkeeping every $b \in B^\ell_{<>}[*]$ is considered for addition to $A^\ell_{<>}[*]$, see 1.35, clause $(\gamma)^1$, subclause (b)(i), and for $\langle\rangle$ there is nothing to stop us.]

$\bigotimes_{10}$ If $\eta \in I^* \setminus \{\langle\ \rangle\}$ and $p \in S(A^\ell_\eta[*])$ is regular orthogonal to $A^\ell_{\eta^-}[*]$, then $\frac{B^\ell_\eta[*]}{A^\ell_\eta[*]} \perp p$.

[Why? If not, as $A^\ell_\eta[*] \subseteq_{\rm na} B^\ell_\eta[*]$ by [BeSh 307, Th. B, p. 277] there is $c \in B^\ell_\eta[*] \backslash A^\ell_\eta[*]$ such that: $\frac{c}{A^\ell_\eta[*]}$ is p. As $c \in B^\ell_\eta[*] = \bigcup_{n<\omega} B^\ell_\eta[Y_n]$, for every $n < \omega$ large enough $c \in B^\ell_\eta[Y_n]$, and p does not fork over $A^\ell_\eta[Y_n]$. So for some such n the triple (i_n, ℓ_n, m_n) is such that $\ell_n = \ell, m_n = \ell g(\eta)$ and $b^\ell_{\eta,i_n} = c$, so by clause $(\gamma)^1$(b)(ii) of 1.35 we have $c \in A^\ell_\eta[Y_n] \subseteq A^\ell_\eta[*]$.]

$\bigotimes_{11}$ If $\eta \in I^*, \ell \in \{1, 2\}$, <u>then</u> $\{a_{\eta\hat{}<\alpha>} : \eta\hat{}\langle\alpha\rangle \in I^*\}$ is a maximal subset of $\{c \in M^\ell : \frac{c}{A^\ell_\gamma[*]}$ regular, $\perp A^\ell_{\eta^-}[*]$ when meaningful$\}$ independent over $A^\ell_\eta[*]$.

[Why? If not, then for some $c \in M, \{a^\ell_{\eta\hat{}\langle\alpha\rangle} : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup \{c\}$ is independent over $A^\ell_\eta[*]$ and $\mathrm{tp}(c, A^\ell_\eta[*])$ is regular (and stationary). Hence by $\otimes_{10}$ we have $\{a^\ell_\eta[Y_n] : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup \{c\}$ is independent over $(A^\ell_\eta[*], B^\ell_\eta[*])$. Now for large enough n we have $c \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\!\bigcup} A^\ell_\eta[*]$ and by $\otimes_{10}$ we have $c \underset{A^\ell_\eta[Y_n]}{\bigcup\!\!\!\!\bigcup} B^\ell_\eta[*]$, hence $c \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\!\bigcup} B^\ell_\eta[Y_n]$, $\{c\} \cup \{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \eta\hat{}\langle\alpha\rangle \in I[Y_n]\}$ is not independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$, but $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \eta\hat{}\langle\alpha\rangle \in I[Y_n]\}$ is independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$. So there is a finite set w of ordinals such that $\alpha \in w \Rightarrow \eta\hat{}\langle\alpha\rangle \in I[Y_n]$ and $\{c\} \cup \{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \alpha \in w\}$ is not independent over $(A^\ell_\eta[Y_n], B^\ell_\eta[Y_n])$, and without loss of generality w is minimal. Let $n_1 \in [n,\omega)$ be such that $\alpha \in w \& a^\ell_{\eta\hat{}<\alpha>} \in B^\ell_\eta[*] \Rightarrow a^\ell_\eta \in B^\ell_\eta[Y_{n_1}]$; these clearly exist as w is finite and let $u = \{\alpha \in w : a^\ell_{\eta\hat{}<\alpha>} \notin B^\ell_\eta[*]\}$; clearly $\alpha \in u \Rightarrow \eta\hat{} < \alpha > \in I^*$. Now $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[*] : \eta\hat{}\langle\alpha\rangle \in I^*\} \cup B^\ell_\eta[*]$ includes $\{a^\ell_{\eta\hat{}\langle\alpha\rangle}[Y_n] : \alpha \in w\}$; easy contradiction to the second sentence above.]

$\bigoplus_{12}$ $f^*_\eta = \bigcup_{m<\omega} f_\eta[Y_n]$ (for $\eta \in I^*$) is an elementary map from $A^1_\eta[*]$ onto $A^2_\eta[*]$. [Easy.]

$\bigoplus_{13}$ $f^* =: \bigcup_{\eta\in I^*} f^*_\eta$ is an elementary mapping from $\bigcup_{\eta\in I^*} A^1_\eta[*]$ onto $\bigcup_{\eta\in I^*} A^2_\eta[*]$.

[Clear using $\otimes_5 + \otimes_6 + \otimes_{12}$ and non-forking calculus.]

$\bigoplus_{14}$ We can find $\langle d^\ell_\alpha : \alpha < \alpha(*)\rangle$ such that:

(a) $d^\ell_\alpha \in M^\ell, \beta < \alpha \Rightarrow d^\ell_\beta \neq d^\ell_\alpha$, $\text{tp}(d^\ell_\alpha, \bigcup_{\eta\in I[*]} A^\ell_\eta[*] \cup \{d^\ell_\beta : \beta < \alpha\})$ is $\aleph_\epsilon$-isolated and $\mathbf{F}^\ell_{\aleph_0}$-isolated, and

(b) $g_\alpha = \bigcup_{\eta\in I^*} f^*_\eta \cup \{\langle(d^1_\alpha, d^2_\alpha) : \alpha < \alpha(*)\rangle\}$ is an elementary mapping,

(c) $\alpha(*)$ is maximal, i.e., we cannot find $d^1_{\alpha(*)}$ such that the demand in (a) holds for $\alpha(*)+1$.

[Why? We can try to choose, by induction on α, a member d^1_α of $M^1 \setminus \bigcup_{\eta\in I[*]} \cup \{d^1_\beta : \beta < \alpha\}$ such that $\text{tp}(d^1_\alpha, \bigcup_{\eta\in I[*]} A^\ell_\eta[*] \cup \{d^1_\beta : \beta < \alpha\})$ is $\aleph_\varepsilon$-isolated and $\mathbf{F}^\ell_{\aleph_0}$-isolated. So for some $\alpha(*)$, d^1_α is well defined iff $\alpha < \alpha(*)$ (as $\beta < \alpha \Rightarrow d^1_\beta \neq d^1_\alpha \in M^1$). Now choose, by induction on $\alpha < \alpha(*), d^2_\alpha \in M^2$ as required above, possible by "M^2_i being $\aleph_\varepsilon$-saturated" (see [Sh:c, XII, 2.1, p. 591], [Sh:c, IV, 3.10, p. 179].]

$\bigotimes_{15}$ $\text{Dom}(g_{\alpha(*)})$, $\text{Rang}(g_{\alpha(*)})$ are universes of elementary submodels of M^1, M^2, called M'_1, M'_2 respectively.

[Why? See [Sh:c, XII, 1.2(2), p. 591] and the proof of $\otimes_{14}$.

Alternatively, choose a formula $\psi(x,\bar{a})$ such that:

(a) $\bar{a} \subseteq \text{Dom}(g_{\alpha(*)})$ and $\models \exists x \psi(x,\bar{a})$ but no $b \in \text{Dom}(g_{\alpha(*)})$ satisfy $\varphi(x,\bar{a})$;

(b) under clause (a), $\text{Rk}(\psi(x,\bar{a}), \mathbb{L}_{\tau|T|}, \infty)$ is minimal (or just has no extension in $S(\text{Dom}(g_{\alpha(*)}))$ forking over $\bar{a}$).

Let $\{\varphi_\ell(x,\bar{y}_\ell) : \ell < \omega\}$ list that $\mathbb{L}_{\tau(T)}$-formula and we choose by induction on ℓ as formula $\psi_n(x,\bar{a}_n)$ such that:

(i) $\bar{a} \subseteq \mathrm{Dom}(g_{\alpha(*)})$,

(ii) $\models (\exists x)\psi_n(x, \bar{a}_n)$,

(iii) $\psi_{n+1}(x, \bar{a}_{n+1}) \vdash \psi_n(x, \bar{a}_n)$,

(iv) $\psi_0(x, \bar{a}_0) = \psi(x, \bar{a})$,

(v) for any formula $\psi'(x, \bar{a}')$ satisfying the demands on $\psi_{n+1}(x, \bar{a}_{n+1})$ we have $\mathrm{Rk}(\psi_{n+1}(x, \bar{a}_{n+1}), \{\varphi_n(x, \bar{y}_n)\}, 2) < \mathrm{Rk}(\psi'(x, \bar{a}), \{\varphi_n(x, \bar{y})\}, 2)$ (on this rank see [Sh:c, II, §2]).

So $p = \{\psi_n(x, \bar{a}_n) : n < \omega\}$ has an extension in $S(\mathrm{Dom}(g_{\alpha(*)}))$; call it q. Now q is $\aleph_\varepsilon$-isolated because $\psi(x, \bar{a}) \in q \in S(\mathrm{Dom}(g_{\alpha(*)}))$. For every $n, \psi_{n+1}(x, \bar{a}_n) \vdash q \restriction \{\varphi_n(x, \bar{y}_n)\}$ by clause (v) above, so as $\psi_{n+1}(x, \bar{a}_n) \in q$ and this holds for every n clearly q is $\mathbf{F}^\ell_{\aleph_0}$-isolated.

$\bigotimes_{16}$ If $M^\ell \neq M'_\ell$; then for some $d \in M_\ell \backslash M'_\ell$, $\frac{d}{M'_\ell}$ is regular.

[Why? By [BeSh 307, Th. 5.9, p. 298] as $N^\ell_\eta \subseteq_{\mathrm{na}} M^\ell$ by $\bigotimes_7$.]

$\bigotimes_{17}$ If $M^\ell \neq M'_\ell$, then for some $\eta \in I^*$, there is $d \in M^\ell \backslash M'_\ell$ such that $\frac{d}{A^\ell_\eta[*]}$ is regular, $d \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\!\smile} M'_\ell$ and $[\ell g(\eta) > 0 \Rightarrow \frac{d}{A^\ell_\eta[*]} \perp A^\ell_{\eta^-}[*]]$.

[Why? By [Sh:c, XII, 1.4, p. 529] every non-algebraic $p \in S(M'_\ell)$ is not orthogonal to some $A^\ell_\eta[*]$, so by $\otimes_{16}$ we can choose $\eta \in I^*$ and $d \in M^\ell \backslash M'_\ell$ such that $\frac{d}{M'_\ell}$ is regular $\pm A^\ell_\eta[*]$. Without loss of generality $\ell g(\eta)$ is minimal; now $A^\ell_\eta[*] \subseteq_{\mathrm{na}} M^\ell$ and by [BeSh 307, 4.5, p. 290] without loss of generality $d \underset{A^\ell_\eta[*]}{\bigcup\!\!\!\!\smile} M'_\ell$; the last clause is by "$\ell g(\eta)$ minimal".]

$\bigoplus_{18}$ $M_\ell = M'_\ell$.

[Why? By $\bigoplus_{11} + \bigoplus_{17}$.]

$\bigoplus_{19}$ There is an isomorphism from M_1 onto M_2 extending $\bigcup_{\eta \in I^*} f^*_\eta$.

[Why? By $\bigoplus_{14} + \bigotimes_{15}$ we have $M'_1 \cong M'_2$, so by $\otimes_{18}$ we are done.] $\blacksquare_{1.36}$ $\blacksquare_{1.30}$

1.37 Lemma: *Assume $B \underset{A}{\bigcup\!\!\!\!\smile} C, A = ac\ell(A) = B \cap C$ and A, B, C are ϵ-finite, $A \cup B \cup C \subseteq M, M$ an $\aleph_\epsilon$-saturated model of T. For notational simplicity make A a set of individual constants.*

__Then__ $\mathrm{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(\mathrm{d.q.})}(B + C; M) = \mathrm{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)}(B; M) + \mathrm{tp}_{\mathbb{L}_{\infty,\aleph_\epsilon}(d.q.)}[C; M]$ where

1.38 Definition: (1) For any logic $\mathbb{L}$ and $\bar{b}$ a sequence from a model M, let

$$\mathrm{tp}_{\mathcal{L}}(\bar{b}; M) = \{\varphi(\bar{x}) : M \models \varphi[B], \varphi \text{ a formula in the vocabulary of } M, \text{ from the logic } \mathcal{L} \text{ (with free variables from } \bar{x}, \text{ where } \bar{x} = \langle x_i : i < \ell g(\bar{b})\rangle)\}.$$

(2) Replacing $\bar{b}$ by a set B means we use the variables $\langle x_b : b \in B \rangle$.
(3) Saying $p_1 = p_2 + p_3$ in 1.37 means that we can compute p_1 from p_2 and p_3 (and knowledge as to how the variables fit and knowledge of T, of course).

Proof of the Lemma 1.37: It is enough to prove:

1.39 CLAIM: *Assume*
(a) M^1, M^2 *are* $\aleph_\epsilon$*-saturated and*
(b) $A_1^i \underset{A_0^i}{\bigcup\!\!\!\!\bigcup} A_2^i$ *for* $i = 1, 2$,
(c) $A_0^i = ac\ell(A_0^i)$ *and* A_m^i *is* ϵ*-finite for* $i = 1, 2$ *and* $m < 3$,
(d) *for* $m = 0, 1, 2$ *we have* $f_m : A_m^1 \xrightarrow{\text{onto}} A_m^2$ *is an elementary mapping preserving* tp_∞ *(in* M^1, M^2 *respectively) and*
(e) $f_0 \subseteq f_1, f_2$.
Then *there is an isomorphism from* M^1 *onto* M^2 *extending* $f_1 \cup f_2$.

Proof of 1.39: Repeat the proof of 1.5, but starting with Y_0 such that $A^\ell_{<>}[Y_0] = A^\ell_0$, $A^\ell_{<>}[Y_0] = A^\ell_1$, $A^\ell_{<1>}[Y_0] = A^\ell_2$, $f^{Y_0}_{<>} = f_0$, $f^{Y_0}_{<0>} = f_1$, $f^{Y_0}_{<1>} = f_2$ and that $\langle\rangle, \langle 0 \rangle, \langle 1 \rangle$ belong to all $I[Y_0]$. During the construction we preserve $\langle 0 \rangle, \langle 1 \rangle \in I[Y_n]$ and for helping to preserve this we add also the demand
$\circledast_{2,m}$ $B^\ell_{<>}[Y_n] \underset{A_0^\ell}{\bigcup\!\!\!\!\bigcup} A^\ell_1 \cup A^\ell_2$.
During the proof, when we have to increase $B^\ell_{<>}$, we use 1.18(1) + 1.16(1). $\blacksquare_{1.39}$

DISCUSSION: A natural version of 1.39 is the conclusion only that

$$\text{tp}_\alpha[\begin{pmatrix} A^1_a \cup A^1_2 \\ A^1_0 \end{pmatrix}, M^1] = \text{tp}_\alpha[\begin{pmatrix} A^2_1 \cup A^2_2 \\ A^2_0 \end{pmatrix}, M^2]$$

and to prove this by induction on α. The case $\alpha = 0$ and α limit are obvious. If $\alpha = \beta + 1$, for the condition of $\leq_{\text{a}}$, we use the induction hypothesis and claim 1.27(1). The condition involving $\leq_{\text{b}}$ is similar but harder. $\blacksquare_{1.39}$

2. Finer types

We shall use here alternative types showing us probably a finer way to manipulate tp.

2.1 CONVENTION: T is superstable, NDOP; M, N are $\aleph_\epsilon$-saturated $\prec \mathfrak{C}^{\text{eq}}$.

2.2 Definition: $\Gamma_3 = \{\binom{\bar b}{\bar a} : \bar a \subseteq \bar b$ are ϵ-finite$\}$,

$$\Gamma_1 = \left\{ \binom{p}{\bar a} : \bar a \text{ is } \epsilon\text{-finite}, p \in S(\bar a) \text{ is regular (so stationary)} \right\},$$

$$\Gamma_2 = \left\{ \binom{p,r}{\bar a} : \bar a \text{ is } \epsilon\text{-finite}, p \text{ is a regular type of depth} > 0, \right.$$

$$p \pm \bar a \text{ (really only the equivalence class } p/\pm \text{ matters)},$$

$$r = r(x,\bar y) \in S(\bar a) \text{ is such that for } (c,\bar b) \text{ realizing } r,$$

$$\left. c/(\bar a + \bar b) \text{ is regular } \pm p, \text{ and } \frac{\bar b}{\bar a} = (r \restriction \bar y) \perp p \right\}.$$

We may add (to Γ_x) superscripts:

(α) f if $\bar a$ (or $\bar a\hat{\ }\bar b$) is finite,

(β) s: for Γ_3 if $\frac{\bar b}{\bar a}$ is stationary, for Γ_1 if p is stationary which holds always, and for Γ_2 if r is stationary and every automorphism of $\mathfrak{C}$ over $\bar a$ fixes $p/\pm$,

(γ) c if $\bar a$ (or $\bar a, \bar b$) are algebraically closed.

2.3 CLAIM: *If p is regular of depth > 0 and $p \pm \bar a$ and $\bar a$ is ϵ-finite, <u>then</u> for some $\bar a', \bar a \subseteq \bar a' \subseteq ac\ell(\bar a)$ and for some q <u>we have</u> $\binom{p,q}{\bar a'} \in \Gamma_2^s$.*

Proof: Use, e.g., [Sh:c, V, 4.11, p. 272]; assume $\frac{\bar b}{\bar a} \pm p$. We can define inductively equivalence relations E_n, with parameters from $ac\ell(\bar a^\ell)$,

$$\bar a^\ell = \bar a\hat{\ }(\bar b/E_0)\hat{\ }\cdots\hat{\ }(\bar b/E_{n-1}),$$

such that $\text{tp}(\bar b/E_n, ac\ell(\bar a^n))$ is semi-regular. By superstability this stops for some n, hence $\bar b \subseteq ac\ell(\bar a^n)$. For some first m, $\text{tp}(\bar b/E_m, ac\ell(\bar a^n))$ is $\pm p$; by [Sh:c, X, 7.3(5), p. 552] the type is regular (because p is trivial having depth > 0; see [Sh:c, X, 7.2, p. 551]). $\blacksquare_{2.3}$

2.4 Definition: We define by induction on an ordinal α the following (simultaneously) [note — if a definition of something depends on another which is not well defined, neither is the something]:

$$\text{tp}^1_\alpha\left[\binom{p}{\bar a}, M\right] \quad \text{for } \binom{p}{\bar a} \in \Gamma_1, \bar a \subseteq M,$$

$$\text{tp}^2_\alpha\left[\binom{p,r}{\bar a}, M\right] \quad \text{for } \binom{p,r}{\bar a} \in \Gamma_1, \bar a \subseteq M,$$

$$\text{tp}^3_\alpha\left[\binom{\bar b}{\bar a}, M\right] \quad \text{for } \binom{\bar b}{\bar a} \in \Gamma_3^c, \bar a \subseteq \bar b \subseteq M.$$

CASE A, $\alpha = 0$: $\mathrm{tp}^1_\alpha[\binom{p}{\bar a}, M]$ is $\mathrm{tp}((c,\bar a),\emptyset)$ for any c realizing p.

$$tp^2_\alpha\left[\binom{p,r}{\bar a}, M\right] \text{is } \mathrm{tp}((c,\bar b,\bar a),\emptyset) \text{ for any } (c,\bar b) \text{ realizing } r.$$

$$\mathrm{tp}^3_\alpha\left[\binom{\bar b}{\bar a}, M\right] \text{is } \mathrm{tp}((\bar b,\bar a),\emptyset)$$

(i.e., the type and the division of the variables between the sequences).

CASE B, $\alpha = \beta + 1$:
(a) $\mathrm{tp}^1_\alpha[\binom{p}{\bar a}, M]$ is:

SUBCASE a1: If p has depth zero, it is $w_p(M/\bar a)$ (the p-weight, equivalently, the dimension).

SUBCASE a2: If p has depth > 0 (hence is trivial), then it is $\{\langle y, \lambda^y_{\bar a,p}\rangle : y\}$ where

$$\lambda^y_{\bar a,p} = \dim(\mathbf{I}^y_{\bar a,p}[M], a)$$

where $\mathbf{I}^y_{\bar a,p}[M] = \{c \in M : c$ realizes p and $y = \mathrm{tp}^3_\beta[\binom{ac\ell(\bar a+c)}{ac\ell(\bar a)}, M]$ where $\bar a^*$ lists $acl(\bar a)$ and $\bar c^*$ lists $acl(\bar a + c)\}$; an alternative probably more transparent and simpler in use is:

$$\lambda^y_{\bar a,p} = \dim\Big\{c \in M\colon c \text{ realizes } p \text{ and}$$

$$y = \{\mathrm{tp}^3_\beta[\binom{ac\ell(\bar a + c')}{ac\ell(\bar a)}, M] : c' \in p(M) \text{ and } c' \bigcup_{\bar a} c\},$$

pedantically $y = \{\mathrm{tp}^3_\beta \left(\begin{matrix} < c' > {}^\frown\bar a^{*\frown}\bar c^* \\ \bar a^\frown\bar a^*\end{matrix}\right), M]$, where

$\bar a^*$lists $acl(\bar a)$ and

$\bar c^*$lists $acl(\bar a + c'), c' \in p(M)$ and $c' \bigcup_{\bar a} c\}\Big\}$.

(b) $\mathrm{tp}^2_\alpha[\binom{p,r}{\bar a}, M]$ is:
$\mathrm{tp}^1_\alpha[\binom{c/\bar b^+}{\bar b^+}, M]$ for any $(c,\bar b)$ realizing $r, \bar b^+ = ac\ell(\bar a+\bar b)$, i.e., $\bar b^+$ lists $acl(\bar a+\bar b)$ (so not well defined if we get at least two different cases; so remember $c/b^+ \in S(\bar b^+)$).
(c) $\mathrm{tp}^3_\alpha[\binom{\bar b}{\bar a}, M]$ is $\{\langle p, \mathrm{tp}^2_\alpha[\binom{p,r}{\bar b}, M]\rangle : \binom{p,r}{\bar b} \in \Gamma^s_2$ and $p \perp \bar a\}$.

CASE C, α LIMIT: For any $\ell \in \{1,2,3\}$ and suitable object OB:

$$\mathrm{tp}^\ell_\alpha[OB, M] = \langle \mathrm{tp}^\ell_\beta[OB, M] : \beta < \alpha\rangle.$$

2.5 Definition: (1) For $\binom{p}{\bar a} \in \Gamma_1$ where $\bar a \in M$, let (remembering 1.14(8)):

$\mathcal{P}^M_{\binom{p}{\bar a}} = \{q \in S(M)$: q regular and : $q \pm p$ or for some $c \in p(M)$ we have $q \in \mathcal{P}^M_{\binom{c}{\bar a}}\}$.

(2) For $\binom{p,r}{\bar a} \in \Gamma_2$ let

$\mathcal{P}^M_{\binom{p,r}{\bar a}} = \{q \in S(M)$: q regular and : $q \pm p$ or for some $(c, \bar b) \in r(M), q \in \mathcal{P}^M_{\binom{c}{\bar a + \bar b}}\}$.

(3) For a set $\mathcal{P}$ of (stationary) regular types not orthogonal to M_1, let $M_1 \leq_{\mathcal{P}} M_2$ mean $M_1 \prec M_2$ and for every $p \in \mathcal{P}$ and $\bar c \in M_2$, $\frac{\bar c}{M_1} \perp p$.
(4) If (in (3)) $\mathcal{P} = \mathcal{P}^{M_1}_{\binom{p}{\bar a}}$ we may write $\binom{p}{\bar a}$ instead $\mathcal{P}$; similarly, if $\mathcal{P} = \mathcal{P}^{M_1}_{\binom{p,r}{\bar a}}$ we may write $\binom{p,r}{\bar a}$.

2.6 CLAIM:
(1) *From* $\mathrm{tp}^1_\alpha[\binom{p}{\bar a}, M]$ *we can compute* $\mathrm{tp}^1_\infty[\binom{p}{\bar a}, M]$ *if* $\mathrm{Dp}(p) < \alpha$.
(2) *From* $\mathrm{tp}^2_\alpha[\binom{p,q}{\bar a}, M]$ *we can compute* $\mathrm{tp}^2_\infty[\binom{p,q}{\bar a}, M]$ *if* $\mathrm{Dp}(p) < \alpha$.
(3) *From* $\mathrm{tp}^3_\alpha[\binom{\bar b}{\bar a}, M]$ *we can compute* $\mathrm{tp}^3_\infty[\binom{b}{\bar a}, M]$ *if* $\mathrm{Dp}(\bar b/\bar a) < \alpha$.
(4) *In Definition 2.5(2) we can replace "some* $(c, \bar b) \in r(M)$*" by "every* $(c, \bar b) \in r(M)$*".*

Proof: (1), (2), (3) We prove this by induction on α. By the definition.
(4) Left to the reader.

2.7 OBSERVATION: From $\mathrm{tp}^\ell_\alpha(OB, M)$ we can compute $\mathrm{tp}^\ell_\beta[OB, M]$, and $\mathrm{tp}^\ell_\beta[OB, M]$ is well defined if $\beta \leq \alpha$ and the former is well defined.

2.8 LEMMA: *For every ordinal* α *the following holds:*
(1) tp^1_α *is well defined.*[3]
(2) tp^2_α *is well defined.*
(3) tp^3_α *is well defined.*
(4) *If* $\bar a \in M_1, \binom{p}{\bar a} \in \Gamma_1, M_1 \leq_{\binom{p}{\bar a}} M_2$, *then* $\mathrm{tp}^1_\alpha[\binom{p}{\bar a}, M_1] = \mathrm{tp}^1_\alpha[\binom{p}{\bar a}, M_2]$.
(5) *If* $\bar a \in M_1, \binom{p,r}{\bar a} \in \Gamma^s_2, M_1 \leq_{\binom{p}{\bar a}} M_2$, *then* $\mathrm{tp}^2_\alpha[\binom{p,r}{\bar a}, M_1] = \mathrm{tp}^2_\alpha[\binom{p,r}{\bar a}, M_2]$.
(6) *If* $\bar a \subseteq \bar b \subseteq M_1, \binom{\bar b}{\bar a} \in \Gamma^c_3, M_1 \leq_{\binom{\bar b}{\bar a}} M_2$, *then* $\mathrm{tp}^3_\alpha[\binom{\bar b}{\bar a}, M_1] = \mathrm{tp}^3_\alpha[\binom{\bar b}{\bar a}, M_2]$.

Proof: We prove it, by induction on α, simultaneously (for all clauses and parameters).

If α is zero, they hold trivially by the definition.

If α is limit, they hold trivially by the definition and induction hypothesis. So for the rest of the proof let $\alpha = \beta + 1$.

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3 I.e., in all the cases we have tried to define it in Definition 2.9.

Proof of $(1)_\alpha$: If p has depth zero — check directly.

If p has depth > 0 — by $(3)_\beta$ (i.e., induction hypothesis) no problem.

Proof of $(2)_\alpha$: Like 1.27 (and $(4)_\alpha$).

Proof of $(3)_\alpha$: Like $(2)_\alpha$.

Proof of $(4)_\alpha$: Like 1.26 (and $(3)_\beta, (6)_\beta$).

Proof of $(5)_\alpha$: By $(2)_\alpha$ we can look only at $(c, \bar{b}^+)$ in M_1, then use $(4)_\alpha$.

Proof of $(6)_\alpha$: By $(5)_\alpha$. $\blacksquare_{2.8}$

2.9 LEMMA: *For an ordinal α, restricting ourselves to the cases (the types p, p_1 being) of depth $< \alpha$:*

(A1) *Assume $\binom{p}{\bar{a}} \in \Gamma_1, \bar{a} \subseteq \bar{a}_1 \subseteq M, \bar{a}_1$ is ϵ-finite, $\frac{\bar{a}_1}{\bar{a}} \perp p$ and p_1 is the stationarization of p over $\bar{a}_1$.*
__Then__ from $\mathrm{tp}^1_\alpha[\binom{p}{\bar{a}}, M]$ we can compute $\mathrm{tp}^1_\alpha[\binom{p_1}{\bar{a}_1}, M]$.

(A2) *Under the assumption of* (A1) *also the inverse computations are O.K.*

(A3) *Assume $\binom{p_\ell}{\bar{a}} \in \Gamma_1$ for $\ell = 1, 2, \bar{a} \subseteq M$ and $p_1 \pm p_2$.*
__Then__ from $\mathrm{tp}^1_\alpha[\binom{p_1}{\bar{a}}, M]$ (and $\mathrm{tp}((\bar{a}, c_1, c_2), \emptyset)$ where c_1, c_2 realizes p_1, p_2 respectively, of course) we can compute $\mathrm{tp}^1_\alpha[\binom{p_2}{\bar{a}}, M]$.

(B1) *Assume $\binom{p_\ell, r_\ell}{\bar{a}} \in \Gamma^{sc}_2$ for $\ell = 1, 2, \bar{a} \in M$ and $p_1 \pm p_2$.*
__Then__ (from the first order information on $\bar{a}, p_1, p_2, r_1, r_2$, of course, and $\mathrm{tp}^2_\alpha[\binom{p_1, r_2}{\bar{a}_1}, M]$) we can compute $\mathrm{tp}^2_\alpha[\binom{p_2, r_2}{\bar{a}}, M]$.

(B2) *Assume $\bar{a} \subseteq \bar{a}_1 \subseteq M$, $\frac{\bar{a}_1}{\bar{a}} \perp p$, $\binom{p, r}{\bar{a}} \in \Gamma^{cs}_2, r \subseteq r_1 \in S(\bar{a}_1), r_1$ does not fork over $\bar{a}$, (so $\binom{p, r_1}{\bar{a}_1} \in \Gamma_2$).*
__Then__ from $\mathrm{tp}^2_\alpha[\binom{p, r_1}{\bar{a}}, M]$ we can compute $\mathrm{tp}^2_\alpha[\binom{p, r_2}{\bar{a}}, M]$.

(B3) *Under the assumption of* (B2)*, the inverse computation is O.K.*

(C1) *Assume $\binom{\bar{b}}{\bar{a}} \in \Gamma^c_3, \bar{a} \subseteq \bar{b} \subseteq M, \bar{a} \subseteq \bar{a}_1, \bar{b} \bigcup_{\bar{a}} \bar{a}_1, \bar{b}_1 = ac\ell(\bar{a}_1 + \bar{b})$.*
__Then__ from $\mathrm{tp}^3_\alpha[\binom{\bar{b}}{\bar{a}}, M]$ we can compute $\mathrm{tp}^3_\alpha[\binom{\bar{b}_1}{\bar{a}_1}, M]$.

(C2) *Under the assumptions of* (C1) *the inverse computation is O.K.*

(C3) *Assume $\binom{\bar{b}}{\bar{a}} \in \Gamma_3, \bar{b} \subseteq b^*, \frac{\bar{b}^*}{\bar{b}} \perp_a \bar{a}, \bar{b}^* = ac\ell(\bar{b}^*)$.*
__Then__ from $\mathrm{tp}^3_\alpha[\binom{\bar{b}}{\bar{a}}, M]$ we can compute $\{\mathrm{tp}^3_\alpha[\binom{\bar{b}'}{\bar{a}}, M] : \bar{b} \subseteq b' \subseteq M$ and $\frac{\bar{b}'}{\bar{b}} = \frac{\bar{b}^*}{\bar{b}}\}$.

Proof: We prove it, simultaneously, for all clauses and parameters, by induction on α and the order of the clauses.

For $\alpha = 0$: easy.

For α limit: very easy.

So assume $\alpha = \beta + 1$.

Proof of $(\mathrm{A1})_\alpha$: As p is stationary $\perp \frac{\bar a_1}{\bar a}$, for every $c \in p(M)$, $\frac{c}{\bar a} \vdash \frac{c}{\bar a_1}$, which necessarily is p_1, hence $p(M) = p_1(M)$. Also, the dependency relation on $p(M)$ is the same over $\bar a_1$, hence dimension. So it suffices to show:

$(*)$ for $c \in p(M)$, from $\mathrm{tp}^3_\beta[\binom{ac\ell(\bar a + c)}{ac\ell\bar a}, M]$ we can compute $\mathrm{tp}^3_\beta[\binom{ac\ell(\bar a_1 + c)}{ac\ell\bar a_1}, M]$. But this holds by $(\mathrm{C1})_\beta$.

Proof of $(\mathrm{A2})_\alpha$: Similar using $(\mathrm{C2})_\beta$.

Proof of $(\mathrm{A3})_\alpha$: If p_1 (equivalently p_2) has depth zero — the dimensions are equal. Assume they have depth > 0, hence are trivial, and dependency over $\bar a$ is an equivalence relation on $p_1(M) \cup p_2(M)$.

Now for $c_1 \in p_1(M)$, from $\mathrm{tp}^3_\beta[\binom{ac\ell(a + c_1)}{ac\ell(\bar a)}, M]$ we can compute for every complete type over $ac\ell(\bar a + c_1)$ not forking over $\bar a$, and $\bar d$ realizing r, $\mathrm{tp}^3_\beta[\binom{ac\ell(\bar a + \bar d + c_1)}{ac\ell(\bar a + \bar d)}, M]$ — by $(\mathrm{C1})_\beta$; then we can compute for each such $r, \bar d$,

$$\left\{ \mathrm{tp}^3_\beta\left[\begin{pmatrix} ac\ell(\bar a + \bar d + c_2) \\ ac\ell(\bar a + \bar d)\end{pmatrix}, M\right] : c_2 \in p_2(M) \text{ and } \frac{c_2}{ac\ell(\bar a + \bar d + c_1)} \perp_a (\bar a + \bar d) \quad (\text{necessarily } c_2 \underset{\bar a}{\bigcup} \bar d) \right\}$$

(this by $(\mathrm{C3})_\beta$).

Proof of $(\mathrm{B1})_\alpha$: As in earlier cases we can restrict ourselves to the case $\mathrm{Dp}(p_\ell) > 0$. We can find $(c_\ell, \bar b_\ell) \in r_\ell(M), \bar b_1 \underset{\bar a}{\bigcup} \bar b_2, c_1\bar b_1 \underset{\bar a}{\bigcup} \bar b_2$ (by [Sh:c, X, 7.3(6)]]. By 2.8(2) (and the definition) from $\mathrm{tp}^2_\alpha[\binom{p_1, r_1}{\bar a}, M]$ we can compute that it is equal to $\mathrm{tp}^1_\alpha[\binom{c_1/ac\ell(\bar a + \bar b_1)}{ac\ell(\bar a + b_1)}, M]$.

By $(\mathrm{A1})_\alpha$ we can compute $\mathrm{tp}^1_\alpha[\binom{c_1/ac\ell(\bar a + \bar b_1 + \bar b_2)}{ac\ell(\bar a + b_1 + \bar b_2)}, M]$, hence by $(\mathrm{A3})_\alpha$ we can compute $\mathrm{tp}^1_\alpha[\binom{c_2/ac\ell(\bar a + \bar b_1 + \bar b_2)}{ac\ell(\bar a + \bar b_1 + \bar b_2)}, M]$.

Now use $(\mathrm{A2})_\alpha$ to compute $\mathrm{tp}^1_\alpha[\binom{c_2/ac\ell(\bar a + \bar b_2)}{ac\ell(\bar a + \bar b_2)}, M]$ and by 2.8(2), 2.4(2) it is equal to $\mathrm{tp}^2_\alpha[\binom{p, r}{\bar a}, M]$.

Proof of $(\mathrm{B2})_\alpha$: Choose $(c, \bar b) \in r(M)$ such that $c\bar b \underset{\bar a}{\bigcup} \bar a_1$.

From $\mathrm{tp}^2_\alpha[\binom{p, r_1}{\bar a}, M]$ we can compute $\mathrm{tp}^1_\alpha[\binom{c/(\bar a + \bar b)}{\bar a + \bar b}, M]$ (just see 2.8(2) and Definition 2.4), from it we can compute $\mathrm{tp}^1_\alpha[\binom{c/(\bar a + \bar b + \bar a_1)}{(\bar a + \bar b + \bar a_1)}, M]$ (by $(\mathrm{A1})_\alpha$); from it we can compute $\mathrm{tp}^2_\alpha[\binom{p, r_2}{\bar a_2}, M]$ (see 2.8(2) and Definition 2.4).

Proof of $(\mathrm{B3})_\alpha$: Let $\binom{p, r}{\bar b_1} \in \Gamma^s_r, p \perp \bar a_1$ be given. So necessarily $\frac{\bar a_1}{\bar a} \pm p$ (this to enable us to use (B2,3). It suffices to compute $\mathrm{tp}^2_\alpha[\binom{p, r}{\bar b_1}, M]$ and we can discard the case $\mathrm{Dp}(p) = 0$.

So p is regular $\pm\bar{b}_1, \perp \bar{a}_1$, hence $p \pm \bar{b}, p \perp \bar{a}$, and as $\bar{a} \subseteq \bar{b}, \bar{b} = ac\ell(\bar{b})$ we can find $r, \binom{p,r_1}{\bar{b}} \in \Gamma_2$, (see 2.3) and we know $\mathrm{tp}^2_\alpha[\binom{p,r_1}{\bar{b}}, M]$, and we can find r_2, a complete type over $\bar{b}_1$ extending r_1 which does not fork over $\bar{b}_1$. From $\mathrm{tp}^2_\alpha[\binom{p,r_1}{\bar{b}}, M]$ we can compute $\mathrm{tp}^2_\alpha[\binom{p,r_2}{\bar{b}_1}, M]$ by $(\mathrm{B2})_\alpha$, and from it $\mathrm{tp}^2_\alpha[\binom{p,r}{\bar{b}_1}, M]$ by $(\mathrm{B1})_\alpha$.

Proof of $(\mathrm{C2})_\alpha$: Similarly, use $(\mathrm{B3})_\alpha$ instead of $(\mathrm{B2})_\alpha$.

Proof of $(\mathrm{C3})_\alpha$: Without loss of generality $\frac{\bar{b}^*}{\bar{b}}$ is semi-regular; let p^* be a regular type not orthogonal to it and, without loss of generality, $\mathrm{Dp}(p^*) > 0 \Rightarrow \frac{\bar{b}^*}{\bar{b}}$ regular (as in 2.3).

If p^* has depth zero, then the only p appearing in the definition $\mathrm{tp}^3_\alpha([\frac{\bar{b}}{\bar{a}}], M)$ is p^* (up to $\pm$) and this is easy. Then tp^2_α is just the dimension and we have no problem.

So assume p^* has depth > 0. We can by $(\mathrm{B1})_\alpha, (\mathrm{B2})_\alpha$ compute $\mathrm{tp}^2_\alpha[\binom{p',q'}{\bar{b}^*}, M]$ when $p' \pm \bar{b}, p' \pm p^*$ (regardless of the choice of $\bar{b}^*$). Next assume $p' \pm p^*$; by $(\mathrm{B1})_\alpha$, without loss of generality, q' does not fork over $\bar{b}$. As $\mathrm{Dp}(p^*) > 0$, it is trivial (and we assume $w_p(\bar{b}^*, \bar{b}) = 1$), hence $\bar{b}^*/\bar{b}$ is regular, so in $\mathrm{tp}^2_\alpha[\binom{p,q'}{\bar{b}^*}, M]$ we just lose a weight 1 for one specific tp^3_β type: the one $\bar{b}^*$ realizes concerning which we have a free choice. We are left with the cases $p' \pm \bar{b}, p' \pm p^*$; well, we know tp^3_β but we have to add tp^3_α. Use Claim 2.6(3) (and $(\mathrm{A1})_\alpha$ as we add a parameter). $\blacksquare_{2.9}$

2.10 Claim: *$\mathrm{tp}^3_\gamma[\binom{\bar{b}}{\bar{a}}, M], \mathrm{tp}^3_\gamma[\bar{a}, M], \mathrm{tp}^3_\gamma[M]$ are expressible by formulas in* $\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}$(d.q.).

By 2.9 we have

2.11 Conclusion: If $\mathrm{Dp}(T) < \infty$ then:
(1) From $\mathrm{tp}^3_\infty[\binom{B}{A}, M]$ we can compute $\mathrm{tp}_\infty[\binom{B}{A}, M]$ (the type from §1).
(2) Similarly, from $\mathrm{tp}^3_\infty[A, M]$ we can compute $\mathrm{tp}_\infty[(A), M]$.

From 2.6, 2.10, 2.11 and 1.30 we get

2.12 Corollary: *If $\gamma = \mathrm{Dp}(T)$ and M, N are $\aleph_\epsilon$-saturated, then*

$$M \cong N \Leftrightarrow \mathrm{tp}^3_\gamma[M] = \mathrm{tp}^3_\gamma[N] \Leftrightarrow M \cong_{\mathbb{L}^\gamma_{\infty,\aleph_\epsilon}(d.q.)} N.$$

Appendix

The following clarifies several issues raised by Baldwin. A consequence of

⊗ the existence of nice invariants for characterization up to isomorphism (or characterization of the models up to isomorphism by their $\mathcal{L}$-theory for suitable logic $\mathcal{L}$)

naturally give absoluteness, e.g., extending the universe, say, by nice forcing preserves non-isomorphism. So negative results for

(*) is non-isomorphism (of models of T) preserved by forcing by "nice forcing notions"?

implies that we cannot characterize models up to isomorphism by their $\mathcal{L}$-theory when the logic $\mathcal{L}$ is "nice", i.e., when $Th_{\mathcal{L}}(M)$ is preserved by nice forcing notions. So coding a stationary set by the isomorphism type can be interpreted as strong evidence of "no nice invariants"; see [Sh 220]. Baldwin, Laskowski and Shelah [BLSh 464] show that not only for every unsuperstable, but also for some quite trivial superstable (with NDOP, NOTOP) countable T, there are non-isomorphic models which can be made isomorphic by some ccc (even σ-centered) forcing notion. This shows that the lack of a really finite characterization is serious.

Can we still get from the characterization in this paper an absoluteness result? Note that for preserving $\aleph_\epsilon$-saturation (for simplicity, for models of countable T) we need to add no reals,[4] and in order not to erase distinction of dimensions we want not to collapse cardinals, so the following questions are natural, for a first order (countable) complete T:

$(*)^1_T$ Assume $v_1 \subseteq v_2$ are transitive models of ZFC with the same cardinals and reals, the theory $T \in V_1$. If the models M_1, M_2 are from v_1 and they are models of T not isomorphic in v_1, must they still be not isomorphic in V_2?[5]

$(*)^2_T$ Like $(*)^1_T$, we assume in addition $\mathcal{P}(|T|)^{V_1} = \mathcal{P}(|T|)^{V_2}$.

Of course, for countable T the answer is negative even for $\aleph_\epsilon$-saturated models except for superstable, NDOP, NOTOP theories, so we restrict ourselves to these. It should be quite transparent that $\mathbb{L}_{\infty,\aleph_\epsilon}(q.d.)$-theory is preserved from v_1 to v_2 (as well as the set of sentences in the logic), hence for the class of $\aleph_\epsilon$-saturated models (of superstable NDOP, NOTOP theory T) the answer to $(*)^2_T$ is: yes.

4 The set of $\{ac\ell(\bar{a}) : \bar{a} \in {}^{\omega>}M\}$ is absolute but the set of their enumeration and of the $\{f \restriction (ac\ell(\bar{a})) : f \in \mathrm{AUT}(\mathfrak{C}), f(\bar{a}) = \bar{a}\}$ is not.

5 Note we did not say they have the same ω-sequences of ordinals; e.g., if $V_2 = V_1^P$, P Prikry forcing, then the assumption of $(*)_T$ holds though a new ω-sequence of ordinals was added. So for $V_1 \subseteq V_2$ as in $(*)_T$, the $\mathcal{L}_{\infty,\aleph_1}$-theory is not necessarily preserved.

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