

Dependent theories and the generic pair conjecture

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We try to understand complete types over a somewhat saturated model of a complete first-order theory which is dependent (previously called NIP), by “decomposition theorems for such types”. Our thesis is that the picture of dependent theory is the combination of the one for stable theories and the one for the theory of dense linear order or trees (and first, we should try to understand the quite saturated case). As a measure of our progress, we give several applications considering some test questions; in particular, we try to prove the generic pair conjecture and do it for measurable cardinals.

Keywords: Model theory; first-order theories; classification theory; dependent theories; the generic pair conjecture.

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Annotated Content

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| 2 | The Type Decomposition Theorem [For κ -saturated $M \prec \mathfrak{C}$ and $\bar{d} \in \mathfrak{C}$ of length $< \theta^+$ we try to analyze the type $\text{tp}(\bar{d}, M)$ in two steps — pseudo-stable and tree-like one. This is the main aim of the section (and a major part of the paper). It is done | 19 |

by looking at K_ℓ and $\text{mxK}_{\lambda, \kappa, \theta}^\ell$. A consequence which fulfilled to some extent the aim is the Type Decomposition Theorem (Theorem 2.4). As a second consequence we give a characterization of “ M is exactly κ -saturated, $\kappa > \text{cf}(\kappa) > |T|$ ”, see Lemma 2.2. In fact, we deal a little with singular exact saturation per se. “Unfortunately” there are independent (complete first-order theories) T 's which have no model with singular exact saturation, see Example 2.23. But the existence of an indiscernible set for dependent T suffices (see Claim 2.26 under instances of G.C.H.) and has a neat characterization. Also, if p is a complete 1-type over a model M of T which is quite saturated then p has a spectrum in a suitable sense, see Claim 2.31.]

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 [For example, here complete type over a saturated model M of cardinality κ , a measurable cardinal, is analyzed. What we get is a better decomposition theorem (the strict one).]

4 Consequences of Strict Decomposition 46
 [We start by sufficient conditions for a sequence being indiscernible. For a measurable cardinal $\kappa (> |T|)$ we confirm the structure half of the generic pair conjecture. Toward this, if we have the consequences of Sec. 3 we can analyze generic pairs of models of T in κ . In a slightly simplified formulation this means: if $2^\kappa = \kappa^+$, ($\kappa = \kappa^{<\kappa} > |T|$), M_α , a model of T of cardinality κ for $\alpha < \kappa^+$ is \prec -increasing continuous, $M = \cup\{M_\alpha : \alpha < \kappa^+\}$ is κ^+ -saturated, then for a club E of κ^+ for all $\alpha < \beta$ belonging to $\{\delta \in E : \delta \text{ has cofinality } \kappa\}$ the pair (M_β, M_α) has the same isomorphism type. In fact for $\kappa_1 < \kappa_2$ we get $\mathbb{L}_{\infty, \kappa_1}(\tau_T)$ -equivalence, so we have a derived first-order theory. For the proof we show that an increasing (short) sequence of so-called strict (κ, θ) -decompositions has a limit.]

0. Introduction

We first give a page of introductory remarks for non-logicians to clarify notions and to motivate working on dependent theories. The classical center of model theory is investigating elementary classes, i.e. we fix a vocabulary τ (i.e. a set of predicates and function symbols), for a τ -structure M let $\text{Th}(M)$ be the set of first-order sentences which M satisfies, a complete first-order theory T is $\text{Th}(M)$ for some τ -model M . We fix T and $\tau = \tau_T$ and investigate τ -models of T , i.e. τ -structures M such that $T = \text{Th}(M)$; about other contexts, see, e.g., [15].

Let M, N denote such structures and they are called models (of T). Let \bar{a} , \bar{b} , \bar{c} , \bar{d} denote sequences of elements of such models and $\varphi(\bar{x})$ or $\varphi(\bar{x}, \bar{y})$ denote

members of $\mathbb{L}(\tau)$, i.e. the set of first-order formulas in this vocabulary but we allow \bar{x} to be infinite though the formula is finite so only finitely many x_ℓ, y_j are relevant.

Let $M \models \varphi[\bar{a}]$ mean that the model M satisfies the formula $\varphi(\bar{x})$ under the substitution $\bar{x} \mapsto \bar{a}$ (so \bar{a}, \bar{x} have the same length).

The right notion of sub-models is \prec , being elementary submodel where $M \prec N$ if and only if $M \subseteq N$ and for every $\varphi(\bar{x}) \in \mathbb{L}(\tau)$ and $\bar{a} \in {}^{\text{lg}(\bar{x})}M$ we have $M \models \varphi[\bar{a}]$ if and only if $N \models \varphi[\bar{a}]$.

Recall that an ordinal is the isomorphism type of a well ordering (which is a linear order for which every non-empty set has a first member). But we identify an ordinal with the set of smaller ordinals. Also a cardinal is an ordinal λ with no smaller ordinal of the same power. Here saying “ x is a cardinal” means “ x is an infinite cardinal” if not said otherwise. Let \aleph_α be the α th infinite cardinal and the cardinality $|\mathcal{U}|$ of a set \mathcal{U} is the minimal ordinal of the same power.

Let the successor λ^+ of a cardinal λ be $\aleph_{\alpha+1}$ when $\lambda = \aleph_\alpha$.

We say E is a closed subset of the limit cardinal γ when $E \subseteq \gamma$ and $\delta < \gamma \wedge \delta = \sup(\delta \cap E) \Rightarrow \delta \in E$ and E is called unbounded when $(\forall \alpha < \gamma) (\exists \beta)(\alpha \leq \beta \in E)$, “ E is a club of γ ” is the shorthand for “ E is a closed unbounded subset of γ ”.

For an ordinal α let $\text{cf}(\alpha) = \min\{|C| : C \text{ an unbounded subset of } \alpha\} = \min\{\text{otp}(C) : C \text{ a closed unbounded subset of } \alpha\}$; we say α is regular if $\alpha = \text{cf}(\alpha)$ is infinite (hence is a cardinal), now recall (see, e.g., [4]) that if α is a limit ordinal (e.g., a cardinal) then $\text{cf}(\alpha)$ is regular, and every cardinal of the form λ^+ is regular. When $\text{cf}(\delta) > \aleph_0$ we say “ $S \subseteq \delta$ is stationary” when $S \cap E \neq \emptyset$ for every club E of δ .

A central notion is type; for $A \subseteq M$ and \bar{a} a sequence from M let $\text{tp}(\bar{a}, A, M)$ be the set $\{\varphi(\bar{x}, \bar{b}) : \bar{\varphi}(\bar{x}, \bar{y}) \in \mathbb{L}(\tau), \bar{b}$ a sequence from A and $M \models \varphi[\bar{a}, \bar{b}]\}$. We may write a instead of $\langle a \rangle$.

Let

$$\mathbf{S}^\alpha(A, M) = \{\text{tp}(\bar{a}, A, N) : \text{for some } N, \bar{a} \text{ we have } M \prec N, \bar{a} \text{ a sequence of length } \alpha \text{ from } N\},$$

$$\mathbf{S}^\alpha(M) = \mathbf{S}^\alpha(M, M).$$

By this we can define another central notion. M is κ -saturated if and only if for every $A \subseteq M, |A| < \kappa$ and $p \in \mathbf{S}^1(A, M)$ some $a \in M$ realizes p in M which means $p = \text{tp}(a, A, M)$. We say the model M is saturated when it is κ -saturated and of cardinality κ for some κ . Let $\text{EC}_\lambda(T)$ be the class of models of T of cardinality λ .

It is classically known that for $\lambda \geq |T|$ (assuming $2^\lambda = \lambda^+$, mostly done here for transparency) there is a saturated member of $\text{EC}_{\lambda^+}(T)$, it is unique up to isomorphism, and the union of an \prec -increasing chain of saturated members of $\text{EC}_{\lambda^+}(T)$ of length λ^+ is a saturated member of $\text{EC}_{\lambda^+}(T)$. On the background so far, see, e.g., [1].

A major theme of the author's work is trying to find natural dividing lines (i.e. properties) in the family of first-order complete T , a criterion for natural is having both "inside definition" by formulas and "outside definition" by properties of the class of its models. That is, such a property is interesting as a dividing line when we have consequences for those with the property and for those without it; see, e.g., [15, Sec. (1A)].

A major such dividing line is " T is stable"; recall that T is stable if and only if $(*)^1_T$ if and only if $(*)^2_T$ where

- $(*)^1_T$ for some $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, model M of T and $\bar{a}_n \in {}^{(\ell g(\bar{x}))}M, \bar{b}_n \in {}^{(\ell g(\bar{y}))}M$ for $n < \omega$ we have $n < m \Leftrightarrow M \models \varphi[\bar{a}_n, \bar{b}_m]$,
- $(*)^2_T$ for every $\lambda \geq |T|$ and limit ordinal $\delta \leq \lambda$ of cofinality $> |T|$, the union of any \prec -increasing chain of length δ of saturated models of T of cardinality λ is saturated.

Another major dividing line is " T is superstable" which holds if and only if

- $(*)^3_T$ like $(*)^2_T$ allowing any limit ordinal δ .

On this and the relevant history, see, e.g., [12].

The property we deal with here is " T is dependent", also called " T is NIP", where its negation, " T is independent" or " T has the independence property" means

- $(*)^4_T$ there are $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, a model M of T and $\bar{a}_u \in {}^{(\ell g(\bar{x}))}M, \bar{b}_n \in {}^{(\ell g(\bar{y}))}M$ for $u \subseteq \omega, n < \omega$ such that $n \in u \Leftrightarrow M \models \varphi[\bar{a}_u, \bar{b}_n]$.

What is the motivation to investigate this dividing line? First, it has a nice, simple definition, parallel to the one for stable theories. Second, it is a much wider class than that of the stable theories; also, extremely important for many, whereas infinite fields with stable first-order complete theory are hard to come by (algebraically closed and separably closed are the only known ones), there are many important fields with dependent first-order complete theory (the p -adics and many of the power series fields). Third, there are some results on it indicating it is not unreasonable to hope there is a rich theory on it to be discovered.

On history and background on dependent theories, see [13, 17].

Let T be a fixed first-order complete theory. For transparency, till Context 0.1, we assume G.C.H., i.e. $2^\kappa = \kappa^+$ for every infinite cardinal κ and consider only λ regular $> |T|$. Let $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ be an \prec -increasing continuous sequence of models of T of cardinality λ with M being saturated where $M := \cup\{M_\alpha : \alpha < \lambda^+\}$. Now M is unique (up to isomorphism, for each λ) and though for a given M, \bar{M} is not unique, for any two such sequences \bar{M}', \bar{M}'' there is a closed unbounded subset E of λ^+ and isomorphism f from $M' = \cup\{M'_\alpha : \alpha < \lambda^+\}$ onto $M'' = \cup\{M''_\alpha : \alpha < \lambda^+\}$ such that f maps M'_δ onto M''_δ for every $\delta \in E$.

So it is natural to ask ($\lambda > |T|$ regular and E varies on closed unbounded subsets of λ^+)

⊙₁ what^a is $\mathbf{n}_\lambda(T) := \text{Min}_E |\{M_\delta / \cong : \delta \in E\}|$? where M_δ / \cong is the isomorphism type of M_δ . When is $\mathbf{n}_\lambda(T)$ equal to one?

Now (see [19]):

⊙₂ $\mathbf{n}_\lambda(T) = 1$ if and only if T is superstable,

⊙₃ for countable T , $\mathbf{n}_\lambda(T) = 2$ if and only if T is strictly stable (i.e. T is stable, not superstable),

⊙₄ given an ordinal γ , for λ large enough $\mathbf{n}_\lambda(T) = |\gamma + 1|$ if T is stable and $\kappa(T) = \aleph_\gamma$ (recalling that for a stable T , $\kappa(T)$ is cardinal $\leq |T|^+$, so for countable T it is \aleph_0 or \aleph_1),

⊙₅ if T is unstable, $\lambda = \aleph_\gamma$ then $\mathbf{n}_\lambda(T) \geq |\gamma + 1|$.

[Why? Because for some closed unbounded subset E of λ^+ , if $\delta \in E$ then M_δ is $\text{cf}(\delta)$ -saturated but not $(\text{cf}(\delta))^+$ -saturated hence $[\delta_1, \delta_2 \in E \wedge \text{cf}(\delta_1) \neq \text{cf}(\delta_2)] \Rightarrow M_{\delta_1} \not\cong M_{\delta_2}$.]

Hence it is natural to replace $\mathbf{n}_\lambda(T)$ by:

⊙₆ let $\mathbf{n}_{\lambda, \kappa}(T) = \text{Min}_E |\{M_\delta / \cong : \delta \in E \text{ and } \text{cf}(\delta) = \kappa\}|$ when $\lambda > \kappa = \text{cf}(\kappa)$, (as above E varies on the clubs of λ^+).

Below we use $\mathbf{n}_{\lambda, \kappa}(T)$ only when $\lambda = \text{cf}(\lambda) > |T| + \kappa \wedge \kappa = \text{cf}(\kappa)$ and remember that for simplicity we are assuming G.C.H.

Now (see [19]):

⊙₇ if T is stable then $\mathbf{n}_{\lambda, \kappa}(T) = 1$.

It is natural to ask whether this characterizes stable theories. The answer is no, in fact, by an example everyone knows (by [22, §1]):

⊙₈ $\mathbf{n}_{\lambda, \kappa}(T) = 1$ for $T = \text{Th}(\mathbb{Q}, <)$, the theory of dense linear orders with neither first nor last element, so $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$.

During the proof we analyze $p \in \mathbf{S}(M_\alpha)$, M_α saturated, of course, only when $p \neq \text{tp}(a, M_\alpha, M_\alpha)$ for $a \in M_\alpha$. So M_α is a linear order and p induces a cut (C_p^-, C_p^+) of M_α , i.e. $C_p^- = \{a \in M_\alpha : (a < x) \in p\}$ is an initial segment of M_α and its complement, $\{a \in M_\alpha : (a < x) \notin p\}$ is an end segment. This gives a pair of cofinalities, (μ_p^-, μ_p^+) , μ_p^- the cofinality of the linear order C_p^- and μ_p^+ the cofinality of the inverse of C_p^+ .

^aWe can present the problem differently, about the existence of (variations of) (λ, κ) -limit models (so $2^\lambda = \lambda^+$ is no longer necessary, by forcing this is equivalent). Also, instead of the function \mathbf{n} getting the value λ^+ we can consider saying for some club no two relevant cases are isomorphic. This does not make a real difference but we find the present choice has more transparent presentation.

Now

- (*)_{8.1} if $\mu_p := \min\{\mu_p^-, \mu_p^+\} < \lambda$, then the type is determined by any subset of M_α of cardinality μ_p such that:
 - the set is unbounded in C_p^- if $\mu_p = \mu_p^-$ and
 - the set is unbounded from below in C_p^+ if $\mu_p = \mu_p^+$;
- (*)_{8.2} if $\mu_p = \lambda$, and we expand M_α by the (unary) relation C_p^- , we still get a saturated model.

Next considering $\odot_7 + \odot_8$ you may think that for every T we get $\mathbf{n}_{\lambda,\kappa}(T) = 1$, but ([22, 2.3(2)] implies directly that):

\odot_9 $\mathbf{n}_{\lambda,\kappa}(T) = \lambda^+$ if T is Peano arithmetic

moreover, this holds for quite many theories T (by [22, Sec. 2]):

\odot_{10} $\mathbf{n}_{\lambda,\kappa}(T) = \lambda^+$ if T has the strong independence property (see [9], i.e. for some first-order formula $\varphi(x, y)$, $\langle \varphi(M, a) : a \in M \rangle$ is an independent sequence of subsets of M , see Definition 0.6).

For the author this rings a bell and strengthens a suspicion — maybe the dividing line is T independent/ T dependent, indeed (by [22, Sec. 2]):

\odot_{11} $\mathbf{n}_{\lambda,\kappa}(T) = \lambda^+$ if T is independent, λ a successor cardinal.

We try here to address the complement, the structure side. This calls for analyzing appropriate \prec -increasing continuous sequence $\bar{M} = \langle M_i : i \leq \kappa \rangle$ of models of T of cardinality λ . Clearly in the relevant cases they “increase fast enough” and M_i is saturated for i non-limit. Now among such sequences, is it not reasonable to first deal with the case of length 2?

This leads to the generic pair conjecture which says that for $\lambda = \lambda^{<\lambda} > |T|$, we have T is independent if and only if $\mathbf{n}_{\lambda,2}(T) = \lambda^+$ where:

\odot_{12} $\mathbf{n}_{\lambda,2}^*(T) := \text{Min}_E\{(M_\beta, M_\alpha) / \cong : \alpha < \beta \text{ belongs to } E \text{ and } \text{cf}(\alpha) = \lambda = \text{cf}(\beta)\}$.

Note that in defining $\mathbf{n}_{\lambda,\kappa}(T)$, $\kappa \in \text{Reg} \cap [\aleph_0, \lambda]$ we speak on models of T , i.e. $\delta \in E$, $\text{cf}(\delta) = \kappa$ whereas here we deal with pairs of models. However, to analyze M_δ for $\delta \in E \wedge \text{cf}(\delta) = \kappa$, E a small enough club of λ^+ , it is natural to assume $\delta = \sup\{\alpha \in E : \text{cf}(\alpha) = \lambda \text{ and } \alpha < \delta\}$ and choose $\bar{\alpha} \in \text{Seq}_{E,\kappa,\delta}$ which means $\bar{\alpha}$ is an increasing continuous sequence $\langle \alpha_i : i < \kappa \rangle$ of ordinals with limit δ such that $i < \kappa$ non-limit $\Rightarrow \text{cf}(\alpha_i) = \lambda$. So a sufficient condition for $\mathbf{n}_{\lambda,\kappa}(T) = 1$ is $\mathbf{n}_{\lambda,\kappa}^*(T) = 1$ where $\mathbf{n}_{\lambda,\kappa}^* = \text{Min}_E\{M_{\bar{\alpha}} / \cong : \bar{\alpha} \in \text{Seq}_{E,\kappa,\delta}\}$, E varying on the clubs of λ^+ . Now though it is not clear if this is also a necessary condition it seems more approachable and is natural. Anyhow it seems reasonable to consider $\mathbf{n}_{\lambda,2}^*(T) = 1$, i.e. the generic pair conjecture.

This connects us to the long term goal of classifying first-order theories by “good” dividing lines, ones in which we find outside properties (like here investigating $\mathbf{n}_{\lambda,\kappa}(T)$ or just $\mathbf{n}_{\lambda,\lambda}(T)$, trying to characterize it) with “inside” definitions (like being dependent), and developing an inside theory; here — looking at decomposition (in Sec. 1 decompositions of models, in Sec. 2 decomposition of types, in Secs. 3, and 4 strict decomposition of types). Moreover, for this we have to analyze types. In Sec. 1 we make a first attempt; more exactly see Claim 1.8 and Theorem 1.9. We try to analyze a model $N := M_\beta$ over^b $M := M_\alpha$ by trying to find models M_1, N_1 such that:

- ⊞₁ $M_\alpha = M \prec M_1 \prec N_1$ and $M_\alpha = M \prec N = M_\beta \prec N_1$,
- ⊞₂ for every $\bar{a} \in {}^{\omega>}(M_1)$ for some $B_\alpha \in [M]^{<\lambda}$ the type $\text{tp}(\bar{a}, M, M_1)$ is definable over B_α in a weak sense, i.e. does not split over B_α , this means that if $n < \omega$ and $\bar{b}, \bar{c} \in {}^n(M_\alpha)$ realize the same type over B_α then so does $\bar{a} \hat{\ } \bar{b}, \bar{a} \hat{\ } \bar{c}$ (this is parallel to $(*)_{8.1}$ from \odot_8); it follows that for any sequence $\bar{a} \in {}^{\kappa>}(M_1)$ a similar statement holds,
- ⊞₃ $\text{tp}(N_1, M_1, N_1)$ is weakly orthogonal to every $q \in \mathbf{S}^{<\omega}(M_1)$ which does not split over some $B \in [M_1]^{<\lambda}$; the weakly orthogonal means that q has a unique extension in $\mathbf{S}^n(N_1)$ wherever $q \in \mathbf{S}^n(M_1)$.

In Sec. 2 we try to analyze a type rather than a pair of models, also we find it better to deal with θ -types, $\theta \geq |T|$, as during the analysis we add more variables. So for a κ -saturated model $M \prec \mathfrak{C}$ and sequence \bar{d} of length $< \theta^+$ we try to analyze $\text{tp}(\bar{d}, M, \mathfrak{C})$ in two steps. The first is to add \bar{c} of length $< \theta^+$ such that

- ⊞₄ $\text{tp}(\bar{c}, M, \mathfrak{C})$ does not split over some $B \subseteq M \prec \mathfrak{C}$ of cardinality $< \kappa$.

This corresponds to the stable type (“unfortunately” but unavoidably depending on κ), so for the theory of dense linear orders it corresponds to types $p \in \mathbf{S}(M)$ with $\mu_p < \kappa$, see $(*)_{8.1}$ above. True, they are not really definable, but non-splitting is a weak form of being definable. The second step is

- ⊞₅ $\text{tp}(\bar{d}, M + \bar{c}, \mathfrak{C})$ is tree like, i.e. if $A \subseteq M \prec \mathfrak{C}$ and $|A| < \kappa$ then for some $\bar{e} \in {}^{\theta^+}M$ we have $\text{tp}(\bar{d}, \bar{c} + \bar{e}) \vdash \text{tp}(\bar{d}, A + c)$.

This property holds for $T = \text{Th}(\mathbb{Q}, <), p \in \mathbf{S}(M)$ when $\mu_p \geq \kappa!$, i.e. when both cofinalities are $\geq \kappa$. This is the Type Decomposition Theorem (Theorem 2.4).

A consequence is some clarification of models of M of a dependent theory which are exactly κ -saturated for singular κ . We deal with this question to some extent per se.

In Sec. 3, we get a better decomposition — strict decomposition. But at present with a price, assuming, e.g., $\kappa = \|M\|$ is a measurable cardinal. The main point appears in Sec. 4, the existence of limits of increasing sequences of strict decompositions.

^bPedantically, when $1 < \alpha < \beta$.

Using this we are able to prove the pair genericity conjecture, the structure side for the case of a measurable cardinal. The measurability assumption seems undesirable. Describing this to Udi Hrushovski he was more concerned about also having the non-structure side for independent T . Now at the time in [22] it was remarked that a similar proof should work for the strongly inaccessible, but the author was not motivated enough to really look into it. Subsequently, [18] completes it.

The order of the sections is by their conceptions, so there are some repetitions. In [7] Kaplan and the author start to continue this work as well as (with Lavi) in [24]. The author continues it in [20] concentrating on saturated models but it works just as well for special models (in singular strong limit cardinals, see, e.g., [1]).

Context 0.1. (1) T is complete first-order theory.

(2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster model for T , omitting T when no confusion arises; i.e. $\bar{\kappa}$ is a large enough cardinal, \mathfrak{C} is a $\bar{\kappa}$ -saturated model such that we deal only with models $M \prec \mathfrak{C}$, sets $A \subseteq \mathfrak{C}$ of cardinality $< \bar{\kappa}$ and sequences $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}$ from ${}^\alpha \mathfrak{C}$ for some $\alpha < \bar{\kappa}$. So $\text{tp}(\bar{c}, A)$ means $\text{tp}(\bar{c}, A, \mathfrak{C})$.

(3) We may not pedantically distinguish a model M and its universe, the cardinality $\|M\|$ of M is that of its universe.

Notation 0.2. (1) For $M \prec \mathfrak{C}$ and $\bar{a} \in {}^\alpha M$ or just $\bar{a} \in {}^\alpha \mathfrak{C}$ let $M_{[\bar{a}]}$ be the expansion of M by every relation $R_{\varphi(\bar{x}, \bar{a})} = \varphi(M, \bar{a})$ where $\varphi(M, \bar{a}) := \{\bar{b} \in {}^{\ell g(\bar{x})} M : \mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$ for $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ such that $\ell g(\bar{y}) = \alpha, \ell g(\bar{x}) < \omega$ or pedantically $\varphi(\bar{x}, \bar{y} \upharpoonright u)$ for \bar{x}, \bar{y} as above, $u \subseteq \alpha$ finite. We define $M_{[A]}$ similarly, i.e. as the expansion of M by $R_{\varphi(\bar{x}, \bar{a})} = \varphi(M, \bar{a})$ for every $\bar{a} \in {}^{\ell g(\bar{y})} A$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$.

(1A) For $p(\bar{x}) \in \mathbf{S}^\alpha(M)$ let $M_{[p]}$ be $M_{[\bar{a}]}$ whenever $\bar{a} \in {}^\alpha \mathfrak{C}$ realizes $p(\bar{x})$.

(1B) We say the sequence $\langle \varphi_s(\bar{x}, \bar{a}_s) : s \in I \rangle$ of formulas from $\mathbb{L}(\tau_M)$ with \bar{a}_s from M is independent in the model M when every finite non-trivial Boolean combination of sets from $\varphi_s(M, \bar{a}_s)$ is non-empty.

(2) Writing $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$, φ here is always first-order but \bar{x} and \bar{y} may be infinite, though sometimes are finite (said or clear from the context). Let $p(\bar{x}), q(\bar{x}), r(\bar{x})$ denote types over some $A \subseteq \mathfrak{C}$, i.e. set of formulas of the form $\varphi(\bar{x}, \bar{b}), \bar{b} \in ({}^{\ell g(\bar{b})} A)$.

(3) $\text{EC}_\lambda(T)$ is the class of models M of T (so $M \prec \mathfrak{C}$) of cardinality λ and $\text{EC}_{\lambda, \kappa}(T)$ is the class of κ -saturated $M \in \text{EC}_\lambda(T)$.

(4) $A + \bar{c}$ is $A \cup \text{Rang}(\bar{c})$, etc.

(5) Let $\text{tp}(A, B)$ be $\text{tp}(\bar{a}, B)$ where \bar{a} is the identity function on A .

Definition 0.3. (1) If $\bar{a}_t \in {}^\gamma \mathfrak{C}$ for $t \in I$ and \mathcal{D} is a filter on I and $\bar{x} = \langle x_i : i < \gamma \rangle$ and $A \subseteq \mathfrak{C}$ then $\text{Av}(\langle \bar{a}_t : t \in I \rangle / \mathcal{D}, A) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^{>\omega} A \text{ and the set } \{t \in I : \mathfrak{C} \models \varphi[\bar{a}_t, \bar{b}]\} \text{ belongs to } \mathcal{D}\}$. Note that if T is dependent, I is a linear order with no last members and $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence, see below then the result $\in \mathbf{S}^\gamma(A)$. Also note that if \mathcal{D} is an ultrafilter on I then $\text{Av}(\langle \bar{a}_t : t \in I \rangle / \mathcal{D}, A)$ belongs to $\mathbf{S}^\gamma(A)$.

- (1A) Recall that if \mathcal{D} is a filter on $\{\bar{a}_t : t \in I\} \subseteq {}^\alpha\mathfrak{C}$ and $A \subseteq \mathfrak{C}$ we define $\text{Av}(\mathcal{D}, A)$ similarly and if I a linear order and \mathcal{D} is the filter of co-bounded subsets of I we may omit it.
- (2) If $p(\bar{x}), q(\bar{y})$ are complete types over A we say $p(\bar{x}), q(\bar{y})$ are weakly orthogonal *when* for every \bar{a}_1, \bar{a}_2 realizing $p(\bar{x})$ and \bar{b}_1, \bar{b}_2 realizing $q(\bar{y})$ we have $\text{tp}(\bar{a}_1 \hat{\ } \bar{b}_1, A) = \text{tp}(\bar{a}_2 \hat{\ } \bar{b}_2, A)$.
- (3) For a linear order I , $\langle \bar{a}_s : s \in I \rangle$ is an indiscernible sequence over B *when*: $\ell g(\bar{a}_s)$ is constant and if $s_0 <_I \dots <_I s_{n-1}$ and $t_0 <_I \dots <_I t_{n-1}$ then the sequences $\bar{a}_{s_0} \hat{\ } \dots \hat{\ } \bar{a}_{s_{n-1}}$ and $\bar{a}_{t_0} \hat{\ } \dots \hat{\ } \bar{a}_{t_{n-1}}$ realize the same type over B .

Recall also (see [12, Chap. II, Sec. 4]) the following fact.

Fact 0.4. If T is dependent *then* for any formula $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ there is $n = n_\varphi < \omega$ (depending on T) such that:

- (a) for no $\bar{c} \in {}^{\ell g(\bar{z})}\mathfrak{C}$ and $\bar{b}_i \in {}^{\ell g(\bar{y})}\mathfrak{C}$ for $i < n$, is the sequence $\langle \varphi(\bar{x}, \bar{b}_i, \bar{c}) : i < n \rangle$ independent, i.e. every non-trivial Boolean combination of the sets $\varphi(M, \bar{b}_i, \bar{c}) = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}, \bar{b}_i, \bar{c}]\}$ for $i < n$ is non-empty.
- (b) If $\langle \bar{b}_i : i < n \rangle$ is an indiscernible sequence over C , $\ell g(\bar{b}_i) = \ell g(\bar{y})$, $\bar{c} \in {}^{\ell g(\bar{z})}C$ (all in \mathfrak{C}) then for no $\bar{a} \in {}^{\ell g(\bar{a})}M$ do we have $\mathfrak{C} \models \varphi[\bar{a}, \bar{b}_i, \bar{c}]^{\text{if } (\ell \text{ even})}$ for $\ell < n$.
- (c) Also there is a finite $\Delta_\varphi \subseteq \mathbb{L}(\tau_T)$ such that in clause (b) it is enough to demand that $\langle \bar{b}_i : i < n \rangle$ is a Δ_φ -indiscernible sequence.

Lastly, we quote Erdős–Rado [3].

Fact 0.5. The Δ -System Lemma for finite sets.

For every natural numbers k, n there is a natural number m such that: if u_i is a finite set with $\leq k$ elements for $i < m$ *then* there are sets $w \subseteq \{0, \dots, m-1\}$ with $|w| = n$ and u_* such that $\langle u_i : i \in w \rangle$ is a Δ -system with heart u_* , which means that $i \neq j \in w \Rightarrow u_i \cap u_j = u_*$.

- Definition 0.6.** (1) A partial order I is κ -directed *when* every set $J \subseteq I$ of cardinality $< \kappa$ has an upper bound $t \in I$ which means that $(\forall s)(s \in J \Rightarrow s \leq_I t)$.
- (2) A sequence $\langle A_s : s \in I \rangle$ is an independent sequence of subsets of A_* *when* ($A_s \subseteq A_*$ for $s \in I$ and) $\bigcap_{s \in u} A_s \setminus \bigcup_{t \in v} A_t$ is non-empty for every disjoint finite $u, v \subseteq I$.

1. Non-Splitting Constructions

On such constructions including $\mathbf{F}_\kappa^{\text{nsp}}$ see [12, Chap. IV, Secs. 1 and 3] but \mathbf{F}^{nsp} here is \mathbf{F}^p there; and see [13, 4.23–4.26], however this section is self-contained.

We try here to analyze κ -saturated models $M \prec N$, e.g., by finding M_1, N_1 such that $M \prec M_1 \prec N_1, N \prec N_1$ where M_1 is $\mathbf{F}_\kappa^{\text{nsp}}$ -constructible over M , see below and $\text{tp}(N_1, M_1)$ is weakly orthogonal to any type over M_1 realized in some $\mathbf{F}_\kappa^{\text{nsp}}$ -construction over it, see Theorem 1.9(B) noting that M, N, N_1, M_1 here stand

for A, A^+, M, N there. We first recall the definition of non-splitting and some of its properties.

Definition 1.1. We say $p(\bar{x})$ does not split over A when: if $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p(\bar{x})$ then $\text{tp}(\bar{b}, A) \neq \text{tp}(\bar{c}, A)$.

Fact 1.2. (1) If $\otimes_{A,B,C}$ below holds and $p(\bar{x}) \in \mathbf{S}^m(B)$ does not split over A , then there is one and only one $q(\bar{x}) \in \mathbf{S}^m(C)$ extending $p(\bar{x})$ and not splitting over A (also called the non-splitting extension of $p(\bar{x})$ over C), where:

- $\otimes_{A,B,C}$ (a) $A \subseteq B \subseteq C$,
- (b) for every $\bar{c} \in {}^\omega C$ there is $\bar{b} \in {}^{\ell g(\bar{c})} B$ realizing $\text{tp}(\bar{c}, A)$.

- (2) Let I be a linear order. If $\text{tp}(\bar{a}_t, B \cup \bigcup\{\bar{a}_s : s <_I t\})$ does not split over B and increases with $t \in I$ then $\langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence over B .
- (3) If $\text{tp}(\bar{a}, B)$ does not split over A , the sequence $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over A and $\bar{b}_t \subseteq B$ for $t \in I$ then $\langle \bar{b}_t : t \in I \rangle$ is an indiscernible sequence over $A \cup \bar{a}$.
- (4) If $A \subseteq B$ then the number of $p \in \mathbf{S}^\theta(B)$ which does not split over A is $\leq 2^{2^{|A|+|T|+\theta}}$, moreover if T is dependent the number is $\leq 2^{|A|+|T|+\theta}$.
- (5) If $A \subseteq B$ and $p \in \mathbf{S}^\alpha(B)$ is finitely satisfiable in A then p does not split over A .

Proof. (1) By [8] or see [13] or see [16, 1.10] for uniqueness.

(2) By [12, I] or [11, I] or [16, 3.2].

(3) By the definitions.

(4) The first conclusion is easy and see [8] or [16, Sec. 1], the second holds by [17, 5.26].

(5) Easy, too. □_{1.2}

Fact 1.3. (Assume T is dependent.) If $p(\bar{x})$ is an α -type over $B \subseteq A$ then we can find $q(\bar{x}) \in \mathbf{S}^\alpha(A)$ extending $p(\bar{x})$ such that for some $C \subseteq A$ of cardinality $\leq |T| + |\alpha|$ the type $q(\bar{x})$ does not split over $B \cup C$.

Proof. [12, III, 7.5, p. 140] or see [13, 4.24]. □_{1.3}

Observation 1.4. For κ regular.

- (1) If $A \subseteq B, |A| < \kappa$ and $\bar{a} \in {}^{\kappa} \mathfrak{C}$ and $\text{tp}(\bar{a}, B)$ is finitely satisfiable in A then it does not split over A .
- (2) If $A \subseteq B, \bar{c} \in {}^{\kappa} \mathfrak{C}$ and $\text{tp}(\bar{c}, B)$ does not split over A and $i < \ell g(\bar{c})$ then $\text{tp}(c_i, B \cup \{c_j : j < i\})$ does not split over $A \cup \{c_j : j < i\}$. Similarly for $\langle \bar{c}_j : j < i \rangle$ when $j \leq i \Rightarrow \text{Rang}(\bar{c}_j) \subseteq \text{Rang}(\bar{c})$.
- (3) If $\text{tp}(\bar{c}_k, B + \bar{c}_0 + \dots + \bar{c}_{k-1})$ does not split over $A \subseteq B$ for $k < n$ then $\text{tp}(\bar{c}_0 \hat{\ } \dots \hat{\ } \bar{c}_{n-1}, B)$ does not split over A .
- (4) If $A \subseteq A_1 \subseteq B_1 \subseteq B$ and $\text{Rang}(\bar{c}_1) \subseteq \text{Rang}(\bar{c})$ and $\text{tp}(\bar{c}, B)$ does not split over A then $\text{tp}(\bar{c}_1, B_1)$ does not split over A_1 .

- (5) If $\bar{c} \in {}^{\kappa} \mathfrak{C}$ and for every finite $u \subseteq \text{lg}(\bar{c})$ and finite $B_1 \subseteq B$ the type $\text{tp}(\bar{c} \upharpoonright u, B_1)$ does not split over A then $\text{tp}(\bar{c}, B)$ does not split over A .

Proof. Easy (or see [12, Chap. IV]).

□_{1.4}

As in [12, IV].

Definition 1.5. (1) \mathcal{A} is an $\mathbf{F}_{\kappa}^{\text{nsp}}$ -construction when:

- (a) $\mathcal{A} = (A, \bar{a}, \bar{B}, \bar{A}, \alpha) = (A^{\mathcal{A}}, \bar{a}^{\mathcal{A}}, \bar{B}^{\mathcal{A}}, \bar{A}^{\mathcal{A}}, \alpha^{\mathcal{A}})$,
- (b) $\bar{a} = \langle a_{\beta} : \beta < \alpha \rangle = \langle a_{\beta}^{\mathcal{A}} : \beta < \alpha \rangle$,
- (c) $\bar{B} = \langle B_{\beta} : \beta < \alpha \rangle = \langle B_{\beta}^{\mathcal{A}} : \beta < \alpha \rangle$,
- (d) $\bar{A} = \langle A_{\beta} : \beta \leq \alpha \rangle = \langle A_{\beta}^{\mathcal{A}} : \beta \leq \alpha \rangle$,
- (e) $A_{\beta} = A \cup \{a_{\gamma} : \gamma < \beta\}$,
- (f) $B_{\beta} \subseteq A_{\beta}$ and $|B_{\beta}| < \kappa$,
- (g) $\text{tp}(a_{\beta}, A_{\beta})$ does not split over B_{β} .

- (2) We let $\text{lg}(\mathcal{A}) = \alpha^{\mathcal{A}}$ and writing \mathcal{A} we may omit $\bar{A}^{\mathcal{A}}, \alpha^{\mathcal{A}}$ as they are determined by the others so may write $\mathcal{A} = (A, \bar{a}, \bar{B})$ or $\mathcal{A} = (A, \langle (a_{\beta}, B_{\beta}) : \beta < \alpha \rangle)$. We may replace a_{β} by a finite sequence \bar{a}_{β} with no real change.

- (3) We say the $\mathbf{F}_{\kappa}^{\text{nsp}}$ -construction \mathcal{A} is μ -full when $\text{cf}(\text{lg}(\mathcal{A})) \geq \kappa$ and if $q \in \mathbf{S}(A_{\text{lg}(\mathcal{A})}^{\mathcal{A}})$ does not split over B where $B \subseteq A_{\text{lg}(\mathcal{A})}^{\mathcal{A}}$ has cardinality $< \kappa$, then $\{\beta < \text{lg}(\mathcal{A}) : a_{\beta} \text{ realizes } q \upharpoonright A_{\beta}^{\mathcal{A}} \text{ and } B \subseteq A_{\beta}^{\mathcal{A}}\}$ is unbounded in $\alpha^{\mathcal{A}}$ and has order type divisible by μ .

- (4) We say C is $\mathbf{F}_{\kappa}^{\text{nsp}}$ -constructible over A when there is an $\mathbf{F}_{\kappa}^{\text{nsp}}$ -construction \mathcal{A} such that $A = A^{\mathcal{A}} = A_0^{\mathcal{A}}$ and $C = A_{\text{lg}(\mathcal{A})}^{\mathcal{A}}$.

Definition 1.6. (1) Let $A \leq_{\kappa} C$ mean that C is $\mathbf{F}_{\kappa}^{\text{nsp}}$ -constructible over A .

- (2) We say that (A^+, A) is κ -reduced when: if $A \leq_{\kappa} C$ and $\bar{c} \in {}^{\kappa} (A^+)$ then $\text{tp}(\bar{c}, A)$ has a unique extension to a complete type over C .

- (3) We say the (N, M) is κ -nice when:

- (a) (N, M) is κ -reduced and $M \prec N$,
- (b) M is κ -saturated,
- (c) N is κ -saturated,
- (d) if $M \leq_{\kappa} M^+$ then $M_{[N]} \prec M_{[N]}^+$, see below.

- (3A) Recall $M_{[B]}$ is M expanded by $R_{\varphi(\bar{x}, \bar{a})} = \{\bar{b} \in {}^{\text{lg}(\bar{x})} M : \mathfrak{C} \models \varphi[\bar{b}, \bar{a}]\}$ for $\varphi(\bar{x}, \bar{y}) \in \mathbf{L}(\tau_T)$ (with \bar{x} finite of course), $\bar{a} \in {}^{\text{lg}(\bar{y})} N$ and recall $\text{Th}(M_{[B]})$ is dependent by [17, Sec. 1].

- (4) We say that (M, A) is pseudo- κ -reduced when: if $\bar{c} \in {}^{\omega} A$, $\|M_1\| < \kappa$, $M_1 \subseteq M$, $q(\bar{x}) \in \mathbf{S}^{<\omega}(M)$ is finitely satisfiable in M_1 then $q(\bar{x})$, $\text{tp}(\bar{c}, M)$ are weakly orthogonal.

Observation 1.7. For κ regular:

- (1) \leq_{κ} is a partial order.
- (2) If $\langle A_i : i \leq \alpha \rangle$ is increasing continuous and $i < \alpha \Rightarrow A_i \leq_{\kappa} A_{i+1}$ then $A_0 \leq_{\kappa} A_{\alpha}$.

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- (3) In Definition 1.6(2) it is enough to consider $\bar{c} \in {}^{\omega>}(C \setminus A)$.
- (4) If $A \leq_{\kappa} B$ and $\bar{c} \in {}^{\kappa>}B$ then $\text{tp}(\bar{c}, A)$ does not split over some $A' \subseteq A$ of cardinality $< \kappa$.
- (5) If the pair (M, C) is κ -reduced then (M, C) is pseudo- κ -reduced.
- (6) If $\text{tp}(\bar{a}, A)$ does not split over B and $B \subseteq A$ has cardinality $< \kappa$ then $A \leq_{\kappa} A + \bar{a}$.

Proof. Easy; e.g., part (6) by Observation 1.4(2) and part (4) by Fact 1.3((3),(4)) and part (5) by Fact 1.2(5). □_{1.7}

- Claim 1.8 (T is dependent and $\kappa = \text{cf}(\kappa) > |T|$).** (1) For every A there is a κ -saturated C such that $A \leq_{\kappa} C$ and $|C| \leq (|A| + |T|)^{<\kappa}$.
- (2) If in addition $\mu \leq (|A| + |T|)^{<\kappa}$ then we can add C is “ μ -full κ -saturated”; clearer if $|C| \leq (|A| + |T|)^{<\kappa} + 2^{2^{\kappa}}$.

Proof. (1) By Fact 1.3 + Observation 1.7(2) and Observation 1.7(6).
 (2) The proof is similar (by Fact 1.2(4)). □_{1.8}

Now we arrive to the first result giving a decomposition. The type $\text{tp}(A^+, A)$ is decomposed in Theorem 1.9 by finding M such that $A \leq_{\kappa} M$ (so the complete types over A realized in M are somewhat definable) and (A^+, M) is κ -reduced, so the type $\text{tp}(A^+, M)$ is weakly orthogonal to types in $\mathbf{S}^{<\omega}(M)$ not splitting over subsets of M of cardinality $< \kappa$.

Theorem 1.9 (The Density of Reduced Pairs Theorem [T dependent]).
 For any $A \subseteq A^+$ and $\kappa = \text{cf}(\kappa) > |T|$ and λ satisfying^d $\theta < \kappa \Rightarrow \lambda = \lambda^{\theta} \geq |A^+|$,

- (A) we can find M such that M is a model of cardinality λ satisfying $A \leq_{\kappa} M$ and (A^+, M) is κ -reduced,
- (B) for some M as in clause (A) and N the pair (N, M) is κ -reduced and even κ -nice and $A^+ \subseteq N$.

Proof. *Proof of (A):* Our intention is to try to do a construction as described in \boxtimes below. Having carried the induction the proof is divided to two cases. In the first we get the desired conclusion. In the second, we get a contradiction to T being dependent; formally to the maximality of the k chosen in $(g)(\beta)$ of \boxtimes .

We choose M_i, B_i, j_i, \bar{c}_i by induction on $i \leq \lambda^+$ such that

- \boxtimes (a) M_i is \prec -increasing continuous, M_i of cardinality $\leq \lambda + |i|$
- (b) $j_i \leq i, B_i \prec M_i, \|B_i\| < \kappa,$
- (c) M_0 is $\mathbf{F}_{\kappa}^{\text{nspp}}$ -constructible over $A,$
- (d) M_{i+1} is $\mathbf{F}_{\kappa}^{\text{nspp}}$ -constructible over M_i and M_{i+1} is κ -saturated,
- (e) $\bar{c}_i \in {}^{\omega>}(M_{i+1})$ and $B_i \subseteq M_{j_i}$ has cardinality $< \kappa,$

^cWe could have chosen this as the definition. This changes the places we really need “ κ regular”.

^dNo real harm if we replace “ $\theta < \lambda \Rightarrow \lambda = \lambda^{\theta} \geq |A^+|$ ” by $\theta < \kappa \Rightarrow \lambda = \lambda^{\theta} + 2^{2^{\theta+|T|}} \geq |A^+|$ and then we can use only the first version of Fact 1.2(4).

- (f) $\text{tp}(\bar{c}_i, M_i)$ does not split over B_i ,
 (g) if $\text{cf}(i) \geq \kappa$ and subclause (α) below holds then subclause (β) holds where:

Subclause (α) : There are $j < i, m < \omega, B \subseteq M_j$ of cardinality $< \kappa$ and $p(\bar{x}) \in \mathbf{S}^m(M_i)$ which does not split over B and $p(\bar{x})$ has ≥ 2 extensions in $\mathbf{S}^m(M_i \cup A^+)$.

Subclause (β) : There are^e $m = m_i < \omega, k = k_i < \omega$ and $\varphi(\bar{x}, \bar{y}) = \varphi_i(\bar{x}_i, \bar{y}_i) \in \mathbb{L}(\tau_T)$ with \bar{x}, \bar{y} finite and $\bar{b} = \bar{b}_i \in {}^{(\ell g(\bar{y}))}(M_j \cup A^+)$ and $\varepsilon_0 < \dots < \varepsilon_{k-1}$ from the interval $[j_i, i)$ such that:

- $B_i \subseteq M_{j_i}$,
- $\text{tp}(\bar{c}_i, M_i) \cup \{\varphi_i(\bar{x}, \bar{b}_i)\}$ and $\text{tp}(\bar{c}_i, M_i) \cup \{\neg\varphi_i(\bar{x}, \bar{b}_i)\}$ are consistent,
- $\text{tp}(\bar{c}_{\varepsilon_\ell}, M_{\varepsilon_\ell}) = \text{tp}(\bar{c}_i, M_{\varepsilon_\ell})$,
- $\mathfrak{C} \models \varphi[\bar{c}_{\varepsilon_\ell}, \bar{b}]^{\text{if } (\ell \text{ even})}$,
- k is maximal for the given $\varphi(\bar{x}, \bar{y}), \bar{b}_i, j_i$ (see \circledast below; k is well defined as T is dependent, see \circledast below),
- $\mathfrak{C} \models \varphi[\bar{c}_i, \bar{b}]^{\text{if } (k \text{ is even})}$.

So in stage i we first choose M_i : if $i = 0$ by clause (c), such model M_i exists by Claim 1.8(1), if i is a limit ordinal we choose M_i as $\cup\{M_j : j < i\}$ and if $i = j + 1$ (so \bar{c}_j has already been defined) then choose M_i such that $M_j \cup \bar{c}_j \leq_\kappa M_{j+1}$ and $M_i = M_{j+1}$ is κ -saturated of cardinality λ (and λ -full if you like), possible by Claim 1.8(1). Note that $M_j \leq_\kappa M_j \cup \bar{c}_j$ by clause (f) and Observation 1.7(6) hence $M_j \leq_\kappa M_{j+1}$ recalling Observation 1.7(2).

Note

- \circledast there is $n = n_{\varphi(\bar{x}, \bar{y})}$ depending on $\varphi(\bar{x}, \bar{y})$ and T only such that in subclause (β) we have $\varphi_i(\bar{x}_i, \bar{y}_i) = \varphi(\bar{x}, \bar{y}) \Rightarrow k_i \leq n$.

[Why? As by clause (f) in subclause (g)(β) of $\boxplus(f)$, by Fact 1.2(2) the sequence $\langle \bar{c}_{\varepsilon_\ell} : \ell < k \rangle$ is an indiscernible sequence, so by T being dependent we are done by Fact 0.4.]

Second, why can we choose $(m_i, j_i, B_i, \varphi_i, \text{tp}(\bar{c}_i, M_i))$ as required in clause (g)? If $\text{cf}(i) < \kappa$ or the antecedent of clause (g), i.e. subclause (g)(α) fails then trivially yes (choose, e.g., \bar{c}_i as the empty sequence). Otherwise let $j < i, B \subseteq M_j$ be of cardinality $< \kappa, m < \omega$ and $p(\bar{x}) \in \mathbf{S}^m(M_i)$ which does not split over B and which has extensions $p_0(\bar{x}) \neq p_1(\bar{x})$ in $\mathbf{S}^m(M_i \cup A^+)$ with $p_0 \upharpoonright (M_i \cup A^+) \neq p_1 \upharpoonright (M_i \cup A^+)$, so $p_0 \upharpoonright M_i = p = p_1 \upharpoonright M_i$ does not split over B .

Hence for some $\bar{b} \in {}^{\omega>}(M_i \cup A^+)$ and $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ we have $\varphi(\bar{x}, \bar{b}) \in p_1(\bar{x}), \neg\varphi(\bar{x}, \bar{b}) \in p_0(\bar{x})$; as i is a limit ordinal without loss of generality

^eWe can add:

- not only j_i satisfies the demand on j in subclause (α) , but it is the minimal such j .

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$\bar{b} \in \omega^{\gt}(M_j \cup A^+)$. We now try to choose ε_ℓ by induction on $\ell \leq n_{\varphi(\bar{x}, \bar{y})}$ such that:

- ⊙ (a) $j \leq \varepsilon_\ell < i$ and $k < \ell \Rightarrow \varepsilon_k < \varepsilon_\ell$,
- (b) $\bar{c}_{\varepsilon_\ell}$ realizes $p(\bar{x}) \upharpoonright M_{\varepsilon_\ell}$,
- (c) $\mathfrak{C} \models \varphi[\bar{c}_{\varepsilon_\ell}, \bar{b}]^{\text{if } (\ell \text{ is even})}$,
- (d) ε_ℓ is minimal under (a)+(b)+(c).

So by \circledast for some $k \leq n_{\varphi(\bar{x}, \bar{y})}$ we have: ε_ℓ is well defined if and only if $\ell < k$. At last we choose:

- (*) (a) $B_i = B$,
- (b) $\varphi_i = \varphi$,
- (c) $j_i = j$,
- (d) $k_i = k$,
- (e) \bar{c}_i realizes $p_1(\bar{x})$ if k is even and realizes $p_0(\bar{x})$ if k is odd.

So at last we are in a situation where the construction from \boxtimes (a)–(g) having been carried out. So now comes the division to cases.

Let $S_* = \{i < \lambda^+ : \text{cf}(i) \geq \kappa \text{ and subclause } (\alpha) \text{ of clause (g) holds for } i\}$.

Recall that S is a stationary subset of an ordinal δ of cofinality $> \aleph_0$ (e.g., a regular uncountable cardinal) when it is not disjoint to any closed unbounded subset E of δ .

Case 1: S_* is a stationary subset of λ^+ . Hence for $i \in S_*$, there are $j_i, B_i, \varphi_i(\bar{x}_i, \bar{y}_i), \bar{b}_i$ and $k_i < \omega$ and $\varepsilon_0(i) < \dots < \varepsilon_{k_i-1}(i) < i$ as in subclause (β) of \boxtimes (g) and by Fodor's lemma (see, e.g., [4]) for some $m_* < \omega, j < \lambda^+, B, \varphi(\bar{x}, \bar{b}), k_*, \langle \varepsilon_i : i < k_* \rangle$ and a stationary subset S of $S_* = \{\delta < \lambda^+ : \text{cf}(\delta) \geq \kappa\}$ we have $\delta \in S \Rightarrow j_\delta = j \wedge B_\delta = B \wedge \ell g(\bar{c}_\delta) = m_* \wedge \varphi_\delta(\bar{x}, \bar{b}_\delta) = \varphi(\bar{x}, \bar{b}) \wedge k_\delta = k_* \wedge \bigwedge_{\ell < k_*} \varepsilon_\ell(\delta) = \varepsilon_\ell$. Also without loss of generality by Fact 1.2(4) we have $\delta \in S \Rightarrow \text{tp}(\bar{c}_\delta, M_{\min(S)}) = \text{tp}(\bar{c}_{\min(S)}, M_{\min(S)})$ recalling that the number of such types is $\leq 2^{|B|+|T|}$. Choose $\delta(0) < \delta(1)$ from S so both have cofinality $\geq \kappa$ and $B_{\delta(0)} = B_{\delta(1)}, \text{tp}(\bar{c}_{\delta(0)}, M_{\delta(0)}) \subseteq \text{tp}(\bar{c}_{\delta(1)}, M_{\delta(1)})$ by Fact 1.2(1) and $\bar{b}_{\delta(1)} = \bar{b}_{\delta(0)}$ and $\varepsilon_i(\delta(0)) = \varepsilon_i(\delta(1))$ for $i < k_*$. But we could have chosen in stage $\delta(1), \varepsilon_k$ for $k < k_*$ and $k'_{\delta(1)} = k_* + 1$ and $\varepsilon'_{k_*}(\delta(1)) = \delta(0)$, contradiction to the maximality of k in subclause (β) of \boxtimes (g).

Case 2: Not Case 1. Then for a club of $i < \lambda^+$ if $\text{cf}(i) \geq \kappa$ then subclause (α) of clause (g) fails for i hence M_i exemplifies that we have gotten the desired conclusion in (A) of Theorem 1.9. By the proof of (4), note:

\boxplus for every $A \subseteq B$ and $\lambda = \lambda^{<\kappa} \geq |B| + |T|$ there is $\mathcal{M} = \mathcal{M}_{A,B} \subseteq \{M : A \leq_\kappa M \prec \mathfrak{C}_T \text{ and } \|M\| = \lambda\}$ such that:

- (a) if $M_1 \leq_\kappa M_2$ are from \mathcal{M} then $(M_1)_{[B]} \prec (M_2)_{[B]}$,
- (b) if $M_1 \in \mathcal{M}_{A,B}$ and $M_1 \leq_\kappa M_2 \prec \mathfrak{C}, \|M_2\| \leq \lambda$ then there is $M_3 \in \mathcal{M}_{A,B}$ such that $M_2 \leq_\kappa M_3$,

- (c) $\mathcal{M}_{A,B}$ is closed^f under increasing union of length $< \lambda^+$.
 (d) $\mathcal{M}_{A,B}$ is \leq_κ -dense above A that is: if $A \leq_\kappa A_1 \subseteq \mathfrak{C}$ and $|A_1| \leq \lambda$ then there is $M \in \mathcal{M}_{A,B}$ satisfying $A_1 \leq_\kappa M$.

Proof of (B): By induction on $i < \lambda^+$ we choose $M_i, N_i, B_i, j_i, \bar{c}_i$ such that

\boxtimes' Clauses (a), (c), (d) of \boxtimes and

- (h) $\langle N_j : j \leq i \rangle$ is \prec -increasing continuous and $A^+ \subseteq M_0^+$,
 (i) $M_i \prec N_i$ and N_i has cardinality λ and if i is non-limit then M_i, N_i are κ -saturated,
 (j) if $\text{cf}(i) \geq \kappa$ and there are $j_i < i, m < \omega, B \prec M_{j_i}$ of cardinality $< \kappa$ and $p \in \mathbf{S}^m(M_i)$ which does not split over B and has ≥ 2 extensions in $\mathbf{S}^m(N_j)$ then subclause (β) of clause $\boxtimes(g)$ above holds (with $\bar{b} \in \ell g(\bar{y})(N_j)$),
 (k) if $\text{cf}(i) \geq \kappa$ and (M_i, N_i) is κ -reduced and $j \leq i$ then there is $M_{i,j}^+$ such that $M_i \leq_\kappa M_{i,j}^+ \leq_\kappa M_{i+1}$ and $M_{i,j}^+ \in \mathcal{M}_{M_j, N_j}$.

The rest of the proof is similar to that of (A). $\square_{1.9}$

For the rest of the section we shall assume the following (as we use it all the time).

Hypothesis 1.10. T is dependent.

Definition 1.11. (1) $\mathbf{S}_{<\kappa}^{\text{nsf}}(A) = \{p \in \mathbf{S}(A) : p \text{ does not split over some } B \subseteq A \text{ of cardinality } < \kappa\}$.

(2) $\mathbf{S}_{<\kappa}^{\text{nsf}, \alpha}(A) \subseteq \mathbf{S}^\alpha(A)$ and $\mathbf{S}_{<\kappa}^{\text{nsf}, <\alpha}(A) \subseteq \bigcup_{\beta < \alpha} \mathbf{S}^\beta(A)$ are defined similarly.

(3) $\mathbf{S}_{\geq \kappa}^{\text{nsf}}(A) = \{p \in \mathbf{S}(A) : p \text{ is weakly orthogonal to } r \text{ for every } r \in \mathbf{S}_{<\kappa}^{\text{nsf}}(A)\}$.

(4) $\mathbf{S}_{\geq \kappa}^{\text{nsf}, \alpha}(A), \mathbf{S}_{\geq \kappa}^{\text{nsf}, <\alpha}(A)$ are defined similarly.

We may note the following.

Observation 1.12. (1) If $\text{tp}(\bar{c}_1, A)$ belongs to $\mathbf{S}_{\geq \kappa}^{\text{nsf}, \alpha}(A)$ and $\bar{c}_2 \in {}^\beta \mathfrak{C}$ and $\text{Rang}(\bar{c}_2) \subseteq \text{Rang}(\bar{c}_1)$ then $\text{tp}(\bar{c}_2, A)$ belongs to $\mathbf{S}_{\geq \kappa}^{\text{nsf}, \beta}$.

(2) $\text{tp}(\bar{c}, A) \in \mathbf{S}_{\geq \kappa}^{\text{nsf}, \alpha}(A)$ if and only if $\text{tp}(\bar{c} \upharpoonright u, A) \in \mathbf{S}_{\geq \kappa}^{\text{nsf}, |u|}(A)$ for every finite $u \subseteq \ell g(\bar{c})$.

(3) If $\text{tp}(\bar{a}, A) \in \mathbf{S}^m(A)$ is weakly orthogonal to $\text{tp}(\bar{c}, A)$ and does not split over $B \subseteq A$ and every $q \in \mathbf{S}^{<\omega}(B)$ is realized in A then $\text{tp}(\bar{a}, A + \bar{c})$ does not split over B .

Proof. Straightforward. $\square_{1.12}$

Observation 1.13. If $\kappa = \text{cf}(\kappa) > |T|$, the model M is κ -saturated and $p \in \mathbf{S}^m(M)$, then we can find N, q such that:

$$\otimes_N^1 \text{ (a) } \|N\| = \|M\|^{<\kappa},$$

^fWe may use $\prec_{\perp, \kappa}$ but then we have to say “increasing union of length $\delta < \lambda^+$ when $\text{cf}(\delta) \geq \kappa$ ”.

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- (b) $q \in \mathbf{S}^m(N)$ extends p ,
- (c) N is $\mathbf{F}_{<\kappa}^{\text{nsp}}$ -constructible over M ,

- $\otimes_{N,q,\kappa}^2$ (a) N is κ -saturated and $q \in \mathbf{S}^m(N)$.
- (b) if $r \in \mathbf{S}_{<\kappa}^{\text{nsp}, <\omega}(N)$ then r, q are weakly orthogonal, i.e. $q \in \mathbf{S}_{\geq\kappa}^{\text{nsp}, m}(N)$.

Proof. Let \bar{c} realize $p(\bar{x})$ and let $C = \text{Rang}(\bar{c})$, now we apply clause (A) of Theorem 1.9 with $M, M \cup C, N$ here standing for A, A^+, M there. □_{1.13}

Theorem 1.14 (The Tree-like Type Theorem). Assume $q(\bar{x}) \in \mathbf{S}_{\geq\kappa}^{\text{nsp}, \alpha}(N)$ and N is κ -saturated and $\kappa > \theta = |T| + |\alpha|$ and let $\bar{z} = \langle z_\alpha : \alpha < \theta \rangle$. Then we can find a sequence $\bar{\psi} = \langle \psi_{\varphi(\bar{x}, \bar{y})}(\bar{x}, \bar{z}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T) \rangle$ of formulas such that for every $A \subseteq M$ of cardinality $< \kappa$ there is $\bar{c} \in {}^\theta M$ such that:

- (a) $\{\psi_{\varphi(\bar{x}, \bar{y})}(\bar{x}, \bar{c}) : \varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)\} \subseteq q \upharpoonright \text{Rang}(\bar{c}) \subseteq q$,
- (b) for each $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_T)$ we have $\psi_{\varphi(\bar{x}, \bar{y})}(\bar{x}, \bar{c}) \vdash \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^{\ell g(\bar{y})} A \text{ and } \varphi(\bar{x}, \bar{b}) \in q\}$.

Proof. This follows from Claims 1.15 and 1.16 below. □_{1.14}

Claim 1.15. (1) Assume that $\otimes_{N,q,\kappa}^2$ from the Claim 1.13 holds, which means N is κ -saturated and $q \in \mathbf{S}_{\geq\kappa}^{\text{nsp}}(N)$. Then

- $\otimes_{N,q,\kappa}^3$ if $M \prec N$ has cardinality $< \kappa$ and $\varphi(x, y)$ is a formula with parameters from N , then for some $\psi(x, \bar{d}) = \psi_{\varphi(x, y), M}(x, \bar{d}_{\varphi(x, y), M}) \in q$ and $\eta \in {}^M 2$ we have $\psi(x, \bar{d}) \vdash p_{\varphi(x, y)}^{M, \eta}$ where $p_{\varphi(x, y)}^{M, \eta} = \{\varphi(x, b)^{\eta(b)} : b \in M\}$; so it is included in q .
- (2) Part (1) works also for $q \in \mathbf{S}^m(N)$, i.e. $q \in \mathbf{S}_{\geq\kappa}^{\text{nsp}, m}(N)$ and $\varphi = \varphi(\bar{x}, \bar{y})$ where $\ell g(\bar{x}) = m, \ell g(\bar{y}) < \omega$.

Proof. Fix M such that $M \prec N$ is of cardinality $< \kappa$.

(1) First note that

(*)₁ if D is an ultrafilter on M then $q(x)$ is weakly orthogonal to $r_D = \text{Av}(D, N)$.

[Why? As r_D does not split over M , by Observation 1.4(1).]

Second, note that

(*)₂ the following type cannot be finitely satisfiable in M :

$$r^*(y) = \{(\exists x_1)(\psi(x_1, \bar{d}) \wedge \varphi(x_1, y)) : \psi(x, \bar{d}) \in q\} \\ \cup \{(\exists x_2)(\psi(x_2, \bar{d}) \wedge \neg\varphi(x_2, y)) : \psi(x, \bar{d}) \in q\}.$$

[Why? Otherwise for some ultrafilter D on M we have $\vartheta(y, \bar{d}) \in r^*(y) \Rightarrow \vartheta(M, \bar{d}) \in D$. Let $b \in \mathfrak{C}$ realize $\text{Av}(D, N)$ so as $q(\bar{x})$ is closed under conjunctions, $q(x) \cup$

$\{\varphi(x, b)\}$ and $q(x) \cup \{\neg\varphi(x, b)\}$ are finitely satisfiable in \mathfrak{C} , and we get a contradiction to $(*)_1$.]

$(*)_3$ there is $\psi(x, \bar{d}) \in q$ such that $\{(\exists x_1)(\psi(x_1, \bar{d}) \wedge \varphi(x_1, y)), (\exists x_2)(\psi(x_2, \bar{d}) \wedge \neg\varphi(x_2, y))\}$ is satisfied by no $b \in M$.

[Why? By the monotonicity in $\psi(x, \bar{d})$ and q being closed under conjunctions this follows from $(*)_2$.]

$(*)_4$ let $\psi_{\varphi(x,y),M}(x, \bar{d}_{\varphi(x,y),M}) = \psi(x, \bar{d})$, from $(*)_3$,

$(*)_5$ for every $b \in M$ we have $N \models “(\forall x)(\psi(x, \bar{d}) \rightarrow \varphi(x, b))”$ or $N \models “(\forall x)(\psi(x, \bar{d}) \rightarrow \neg\varphi(x, \bar{b}))”$.

[Why? By logic this follows by $(*)_3$.]

$(*)_6$ there is $\eta \in {}^M 2$ such that for every $b \in M$ we have $M \models “(\exists x)(\psi(x, \bar{d}) \wedge \varphi(x, b))”$ if and only if $M \models “\neg(\exists x)(\psi(x, y)) \wedge \neg\varphi(x, b)”$ if and only if $\eta(b) = 1$.

[Why? By $(*)_5 + (*)_3$.]

So we are done.

(2) The proof is similar. □_{1.15}

Claim 1.16. (1) In Claim 1.15, fixing q , if $\text{cf}(\kappa) > |T|$ then ψ depends on φ but does not depend on M though \bar{d} in general does, i.e. given $\varphi(\bar{x}, \bar{y})$ we may assume without loss of generality that $\psi_{\varphi(\bar{x}, \bar{d})} = \psi_{\varphi(\bar{x}, \bar{y})}(x, \bar{d}_{\varphi(\bar{x}, \bar{y}), M})$.

(2) Assume $\otimes_{N, q, \kappa}^2$ from Claim 1.13, i.e. N is κ -saturated and $q \in \mathbf{S}_{\geq \kappa}^{\text{nsp}, m}(N)$. Then the following partial order is κ -directed

- (a) elements: $q \upharpoonright B$ for $B \subseteq N$ of cardinality^s $\leq |T|$
- (b) order: $p_1 \leq p_2$ if $p_2 \vdash p_1$.

Proof. (1) As if $N_1 \prec N_2 \prec N$ and $\|N_2\| < \kappa$, then $\psi_{\varphi(\bar{x}, \bar{y}), N_2}(x, \bar{d}_{\varphi(\bar{x}, \bar{y}), N_2})$ can serve as $\psi_{\varphi(\bar{x}, \bar{y}), N_1}(\bar{x}, \bar{d}_{\varphi(\bar{x}, q), N_2})$. As the number of possible ψ 's is $< \kappa = \text{cf}(\kappa)$ we are easily done.

(2) Easy. □_{1.16}

As a conclusion we can now show that a key fact in [22] for the theory $T = \text{Th}(\mathbb{Q}, <)$ has a parallel for every dependent T .

Conclusion 1.17 (The Saturated Expansion Conclusion). Assume

- (a) N is κ -saturated,
- (b) $|T| \leq |A| < \kappa$,
- (c) if $\bar{a} \in A$ then $q_{\bar{a}} = \text{tp}(\bar{a}, N) \in \mathbf{S}_{\geq \kappa}^{\text{nsp}, < \omega}(N)$, see Definition 1.11(3),
- (d) $N_{[A]}$ has elimination of quantifiers.

^sFrom some form of strongly dependent we should be able to get “essentially finite”.

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Then $N_{[A]}$ is κ -saturated.

Remark 1.18. (1) Recall $N_{[A]}$ is N expanded by $R_{\varphi(\bar{x}, \bar{b})} = \{\bar{a} \in {}^{\ell g(\bar{x})}N : \mathfrak{C} \models \varphi[\bar{a}, \bar{b}]\}$ for $\varphi(\bar{x}, \bar{b})$ a formula with parameters from A , see Definition 1.6(3A) or [17, Sec. 1].

(2) We can omit assumption (d) in Conclusion 1.17, but then get κ -saturated only for quantifier free types.

Proof. Without loss of generality, κ is regular, this as it is enough to prove λ^+ -saturation for every $\lambda \in [|A|, \kappa)$. Let $M \prec N$ be such that $\|M\| < \kappa$ and assume $p = p(\bar{y}) \in \mathbf{S}^m(|M|, N_{[A]})$ and we shall prove that some $\bar{c} \in {}^m N$ realizes $p(\bar{y})$. Actually, without loss of generality, $M_{[A]} \prec N_{[A]}$ and by assumption (d), equivalently $p(\bar{y}) \in \mathbf{S}^m(M \cup A) = \mathbf{S}^m(M \cup A, \mathfrak{C})$ is finitely satisfiable in M . Let $\bar{c} = \langle c_\alpha : \alpha < \alpha^* \rangle$ list A so $\alpha^* < \kappa$ and for $u \subseteq \alpha^*$ let $\bar{c}_u = \langle c_\alpha : \alpha \in u \rangle, \bar{x}_u = \langle x_\alpha : \alpha \in u \rangle$.

Next note that by Claims 1.16(1) and 1.15(2) (here clause (c) of the assumption is used) applied to $\text{tp}(\bar{c}_u, N)$ noting \bar{y} of length m is fixed and letting $\bar{x}_u = \langle x_\alpha : \alpha \in u \rangle$, we have:

(*)₁ for every finite $u \subseteq \alpha^*$ and formula $\varphi = \varphi(\bar{x}_u, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ there are $\psi_{\varphi(\bar{x}_u, \bar{y}, \bar{z})}(\bar{x}_u, \bar{d}_{\varphi(\bar{x}_u, \bar{y}, \bar{z})}, M) \in \text{tp}(\langle c_\alpha : \alpha \in u \rangle, N)$ so $\bar{d}_{\varphi(\bar{x}_u, \bar{y}, \bar{z})}, M \in {}^{\omega} N$ and η a function from ${}^{\ell g(\bar{y}) + \ell g(\bar{z})} M$ to $\{0, 1\}$ such that:

$$\psi_{\varphi(\bar{x}_u, \bar{y}, \bar{z})}(\bar{x}_u, \bar{d}_{\varphi(\bar{x}_u, \bar{y}, \bar{z})}, M) \vdash \{\varphi(\bar{x}_u, \bar{b}, \bar{c})^{\eta(\bar{b} \hat{\ } \bar{c})} : \bar{b} \in {}^{\ell g(\bar{y})} M \text{ and } \bar{c} \in {}^{\ell g(\bar{z})} M\}.$$

Clearly $|p(\bar{y})| < \kappa$ so there are $\zeta^* < \kappa$ and a sequence $\langle (\varphi_\zeta(\bar{x}_{u_\zeta}, \bar{y}, \bar{z}_\zeta), u_\zeta) : \zeta < \zeta^* \rangle$ listing the pairs $(\varphi(\bar{x}_u, \bar{y}, \bar{z}), u)$ as above so we have

$$p(\bar{y}) = \{\varphi_\zeta(\bar{c} \upharpoonright u_\zeta, \bar{y}, \bar{e}) : \zeta < \zeta^* \text{ and } \bar{e} \in {}^{\ell g(\bar{z}_\zeta)} M\},$$

so $u_\zeta \subseteq \alpha^*$ is finite.

For each $\zeta < \zeta^*$ we choose $\psi_\zeta(\bar{x}_{u_\zeta}, \bar{d}_\zeta)$ as guaranteed by (*)₁ above (for $\varphi_\zeta(\bar{x}_{u_\zeta}, \bar{y}, \bar{z}_\zeta)$).

Let

$$p'(\bar{y}) := \{(\forall \bar{x}_{u_\zeta})[\psi_\zeta(\bar{x}_{u_\zeta}, \bar{d}_\zeta) \rightarrow \varphi_\zeta(\bar{x}_{u_\zeta}, \bar{y}, \bar{e})] : \zeta < \zeta^* \text{ and } \bar{e} \in {}^{\ell g(\bar{z}_\zeta)} M\}.$$

Now

(*)₂ $p'(\bar{y})$ is finitely satisfiable in M .

[Why? As $p(\bar{y})$ finitely satisfiable in M , using the translation and the choice of $\psi_\zeta(\bar{x}_{u_\zeta}, \bar{d}_\zeta)$. That is, let $p''(\bar{y})$ be a finite subset of $p'(\bar{y})$ so for some $k < \omega, \zeta_\ell = \zeta(\ell) < \zeta^*, \bar{c}_\ell \in {}^{\ell g(\bar{z}_{\zeta_\ell})} M$ for $\ell < k$ we have $p''(\bar{y}) = \{(\forall \bar{x}_{u_{\zeta_\ell}})[\psi_{\zeta_\ell}(\bar{x}_{u_{\zeta_\ell}}, \bar{d}_{\zeta_\ell}) \rightarrow \varphi_{\zeta_\ell}(\bar{x}_{u_{\zeta_\ell}}, \bar{y}, \bar{e}_\ell)] : \ell < k\}$. Now $\{\varphi_{\zeta_\ell}(\bar{c} \upharpoonright u_{\zeta_\ell}, \bar{y}, \bar{e}_\ell) : \ell < k\}$ is a finite subset of $p(\bar{y})$ hence is realized by some $\bar{b} \in {}^m M$, hence by (*)₁ the sequence \bar{b} realizes $p''(\bar{y})$.]

(*)₃ the type $p'(\bar{y})$ is over $\cup\{\bar{d}_\zeta : \zeta < \zeta^*\} \cup M \subseteq N$.

[Why? Check.]

(*)₄ $p'(\bar{y})$ has cardinality $\leq |A| + |T| + \|M\| < \kappa$.

[Why? Obvious.]

(*)₅ there are M^+ and \bar{b} such that:

- (a) $M \prec M^+ \prec N$,
- (b) $\text{tp}(M^+, M \cup \bigcup\{\bar{d}_\zeta : \zeta < \zeta^*\})$ is finitely satisfiable in M ,
- (c) $\bar{b} \in M^+$ realizing $p'(\bar{y})$.

[Why? Easy, e.g., using ultrapower, “ N is κ -saturated” and (*)₂ + (*)₃.]

(*)₆ \bar{b} realizes $p(\bar{y}) \in \mathbf{S}^m(N \cup A)$ and $\bar{b} \in {}^{\ell g(\bar{y})}(M^+) \subseteq {}^{\ell g(\bar{y})}N$.

[Why? Follow the translations.]

So we are done. □_{1.17}

Question 1.19. (1) Can we waive assumption (d) in Conclusion 1.17?

(2) Is the family of (N, A) as in Conclusion 1.17 “dense under \leq_κ ”?

Discussion 1.20. (1) Assume $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa) > |T|$ and we try to prove that there is a (λ, κ) -limit model.

So let $M \in \text{EC}_\lambda(T)$ be saturated and we try to analyze the class of $N, M \prec N \in \text{EC}_\lambda$, which are “close enough”, in the sense of (λ, κ) -limit model.

So if $p \in \mathbf{S}_{<\lambda}^{\text{nsf}}(M)$, say p does not split over B , for some $B \subseteq M$ of cardinality $< \lambda$, then we can assume that in N there are “enough elements” realizing “types not-splitting over B ” extensions of p . So hopefully we can analyze such N by $\mathcal{P} \subseteq \mathbf{S}_{\geq\lambda}^{\text{nsf}}(M)$ pairwise perpendicular or $\mathcal{P} \subseteq \mathbf{S}_{\geq\lambda}^{\text{nsf}, |T|}(M)$ such that for each $p \in \mathcal{P}$ the model $M_{[p]}$ from Notation 0.2(1A) has elimination of quantifiers and is saturated, it is reasonable that this holds if we can expand M by definition of $< \lambda$ types $p \in \mathcal{P}$.

What we need, i.e. what is necessary for this line of attack (but not yet clear if sufficient to carry it), is:

(*) if \mathcal{P}_ℓ above has cardinality λ and is quite dense (e.g., using \mathbf{F} 's for $\ell = 1, 2$) then there is an automorphism of M which maps \mathcal{P}_1 onto \mathcal{P}_2 .

This leads to the generic pair conjecture.

(2) About $\mathbf{S}_{<\lambda}^{\text{nsf}}(M)$ recall Definition 1.11(1).

2. The Type Decomposition Theorem

Context 2.1. (1) T is a complete first-order theory; dependent if not said otherwise.

(2) $\mathfrak{C} = \mathfrak{C}_T$ is a monster for T , etc. as in Context 0.1.

Here we try to analyze a type $p \in \mathbf{S}^{\leq\theta}(M)$ for κ -saturated $M \prec \mathfrak{C}$ where $\kappa > \theta \geq |T|$, the characteristic case being $\kappa \gg \theta$ (θ may be \aleph_0 , if T is countable). In the case of $\theta < |T|$, or even better $\theta < \aleph_0$ we know less but mention it. We look

at “ T being stable” as our dream, our paradise. The hard reality is T being just dependent. In some sense T dependent should be like stable but we allow order, e.g., $\text{Th}(\mathbb{Q}, <)$ or trees. What we actually do is investigate the K_ℓ (see Definition 2.6).

How helpful is this analysis? We present two consequences. The first to some extent accomplished the professed aim: the Type Decomposition Theorem (Theorem 2.4).

What is its meaning? If M is κ -saturated, $\bar{d} \in {}^{\theta^+}\mathfrak{C}$ and $\kappa > \theta \geq |T|$ then we try to analyze the type $\text{tp}(\bar{d}, M)$ in two steps: for some \bar{c}, B :

- (a) $B \subseteq M$ has cardinality $< \kappa$, say $B = |N|$,
- (b) the type in the first step is similar to the types of stable theories, i.e. $\text{tp}(\bar{c}, M)$ does not split over $B \subseteq M$ (we can even demand $\text{tp}(\bar{c}, M)$ is finitely satisfiable in B),
- (c) the type in the second step, $\text{tp}(\bar{d}, M + \bar{c})$ behaves as in trees; e.g., letting $\mathbf{x} = (M, B, \bar{c}, \mathbf{d})$ we have: on ${}^\theta M$ the partial order $\leq_{\mathbf{x}}$ is κ -directed (see Definition 0.6) where we let $\bar{a}_1 \leq_{\mathbf{x}} \bar{a}_2$ if and only if $\text{tp}(\bar{d}, \bar{c} \hat{\ } \bar{a}_2) \vdash \text{tp}(\bar{d}, \bar{c} \hat{\ } \bar{a}_1)$.

The reader may say that clause (b) is not a true parallel to a stable case, as $|B|$ is not bounded by $\theta + |T|$ (but this is impossible even for the theory of dense linear order). Still a type not splitting over a set is a weak form of definability. Also we may wonder, what is the meaning, when T is $\text{Th}(\mathbb{Q}, <)$? If M is κ -saturated each $p \in \mathbf{S}(M)$ actually stands for a cut of M . Now the cuts are divided to those which have cofinality $\geq \kappa$ from both sides (falling under (c)), and those which do not (hence fall under (b)).

The second consequence deals with singular μ of cofinality $> |T|$. We ask: is there $M \prec \mathfrak{C}$ which is exactly μ -saturated, i.e. is μ -saturated but not μ^+ -saturated. Now if $T = \text{Th}(\mathbb{Q}, <)$ this is impossible, that is, there is no such M . If T is stable there is no problem to find such M , the main case being $\text{cf}(\mu) > |T|$ (or just $\text{cf}(\mu) \geq \kappa(T)$, see [12, Chap. III]) and let M be μ -prime over an indiscernible set of cardinality μ . The result says that for dependent T there is something like that, this is Lemma 2.2.

Lemma 2.2 (Singular Exact Saturation Lemma). *Assume T is a dependent theory, κ is singular of cofinality $> |T|$ and $M \prec \mathfrak{C}$ is an exactly κ -saturated model, i.e. is κ -saturated but not κ^+ -saturated. Then there are N and A such that:*

- (a) $N \prec M$ of cardinality $< \kappa$ and $A \subseteq M$ of cardinality κ and M omits some $p \in \mathbf{S}(A)$ which does not split over N ;

moreover

- (b) there is $q \in \mathbf{S}(M)$ which does not split over N such that $p = q \upharpoonright A$,
- (c) there is an indiscernible sequence $\langle \bar{a}_\alpha : \alpha < \kappa \rangle$ over N of θ -tuples from M such that $\text{Av}(\langle \bar{a}_\alpha : \alpha < \kappa \rangle, N \cup \{\bar{a}_\alpha : \alpha < \kappa\})$ is omitted by M (equivalently, we cannot choose $\bar{a}_\kappa \in M$) and does not split over N .

Remark 2.3. We can add in (c):

(c)⁺ moreover we can demand that there is an ultrafilter \mathcal{D} on N such that M omits $p = \text{Av}(\mathcal{D}, A)$ where $A := \cup\{\bar{a}_\alpha : \alpha < \kappa\} \cup N$ and $p(\bar{x})$ is as in clause (c).

Theorem 2.4 (The Type Decomposition Theorem). Assume $\text{cf}(\kappa) > \theta \geq |T|$, M is κ -saturated and $\bar{\mathbf{d}} \in {}^\theta \geq \mathfrak{C}$. Then for some $\bar{\mathbf{c}} \in {}^\theta \mathfrak{C}$, recalling Definition 1.11 we have $\text{tp}(\bar{\mathbf{c}}, M) \in \mathbf{S}_{< \kappa}^{\text{nsf}, \theta}(M)$ and $(\mathbf{P}, \leq_{\mathbf{P}})$ is a κ -directed partial order where:

- (a) $\mathbf{P} = \{\text{tp}(\bar{\mathbf{d}}, A \cup \bar{\mathbf{c}}) : A \subseteq M \text{ has cardinality } \leq \theta\}$,
- (b) $p_1 \leq_{\mathbf{P}} p_2$ if and only if $p_2 \vdash p_1$.

Remark 2.5. (1) In fact $(\mathbf{P}, \leq_{\mathbf{P}})$ is $(\mathbf{P}_{\mathbf{x}, \theta}, \leq_{\mathbf{x}, \theta})$ from the Definition 2.6(8) below.
 (2) Note that being θ^+ -directed is obvious.
 (3) Would it be more transparent to use the notation $p_2 \vdash p_1$ instead of $p_1 \leq_{\mathbf{P}} p_2$?
 A matter of taste, the author feels that not.

Definition 2.6. (1) Let $K = K_1$ be the family of \mathbf{x} satisfying

- (a) $\mathbf{x} = (A, B, \bar{\mathbf{c}}, \bar{\mathbf{d}})$ but if $A = |M|$, as usual, we may write M instead of A and if $B = \emptyset$ we may omit it,
- (b) $B \subseteq A$,
- (c) I a linear order,
- (d) $\bar{\mathbf{c}} = \langle \bar{c}_{t,n} : n < n_t, t \in I \rangle$ where $n_t \leq \omega$, each $\bar{c}_{t,n}$ is a finite^h sequence and letⁱ $\bar{c}_t = \bar{c}_{t,0} \hat{\ } \bar{c}_{t,1} \hat{\ } \cdots \hat{\ } \bar{c}_{t,n_t-1}$,
- (e) $\langle \bar{c}_{t,n} : n < n_t \rangle$ is an indiscernible sequence over $A \cup \{\bar{c}_s : s \in I \setminus \{t\}\}$, so if $n_t = 1$ this is an empty statement,
- (f) if $t \in I$ then^j $\text{tp}(\bar{c}_t, \{\bar{c}_s : s <_I t\} \cup A)$ does not split over B ,
- (g) $\bar{\mathbf{d}}$ is a sequence of elements or finite sequences from \mathfrak{C} .

- (2) Let K_0 be defined similarly omitting clause (f).
- (3) For $\lambda \geq \kappa$, $\text{cf}(\lambda) \geq \theta$ (or just $\lambda \geq \theta$), $\text{cf}(\kappa) \geq \theta$ and $\ell \in \{0, 1\}$ let $K_{\lambda, \kappa, < \theta}^\ell = \{(M, B, \bar{\mathbf{c}}, \bar{\mathbf{d}}) \in K_\ell : M \text{ is } \lambda\text{-saturated, } |B| < \kappa \text{ and } |\ell g(\bar{\mathbf{d}})| + |\ell g(\bar{\mathbf{c}})| < \theta\}$; omitting ℓ means 1. If $\theta = \sigma^+$ instead of “ $< \theta$ ” we may write σ .
- (4) (a) let $\mathbf{x} = (A_{\mathbf{x}}, B_{\mathbf{x}}, \bar{\mathbf{c}}_{\mathbf{x}}, \bar{\mathbf{d}}_{\mathbf{x}})$ for $\mathbf{x} \in K_0$ (or $M_{\mathbf{x}}$ instead of $A_{\mathbf{x}}$), $I = I_{\mathbf{x}}$, $\bar{c}_{\mathbf{x},t} = \bar{c}_t, n_t = n_{\mathbf{x},t}$ and $C_{\mathbf{x}} = \cup\{\text{Rang}(\bar{c}_{t,n}) : t \in I \text{ and } n < n_{\mathbf{x},t}\}$,
- (b) we may^k replace $\bar{\mathbf{d}}$ by $D_{\mathbf{x}} = \text{Rang}(\bar{\mathbf{d}})$,
- (c) we may omit κ if $\kappa = \lambda$.

^hWe remark when it matters.

ⁱBut abusing our notation, $\text{Rang}(\bar{\mathbf{c}})$ is the set of elements of \mathfrak{C} appearing in it; similarly in other cases.

^jOf course, this implies that clause (e) follows from a weak version, see Claim 2.7(5) but see part (2).

^kWe may write $C_{\mathbf{x}} = \text{Rang}(\bar{\mathbf{c}})$, $\bar{c}_{\mathbf{x},t,n} = \langle c_{t,n,m}^{\mathbf{x}} : m < \ell g(c_{\mathbf{x},t,n}) \rangle$ so in fact $\bar{c}_{\mathbf{x}} = \langle c_{t,n,m}^{\mathbf{x}} : (t, n, m) \in J \rangle$ for the natural $J = J_{\mathbf{x}}$.

For $\lambda = \aleph_0$, let “ $< \lambda$ ” mean $A_{\mathbf{x}}$ is the universe of $M_{\mathbf{x}} \prec \mathfrak{C}$ (no saturation demand).

- (5) We define a two-place relation \leq_0 on K_0 : $\mathbf{x} \leq_0 \mathbf{y}$ if and only if $A_{\mathbf{x}} \subseteq A_{\mathbf{y}}, B_{\mathbf{x}} \subseteq B_{\mathbf{y}}, I_{\mathbf{x}} \subseteq I_{\mathbf{y}}, \bar{\mathbf{c}}_{\mathbf{x}} = \bar{\mathbf{c}}_{\mathbf{y}} \upharpoonright I_{\mathbf{x}}$, i.e. $t \in I_{\mathbf{x}} \Rightarrow \bar{c}_{\mathbf{y},t} = \bar{c}_{\mathbf{x},t}$ moreover $t \in I_{\mathbf{x}} \Rightarrow n_{\mathbf{y},t} = n_{\mathbf{x},t}$ and $t \in I_{\mathbf{x}} \wedge n < n_{\mathbf{x},t} \Rightarrow \bar{c}_{\mathbf{y},t,n} = \bar{c}_{\mathbf{x},t,n}, \bar{\mathbf{d}}_{\mathbf{x}} \leq \bar{\mathbf{d}}_{\mathbf{y}}$ and $\text{tp}(\bar{\mathbf{c}}_{\mathbf{x}}, A_{\mathbf{y}})$ does not split over $B_{\mathbf{x}}$ hence “ $\text{tp}(\bar{\mathbf{c}}_{\mathbf{x}}, A_{\mathbf{x}})$ does not split over $B_{\mathbf{x}}$ ” follows.
- (5A) $\mathbf{x} \leq_1 \mathbf{y}$ mean $\mathbf{x} \leq_0 \mathbf{y}$ and $A_{\mathbf{y}} = A_{\mathbf{x}}$.
- (6) We define a two-place relation \leq_2 on K_0 : $\mathbf{x} \leq_2 \mathbf{y}$ if and only if $\mathbf{x} \leq_1 \mathbf{y} \wedge \bar{\mathbf{d}}_{\mathbf{x}} = \bar{\mathbf{d}}_{\mathbf{y}}$.
- (7) $\mathbf{x} \in K_{\lambda,\kappa,\theta}^0$ is called normal when $\text{Rang}(\bar{\mathbf{c}}_{\mathbf{x}}) \subseteq \text{Rang}(\bar{\mathbf{d}}_{\mathbf{x}})$.
- (8) For $\mathbf{x} \in K$, let $\mathbf{P}_{\mathbf{x},\theta} = (\mathbf{P}_{\mathbf{x},\theta} \leq_{\mathbf{x},\theta})$ be defined by:
 - (a) $\mathbf{P}_{\mathbf{x}} = \{\text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, A + \bar{\mathbf{c}}_{\mathbf{x}}) : A \subseteq A_{\mathbf{x}} \text{ has cardinality } \leq \theta\}$,
 - (b) $\leq_{\mathbf{x},\theta}$ is the following two-place relation on $\mathbf{P}_{\mathbf{x},\theta}$: $p_1(\bar{x}_{\bar{\mathbf{d}}_{\mathbf{x}}}) \leq_{\mathbf{x},\theta} p_2(\bar{x}_{\bar{\mathbf{d}}_{\mathbf{x}}})$ if and only if $p_2 \vdash p_1$.
- (9) If $\theta = |T| + |\ell g(\bar{\mathbf{d}}_{\mathbf{x}})| + |\ell g(\bar{\mathbf{c}}_{\mathbf{x}})|$, i.e. we use “ $< \theta^+$ ”, we may omit it.

Claim 2.7. (1) $K_1 \subseteq K_0$.

- (2) \leq_i is a partial order on K_0 for $i = 0, 1, 2$.
- (3) If $i \in \{0, 1, 2\}$, $\langle \mathbf{x}_{\alpha} : \alpha < \delta \rangle$ is \leq_i -increasing in $K_{\lambda,\kappa,\theta}^0$ where δ is a limit ordinal, $[\text{cf}(\delta) \geq \theta^+ \Rightarrow \bigwedge_{j < \delta} \bar{\mathbf{c}}_{\mathbf{x}_j} = \bar{\mathbf{c}}_{\mathbf{x}_0}]$ and¹ $[i \leq 1 \wedge \text{cf}(\delta) \geq \theta^+ \Rightarrow (\bigwedge_{\alpha < \delta} \bar{\mathbf{d}}_{\mathbf{x}_{\alpha}} = \bar{\mathbf{d}}_{\mathbf{x}_0})]$, $[i = 0 \Rightarrow \lambda \leq \text{cf}(\delta)]$ and $\delta < \text{cf}(\kappa) \vee (\bigwedge_{\alpha < \delta} B_{\mathbf{x}_{\alpha}} = B_{\mathbf{x}_i})$, then it has a \leq_i -lub $\mathbf{x}_{\delta} := \cup\{\mathbf{x}_{\alpha} : \alpha < \delta\} \in K_{\lambda,\kappa,\theta}^0$ defined by $A_{\mathbf{x}} = \cup\{A_{\mathbf{x}_{\alpha}} : \alpha < \delta\}, B_{\mathbf{x}} = \cup\{B_{\mathbf{x}_{\alpha}} : \alpha < \delta\}, I_{\mathbf{x}} = \cup\{I_{\mathbf{x}_{\alpha}} : \alpha < \delta\}, \bar{\mathbf{c}}_{\mathbf{x}} = \cup\{\bar{\mathbf{c}}_{\mathbf{x}_{\alpha}} : \alpha < \delta\}$, i.e. $\bar{c}_{\mathbf{x},t} = \bar{c}_{\mathbf{x}_{\alpha},t}$ when $t \in I_{\mathbf{x}_{\alpha}}$ and $\bar{\mathbf{d}}_{\mathbf{x}} = \cup\{\bar{\mathbf{d}}_{\mathbf{x}_{\alpha}} : \alpha < \delta\}$.
- (3A) In part (3), if $\alpha < \delta \Rightarrow \mathbf{x}_{\alpha} \in K_{\lambda,\kappa,\theta}^1$ then $\mathbf{x}_{\delta} \in K_{\lambda,\kappa,\theta}^1$.
- (4) If $\bar{\mathbf{d}} \in \theta^+ > \mathfrak{C}$ and M is κ -saturated and $\kappa > \theta$ then $\mathbf{x} = (M, \emptyset, \langle \rangle, \bar{\mathbf{d}}) \in K_{\kappa,\theta}^{\ell}$ for $\ell = 0, 1$.
- (5) In the definition of $\mathbf{x} \in K_1$: in clause (e) it suffices to demand that: if $n_t > 1$ then $\langle \bar{c}_{t,n} : n < n_t \rangle$ is indiscernible over $A \cup \{\bar{c}_{s,m} : s <_I t, m < n_s\}$.
- (6) For every $\mathbf{x} \in K_{\lambda,\kappa,\theta}$ there is a normal $\mathbf{y} \in K_{\lambda,\kappa,\theta}$ satisfying $\mathbf{x} \leq_1 \mathbf{y}, \bar{\mathbf{c}}_{\mathbf{x}} = \mathbf{c}_{\mathbf{y}}$ and $\text{Rang}(\bar{\mathbf{d}}_{\mathbf{y}}) = \text{Rang}(\bar{\mathbf{d}}_{\mathbf{x}}) \cup \text{Rang}(\bar{\mathbf{c}}_{\mathbf{x}})$. Hence $\mathbf{y} \in \text{mx}K_{\lambda,\kappa,\theta} \Leftrightarrow \mathbf{x} \in \text{mx}K_{\lambda,\kappa,\theta}$, see Definition 2.8 below.
- (7) If $i = 0$ and $\langle \mathbf{x}_{\alpha} : \alpha < \delta \rangle$ is \leq_i -increasing in $K_{\lambda,\kappa,\theta}^0$ and $\text{cf}(\delta) < \theta^+, \delta < \text{cf}(\kappa)$, then the sequence has a \leq_i -upper bound $\mathbf{x}_{\delta} \in K_{\lambda,\kappa,\theta}^0$, note that we have not said “lub”.
- (7A) In part (7), if $\alpha < \delta \Rightarrow \mathbf{x}_{\alpha} \in K_{\lambda,\kappa,\theta}^1$ then we can add $\mathbf{x}_{\delta} \in K_{\lambda,\kappa,\theta}^1$.

Proof. Easy e.g., we prove the clauses (7) and (7A).

(7) and (7A) The problem is when part (3) does not cover it, so $\lambda > \aleph_0$. It is clear how to choose $\bar{\mathbf{c}}_{\mathbf{x}_{\delta}}, \bar{\mathbf{d}}_{\mathbf{x}_{\delta}}$ and $B_{\mathbf{x}_{\delta}}$, but we should choose a λ -saturated $M_{\mathbf{x}_{\delta}}$.

¹This follows from “ $\langle \mathbf{x}_{\alpha} : \alpha < \delta \rangle$ is \leq_i -increasing” when $i = 2$.

Let $B = \cup\{B_{\mathbf{x}_\alpha} : \alpha < \delta\}$, $I = \cup\{I_{\mathbf{x}_\delta} : \alpha < \delta\}$ and $\bar{\mathbf{c}} = \langle \bar{c}_t : t \in I \rangle$ with $\bar{c}_t = \bar{c}_{\mathbf{x}_\alpha, t}$ when $\alpha \in I_{\mathbf{x}_\alpha}$; similarly $\bar{\mathbf{d}}$.

First, choose a λ -saturated M extending $\cup\{M_{\mathbf{x}_\alpha} : \alpha < \delta\}$ but what about “ $\text{tp}(\bar{\mathbf{c}}_{\mathbf{x}_\alpha}, M)$ does not split over $B_{\mathbf{x}_\alpha}$ for each $\alpha < \delta$ ”?

Now for each $\alpha < \gamma < \delta$, $\text{tp}(\bar{\mathbf{c}}_{\mathbf{x}_\alpha}, M_\gamma)$ does not split over $B_{\mathbf{x}_\alpha}$ which means $p_{\alpha, \gamma}(\bar{x}) = \text{tp}(\bar{\mathbf{c}}_{\mathbf{x}_\alpha}, M_\gamma)$ does not split over $B_{\mathbf{x}_\alpha}$ hence $p_\alpha(\bar{x}) := \cup\{p_{\alpha, \beta}(\bar{x}) : \beta \in (\alpha, \delta)\} = \text{tp}(\bar{\mathbf{c}}_{\mathbf{x}_\alpha}, \bigcup_{\beta < \delta} M_\beta)$ does not split over $B_{\mathbf{x}_\alpha}$. Hence for $\alpha < \delta$ by Fact 1.2(1) there is $\bar{\mathbf{c}}'_\alpha$ such that $\text{tp}(\bar{\mathbf{c}}'_\alpha, M)$ does not split over $B_{\mathbf{x}_\alpha}$ and extends p_α . As $M_{\mathbf{x}_\alpha} \supseteq B_{\mathbf{x}_\alpha}$ is λ -saturated and $\lambda \geq \kappa$ by Definition 2.6(3) clearly the model $M_{\mathbf{x}_\beta}$ is κ -saturated and $|B_{\mathbf{x}_\alpha}| < \kappa$ by the definition of $K_{\lambda, \kappa, \theta}^i$. Recalling Fact 1.2(1), by the last two sentences $\bar{\mathbf{c}}'_\beta \upharpoonright I_{\mathbf{x}_\alpha}$ realizes $\text{tp}(\bar{\mathbf{c}}'_\alpha, M)$ for $\alpha < \beta < \delta$ hence without loss of generality $\alpha < \beta < \delta \Rightarrow \bar{\mathbf{c}}'_\beta \upharpoonright I_{\mathbf{x}_\alpha} = \bar{\mathbf{c}}'_\alpha$.

Hence there is an elementary mapping f mapping with domain $\cup\{\bar{\mathbf{c}}'_\alpha : \alpha < \delta\} \cup M$, mapping $\bar{\mathbf{c}}'_\alpha$ to $\bar{\mathbf{c}}_{\mathbf{x}_\alpha}$ for $\alpha < \delta$, and extending $\text{id}_{\cup\{M_{\mathbf{x}_\alpha} : \alpha < \delta\}}$. Now $M_{\mathbf{x}_\delta} := f(M)$ will do, i.e. let $M_{\mathbf{y}} = M$, $B_{\mathbf{y}} = B$, $\bar{\mathbf{c}}_{\mathbf{y}} = \bar{\mathbf{c}}$, $\bar{\mathbf{d}}_{\mathbf{y}} = \bar{\mathbf{d}}$. $\square_{2.7}$

Definition 2.8. (1) For $\ell = 0, 1$ let $\text{mx}K_{\lambda, \kappa, < \theta}^\ell$ be the family of $\mathbf{x} \in K_{\lambda, \kappa, < \theta}^\ell$ which are \leq_2^+ -maximal in $K_{\lambda, \kappa, < \theta}^\ell$, i.e. for no \mathbf{y} do we have $\mathbf{x} <_2^+ \mathbf{y} \in K_{\lambda, \kappa, < \theta}^\ell$, see below; if $\ell = 1$ we may omit it.

(2) For $i = 0, 1, 2$ let \leq_i^+ be the following two-place relation on K_0 : $\mathbf{x} \leq_i^+ \mathbf{y}$ if and only if $\mathbf{x} \leq_i \mathbf{y}$, see Definition 2.6 and if $\mathbf{x} \neq \mathbf{y}$ then for some $t \in I_{\mathbf{y}} \setminus I_{\mathbf{x}}$ satisfying $n_{\mathbf{x}, t} \geq 2$ we have: $\mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{t, 1}, \bar{b}] \wedge \neg \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{t, 0}, \bar{b}]$ for some $\varphi = \varphi_t(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{b} \subseteq A_{\mathbf{x}} \cup \cup\{\bar{c}_{\mathbf{y}, s} : s \in I_{\mathbf{y}} \setminus \{t\}\}$.

(3) Again, if $\theta = \sigma^+$ instead of “ $< \theta$ ”: we may write σ , and if $\kappa = \lambda$ we may omit λ .

(4) Of course, $\mathbf{x} <_i^+ \mathbf{y}$ means $\mathbf{x} \leq_i^+ \mathbf{y} \wedge \mathbf{x} \neq \mathbf{y}$.

Observation 2.9. Let $i = 0, 1, 2$.

(1) For $\ell = 0, 1$ the two-place relation \leq_i^+ is a partial order on K^ℓ .

(2) If $\mathbf{x}_1 \leq_i \mathbf{x}_2 <_i^+ \mathbf{x}_3 \leq_i \mathbf{x}_4$ then $\mathbf{x}_1 <_i^+ \mathbf{x}_4$.

(3) If $\mathbf{x} <_i^+ \mathbf{z}$ are from $K_{\lambda, \kappa, < \theta}^\ell$ then there is $\mathbf{y} \in K_{\lambda, \kappa, < \theta}^\ell$ such that $\mathbf{x} <_2^+ \mathbf{y} \leq_i \mathbf{z}$ and $\bar{\mathbf{d}}_{\mathbf{y}} = \bar{\mathbf{d}}_{\mathbf{x}}$, $I_{\mathbf{y}} \setminus I_{\mathbf{x}}$ is finite.

(4) The parallel of Claim 2.7(3), (3A) holds for $\text{mx}K$.

Proof. For example, we prove clause (3).

(3) Let $t \in I_{\mathbf{z}} \setminus I_{\mathbf{x}}$ and $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{b} \subseteq A_{\mathbf{x}} \cup \cup\{\bar{c}_{\mathbf{y}, s} : s \in I_{\mathbf{y}} \setminus \{t\}\}$ be such that $\mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{y}, t, 1}, \bar{b}] \wedge \neg \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{y}, t, 0}, \bar{b}]$. We choose a finite $I \subseteq I_{\mathbf{y}} \setminus \{t\}$ such that $\bar{b} \subseteq \cup\{\bar{c}_{\mathbf{y}, s} : s \in I\} \cup A_{\mathbf{x}}$. Now define \mathbf{y} by: $M_{\mathbf{y}} = M_{\mathbf{x}}$, $I_{\mathbf{y}} = I_{\mathbf{x}} \cup I \cup \{t\}$, $\bar{\mathbf{c}}_{\mathbf{y}} = \bar{c}_{\mathbf{z}} \upharpoonright I_{\mathbf{y}}$, $\bar{\mathbf{d}}_{\mathbf{y}} = \bar{\mathbf{d}}_{\mathbf{x}}$ and $B_{\mathbf{y}} = B_{\mathbf{x}}$.

Now check.

$\square_{2.9}$

The following claim may be good for digesting the meaning of $\text{mx}K_{\lambda, \kappa, \theta}^\ell$.

Claim 2.10 (The L.S.T. Claim for $\text{mx}K$). If $\mathbf{x} \in \text{mx}K_{\kappa, \theta}$ and $M = M_{\mathbf{x}}$ then for some function F with domain $[M]^{< \kappa}$ satisfying $F(A) \in [M]^{\leq 2^{|A|+1}}$ for

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$A \in \text{Dom}(F)$, we have: if $M_1 \prec M$ is closed under F and contains $B_{\mathbf{x}}$ then $(M_1, B_{\mathbf{x}}, \bar{\mathbf{c}}_{\mathbf{x}}, \bar{\mathbf{d}}_{\mathbf{x}})$ belongs to $\text{mxK}_{\kappa, \theta}^{\ell}$.

Remark 2.11. By Observation 2.9(4), it suffices to consider F with domain $[M]^{< \aleph_0}$.

Proof. We can choose $F(\emptyset) \in [M]^1$ and for notational transparency we fix a set J of cardinality \aleph_0 disjoint to $I_{\mathbf{x}}$.

Note that for every $N \prec M$ satisfying $B_{\mathbf{x}} \subseteq N$ we have $\mathbf{x}_N := (N, B_{\mathbf{x}}, \bar{\mathbf{c}}_{\mathbf{x}}, \bar{\mathbf{d}}_{\mathbf{x}}) \in K_{\kappa, \theta}^{\ell}$; call such N a candidate. So to choose F let us analyze the cases $B_{\mathbf{x}} \subseteq N \prec M$ but $\mathbf{x}_N \notin \text{mxK}_{\kappa, \theta}^{\ell}$. Considering Definition 2.8, it suffices by Observation 2.9(3) to consider the case $\mathbf{x}_N <_2^+ \mathbf{y}$, $I_{\mathbf{y}} \setminus I_{\mathbf{x}}$ is finite and is $\subseteq J$, $t_* \in I_{\mathbf{y}} \setminus I_{\mathbf{x}}$, $\varphi_*(\bar{x}, \bar{y}, \bar{z}), \bar{b}_*$ as there, we can ignore the possibility that also some other $t \in I_{\mathbf{y}} \setminus I_{\mathbf{x}} \setminus \{t_*\}$ works.

We let \bar{b}_0 list $B_{\mathbf{y}}, I_{\mathbf{y}} \setminus I_{\mathbf{x}} = \{t_0, \dots, t_{k-1}\}$, $t_* = t_{\ell(*)}$, $\ell(*) < k$, $m_t > \text{lg}(\bar{c}_{\mathbf{y}, t, 0})$, $n_t = n_{\mathbf{y}, t}$ for $t \in I_{\mathbf{y}}$. Also without loss of generality $\varphi = \varphi(\bar{x}, \bar{y}; \bar{z}_1, \bar{z}_2)$, $\text{lg}(\bar{y}) = \text{lg}(\bar{c}_{\mathbf{y}, t_*, 0})$, $\text{lg}(\bar{x}) = \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}})$, $\bar{b} = \bar{b}_1 \hat{\ } \bar{b}_2$, $\bar{b}_1 \in \omega^{>}(M_{\mathbf{y}}) = \omega^{>}N$, $\text{lg}(\bar{z}_1) = \text{lg}(\bar{b}_1)$ and abusing our notation, $\text{lg}(\bar{z}_2) = \text{lg}(\bar{b}_2)$ where $\bar{b}_2 = \langle \bar{c}_{\mathbf{y}, t, k} : k < n_{\mathbf{y}, t}, t \in I_{\mathbf{y}} \rangle$ and $\bar{b}_{2, < t} = \langle c_{\mathbf{y}, s, k} : k < n_t, s < t \text{ so } t \in I_{\mathbf{y}} \rangle$ and $\bar{b}_{2, \neq t} = \langle \bar{c}_{\mathbf{y}, s, k} : k < n_s, s \in I_{\mathbf{y}} \setminus \{t\} \rangle$, so $\mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{y}, t_*, 1}, \bar{b}_1, \bar{b}_{2, t_*}] \wedge \neg \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{y}, t_*, 0}, \bar{b}_1, \bar{b}_{2, t_*}]$ and let $\text{lg}(\bar{z}_t^*) = \text{lg}(\bar{c}_{\mathbf{y}, t})$, $\text{lg}(\bar{z}_{t, \ell}) = \text{lg}(\bar{c}_{\mathbf{y}, t, \ell})$ so $\bar{z}_t^* = \bar{z}_{t, 0} \hat{\ } \dots \hat{\ } \bar{z}_{t, n_t - 1}$ and $\bar{z}_2 = (\dots, \bar{z}_t^*, \dots)_{t \in I_{\mathbf{y}}}$.

All this information will be called a witness against the candidate N and we denote it by \mathbf{w} .

Let \mathbf{s} consist of the following pieces of information on the witness \mathbf{w} and in this case we shall say that \mathbf{w} materializes \mathbf{s} and \mathbf{s} is a case for N .

- ⊠ (a) $I = I_{\mathbf{y}}$ and $\langle t_{\ell} : \ell < k \rangle$ (so we will write $I_{\mathbf{s}}, t_{\mathbf{s}, \ell}$) and $\ell(*)$,
- (b) n_t, m_t for $t \in I$,
- (c) $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}_1, \bar{z}_2)$ hence $\text{lg}(\bar{b}_1) = \text{lg}(\bar{z}_1)$,
- (d) $\zeta(0) = \text{lg}(\bar{b}_0)$ but not \bar{b}_0 itself,
- (e) $q_0 = \text{tp}(\bar{b}_0, \emptyset)$ and $q_1 = \text{tp}(\bar{b}_0 \hat{\ } \bar{b}_1 \hat{\ } \bar{b}_2, \emptyset)$, so from q_1 we know when $b_{2, \ell_1} = c_{t_{\ell}, n, \ell_2}$,
- (f) the scheme of non-splitting of $\text{tp}(\bar{c}_t, M)$ for $t \in I_{\mathbf{y}}$ from clause (f) of Definition 2.6(1), that is $\Xi_t = \{(\psi(\bar{z}_{\bar{c}_{\mathbf{y}, t}}, \bar{y}'), q(\bar{y}', \bar{y}_{\bar{b}_0}')) : \text{for some } \bar{b} \in {}^{\text{lg}}(\bar{y}')N \text{ we have } \mathfrak{C} \models \psi[\bar{c}_{\mathbf{y}, t}, \bar{b}] \text{ and } q(\bar{y}', \bar{y}_{\bar{b}_0}) = \text{tp}(\bar{b} \hat{\ } \bar{b}_0, \emptyset)\}$.

We shall write $I = I_{\mathbf{s}}, t_* = t_*(\mathbf{s}), \varphi = \varphi_{\mathbf{s}}(\bar{x}_{\mathbf{s}}, \bar{\mathbf{y}}, \bar{z}_{\mathbf{s}, 1}, \bar{z}_{\mathbf{s}, 2}), q_{\mathbf{s}, 0} = q_0$, etc. and let $r_{\mathbf{s}} = \text{tp}(\bar{\mathbf{c}}_{\mathbf{y}}, N)$. We call \mathbf{s} a case when it is a case for some candidate N . If \mathbf{s} is a case and $\bar{b}_0 \in {}^{\zeta(0)}(M_{\mathbf{x}})$ realizes $q_{\mathbf{s}, 0}$, then we can choose $\bar{\mathbf{c}}_{\mathbf{s}, \bar{b}_0} = \langle \bar{c}_{\mathbf{x}, \bar{b}_0, t} : t \in I_{\mathbf{s}} \rangle$ such that $\text{tp}(\bar{\mathbf{c}}_{\mathbf{s}, \bar{b}_0, t}, \cup\{\bar{\mathbf{c}}_{\mathbf{s}, \bar{b}_0, s} : s < I_{\mathbf{s}} t\} \cup M_{\mathbf{x}})$ is defined by the scheme Ξ_t with the parameter \bar{b}_0 , this type is determined by \mathbf{s}, \bar{b}_0 and \mathbf{x} (though not the $\bar{c}_{\mathbf{s}, \bar{b}_0, t}$'s themselves). Without loss of generality $t \in I_{\mathbf{x}} \Rightarrow \bar{c}_{\mathbf{s}, \bar{b}_0, t} = \bar{c}_{\mathbf{x}, t}$.

Now clearly

- (*)₁ for a candidate N we have $\mathbf{x}_N \in \text{mxK}_{\kappa, \theta}$ if and only if for every case \mathbf{s} and $\bar{b}_0 \in {}^{\zeta(0)}N$ realizing $q_{\mathbf{s}, 0}$ there is no $\bar{b}_1 \in {}^{\ell g(\bar{y})}N$ such that $\mathbf{s}_1, \bar{b}_0, \bar{b}_1, \bar{b}_2 = \bar{c}_{\mathbf{s}, \bar{b}_0}$ are as above.

So

- (*)₂ for every case \mathbf{s} and $\bar{b}_0 \in {}^{\zeta(0)}(M_{\mathbf{x}})$ realizing $q_{\mathbf{s}, 0}$ and $\bar{b}_1 \in {}^{\ell g(\bar{z}_{\mathbf{s}, 1})}M$, we cannot choose \bar{c}', \bar{c}'' realizing $\text{tp}(\bar{c}_{\mathbf{s}, \bar{b}_0}, M_{\mathbf{x}})$ such that $\bar{c}' \upharpoonright I_{\mathbf{x}} = \bar{c}_{\mathbf{x}} = \bar{c}'' \upharpoonright I_{\mathbf{x}}$ and $\mathcal{C} \models \varphi_{\mathbf{s}}[\mathbf{d}_{\mathbf{x}}, \bar{c}'_{t_*(\mathbf{s})}, \bar{b}_1, \bar{c}' \upharpoonright (I_{\mathbf{x}} \setminus \{t_*(\mathbf{s})\})] \wedge \neg \varphi_{\mathbf{s}}[\mathbf{d}_{\mathbf{x}}, \bar{c}''_{t_*(\mathbf{s})}, \bar{b}_1, \bar{c}'' \upharpoonright (I_{\mathbf{x}} \setminus \{t_*(\mathbf{s})\})]$.

[Why? As then $\mathbf{x} \notin \text{mxK}_{\kappa, \theta}$.]

Hence

- (*)₃ for every case \mathbf{s} and $\bar{b}_0 \in {}^{\zeta(0)}(M_{\mathbf{x}})$ realizing $q_{\mathbf{s}, 0}$ and $\bar{b}_1 \in {}^{\ell g(\bar{z}_{\mathbf{s}, 1})}(M_{\mathbf{x}})$ there is a finite set $C = C_{\mathbf{s}}(\bar{b}_0, \bar{b}_1) \subseteq M_{\mathbf{x}}$ such that: if N is a candidate which includes \bar{b}_0, \bar{b}_1, C then there is no witness \mathbf{w} against N with $\mathbf{s} = \mathbf{s}_{\mathbf{w}}, \bar{b}_0 = \bar{b}_{\mathbf{w}, 0}, \bar{b}_1 = \bar{b}_{\mathbf{w}, 1}$.

Also (see Fact 1.2(4))

- (*)₄ for $B \subseteq M$ of cardinality $< \kappa$ let $C(B)$ be a subset of M of cardinality $\leq 2^{|B|+|T|}$ in which every $p \in \mathbf{S}^{< \omega}(B)$ is realized.

Lastly, let F be defined for $B \in [M]^{< \kappa}$, $F(B) = \cup \{C_{\mathbf{s}}(\bar{b}_0, \bar{b}_1) : \mathbf{s} \text{ a case and } \bar{b}_0, \bar{b}_1 \text{ suitable sequences from } B\} \cup C(B)$.

Now the number of cases is $\leq \aleph_0 + \aleph_0 + \aleph_0 + |T| + \theta + 2^{|T|+|B|} = 2^{|T|+|B|}$, so $F(B) \in [M]^{\leq \theta}$. So we are done. $\square_{2.10}$

Theorem 2.12 (The Existence Theorem). *If $\ell = 0, 1$ and $\text{cf}(\kappa) > \theta \geq |T|$ and $\mathbf{x} \in K_{\lambda, \kappa, \theta}^{\ell}$ then there is \mathbf{y} such that $\mathbf{x} \leq_2 \mathbf{y} \in \text{mxK}_{\lambda, \kappa, \theta}^{\ell}$.*

Remark 2.13. (1) If we use $K_{\lambda, \kappa, < \theta}^{\ell}$ instead of “ $\theta \geq |T|$ ” we should demand “ $\text{cf}(\theta) > |T|$ ”.

- (2) We may get more. For example, demand $I_1 = I_{\mathbf{x}}, I_2$ is well ordered and $I_{\mathbf{y}} = I_1 \cup I_2, I_1 < I_2$, i.e. $s_1 \in I_1 \wedge s_2 \in I_2 \Rightarrow s_1 <_I s_2$.
- (2A) Also this claim holds (by the same proof) when we replace clause (f) in Definition 2.6(1) by

(f)₂ $\text{tp}(\bar{c}_{\mathbf{x}}, A_{\mathbf{x}})$ is finitely satisfiable in $B_{\mathbf{x}}$.

Then in part (2) of the remark we may add

(*) for $t \in I_2$, $\text{tp}(\bar{c}_t, M_{\mathbf{x}} \cup \{\bar{c}_{\mathbf{y}, s} : s <_{I_{\mathbf{y}}} t\})$ is finitely satisfiable in $B_{\mathbf{y}}$.

See more in Definition 2.33, Claim 2.34 and Claim 2.35.

- (2B) In this case we may say “clause (f)₁, of Definition 2.6(1)” instead of clause (f).

- (3) We can be more relaxed in the demands on $\langle \mathbf{x}_{\alpha} : \alpha < \theta^+ \rangle$ in the proof, e.g. it suffices to demand

⊗' (a) $\mathbf{x}_{\alpha} \in K_{\lambda, \kappa, \theta}^0$,

(b) \mathbf{x}_{α} is \leq_1 -increasing continuous, natural to demand “ \leq_2 -increasing”, that is, $\bar{\mathbf{d}}_{\mathbf{x}_{\alpha}} = \bar{\mathbf{d}}_{\mathbf{x}_0}$ but not necessary,

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(c) for each $\alpha < \theta^+$ (or just for stationarily many $\alpha < \theta^+$) we have $\mathbf{x}_\alpha <_1^+ \mathbf{x}_{\alpha+1}$.

[Why? Let $\bar{\mathbf{d}}_{\mathbf{x}_1} \upharpoonright u_\alpha$ (u_α finite) $t_\alpha, \varphi_\alpha(\bar{x}_\alpha, \bar{y}_\alpha, \bar{b}_\alpha)$ witness $\mathbf{x}_\alpha <_1^+ \mathbf{x}_{\alpha+1}$ when $\alpha \in S_0$ where $S_0 := \{\alpha < \theta^+ : \mathbf{x}_\alpha <_1^+ \mathbf{x}_{\alpha+1}\}$. For $\alpha \in S_0$ let $h(\alpha) = \text{Min}\{\gamma : \bar{\mathbf{d}}_{\mathbf{x}_\alpha} \upharpoonright u_\alpha = \bar{\mathbf{d}}_{\mathbf{x}_\gamma} \upharpoonright u_\alpha, \text{equivalently } u_\alpha \subseteq \text{Dom}(\bar{\mathbf{d}}_{\mathbf{x}_\gamma}) \text{ and } \text{Rang}(\bar{b}_\alpha) \cap (A_{\mathbf{x}_\alpha} + \bar{\mathbf{c}}_{\mathbf{x}_\alpha}) \subseteq A_{\mathbf{x}_\gamma} + \bar{\mathbf{c}}_{\mathbf{x}_\gamma}\}$, clearly $h(\alpha) < \alpha$ for α limit $\in S_0$.

So by Fodor's lemma for some $\beta < \alpha$ and u the set $S = \{\delta \in S_0 : \delta \text{ is a limit ordinal as in clause (c) above } u_\delta = u \text{ and } h(\delta) = \beta\}$ is stationary. As $\theta \geq |T|$, for some^m formula φ the set $S_2 := \{\delta \in S_1 : \varphi_\delta = \varphi\}$ is a stationary subset of θ^+ and we continue as in the proof.

(4) How does part (3) of the remark help? For example, if we like to get $\mathbf{y} \in \text{mxK}_{\lambda, \kappa, \theta}^\ell$ which is normal and $\text{Rang}(\bar{\mathbf{d}}_{\mathbf{y}})$ is the universe of some $N \prec \mathfrak{C}$.

Proof. Assume this fails. We try to choose \mathbf{x}_α by induction on $\alpha < \theta^+$ such that

- ⊗ (a) $\mathbf{x}_\alpha \in K_{\lambda, \kappa, \theta}^\ell$,
- (b) $\mathbf{x}_\beta \leq_2 \mathbf{x}_\alpha$ for $\beta < \alpha$;
- (c) if $\alpha = \beta + 1$ then $\mathbf{x}_\beta <_2^+ \mathbf{x}_\alpha$, i.e. \mathbf{x}_α witness $\mathbf{x}_\beta \notin \text{mxK}_{\lambda, \kappa, \theta}^\ell$ (i.e. is like \mathbf{y} in Definition 2.8);
- (d) $\mathbf{x}_0 = \mathbf{x}$.

For $\alpha = 0$ use clause (d), for $\alpha = \beta + 1$ we use our assumption toward contradiction. For α limit use Claim 2.7(3). Note that $\bar{\mathbf{d}}_{\mathbf{x}_\alpha} = \bar{\mathbf{d}}_{\mathbf{x}}$ for $\alpha < \theta^+$ by clauses (d) + (b).

Having carried the induction, for each $\alpha < \theta^+$ there are $t_\alpha, \varphi_\alpha(\bar{x}, \bar{y}_\alpha, \bar{z}_\alpha), \bar{b}_\alpha$ satisfying:

- (*) _{α} (a) $t_\alpha \in I_{\mathbf{x}_{\alpha+1}} \setminus I_{\mathbf{x}_\alpha}$,
- (b) $\varphi_\alpha(\bar{x}, \bar{y}_\alpha, \bar{z}_\alpha) \in \mathbb{L}(\mathcal{T})$,
- (c) $\bar{b}_\alpha \subseteq M_{\mathbf{x}_\alpha} \cup \bigcup \{\bar{c}_{\mathbf{x}_{\alpha+1}, s} : s \in I_{\mathbf{x}_{\alpha+1}} \setminus \{t_\alpha\}\}$,
- (d) $lg(\bar{b}_\alpha) = lg(\bar{z}_\alpha)$ and $lg(\bar{y}_\alpha) = lg(\bar{c}_{\mathbf{x}_{\alpha+1}, t_\alpha, 0})$,
- (e) $\mathfrak{C} \models \neg \varphi_\alpha[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{x}_{\alpha+1}, t_\alpha, 0}, \bar{b}_\alpha] \wedge \varphi_\alpha[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{\mathbf{x}_{\alpha+1}, t_\alpha, 1}, \bar{b}_\alpha]$ so $lg(\bar{x}) = lg(\bar{\mathbf{d}}_{\mathbf{x}})$ and $n_{t_\alpha} \geq 2$.

Clearly the sequence $\langle t_\alpha : \alpha < \theta^+ \rangle$ is without repetitions. Now let $J_\alpha \subseteq I_{\mathbf{x}_{\alpha+1}} \setminus \{t_\alpha\}$ be finite such that $\bar{b}_\alpha \subseteq M_{\mathbf{x}_\alpha} \cup \bigcup \{\bar{c}_{\mathbf{x}_{\alpha+1}, s} : s \in J_\alpha\}$. We can find a pair $(\varphi(\bar{x}, \bar{y}, \bar{z}), n_*, m_*)$ such that the set $S_0 = \{\delta < \theta^+ : |J_\delta| \leq n_* \text{ and } \varphi_\delta(\bar{x}, \bar{y}_0, \bar{z}_0) = \varphi(\bar{x}, \bar{y}, \bar{z})\}$ is infinite (even stationary). By Ramsey's theorem (or Fodor's lemma) we can find an infinite (and even stationary) set $S_1 \subseteq S_0 \subseteq \theta^+$ such that $\delta \in S_1 \Rightarrow J_\delta \cap \{t_\beta : \beta \in S_1\} = \emptyset$. Note that there are $\leq \theta$ possibilities for φ , not necessarily $\leq |T|$ because though $\varphi_\alpha(\bar{x}, \bar{y}_\alpha, \bar{z}_\alpha)$ depends only on $\bar{x} \upharpoonright u$ for some finite $u \subseteq lg(\bar{x})$ there are $\leq lg(\bar{\mathbf{d}}_{\mathbf{x}}) + \aleph_0$ possibilities for u .

^mIf $\bar{c}_{t_\alpha, n}$ is infinite we let $u_\alpha \subseteq lg(\bar{c}_{t_\alpha, 0}^{\mathbf{x}_\alpha})$ be finite such that $\mathfrak{C} \models \varphi_\alpha[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{c}_{t_\alpha, 0}^{\mathbf{x}_{\alpha+1}} \upharpoonright u_\alpha, \bar{b}_\alpha] \wedge \neg \varphi_\alpha[\bar{\mathbf{d}}_{\mathbf{x}_\alpha}, \bar{c}_{t_\alpha, 1}^{\mathbf{x}_{\alpha+1}}, \bar{b}_\alpha]$ and the rest is the same.

Next we shall prove that in this case $\varphi(\bar{x}; \bar{y}, \bar{z})$ has the independence property (for T), a contradiction.

For every $w, v \subseteq S_1$ and $\eta \in {}^w 2$ let, noting that $w = \text{Dom}(\eta)$:

- (*)₂ (a) $A_{\eta,v} = A_{\mathbf{x}} \cup \{\bar{c}_{\mathbf{x}_{\beta+1},t,n} : t \in J_\beta \text{ and } n < n_{\mathbf{x}_{\beta+1},t} \text{ for some } \beta \in v\} \cup \{\bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,\eta(\alpha)} : \alpha \in w\}$,
- (b) $f_{\eta,v}$ is the function with:
 - (α) domain $A_{\eta,v}$,
 - (β) is the identity on $A_{\mathbf{x}} \cup \{\bar{c}_{\mathbf{x}_{\beta+1},t,n} : t \in J_\beta \text{ and } n < n_{\mathbf{x}_{\beta+1},t} \text{ for some } \beta \in v\}$,
 - (γ) $f_{\eta,v}(\bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,\eta(\alpha)}) = \bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,0}$ for $\alpha \in w = \text{Dom}(\eta)$.

Now

- (*)₃ $f_{\eta,v}$ is an elementary mapping.

[Why? Without loss of generality $w \subseteq v$ are finite so $\subseteq \alpha(*)$ for some $\alpha(*) < \theta^+$ and prove this by induction on $|v|$. We just use: for $\alpha \in v$ the sequence $\langle \bar{c}_{\mathbf{x}_{\alpha(*)},t,n} : n < n_{\mathbf{x}_{\alpha(*)},t} \rangle$ is indiscernible over $A_{\mathbf{x}_{\alpha(*)}} \cup \{\bar{c}_{\mathbf{x}_{\alpha(*)},s} : s \in I_{\mathbf{x}_{\alpha(*)}} \setminus \{t\}\}$, by Definition 2.6(1), clause (e).]

Now let $g_{\eta,v} \in \text{aut}(\mathfrak{C})$ extends $f_{\eta,v}$. So $\alpha \in v \Rightarrow g_{\eta,v}(\bar{b}_\alpha) = f_{\eta,v}(\bar{b}_\alpha) = \bar{b}_\alpha$ and $g_{\eta,v}(\bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,\eta(\alpha)}) = \bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,0}$ for $\alpha \in w$; hence by the choice of J_α so for $\eta \in {}^w 2$ we have

- (*)₄ $\mathfrak{C} \models \varphi[g_{\eta,v}(\bar{\mathbf{d}}_{\mathbf{x}}), \bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,0}, \bar{b}_\alpha]$ if and only if $\eta(\alpha) = 1$.

So $\langle \varphi(\bar{x}, \bar{c}_{\mathbf{x}_{\alpha+1},t_\alpha,0}, \bar{b}_\alpha) : t \in u \rangle$ is an independent sequence of formulas, see Fact 0.4(a); as w is any subset of S_1 we get a contradiction as promised. $\square_{2.12}$

Claim 2.14 (The Weak Orthogonality Claim). Assume $\ell = 0, 1$ and $\mathbf{x} \in \text{mxK}_{\lambda,\kappa,\theta}^\ell$.

- (1) If $m < \omega, B' \subseteq M_{\mathbf{x}}$ andⁿ $|B'| < \kappa$ and $q \in \mathbf{S}^m(M_{\mathbf{x}} \cup C_{\mathbf{x}})$ does not split over B' then $\text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M_{\mathbf{x}} \cup C_{\mathbf{x}})$ is weakly orthogonal to q .
- (1A) Above it suffices^o that $q = \text{tp}(\bar{c}_1 \hat{\ } \bar{c}_2, M_{\mathbf{x}} \cup C_{\mathbf{x}})$ and $\text{tp}(\bar{c}_1, M_{\mathbf{x}})$ does not split over some $B' \in [M]^{<\kappa}$, $\text{tp}(C_{\mathbf{x}}, M_{\mathbf{x}} + \bar{c}_1)$ does not split over $B_{\mathbf{x}}$ and $\text{tp}(\bar{c}_2, M_{\mathbf{x}} + \bar{c}_1 + C_{\mathbf{x}})$ does not split over some $B''' \in [M_{\mathbf{x}}]^{<\kappa}$.
- (2) If $\mathbf{x}_\alpha \in \text{mxK}_{\lambda,\kappa,\theta}^\ell$ for $\alpha < \delta$ is \leq_1 -increasing, $\delta < \theta^+$ and $\mathbf{x}_\delta := \cup\{\mathbf{x}_\alpha : \alpha < \delta\}$ belongs to $K_{\lambda,\kappa,\theta}^\ell$ (recall Claim 2.7(3)) then it also belongs to $\text{mxK}_{\lambda,\kappa,\theta}^\ell$.

Proof. (1) Assume toward contradiction that those types are not weakly orthogonal. Let $q = q(\bar{y}), \bar{y} = \langle y_k : k < m \rangle$ and let $\bar{x} = \langle x_\alpha : \alpha < \alpha_{\mathbf{x}} \rangle$ recalling $\alpha_{\mathbf{x}} = \ell g(\bar{\mathbf{d}}_{\mathbf{x}})$ and $p(\bar{x}) = \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M_{\mathbf{x}} \cup C_{\mathbf{x}})$. So for some formula $\varphi(\bar{x}, \bar{y}, \bar{z})$ and $\bar{e} \in {}^{\ell g(\bar{z})}(M_{\mathbf{x}} \cup C_{\mathbf{x}})$ the type $p(\bar{x}) \cup q(\bar{y})$ does not decide $\varphi(\bar{x}, \bar{y}, \bar{e})$, i.e. $r_t(\bar{x}, \bar{y}) = p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x}, \bar{y}, \bar{e})\}^t$

ⁿThis is just to ensure that M realizes every $q \in \mathbf{S}^{<\omega}(B')$.
^oEven more as we can increase the linear order $I_{\mathbf{x}}$.

is consistent (= finitely satisfiable in \mathfrak{C}) for $\mathbf{t} = 0, 1$ and let \bar{c}'_0, \bar{c}'_1 be such that $\bar{\mathbf{d}} \wedge \bar{c}'_{\mathbf{t}}$ realizes $r_{\mathbf{t}}(\bar{x}, \bar{y})$ for $\mathbf{t} = 0, 1$. Now it cannot be that $\text{tp}(\bar{c}'_{\mathbf{t}}, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$ does not split over B' for both $\mathbf{t} = 0, 1$ (by Fact 1.2(1), as every $p \in \mathbf{S}^{<\omega}(B')$ is realized in M recalling M is κ -saturated and $\kappa > |B'|$). So choose $\bar{c}_0 \in \{\bar{c}'_0, \bar{c}'_1\}$ such that the type $\text{tp}(\bar{c}_0, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}})$ splits over B' .

Now by Fact 1.2(1) there is $\bar{c}_1 \in {}^m\mathfrak{C}$ such that $\text{tp}(\bar{c}_1, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}} \cup \bar{c}_0)$ extends $q(\bar{y})$ and does not split over B' . Hence also $\text{tp}(\bar{c}_1, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$ does not split over B' , hence it is different from $\text{tp}(\bar{c}_0, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$. We can continue and choose \bar{c}_n ($n = 2, 3, \dots$) realizing the complete type over $M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}} \cup \bigcup\{\bar{c}_k : k < n\}$ which extends q and does not split over B' . Hence

(*)₀ for every $n < \omega$, $\text{tp}(\bar{c}_n, M_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bigcup\{\bar{c}_k : k < n\})$ extends $q(\bar{y})$ and does not split over B' .

So by Fact 1.2(2)

(*)₁ $\langle \bar{c}_n : n < \omega \rangle$ is an indiscernible sequence over $M_{\mathbf{x}} \cup C_{\mathbf{x}}$.

Also (by induction on $\gamma \leq \omega$) by Fact 1.2(3) we have:

(*)₂ if $t \in I_{\mathbf{x}}$ then $\langle \bar{c}_{\mathbf{x},t,n} : n < n_t \rangle$ is an indiscernible sequence over $M_{\mathbf{x}} \cup \{\bar{c}_{\mathbf{x},s,m} : s \in I \setminus \{t\}\} \cup \{\bar{c}_n : n < \gamma\}$.

Now we define $\mathbf{y} = (M_{\mathbf{y}}, B_{\mathbf{y}}, \bar{\mathbf{c}}_{\mathbf{y}}, \bar{\mathbf{d}}_{\mathbf{y}})$ by

- ⊗ (a) $M_{\mathbf{y}} = M_{\mathbf{x}}$,
- (b) $B_{\mathbf{y}} = B_{\mathbf{x}} \cup B'$,
- (c) $\bar{\mathbf{d}}_{\mathbf{y}} = \bar{\mathbf{d}}_{\mathbf{x}}$,
- (d) $I_{\mathbf{y}} = I_{\mathbf{x}} \cup \{s(*)\}$ were $[t \in I_{\mathbf{x}} \Rightarrow t < I_{\mathbf{y}} s(*)]$,
- (e) $\bar{\mathbf{c}}_{\mathbf{y}} \upharpoonright I_{\mathbf{x}} = \bar{\mathbf{c}}_{\mathbf{x}}$,
- (f) $n_{\mathbf{y},s(*)} = \omega$ (or any number $\in [2, \omega]$) and $\bar{c}_{\mathbf{y},s(*)} = \bar{c}_n$ for $n < n_{\mathbf{y},s(*)}$.

Now \mathbf{y} contradicts the assumption $\mathbf{x} \in \text{mxK}_{\lambda, \kappa, \theta}^{\ell}$.

(1A) The proof is similar (recalling that we can use $\{s_1\} + I_{\mathbf{x}}$).

(2) Easy.

□_{2.14}

The following claim is a crucial step toward proving the Type Decomposition Theorem (Theorem 2.4).

Claim 2.15. *If $\mathbf{x} \in \text{mxK}_{\lambda, \kappa, \theta}^{\ell}$ (or just $\mathbf{x} \in K$ and ⊗ below) then for every $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ and $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ satisfying $\ell g(\bar{x}) = \ell g(\bar{\mathbf{d}}_{\mathbf{x}}), \ell g(\bar{z}) = \ell g(\bar{c}_{\mathbf{x}}), \ell g(\bar{y}) = m$, there is $\psi(\bar{x}, \bar{e}, \bar{\mathbf{c}}_{\mathbf{x}}) \in \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M_{\mathbf{x}} \cup C_{\mathbf{x}})$ satisfying $\bar{e} \in {}^{\omega}M_{\mathbf{x}}$ such that $\psi(\bar{x}, \bar{e}, \bar{\mathbf{c}}_{\mathbf{x}}) \vdash \{\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} : \bar{b} \in {}^{\ell g(\bar{y})}A \text{ and } \mathbf{t} \in \{0, 1\}\}$ are such that $\mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}}]^{\mathbf{t}}$ where*

- ⊗ if $q(\bar{y}) \in \mathbf{S}^{\ell g(\bar{y})}(M_{\mathbf{x}} \cup \bar{\mathbf{c}}_{\mathbf{x}})$ is finitely satisfiable in some $A \subseteq M_{\mathbf{x}}$ of cardinality $< \kappa$ then $q(\bar{y})$ is weakly orthogonal to $\text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M_{\mathbf{x}} \cup C_{\mathbf{x}})$.

Proof. By Claim 2.14, if $\mathbf{x} \in \text{mxK}_{\lambda, \kappa, \theta}$ then \circledast holds; hence we can in any case assume \circledast .

Let $p(\bar{x}) = \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M \cup C_{\mathbf{x}})$, so $\bar{x} = \langle x_i : i < \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}}) \rangle$ and recalling $\bar{y} = \langle y_\ell : \ell < m \rangle$ we define a set $r = r(\bar{y})$ as follows:

$$r(\bar{y}) := \{(\exists \bar{x})(\varphi(\bar{x}, \bar{y}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} \wedge \psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}})) : \mathbf{t} \in \{0, 1\} \text{ and } \psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \in p(\bar{x}) \text{ and } \bar{a} \in {}^{\omega>}(M_{\mathbf{x}})\}.$$

Now

\circledast_1 $r(\bar{y})$ is not finitely satisfiable in ${}^m A$.

[Why? If $r(\bar{y})$ is finitely satisfiable in ${}^m A$, then there is an ultrafilter \mathcal{D} on ${}^m A$ such that for every $\vartheta(\bar{y}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \in r(\bar{y})$, the set $\{\bar{b} : \bar{b} \in {}^m A \text{ and } \models \vartheta[\bar{b}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}]\}$ belongs to \mathcal{D} . Let $q(\bar{y}) = \text{Av}(\mathcal{D}, M_{\mathbf{x}} \cup \bar{\mathbf{c}}_{\mathbf{x}})$, clearly $q(\bar{y}) \in \mathbf{S}^m(M_{\mathbf{x}} \cup \bar{\mathbf{c}}_{\mathbf{x}})$ is finitely satisfiable in ${}^m A$, $A \subseteq M$ and $|A| < \kappa$. Let $\bar{b}^* \in {}^m \mathfrak{C}$ realize $q(\bar{y})$, so $(\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \in p(\bar{x})) \wedge \mathbf{t} \in \{0, 1\} \Rightarrow \mathfrak{C} \models (\exists \bar{x})(\varphi(\bar{x}, \bar{b}^*, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} \wedge \psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}))$. Why? This holds by the choices of $r(\bar{y})$, \mathcal{D} and \bar{b}^* . As $p(\bar{x})$ is closed under conjunctions it follows that $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{b}^*, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}}\}$ is finitely satisfiable in \mathfrak{C} for $\mathbf{t} = 0, 1$. But this contradicts the assumption \circledast .]

Hence for some $n < \omega$ and $\psi_\ell(\bar{x}, \bar{a}_\ell, \bar{\mathbf{c}}_{\mathbf{x}}) \in p(\bar{x})$ for $\ell < n$ we have

\circledast_2 for no $\bar{b} \in {}^m A$ do we have $(\exists \bar{x})(\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} \wedge \psi_\ell(\bar{x}, \bar{a}_\ell, \bar{\mathbf{c}}_{\mathbf{x}}))$ for $\ell < n$, $\mathbf{t} \in \{0, 1\}$.

Let $\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) = \bigwedge_{\ell < n} \psi_\ell(\bar{x}, \bar{a}_\ell, \bar{\mathbf{c}}_{\mathbf{x}})$, so clearly

\circledast_3 (a) $\bar{a} \in {}^{\omega>}(M_{\mathbf{x}})$,
 (b) $\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \in p(\bar{x})$,
 (c) for no $\bar{b} \in {}^m A$ do we have $\bigwedge_{\mathbf{t}=0}^1 (\exists \bar{x})(\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} \wedge \psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}))$.

So for every $\bar{b} \in {}^m A$ for some $\mathbf{t} = \mathbf{t}(\bar{b}) \in \{0, 1\}$ we have $\mathfrak{C} \models \neg(\exists \bar{x})(\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} \wedge \psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}))$ hence $\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \vdash \neg\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}(\bar{b})}$. As $\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}}) \in p(\bar{x}) = \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M_{\mathbf{x}} \cup \bar{\mathbf{c}}_{\mathbf{x}})$ it follows that $\neg\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}(\bar{b})} \in p(\bar{x})$.

So $\psi(\bar{x}, \bar{a}, \bar{\mathbf{c}}_{\mathbf{x}})$ is as required. $\square_{2.15}$

Claim 2.16. Assume $\mathbf{x} \in \text{mxK}_{\lambda, \kappa, \theta}^\ell$. If $B' \subseteq M_{\mathbf{x}}$, $|B'| < \kappa$, $\bar{d} \in {}^m(D_{\mathbf{x}})$, $\bar{c} \in {}^{\omega>}(C_{\mathbf{x}})$, $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z})$, $\text{lg}(\bar{x}) = \text{lg}(\bar{d})$, $\text{lg}(\bar{z}) = \text{lg}(\bar{c})$ then for some $\psi(\bar{x}, \bar{y}', \bar{z}')$ and $\bar{e} \in {}^{\text{lg}(\bar{y}')} (M_{\mathbf{x}})$ we have $\models \psi[\bar{d}, \bar{e}, \bar{c}]$ and

$$\psi(\bar{x}, \bar{e}, \bar{c}) \vdash \{\varphi(\bar{x}, \bar{b}, \bar{c})^{\mathbf{t}} : \bar{b} \in {}^{\text{lg}(\bar{y}')} (B') \text{ and } \mathfrak{C} \models \varphi[\bar{d}, \bar{b}, \bar{c}]^{\mathbf{t}} \text{ and } \mathbf{t} \in \{0, 1\}\}.$$

Proof. This just reformulates Claim 2.15; see more in Claim 2.20(0). $\square_{2.16}$

Proof of Theorem 2.4 (The Type Decomposition Theorem). By Claim 2.7(4) there is $\mathbf{x} \in \text{K}_{\kappa, \leq \theta}^1$ such that $\mathbf{d}_{\mathbf{x}} = \bar{\mathbf{d}}$ and $M_{\mathbf{x}} = M$. By Theorem 2.12 without loss of generality $\mathbf{x} \in \text{mxK}_{\kappa, \leq \theta}^1$. Clearly $(\mathbf{P}_{\mathbf{x}, \theta}, \leq_{\mathbf{x}, \theta})$ is a partial order. Assume that $\alpha(*) < \kappa$ and $p_\alpha \in \mathbf{P}_{\mathbf{x}, \theta}$ for $\alpha < \alpha(*)$. Let $B = \cup\{\text{Dom}(p_\alpha) : \alpha < \alpha(*)\} \cup B_{\mathbf{x}}$,

so $B \subseteq M_{\mathbf{x}}$ has cardinality $< \kappa$. Hence by Claim 2.16 for every $v \subseteq \text{lg}(\bar{\mathbf{c}}_{\mathbf{x}})$ and $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z})$ satisfying $\text{lg}(\bar{z}) = \text{lg}(\bar{\mathbf{c}}_{\mathbf{x}}), \text{lg}(\bar{x}) = \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}}), \text{lg}(\bar{y}) < \omega$ there is $\psi = \psi_{\varphi}(\bar{x}, \bar{e}_{\varphi}, \bar{\mathbf{c}}_{\mathbf{x}}) \in \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, M \cup \bar{\mathbf{c}}_{\mathbf{x}})$ where $\bar{e}_{\varphi} \in {}^{\text{lg}(\bar{y})}(M_{\mathbf{x}})$ such that $\psi_{\varphi}(\bar{x}, \bar{e}_{\varphi}, \bar{\mathbf{c}}_{\mathbf{x}}) \vdash \{\varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} : \bar{b} \in {}^{\text{lg}(\bar{y})}B \text{ and } \mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}}]^{\mathbf{t}} \text{ and } \mathbf{t} \in \{0, 1\}\}$.

Let $A = \cup\{\text{Rang}(\bar{e}_{\varphi}) : \varphi \text{ as above}\}$, clearly $|A| \leq \theta$ and let $p = \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, A \cup \bar{\mathbf{c}}_{\mathbf{x}})$; it is an upper bound, as required. □_{2.4}

Discussion 2.17. The type decomposition theorems say that we can analyze a type $p \in \mathbf{S}(M)$ in two steps; first $\text{tp}(\bar{c}, M)$ does not split over some “small” $B \subseteq M$. Second, $\text{tp}(\bar{d}, \bar{c} + M)$ is like a type in the theory of trees and lastly without loss of generality by Claim 2.7(6) some initial segment of \bar{d} realizes p . As an example, see [22].

Exercise 2.18. Let $T = T_{\text{ord}}$ be $\text{Th}(\mathbb{Q}, <)$ and if $M \prec \mathfrak{C}_T, M$ is κ -saturated, in the main case and $p \in \mathbf{S}(M)$ then

- (a) p induces a cut \bar{C}_p of M where $\bar{C}_p = \langle C_{p,1}, C_{p,2} \rangle, C_{p,1} = \{a \in M : (a < x) \in p\}$ and $C_{p,2} = M \setminus C_{p,1}$,
- (b) now \bar{C}_p has a pair (κ_1, κ_2) of cofinalities, that is $\kappa_1 = \text{cf}(C_{p,1}, <_M), \kappa_2 = \text{cf}(C_{p,2}, >_M)$, as M is κ -saturated necessarily $\max\{\kappa_1, \kappa_2\} \geq \kappa$,
- (c) now p does not split over some subset B of M of cardinality $\leq \lambda < \kappa$ if and only if $\min\{\kappa_1, \kappa_2\} \leq \lambda$,
- (d) for every $B \subseteq M$ of cardinality $< \min\{\kappa_1, \kappa_2\}$ for some $\varphi(x, \bar{a}) \in p$ we have $\varphi(x, \bar{a}) \vdash p \upharpoonright B$ (i.e. p under \vdash is $\min\{\kappa_1, \kappa_2\}$ -directed); in fact we can add that for some $a_1 \in C_{p,1}$ and $a_2 \in C_{p,2}$ we have $\varphi(x, \bar{a}) = a_1 < x < a_2$,
- (e) so for $\kappa = \min\{\kappa_1, \kappa_2\}$ we have a decomposition which is trivial in some sense: either we have $\bar{c} = \langle \rangle$ or we have $\bar{d} = \bar{c}$,
- (f) if, e.g., B_1 is an unbounded subset of $C_{p,1}$ of cardinality κ_1 and c, d realize in \mathfrak{C} (where $M \prec \mathfrak{C}$) the type p and $\mathfrak{C} \models c < d$, then $\text{tp}(c, M \cup \{d\})$ is finitely satisfiable in B_1 and for every $A \subseteq M$ of cardinality $< \kappa_2$ for some formula $\varphi(x, \bar{a}) \in p$ we have $\varphi(x, \bar{a}) \wedge (c < x) \vdash \text{tp}(d, A \cup \{c\})$.

Discussion 2.19. Note: if T is stable and $\mathbf{x} \in \text{mxK}_{\kappa, \theta}$ is normal then $\bar{\mathbf{d}}_{\mathbf{x}} \subseteq \text{dcl}(\text{Rang}(\bar{\mathbf{c}}_{\mathbf{x}}))$ recalling $\text{dcl}(A) = \{b : b \text{ is definable over } A, \text{ equivalently } \text{tp}(c, A) = \text{tp}(b, A) \Rightarrow c = b\}$.

Claim 2.20. (0) In Claim 2.16 if $\text{cf}(\kappa) > \theta + |T|$ then we can choose $\psi = \psi(\bar{x}, \bar{y}', \bar{z}')$ such that it depends on $\mathbf{x}, \bar{d}, \varphi(\bar{x}, \bar{y}, \bar{z})$ but not on B' .

(0A) In Claim 2.16 if $\text{cf}(\kappa) > 2^{\theta + |T|}$ then we can fix also $q = \text{tp}(\bar{e}, C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$. If $\text{cf}(\kappa) > 2^{\theta + |T| + |B_{\mathbf{x}}|}$ then moreover we can fix $q = \text{tp}(\bar{e}, B_{\mathbf{x}} \cup C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$.

- (1) Assume that $\mathbf{x} \in K_0$ and $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ are such that \bar{y}, \bar{z} are finite and $\text{lg}(\bar{x}) = \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}})$.

Then the following set is finite: $J = J_{\mathbf{x}, \varphi} = \{t \in I_{\mathbf{x}} : \text{there are } \bar{\alpha} = \langle \alpha_{\ell} : \ell < \text{lg}(\bar{y}) \rangle \text{ and a sequence } \bar{b} \in {}^{\text{lg}(\bar{y})}(A_{\mathbf{x}} \cup \{\bar{c}_{\mathbf{x}, s} : s \in I_{\mathbf{x}} \setminus \{t\}\}) \text{ such}$

that $\ell < \text{lg}(\bar{y}) \Rightarrow \alpha_\ell < \text{lg}(\bar{c}_{\mathbf{x},t,0})$ and $\mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \langle (\bar{c}_{\mathbf{x},t,0})_{\alpha_\ell} : \ell < \text{lg}(\bar{y}) \rangle, \bar{b}] \wedge \neg \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \langle (\bar{c}_{\mathbf{x},t,1})_{\alpha_\ell} : \ell < \text{lg}(\bar{y}) \rangle, \bar{b}]$.

- (2) Moreover, there is a bound on $|J|$ depends just on φ and T .
- (3) For any $\mathbf{x} \in K_{<\aleph_0, <\aleph_0}$ and φ there is \mathbf{y} satisfying $\mathbf{x} \leq_2 \mathbf{y} \in K_{<\aleph_0, <\aleph_0}$ such that $I_{\mathbf{y}} \setminus I_{\mathbf{x}}$ is finite and the local version of maximality holds, i.e.

$$\mathbf{y} \leq_2 \mathbf{z} \in K_{<\aleph_0, <\aleph_0} \Rightarrow J_{\mathbf{z},\varphi} = J_{\mathbf{y},\varphi}.$$

- (4) For any $\mathbf{x} \in K_{<\aleph_0, <\aleph_0}$ and sequence $\langle \varphi_n : n < \omega \rangle$ we can find $\langle \mathbf{x}_n : n < \omega \rangle$ such that $\mathbf{x}_0 = \mathbf{x}, \mathbf{x}_n \leq_2 \mathbf{x}_{n+1} \in K_{<\aleph_0, <\aleph_0}, I_{\mathbf{x}_n} \setminus I_{\mathbf{x}}$ finite and $\mathbf{x}_{n+1}, \varphi_n$ satisfies the demands on \mathbf{y}, φ above for every n .

Proof. (0) As there are $\leq \theta + |T| < \text{cf}(\kappa)$ possible choices of ψ and the set of possible B 's is $(\text{cf}(\kappa))$ -directed.

(0A) The proof is similar.

- (1) (2) Let n_1 be minimal such that: there are no $\bar{b}_\ell \in {}^{\ell g(\bar{y})}\mathfrak{C}, \bar{c}_\ell \in {}^{\ell g(\bar{z})}\mathfrak{C}$ for $\ell < n_1$ such that $\langle \varphi(\bar{x}, \bar{b}_\ell, \bar{c}_\ell) : \ell < n_1 \rangle$ an independent sequence of formulas; note that n_1 exists as T is a dependent theory. Let n_2 be minimal such that if $u_i \in [n_2 \setminus \{i\}]^{\ell g(\bar{z})}$ for $i < n_2$ then for some $v \in [n_2]^{n_1}$ we have $i, j \in v \Rightarrow i \notin u_j$ (the Δ -system lemma for finite sets, see Fact 0.5). Now n_2 is a bound as required by the proof of Theorem 2.12.

(3), (4) follow. □2.20

Conclusion 2.21. If $\mathbf{x} \in \text{mxK}_{\lambda, \kappa, \theta}^\ell$ and $\text{cf}(\kappa) > \theta + |T|$ then we can find $\bar{e} \in {}^\theta \mathfrak{C}$ and $\langle \psi_{\varphi(\bar{x}, \bar{y}, \bar{z})}(\bar{x}, \bar{e}, \bar{\mathbf{c}}_{\mathbf{x}}) : \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T) \rangle$ satisfying $\text{lg}(\bar{x}) = \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}}), \text{lg}(\bar{z}) = \bar{\mathbf{c}}_{\mathbf{x}}, \text{lg}(\bar{y}) = \theta$ such that: for each $\varphi(\bar{x}, \bar{y}, \bar{z})$ with $\text{lg}(\bar{x}) = \text{lg}(\bar{\mathbf{d}}_{\mathbf{x}}), \text{lg}(\bar{y}) = \theta, \text{lg}(\bar{z}) = \bar{\mathbf{c}}_{\mathbf{x}}$ (but φ depends just on finitely many variables) we have:

- $\psi_{\varphi(\bar{x}, \bar{y}, \bar{z})}(\bar{x}, \bar{e}, \bar{\mathbf{c}}_{\mathbf{x}}) \in \text{tp}(\bar{\mathbf{d}}_{\mathbf{x}}, \bar{e} \cup \bar{\mathbf{c}}_{\mathbf{x}})$,
- $\psi_{\varphi(\bar{x}, \bar{y}, \bar{z})}(\bar{x}, \bar{e}, \bar{\mathbf{c}}_{\mathbf{x}}) \vdash \{ \varphi(\bar{x}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}})^{\mathbf{t}} : \bar{b} \in {}^{\ell g(\bar{y})}M \text{ and } \mathbf{t} \in \{0, 1\} \text{ such that } \mathfrak{C} \models \varphi[\bar{\mathbf{d}}_{\mathbf{x}}, \bar{b}, \bar{\mathbf{c}}_{\mathbf{x}}]^{\mathbf{t}} \}$
- $\text{tp}(\bar{e}, M_{\mathbf{x}} \cup \bar{\mathbf{c}}_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}})$ is finitely satisfiable in $M_{\mathbf{x}}$.

Proof. By Claims 2.15 and 2.20 and compactness (i.e. saturation of \mathfrak{C}). □2.21

On strongly/strongly² dependent theories see [21].

Remark 2.22. (1) If T is strongly dependent then in Claim 2.20(1), if \mathbf{x} satisfies $t \in I_{\mathbf{x}} \Rightarrow n_t = \omega$ then for each $n < \omega$ even the set $\mathbf{J}_{\mathbf{x}} = \cup \{ \mathbf{J}_{\mathbf{x}, \varphi} : \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T) \text{ as there and } \text{lg}(\bar{z}) < n \}$ is finite.

(2) If T is strongly dependent² then above we can allow $n = \omega$.

(3) The proofs are similar.

We now turn to exact saturation. We first prove Lemma 2.2, second we prove in Example 2.23 that some independent T 's satisfy it, even give a sufficient criterion.

Third, we give a sufficient condition for T to satisfy the theorem — in 2.26 — the existence of a stable indiscernible set (Definition 2.24).

Proof of Lemma 2.2 (The Singular Exact Saturation Lemma). Let $\theta = |T|$. As M is not κ^+ -saturated, there are $A \subseteq M$ of cardinality $\leq \kappa$ and $p \in \mathbf{S}^1(A)$ omitted by M . Let $d \in \mathfrak{C}$ realize p . By Theorem 2.4, there is $\bar{c} \in {}^\theta \mathfrak{C}$ as there for $\langle d \rangle$, so in particular such that $\text{tp}(\bar{c}, M)$ does not split over some $N \prec M$ of cardinality $< \kappa$. Let $\langle B_i : i < \text{cf}(\kappa) \rangle$ be an \subseteq -increasing sequence of sets with union $N \cup A$ such that $i < \text{cf}(\kappa) \Rightarrow |B_i| < \kappa$ and $N \subseteq B_0$. Now we choose $A_i \subseteq M$ by induction on $i < \text{cf}(\kappa)$ such that A_i is of cardinality $\leq \theta$ and $\text{tp}(d, A_i \cup \bar{c}) \vdash \text{tp}(d, B_i^+)$ where $B_i^+ := B_i \cup \bigcup \{A_j : j < i\}$.

The choice is possible as $|B_i^+| < \kappa$ by Theorem 2.4, i.e. by the choice of \bar{c} . Next we can find $A_\kappa \subseteq M$ of cardinality $\leq \theta$ such that $\text{tp}(d, A_\kappa \cup \bar{c}) \vdash \text{tp}(d, \bigcup_{i < \kappa} A_i)$, possible as $|\bigcup_{i < \text{cf}(\kappa)} A_i| \leq \theta + \text{cf}(\kappa) < \kappa$. Let $B^+ = \bigcup \{B_i^+ : i < \text{cf}(\kappa)\} \cup A_\kappa$ so $|B^+| = \kappa$ and we ask the following question:

⊙ is there an elementary mapping f (or automorphism of \mathfrak{C}) such that $f \upharpoonright B^+$ is the identity and $f(\bar{c}) \in {}^\theta M$?

If yes, then let $d' \in M$ realize $f(\text{tp}(d, A_\kappa \cup \bar{c})) \in \mathbf{S}(A_\kappa \cup f(\bar{c}))$ hence there is an elementary mapping g satisfying $g \upharpoonright A_\kappa = \text{id}_{A_\kappa} = f \upharpoonright A_\kappa$, $g(\bar{c}) = f(\bar{c})$ and $g(d) = d'$. Hence easily for each $i < \text{cf}(\kappa)$ the sequence $\langle d' \rangle \wedge f(\bar{c})$ realizes $f(\text{tp}(\langle d \rangle \wedge \bar{c}, A_i))$ hence it realizes also $f(\text{tp}(\langle d \rangle \wedge \bar{c}, B_i^+))$; so d' realizes $f(\text{tp}(d, B_i^+))$. But as B_i^+ increases with i it realizes $f(\text{tp}(d, \bigcup \{B_i^+ : i < \text{cf}(\kappa)\}))$, but $A \subseteq \bigcup \{B_i : i < \text{cf}(\kappa)\} \subseteq \bigcup \{B_i^+ : i < \kappa\}$ hence d' realizes $\text{tp}(d, A)$, but $d' \in M$ contradicting the choice of p, A, d .

So the answer to the question is no, which gives clauses (a) and (b) of the desired conclusion. Let $B_i^* = B_i^+ \cup A_\kappa$ hence $\langle B_i^* : i < \text{cf}(\kappa) \rangle$ is \subseteq -increasing with union B^+ and $|B_i| < \kappa$ for $i < \text{cf}(\kappa)$.

As for clause (c), we choose \bar{c}_ε by induction on $\varepsilon < \kappa$ such that:

- (*) (a) $\bar{c}_\varepsilon \in {}^\theta M$,
- (b) \bar{c}_ε realizes $\text{tp}(\bar{c}, \bigcup \{\bar{c}_\zeta : \zeta < \varepsilon\} \cup N)$,
- (c) for even ε , if possible \bar{c}_ε does not realize $\text{tp}(\bar{c}, N \cup B^+)$ hence for some $\alpha = \alpha_\varepsilon < \text{cf}(\kappa)$, \bar{c}_ε does not realize $\text{tp}(\bar{c}, B_\alpha^*)$,
- (d) for even ε , if α_ε is well defined, it is minimal,
- (e) for odd ε , $\alpha_\varepsilon = \alpha_{\varepsilon-1}$ and \bar{c}_ε realizes $\text{tp}(\bar{c}, B_\alpha^*)$ or even $\text{tp}(\bar{c}, \bigcup \{\bar{c}_\zeta : \zeta < \varepsilon\} \cup N \cup B_{\alpha_\varepsilon}^*)$.

There is no problem in carrying the induction, by Fact 1.2(2) the sequence $\langle \bar{c}_\varepsilon : \varepsilon < \kappa \rangle$ is indiscernible over N . Also obviously the sequence $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ is non-decreasing; if $\alpha_* := \bigcup \{\alpha_\varepsilon : \varepsilon < \kappa\}$ is equal to $\text{cf}(\kappa)$ or α_ε not well defined for some ε we are done; that is letting \bar{c}_κ from ${}^\theta M$ be such that $\langle \bar{c}_\varepsilon : \varepsilon \leq \kappa \rangle$ is an indiscernible sequence over N , necessarily \bar{c}_κ realizes $\text{tp}(\bar{c}, N \cup B_i^*)$ for each $i < \kappa$ hence realizes $\text{tp}(\bar{c}, B^+)$ contradiction to “⊙ fails”. But the non-existence of \bar{c}_κ and the properties of $\langle \bar{c}_i : i < \kappa \rangle$ are as promised in clause (c) of the lemma, so we are indeed done.

So assume $\alpha_* < \text{cf}(\kappa)$, so for each $\varepsilon < \kappa$ there is a formula $\varphi_\varepsilon(\bar{x}, \bar{e}_\varepsilon) \in \text{tp}(\bar{c}, B_{\alpha_{2\varepsilon}}^*) \subseteq \text{tp}(\bar{c}, B_{\alpha_*}^*)$ with $\ell g(\bar{x}) = \ell g(\bar{c})$ such that $\mathfrak{C} \models \neg\varphi_\varepsilon[\bar{c}, \bar{e}_\varepsilon]$, but by clause (e) of (*) necessarily $\mathfrak{C} \models \varphi_\varepsilon[\bar{c}_{2\varepsilon+1}, \bar{e}_\varepsilon]$. As $\kappa > |B_{\alpha_*}^*| + |T|$ for some formula $\varphi(\bar{x}, \bar{e})$ we have $\{\varepsilon < \kappa : \varphi_\varepsilon(\bar{x}, \bar{e}_\varepsilon) = \varphi(\bar{x}, \bar{e}_\varepsilon)\}$ is infinite. But this contradicts T being dependent, so we are done proving clause (c).

Clause (c)⁺, that is “moreover, there is an ultrafilter \mathcal{D} on N ”, follows when we use the version of mxK from Lemma 2.13(2A) or Definition 2.33, Claims 2.34 and 2.35 below. □_{2.2}

We may have hoped that Lemma 2.2 characterize being dependent, but this is not so. Clarification when this property (characterization of exactly κ -saturated $\kappa > \text{cf}(\kappa) > |T|$, as in Lemma 2.2) holds is given by the following.

Example 2.23. (1) There is an independent T such that: if T has an exactly κ -saturated model then κ is regular. In fact, this is a sufficient condition.
 (2) The same holds for exactly κ -compact, $\kappa > \aleph_0$.

Proof. We use T which satisfies “The Chang Trick” from his proof of his two cardinal theorem $(\aleph_1, \aleph_0) \rightarrow (\lambda^+, \lambda)$ when $\lambda = \lambda^{<\lambda}$; the use is not an incident, he uses such T to overcome a related problem in his proof.

The condition is:

- ⊗ for some predicate $R(x, y) \in \tau_T$ written xRy (or just a formula $\varphi_*(x, y) \in \mathbb{L}(\tau_T)$) we have^P:
 - (a) the empty set can be coded, that is $\exists y \forall x (\neg xRy)$,
 - (b) we can add to a coded set one element, that is $(\forall x, y)(\exists y_1)(\forall x_1)[x_1Ry_1 \equiv (x_1Ry \vee x_1 = x)]$.

Note: for any model M , if $R \notin \tau_M$, M an infinite model, let $\langle u_b : b \in M \rangle$ list the finite subsets of M , and we expand M to M^+ by choosing $R^{\mu^+} = \{(a, b) : a \in u_b \text{ and } b \in M\}$, then $\text{Th}(M^+)$ is as required.

So assume $\kappa = \Sigma\{\kappa_i : i < \text{cf}(\kappa)\}$, $\kappa_i < \kappa_j < \kappa$ for $i < j < \text{cf}(\kappa)$, $\text{cf}(\kappa) < \kappa$ and M is κ -saturated. Let $A \subseteq M$, $|A| = \kappa$ and $p \in \mathbf{S}(A, M)$. Let $A = \cup\{A_i : i < \text{cf}(\kappa)\}$, $|A_i| \leq \kappa_i$, A_i increasing with i . Let c_i realize $p \upharpoonright A_i$.

By induction on i , we choose $b_i \in M$ which realizes the type

$$\begin{aligned} p_i(y) = & \{c_jRy : j < \text{cf}(\kappa) \text{ and } j \geq i\} \\ & \cup \{(\forall x)(xRy \rightarrow xRb_j) : j < i\} \\ & \cup \{(\forall x)(xRy \rightarrow \varphi(x, \bar{a})) : \varphi(\bar{x}, \bar{a}) \in p \upharpoonright A_i\}. \end{aligned}$$

Arriving to the i th stage by ⊗ and the induction hypothesis on i , $p_i(y)$ is finitely satisfiable in M .

^PInstead (a) + (b) we can have (a)' $(\forall x_0, \dots, x_{n-1})(\forall z)(zRy = \bigvee_{\ell < n} z = x_\ell)$ for every n .

[Why? Let $p'(y) \subseteq p_i(y)$ be finite so it has the form $\{c_j R y : j \in u\} \cup \{(\forall x)(x R y \rightarrow x R b_j) : j \in v\} \cup \{(\forall x)(x R y \rightarrow \varphi_\ell(x, \bar{a}_\ell)) : \ell < n\}$ where $u \subseteq [i, \text{cf}(\kappa))$ is finite, $v \subseteq i$ is finite and $\varphi_\ell(\bar{x}, \bar{a}_\ell) \in p \upharpoonright A_i$ for $\ell < n$. By \otimes we can find $c \in M$ such that $M \models (\forall x)(x R c \equiv (\bigvee_{j \in v} x = c_j))$, thus c realizes $p'(y)$.]

But $|p_i| < \kappa$ so we can choose b_i .

Now $\{x R b_i : i < \text{cf}(\kappa)\}$ is a set of formulas finitely satisfiable in M of cardinality $< \kappa$ and any element realizing it realizes p . □_{2.23}

A weak complement to Theorem 2.4 is Claim 2.26 but first recall the following.

Definition 2.24 (T not necessarily dependent). (1) $\mathbf{I} \subseteq {}^\alpha \mathfrak{C}$ is a stable indiscernible set when: \mathbf{I} is an infinite indiscernible set and $\text{Av}(\mathbf{I}, \mathfrak{C})$ is well defined, i.e. for any $\varphi(\bar{x}, \bar{y})$, $\text{lg}(\bar{x}) = \alpha$ and $\bar{b} \in {}^{\text{lg}(\bar{y})} \mathfrak{C}$ either $\varphi(\mathbf{I}, \bar{b})$ or $\neg \varphi(\mathbf{I}, \bar{b})$ is finite.
 (2) \mathbf{I} is a dependent indiscernible sequence when: I is a linear order and $\mathbf{I} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence and for every formula $\varphi(\bar{x}, \bar{b})$, $\text{lg}(\bar{x}) = \text{lg}(\bar{a}_t)$, there is a convex equivalent relation E on I with finitely many equivalence classes such that $sEt \Rightarrow \varphi[\bar{a}_t, \bar{b}] \equiv \varphi[\bar{a}_s, \bar{b}]$.

Fact 2.25. If T is dependent, $\mathbf{I} \subseteq {}^\alpha \mathfrak{C}$ is a stable indiscernible set if and only if \mathbf{I} is an infinite indiscernible set.

Proof. By [13, 1.28]. □_{2.25}

Claim 2.26. (1) Assume (T is dependent and) there is an infinite indiscernible set $\mathbf{I} \subseteq \mathfrak{C}$. If $\kappa^+ = 2^\kappa$ and $\kappa > |T|$ then^a T has an exactly κ -saturated model.
 (2) Assume (T not necessarily dependent), $\mathbf{I} \subseteq \mathfrak{C}$ is a stable indiscernible set. Then the conclusion of part (1) holds.
 (3) In parts (1) and (2) the conclusion holds for T if the assumption holds for T^{eq} .

Remark 2.27. (1) Of course, trivially if for some non-zero ordinal α there is an infinite indiscernible set $\mathbf{I} \subseteq {}^\alpha \mathfrak{C}$ then for some $i < \alpha$, $\{(\bar{a})_i : \bar{a} \in \mathbf{I}\}$ is an infinite indiscernible set.
 (2) But we could use below indiscernible set $\mathbf{I} \subseteq {}^\alpha \mathfrak{C}$.
 (3) On indiscernible sets for T dependent see [13, §1], we use it freely.
 (4) Of course: if $\{\bar{a}_t : t \in I\} \subseteq {}^\alpha \mathfrak{C}$ is an infinite indiscernible set/a stable indiscernible set, $u \subseteq \alpha$ and $\langle \bar{a}_t \upharpoonright u : t \in I \rangle$ is not constant then $\{\bar{a}_t \upharpoonright u : t \in I\} \subseteq {}^u \mathfrak{C}$ is an infinite indiscernible set/a stable indiscernible set. So using singletons in Claim 2.26 is not a loss.
 (5) Recall that if I is a linear order and $I_1, I_2 \subseteq I$ are infinite, $\bar{a}_t \in {}^\beta \mathfrak{C}$ for $t \in I$ and $\mathbf{I} = \langle \bar{a}_t : t \in I \rangle$ is an indiscernible sequence then $\mathbf{I} \upharpoonright I_1$ is stable if and only if $\mathbf{I} \upharpoonright I_2$ is stable.

This is easy.

^aIn [25], we try to eliminate the set theoretic assumption “ $2^\kappa = \kappa^+$ ”.

Proof. (1) By part (2) (and Fact 2.25).

(2) Let $\mathbf{I} = \{a_\alpha : \alpha < \kappa\} \subseteq \mathfrak{C}$ be an infinite stable indiscernible set. Now easily

\odot_1 for any $\bar{b} \in {}^{\omega>}\mathfrak{C}$ for some $\mathbf{J} \in [\mathbf{I}]^{\leq|T|}$, the set $\mathbf{I} \setminus \mathbf{J}$ is an indiscernible set over $\bar{b} \cup \mathbf{J}$.

[Why? By the definition but we elaborate. First prove that for any n and $\varphi = \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{b}')$ for some finite \mathbf{J}_φ we have $\mathfrak{C} \models \text{“}\varphi[a_0, \dots, a_{n-1}, \bar{b}'] \equiv \varphi[a'_0, \dots, a'_{n-1}, \bar{b}']\text{”}$ when $a_0, \dots, a_{n-1}, a'_0, \dots, a'_{n-1} \in \mathbf{I} \setminus \mathbf{J}_\varphi$ with no repetitions. Second, use transitivity of equivalence to show it suffices that $a_0, \dots, a_{n-1} \in \mathbf{I} \setminus \mathbf{J}_\varphi$ with no repetition and $a'_0, \dots, a'_{n-1} \in \mathbf{I} \setminus \mathbf{J}_\varphi$ with no repetitions. Lastly, we choose $\mathbf{J}_k \subseteq \mathbf{I}$ by induction on k such that $\|\mathbf{J}_k\| \leq |T|, m < k \Rightarrow \mathbf{J}_m \subseteq \mathbf{J}_k \subseteq \mathbf{I}$ and if $k = m + 1$ and $\varphi = \varphi(\bar{x}_0, \dots, \bar{x}_{n-1}, \bar{b}'), \bar{b}' \subseteq (\cup \mathbf{J}_m) \cup \bar{b}$ then we can above choose $\mathbf{J}_\varphi \subseteq \mathbf{J}_k$. Now $\bigcup_k \mathbf{J}_k$ is as required.]

\odot_2 the following conditions on $\bar{b} \in {}^{\omega>}\mathfrak{C}$ are equivalent:

- (a) $\text{tp}(\bar{b}, \mathbf{I}), \text{Av}(\mathbf{I}, \mathbf{I})$ are weakly orthogonal,
- (b) for some $\mathbf{J} \in [\mathbf{I}]^{\leq|T|}$ we have $\text{tp}(\bar{b}, \mathbf{J}) \vdash \text{tp}(\bar{b}, \mathbf{I})$.

[Why? Easy using \odot_1 .]

\odot_3 let $\mathbf{D} = \mathbf{D}_\mathbf{I} = \{p \in \mathbf{S}^{<\omega}(\mathbf{I}) : p \text{ weakly orthogonal to } \text{Av}(\mathbf{I}, \mathbf{I})\}$.

We define the following:

\odot_4 we say A is a \mathbf{D} -set if $\bar{a} \in {}^{\omega>}A \Rightarrow \text{tp}(\bar{a}, \mathbf{I}) \in \mathbf{D}_\mathbf{I}$,

\odot_5 if $\mathbf{I} \subseteq A$ we let $\mathbf{S}_\mathbf{D}^m(A) = \{\text{tp}(\bar{b}, A) : A \cup \bar{b} \text{ is a } \mathbf{D}\text{-set and } \ell g(\bar{b}) = m\}$.

We note

\odot_6 (a) if A is a \mathbf{D} -set and $\mathbf{I} \subseteq A$ then $\text{Av}(\mathbf{I}, \mathbf{I}) \vdash \text{Av}(\mathbf{I}, A)$; hence if $A = |M|$, then M is not κ^+ -saturated,

(b) A is a \mathbf{D} -set if and only if $A \cup \mathbf{I}$ is a \mathbf{D} -set.

\odot_7 If A is a \mathbf{D} -set of cardinality $< \kappa$ and $p \in \mathbf{S}_\mathbf{D}^m(A \cup \mathbf{I})$ then for some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $\leq |A| + |T|$ we have $p \upharpoonright (A \cup \mathbf{J}) \vdash p$.

[Why? By \odot_2 and \odot_6 .]

\odot_8 if $\langle A_\alpha : \alpha < \delta \rangle$ is an \subseteq -increasing sequence of \mathbf{D} -sets, then $A_\delta := \cup \{A_\alpha : \alpha < \delta\}$ is a \mathbf{D} -set.

[Why? By the definition of a \mathbf{D} -set.]

\odot_9 if $\langle A_\alpha : \alpha \leq \delta \rangle$ is \subseteq -increasing continuous sequence of \mathbf{D} -sets, $\mathbf{I} \subseteq A_0$ and $p \in \mathbf{S}^m(A_\delta)$ then $p \in \mathbf{S}_\mathbf{D}^m(A_\delta) \Leftrightarrow \bigwedge_{\alpha < \delta} p \upharpoonright A_\alpha \in \mathbf{S}_\mathbf{D}^m(A_\alpha)$.

[Why? By the definition of $\mathbf{S}_\mathbf{D}^m(-)$ and \odot_8 .]

Now comes a major point

\odot_{10} if $A \subseteq \mathfrak{C}$ and $|A| < \kappa$ then we can find \mathbf{I}_1 such that: \mathbf{I}_1 is an indiscernible set, $\mathbf{I} \subseteq \mathbf{I}_1$, $|\mathbf{I}_1 \setminus \mathbf{I}| \leq |T| + |A|$ and A is a $\mathbf{D}_{\mathbf{I}_1}$ -set.

[Why? Let $\theta = |A| + |T|$ and we try by induction on $\alpha < \theta^+$ to choose an element a_α of \mathfrak{C} which realizes $\text{Av}(\mathbf{I}, \{a_\beta : \beta < \alpha\} \cup \mathbf{I})$ but α is even if and only if a_α realizes $\text{Av}(\mathbf{I}, \cup\{a_\beta : \beta < \alpha\} \cup A \cup \mathbf{I})$. But if we succeed to carry the induction clearly $\mathbf{I}^+ := \mathbf{I} \cup \{a_\alpha : \alpha < \theta^+\}$ is an indiscernible set, and a stable one (recalling Remark 2.27(5)) hence for some $\mathbf{J} \subseteq \mathbf{I}^+$ of cardinality $\leq |A| + |T|$, also $\mathbf{I}^+ \setminus \mathbf{J}$ is an indiscernible set over $A \cup \mathbf{J}$, but necessarily $\mathbf{J} \subseteq \mathbf{I} \cup \{a_\beta : \beta < \alpha\}$ for some $\alpha < \theta^+$, easy contradiction to the choice of the a_α 's.]

\odot_{11} if $A_1 \subseteq A_2$ are $\mathbf{D}_{\mathbf{I}}$ -sets, $|A_2| \leq \kappa$, $|A_1| < \kappa$ and $p \in \mathbf{S}_{\mathbf{D}}^m(A_1 \cup \mathbf{I})$ then there is $q \in \mathbf{S}_{\mathbf{D}}^m(A_2 \cup \mathbf{I})$ extending p .

[Why \odot_{11} ? By \odot_9 without loss of generality $A_2 = A_1 \cup \{b\}$, so $|A_2| < \kappa$ and let $\theta = |A_2| + |T|$. We can find \bar{c} realizing $p(\bar{y})$ and let $A = A_1 \cup \{b\} \cup \bar{c} = A_2 \cup \bar{c}$. So by \odot_{10} there is \mathbf{I}^+ such that: \mathbf{I}^+ is an indiscernible set, $\mathbf{I} \subseteq \mathbf{I}^+$, $|\mathbf{I}^+ \setminus \mathbf{I}| \leq \theta := |A| + |T| = |A_1| + |T| < \kappa$ and A is a $\mathbf{D}_{\mathbf{I}^+}$ -set. As A_2 is a $\mathbf{D}_{\mathbf{I}}$ -set we can find $\mathbf{J}_1 \subseteq \mathbf{I}$ of cardinality $\leq |A_2| + |T| \leq \theta < \kappa$ satisfying $\text{tp}(A_2, \mathbf{J}_1) \vdash \text{tp}(A_2, \mathbf{I})$. Also $A_1 \cup \bar{c}$ is a $\mathbf{D}_{\mathbf{I}}$ -set (as A_1 is a $\mathbf{D}_{\mathbf{I}}$ -set and \bar{c} realizes $p(\bar{y}) \in \mathbf{S}_{\mathbf{D}}^m(A_1 \cup \mathbf{I})$) hence there is $\mathbf{J}_2 \subseteq \mathbf{I}$ of cardinality $\leq |A_1 \cup \bar{c}| + |T| = \theta < \kappa$ such that $\text{tp}(A_1 \cup \bar{c}, \mathbf{J}_2) \vdash \text{tp}(A_1 \cup \bar{c}, \mathbf{I})$. Lastly, as A is a $\mathbf{D}_{\mathbf{I}^+}$ -set there is $\mathbf{J}_3 \subseteq \mathbf{I}^+$ of cardinality $\leq |A| + |T| = \theta$ such that $\text{tp}(A, \mathbf{J}_3) \vdash \text{tp}(A, \mathbf{I}^+)$.

As $\text{tp}(A_2, \mathbf{J}_1) \vdash \text{tp}(A_2, \mathbf{I})$ necessarily $\text{tp}(A_2, \mathbf{J}_1) \vdash \text{tp}(A_2, \mathbf{I}^+)$. Similarly $\text{tp}(A_1 \cup \bar{c}, \mathbf{J}_2) \vdash \text{tp}(A_2 \cup \bar{c}, \mathbf{I}^+)$. By cardinality considerations there is a permutation h of \mathbf{I}^+ which is the identity on $\mathbf{J}_1, \mathbf{J}_2$ and $\mathbf{J}_3 \cap \mathbf{I}$ and maps $\mathbf{J}_3 \setminus \mathbf{I}$ into \mathbf{I} . As \mathbf{I}^+ is an indiscernible set, h is an elementary mapping (of \mathfrak{C}). As $h \upharpoonright \mathbf{J}_1$ is the identity and $\text{tp}(A_2, \mathbf{J}_1) \vdash \text{tp}(A_2, \mathbf{I}^+)$, see above also $h \cup \text{id}_{A_2}$ is an elementary mapping hence there is an automorphism g of \mathfrak{C} extending $h \cup \text{id}_{A_2}$. As $\text{tp}(A_1 \cup \bar{c}, \mathbf{J}_2) \vdash \text{tp}(A_1 \cup \bar{c}, \mathbf{I}^+)$ and $h \upharpoonright \mathbf{J}_2 = \text{id}_{\mathbf{J}_2}$, $h \upharpoonright A_1 = \text{id}_{A_1}$ (recalling $A_1 \subseteq A_2$) and $h(\mathbf{I}^+) = \mathbf{I}^+$ necessarily $g(\bar{c})$ realizes $\text{tp}(\bar{c}, A_1 \cup \mathbf{I}^+)$ hence it realizes $\text{tp}(\bar{c}, A_1 \cup \mathbf{I})$ which is equal to $p(\bar{y})$. Also $\text{tp}(A_2 \cup \bar{c}, \mathbf{J}_3) \vdash \text{tp}(A_2 \cup \bar{c}, \mathbf{I}^+)$ hence $\text{tp}(A_2 \cup g(\bar{c}), h(\mathbf{J}_3)) \vdash \text{tp}(A_2 \cup h(\bar{c}), \mathbf{I}^+)$ hence $\text{tp}(A_2 \cup g(\bar{c}), h(\mathbf{J}_3)) \vdash \text{tp}(A_2 \cup g(\bar{c}), \mathbf{I})$, but $h(\mathbf{J}_3) \subseteq \mathbf{I}$ has cardinality $< \kappa$. So $A_2 \cup g(\bar{c})$ is a $\mathbf{D}_{\mathbf{I}}$ -set hence $q(\bar{y}) := \text{tp}(g(\bar{c}), A_2 \cup \mathbf{I})$ belongs to $\mathbf{S}_{\mathbf{D}}^m(A_2 \cup \mathbf{I})$ so is as required.]

As $2^\kappa = \kappa^+$, by $\odot_9 + \odot_{11}$ there is $M \supseteq \mathbf{I}$ of cardinality κ^+ which is κ -saturated and is a \mathbf{D} -set hence by \odot_6 is not κ^+ -saturated.

(3) Should be clear. □_{2.26}

Observation 2.28 (Any Complete First-Order T). In \mathfrak{C} there is no infinite indiscernible set if and only if for some n and $\varphi = \varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_T)$, φ is connected and anti-symmetric, i.e. if $a_0, \dots, a_{n-1} \in \mathfrak{C}$ with no repetitions then for

some permutations π_1, π_2 of $\{0, \dots, n-1\}$ we have

$$\mathfrak{C} \models \varphi[a_{\pi_1(0)}, \dots, a_{\pi_1(n-1)}] \wedge \neg \varphi[a_{\pi_2(a)}, \dots, a_{\pi_2(a_{n-1})}].$$

Remark 2.29. (1) The second condition is related to the property (E) of complete first-order theories of Ehrenfeucht [2] which says that the condition holds for some infinite set.

(2) Note that \mathfrak{C} may have no infinite indiscernible set but \mathfrak{C}^{eq} has.

Proof. The implication \Leftarrow is obvious.

So assume the first statement. For $\alpha \leq \omega$ and $\Delta \subseteq \Delta_* := \{\varphi(\bar{x}) : \varphi \in \mathbb{L}(\tau_T), \bar{x} = \langle x_\ell : \ell < n \rangle\}$ let $\Gamma_\Delta^\alpha = \{y_k \neq y_\ell : k < \ell < \alpha\} \cup \{\varphi(y_{k_0}, \dots, y_{k_{n-1}}) \equiv \varphi(y_{\ell_0}, \dots, y_{\ell_{n-1}}) : n < \alpha, \varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_T) \text{ and } k_0, \dots, k_{n-1} < \alpha \text{ without repetitions and } \ell_0, \dots, \ell_{n-1} < \alpha \text{ without repetitions}\}$. Easily $\Gamma_{\Delta_*}^\omega$ is not realized in \mathfrak{C} by the present assumption and $\langle \Gamma_\Delta^k : \Delta \text{ is a finite subset of } \Delta_* \text{ and } k < \omega \rangle$ is \subseteq -increasing with k and Δ with union $\Gamma_{\Delta_*}^\omega$. Hence for some finite $\Delta \subseteq \Delta_*$ and $k < \omega$, the set Γ_Δ^k is not realized in \mathfrak{C} .

Let $\langle \varphi_i(x_0, \dots, x_{n_i-1}) : i < i(*) \rangle$ list Δ so $i(*) < \omega$, so without loss of generality $n_i < k$ for $i < i(*)$. Lastly, we define $\varphi(y_0, \dots, y_{k-1})$, it says: if $\langle y_\ell : \ell < k \rangle$ is without repetitions and $i < i(*)$ is minimal such that $\langle y_\ell : \ell < k \rangle$ is not a $\{\varphi_i(x_0, \dots, x_{n_i-1})\}$ -indiscernible set then $\varphi_i(y_0, \dots, y_{n_i-1})$. Now check. $\square_{2.28}$

Question 2.30. (1) Is there a dependent T such that even in \mathfrak{C}^{eq} there is no infinite indiscernible set but some singular κ of cofinality $> |T|$ there is an exactly κ -saturated model of T ?

- (2) For a dependent theory T , characterize $\{\kappa : \kappa \text{ singular and } T \text{ has exactly } \kappa\text{-saturated models}\}$.
- (3) In both parts we may at least initially restrict ourselves to κ strong limit of large enough cofinality such that $2^\kappa = \kappa^+$.
- (4) Try to eliminate the assumption “ $2^\kappa = \kappa^+$ ” in Claim 2.26 at least when κ is strong limit of cofinality $> |T|$. A natural way is via $\mathcal{P}^-(n)$ -diagrams (as in [12, Chap. XII] and even closer in [10]).

The following in a sense gives a spectrum for $\bar{\mathfrak{d}}/M$.

Claim 2.31. For $\theta \geq |T|$, a model M and sequence $\bar{\mathfrak{d}} \in {}^\theta \geq \mathfrak{C}$, there is a set Θ such that:

- (a) $\Theta \subseteq \Theta_* := \{\kappa : \kappa > \theta \text{ and } M \text{ is } \kappa\text{-saturated}\}$,
- (b) $|\Theta| \leq \theta$,
- (c) if $\kappa \in \Theta$ and $\text{cf}(\kappa) > \theta$ then there is $\mathbf{x} \in \text{mxK}_{\kappa, \leq \theta}$ such that $M_{\mathbf{x}} = M$, $\bar{\mathfrak{d}}_{\mathbf{x}} = \bar{\mathfrak{d}}$ and $|B_{\mathbf{x}}| \leq \theta + \sup(\Theta \cap \kappa)$,
- (d) if $\theta < \kappa \in \Theta$ and $\text{cf}(\kappa) \leq \theta$ then $\sup(\Theta \cap \kappa) = \kappa$.

^rYes: we use singletons y 's.

Proof. Straightforward.

For each $\kappa \in \Theta' := \{\kappa' \in \Theta_* : \kappa' \text{ has cofinality } > \theta\}$ we can find $\mathbf{x}_\kappa \in \text{mxK}_{\kappa, \leq \theta}$ such that $M_{\mathbf{x}} = M, \bar{\mathbf{d}}_{\mathbf{x}} = \bar{\mathbf{d}}$ and for $\kappa \in \Theta_* \setminus \Theta'$ let Θ_κ be a cofinal subset of κ of cardinality $\text{cf}(\kappa) \leq \theta$. Let $f : \Theta' \rightarrow \text{Card}$ be defined by $f(\kappa) = |B_{\mathbf{x}}| + \theta$. Note that Θ' has a maximal member or Θ' has a cofinal subset of cardinality $\leq \theta$. Now we shall choose Θ_n by induction on n such that $\Theta_n \subseteq \Theta_*, |\Theta_n| \leq \theta$ and $n = m+1 \Rightarrow \Theta_m \subseteq \Theta_n$. Let Θ_0 be a cofinal subset of Θ_* of cardinality $\leq \theta$, see above why possible. If $n = m+1$, let $\Theta_n = \{f(\kappa) : \kappa \in \Theta_m \cap \Theta'\} \cup \{\Theta_\kappa : \kappa \in \Theta_m \setminus \Theta'\} \cup \Theta_m$. Now $\cup\{\Theta_n : n < \omega\}$ is as required. □_{2.31}

Discussion 2.32. Note that $\mathbf{P}_{\mathbf{x}}$ in Theorem 2.4 is κ -directed, but in general it is not definable in $M_{\mathbf{x}}$ and even not definable in $(M_{\mathbf{x}})_{[\bar{\mathbf{c}}_{\mathbf{x}}]}$ (or $M_{[B_{\mathbf{x}} + \bar{\mathbf{c}}_{\mathbf{x}}]}$) even by disjunction of types as it depends on $\bar{\mathbf{d}}_{\mathbf{x}}$. So we may consider $\mathbf{P}'_{\mathbf{x}} = \mathbf{P}_{M, \bar{\mathbf{c}}_{\mathbf{x}}, \alpha_{\mathbf{x}}} = \{p : p \in \mathbf{S}^{\alpha(\mathbf{x})}(A \cup \bar{\mathbf{c}}_{\mathbf{x}}) \text{ and } A \subseteq M \text{ has cardinality } < \kappa\}$ ordered as before. Now $\mathbf{P}'_{\mathbf{x}}$ is partially ordered but it is not clear that it is κ -directed. Moreover $(M_{\mathbf{x}})_{[\bar{\mathbf{c}}_{\mathbf{x}}]}$ is not κ -saturated, but is $(\mathbf{D}_{\mathbf{x}}, \kappa)$ -sequence homogeneous for suitable $\mathbf{D}_{\mathbf{x}}$ and $\mathbf{D}_{\mathbf{x}}$ is a good diagram (see, e.g., [8]; see more in [20]). So we can consider the families of such \mathbf{D} 's, fixing $(T \text{ and } \theta)$.

But we can define the order in the κ -saturated $(M_{\mathbf{x}})_{[B_{\mathbf{x}}]}$ which is $\mathbb{L}_{\infty, \kappa}(\tau_T)$ -equivalent to $\mathfrak{C}_{[B_{\mathbf{x}}]}$. In this model we have $\psi(\bar{x}, \bar{y}) \in \mathbb{L}_{\infty, \kappa}$ which is a partial order on the θ -tuples, $\text{lg}(\bar{x}) = \alpha_{\mathbf{x}} = \text{lg}(\bar{y})$.

However, in our case we know more. Letting $\Gamma = \mathbb{L}(\tau_T)$, if $\text{cf}(\kappa) > \theta \geq |T|$ we know that we can find $\bar{\psi} = \langle \psi_\varphi(\bar{x}_\varphi, \bar{y}_\varphi, \bar{z}_\varphi) : \varphi \in \Gamma \rangle$ and the order on the set of $\bar{\mathbf{e}} = (\dots \hat{\bar{e}}_\varphi \hat{\dots})_\varphi$ such that in Claim 2.15 we can choose $\psi = \psi_\varphi$ (easy, see Claim 2.20(0)). If $\text{cf}(\kappa) > 2^\theta$, we can fix there also the type of $\bar{\mathbf{e}}$ over $C_{\mathbf{x}} \cup \bar{\mathbf{d}}_{\mathbf{x}}$.

So

- (*)₁ let $\mathbf{I} = \{\bar{\mathbf{e}} : \bar{\mathbf{e}} \text{ as above}\}$, so \mathbf{I} is type-definable in $(M_{\mathbf{x}})_{[B_{\mathbf{x}}]}$.
- (*)₂ $p_{\bar{\mathbf{e}}} = \{\psi_\varphi(\bar{x}_\varphi, \bar{\mathbf{e}}, \bar{\mathbf{c}}_{\mathbf{x}}) : \varphi \in \Gamma\}$ for $\bar{\mathbf{e}} \in \mathbf{I}$.
- (*)₃ (a) \leq_1 defined by $\bar{\mathbf{e}}_1 \leq \bar{\mathbf{e}}_2$ if $p_{\bar{\mathbf{e}}_1} \vdash p'$ for some p' such that $p_{\bar{\mathbf{e}}_1} \subseteq p' \in \mathbf{S}(B_{\mathbf{x}} \cup \bar{\mathbf{e}}_1)$,
 (b) \mathbf{R} is defined by $\bar{\mathbf{e}}_1 R \bar{\mathbf{e}}_2$ if and only if $\psi_\varphi(\bar{x}, \bar{e}_\varphi^1, \bar{c}) \vdash p' \cap \{\varphi(\bar{x}, \bar{b}, \bar{c}) : \bar{b} \subseteq \text{Rang}(\bar{\mathbf{e}}_1 \cup B_{\mathbf{x}})\}$ for each $\varphi \in \Gamma$ where p' is as above.

There are other variants, we intend to return to this.

We now consider some variants of Definition 2.6.

Definition 2.33. (1) In Definition 2.6 we add and define $K_\ell, K_{\lambda, \kappa, \theta}^\ell$, etc. also for $\ell = 2, 3$ by replacing clause (f)₁ = (f) by (f)_ℓ where:

- (f)₂ if $\ell = 2$ then $\text{tp}(\bar{c}_t, A \cup \{\bar{c}_{s,m} : s <_I t, m < n_s\})$ is finitely satisfiable in B ,

(f)₃ if $\ell = 3$ then for some endless indiscernible sequence $\bar{\mathbf{b}}_t = (\bar{b}_{t,r} : r \in J_t)$ of sequences from B , the sequence $\bar{c}_{t,0}$ realizes^s the type $\text{Av}(A \cup \{\bar{c}_{s,m} : s < I, t, m < n_t\}, \bar{\mathbf{b}}_t)$.

(2) We define $\text{mxK}_{\lambda,\kappa,\theta}^\ell$ similarly.

Claim 2.34. (1) $K_3 \subseteq K_2 \subseteq K_1$.

(2) If $\ell \in \{2, 3\}$ and $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is \leq_1 -increasing in $K_{\lambda,\kappa,<\theta}^\ell$ and $\delta < \text{cf}(\theta)$, $\delta < \text{cf}(\kappa)$ then $\mathbf{x}_\delta = \bigcup_{\alpha < \delta} \mathbf{x}_\alpha$ defined as in Claim 2.7 belongs to $K_{\lambda,\kappa,<\theta}^\ell$ and is a \leq_1 -lub of the sequence.

(3) If $\ell = 0, 1, 2, 3$ and $\bar{\mathbf{d}} \in {}^{\theta^+}\mathfrak{C}$ and M is κ -saturated, $\text{cf}(\kappa) \geq \theta$ then $\mathbf{x} = (M, \emptyset, \langle \rangle, \bar{\mathbf{d}}) \in K_{\kappa,<\theta}^\ell$.

(4) Like Theorem 2.12 for $\ell = 2, 3$, i.e.: if $\text{cf}(\theta) > |T|$, $\text{cf}(\kappa) > \theta$, $\ell \leq 3$ and $\mathbf{x} \in K_{\lambda,\kappa,<\theta}^\ell$ then^t for some \mathbf{y} we have $\mathbf{x} \leq_1 \mathbf{y} \in \text{mxK}_{\lambda,\kappa,<\theta}^\ell$; so in $\text{tp}(\mathbf{c}_\mathbf{x}, M_\mathbf{y} + \mathbf{c}_\mathbf{x})$ we can get $\text{tp}(\bar{c}, M)$ is finitely satisfiable in $M_\mathbf{y} + (C_\mathbf{y} \setminus C_\mathbf{x})$.

(5) If $\mathbf{x} \in K_2$ and $\bar{c} \in C_\mathbf{x}$ then $\text{tp}(\bar{c}, A_\mathbf{x})$ is finitely satisfiable in $B_\mathbf{x}$.

Proof. Similar to the proofs for $\ell = 1$. □_{2.34}

Claim 2.35. (1) In Claim 2.14 we can deal with $K_{\lambda,\kappa,<\theta}^\ell$, $\ell = 2, 3$, i.e. if $\ell = 2$ we should strengthen the assumption to “ q is finitely satisfiable in B ”.

(2) If $\ell = 3$ we should strengthen the assumption to $q = \text{Av}(M_\mathbf{x} \cup C_\mathbf{x}, \mathbf{I}), \mathbf{I}$ an endless indiscernible sequence of cardinality $< \kappa$.

(3) In Claim 2.15 we can deal with $\text{mxK}_{\lambda,\kappa,\theta}^2$.

Proof. Similar to the proof for $\ell = 1$. □_{2.35}

3. Existence of Strict Type Decompositions

We here get a better decomposition, but at a price: using normal ultrafilters (so measurable or supercompact cardinals). Why is the decomposition from Sec. 2 not enough for our purposes? See Explanation 4.5 below.

Hypothesis 3.1. We assume T is dependent, $\mathfrak{C} = \mathfrak{C}_T$ a monster; if not said otherwise, we assume (1) or just (2) where:

- (1) (a) $\lambda = \kappa$ is a measurable cardinal,
- (b) \mathcal{D} is a normal ultrafilter on $I = \lambda$, so I is a linear order,
- (c) $M_\alpha \prec \mathfrak{C}$ is \prec -increasing, $\|M_\alpha\| < \lambda$ for $\alpha < \lambda$,
- (d) $M_\lambda = \bigcup_{\alpha < \lambda} M_\alpha$, by clauses (a) + (c) + (e) necessarily M_λ is saturated,
- (e) M_α is $\|M_\beta\|^+$ -saturated for $\beta < \alpha$,

^sWe may consider \bar{c}_t realizes seeming this makes no difference.

^tSo if $\kappa = \text{cf}(\kappa) > \theta \geq |T|$, M is κ -saturated and $\bar{\mathbf{d}} \in {}^{\theta^+}\mathfrak{C}$ then for some $\bar{c} \in {}^{\theta^+}\mathfrak{C}$ and $B \in [M]^{<\kappa}$ we have $(M, B, \bar{c}, \bar{\mathbf{d}}) \in \text{mxK}_2$.

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(2) (a) I is the following partial order, which is $(< \kappa)$ -directed, and:

- (α) set of elements $\{a \in [\lambda]^{<\kappa} : a \cap \kappa \in \kappa\}$ and
- (β) $s <_I t$ if and only if $s \subseteq t \wedge |s| < \min(\kappa \setminus t)$,
- (b) \mathcal{D} is a fine normal ultrafilter on I and it follows that κ is a measurable cardinal $\leq \lambda$,
- (c) $M_t \prec \mathfrak{C}$, $\|M_t\| < \kappa$ and $s <_I t \Rightarrow M_s \prec M_t$,
- (d) $M_\lambda = \cup\{M_t : t \in I\}$ (by (a)+(c)+(e), M_λ is κ -saturated)
- (e) if $s <_I t$ then M_t is $\|M_s\|^+$ -saturated.

Remark 3.2. (1) So in Hypothesis 3.1 we can define:

- (A) like (2) without the normality and
- (B) (a), (b), (c), (d) of part (2).

(2) Note that we have (1) \Rightarrow (2) \Rightarrow (A) and (2) \Rightarrow (B).

Notation 3.3. (1) In Hypothesis 3.1(1) let $\kappa_I(t) := t$ for $t \in I$, this notation is introduced only for having a uniform treatment of (1) and (2).

(2) In Hypothesis 3.1(2) let $\kappa_I(t) = \text{Min}(\kappa \setminus t)$ for $t \in I$.

Definition 3.4. (Under Hypothesis 3.1(1) or (2) or alternatively (B) from Remark 3.2 so these notions depend on $\langle M_t : t \in I \rangle$.)

(1) For $\mathcal{U} \subseteq I$ (usually $\in \mathcal{D}$), so is a partial order, we say $\langle \bar{a}_t : t \in \mathcal{U} \rangle$ is indiscernible in M_λ over A when ($A \subseteq M_\lambda$ and):

- (a) $\ell g(\bar{a}_t)$ is constant, possibly infinite, and $\bar{a}_t \subseteq M_\lambda$ for $t \in \mathcal{U}$,
- (b) for each n for some p_n for every $t_0 <_I \dots <_I t_{n-1}$ from \mathcal{U} we have $\text{tp}(\bar{a}_{t_0} \hat{\ } \dots \hat{\ } \bar{a}_{t_{n-1}}, A, M_\lambda) = p_n$, see the ninth paragraph of Sec. 0.

(2) We say $\langle \bar{a}_t : t \in \mathcal{U} \rangle$ is fully indiscernible (in M_λ) over A when clauses (a) and (b) above hold and

- (c) if $s <_I t$ are from \mathcal{U} then $\bar{a}_s \subseteq M_t$, recalling \bar{M} is from Hypothesis 3.1,
- (d) if $s \in \mathcal{U}$ then recalling \bar{M} is from Hypothesis 3.1^u the sequence $\langle a_t : t \in \mathcal{U} \cap I_{\geq s} \rangle$ is indiscernible over $M_s \cup A$ where, of course, $I_{\geq s} := \{t \in I : s \leq_I t\}$.

(3) In parts (1) and (2) of the definition we say k -indiscernible when in clause (b) we demand $n \leq k$.

We shall in Theorem 3.10 (see Theorem 3.8) below prove the existence of the following definition.

Definition 3.5. (1) For an infinite linear order J we say $\langle (\bar{c}_t, \bar{d}_t) : t \in J \rangle$ is a strict₁ $(\kappa, < \theta)$ -decomposition over (M, B) (and over M means “for some $B \subseteq M$ of

^uBy normality, i.e. if (1) or (2) or (B) holds, then this follows.

cardinality $< \kappa$) when:

- (a) $B \subseteq M$ is of cardinality $< \kappa$ and $M \prec \mathfrak{C}$ is κ -saturated, but if we write $< \aleph_0$ instead of κ we mean $M \prec \mathfrak{C}$ and if we write 0 instead of κ we replace M by a set $\supseteq B$, but in both cases we omit $|B| < \kappa$
- (b) $\alpha = \ell g(\bar{c}_t), \beta = \ell g(\bar{d}_t)$ are $< \theta$,
- (c) if $t_0 <_J \dots <_J t_k$ then $\text{tp}(\bar{c}_{t_n}, M + \bar{c}_{t_0} \hat{\ } \bar{d}_{t_0} + \dots + \bar{c}_{t_{n-1}} \hat{\ } \bar{d}_{t_{n-1}})$ is increasing with $n \leq k$ and does not split over B ,
- (d) $\langle \bar{c}_t \hat{\ } \bar{d}_t : t \in J \rangle$ is an indiscernible sequence over M ,
- (e) if $s <_J t$ then $\text{tp}(\bar{d}_t, \bar{c}_t + \bar{d}_s) \vdash \text{tp}(\bar{d}_t, \cup \{ \bar{c}_r \hat{\ } \bar{d}_r : r \leq_J s \} \cup \cup \{ \bar{c}_r : r \in J \} \cup M)$,
- (f) for every $A \subseteq M$ of cardinality $< \kappa$ for some $\bar{c} \hat{\ } \bar{d} \in {}^{\alpha+\beta}M$, the sequence $\langle \bar{c} \hat{\ } \bar{d} \rangle \hat{\ } \langle \bar{c}_t \hat{\ } \bar{d}_t : t \in J \rangle$ is an indiscernible sequence over A , so if $\kappa = 0$ this is an empty demand.

(2) We say $\text{strict}_{-1}(\kappa, \theta)$ -decomposition if (in part (1)) we omit clauses (e) and (f).

(3) We say $\text{strict}_0(\kappa, < \theta)$ -decomposition if we omit (f) and weaken (e) to (e)⁻, where

$$(e)^- \text{ if } s <_I t \text{ then } \text{tp}(\bar{d}_t, \bar{c}_t + \bar{d}_s) \vdash \text{tp}(\bar{d}_t, \cup \{ \bar{c}_r \hat{\ } \bar{d}_r : r \leq s \} \cup \bar{c}_t \cup M).$$

(4) Notation:

- If $\theta = \sigma^+$ instead of “ $< \theta$ ” we may write σ .
- If $\kappa = 0$ then M is replaced by a set B , if we write $< \aleph_0$ instead of κ then M is just a model, but in both cases we omit the demand “ $|B| < \kappa$ ”.
- Strict_1 may be written strict .

A natural question about those notions of indiscernibility is about existence results. Now Fact 3.6 is a well-known set-theoretic existence and Theorem 3.8 is existence for dependent theories.

Fact 3.6. (1) If $A \subseteq \mathfrak{C}, |A| < \kappa, \alpha < \kappa, \mathcal{U}_1 \in \mathcal{D}$ and $\bar{a}_t \in {}^\alpha \mathfrak{C}$ for $t \in \mathcal{U}_1$ then for some $\mathcal{U}_2 \subseteq \mathcal{U}_1$ from \mathcal{D} the sequence $\langle \bar{a}_t : t \in \mathcal{U}_2 \rangle$ is indiscernible over A .

(2) If in addition $\bar{a}_t \in {}^\alpha(M_\lambda)$ then we can add “fully indiscernible”.

(3) If $\iota \in \{-1, 0, 1\}$ and $\langle \bar{c}_t, \bar{d}_t \rangle : t \in J$ is a $\text{strict}_\iota(\kappa, < \theta)$ -decomposition over (M, B) and $B \subseteq B_1 \subseteq M, |B_1| < \kappa$ then it is a $\text{strict}_\iota(\kappa, < \theta)$ -decomposition over (M, B_1) . Similarly we can replace M by M' if $B_1 \subseteq M' \subseteq M$ and M_1 satisfies clause (a) in Definition 3.5.

Proof. (1) and (2) By well-known set theory (see [6]).

(3) Obvious. □_{3.6}

Observation 3.7. (1) For some $\mathcal{U}_* \in \mathcal{D}$, for every $t \in \mathcal{U}_*$ the model M_t is $\kappa(t)$ -saturated and $\kappa(t) > |T|$.

(2) If $M \prec \mathfrak{C}$ is κ -saturated, J an infinite linear order $\theta \geq \aleph_0, B \in [M]^{< \kappa}$ and $\bar{c}_t = \langle \rangle = \bar{d}_t$ for $t \in J$ then $\langle \bar{c}_t : \bar{d}_t \rangle : t \in J$ is a $\text{strict}_\iota(\kappa, < \theta)$ -decomposition over (M, B) for $\iota = 1$ hence for $\iota \in \{-1, 0\}$, too.

Proof. Obvious. □3.7

Theorem 3.8. Assume θ satisfies $\kappa > \theta \geq |T|$ and recall $\lambda \geq \kappa$. For every $\gamma(*) < \theta^+$ and $\bar{d} \in \gamma(*)\mathfrak{C}$ there are B and $\langle (\bar{c}_t^\omega, \bar{d}_t^\omega) : t \in \mathcal{U} \cup \{\lambda\} \rangle$ such that:

- ⊠ (a) $\mathcal{U} \in \mathcal{D}$,
- (b) $B \subseteq M_\lambda, |B| < \kappa$,
- (c) $lg(\bar{d}_t^\omega) = \gamma(*) + \theta \cdot \omega$,
- (d) $\bar{d} \trianglelefteq \bar{d}_\lambda^\omega$,
- (e) $\mathbf{x} = (M_\lambda, B, \bar{c}_\lambda^\omega, \bar{d}_\lambda^\omega) \in \text{mxK}_{\kappa, \theta}$,
- (f) $\bar{c}_t^\omega \hat{\ } \bar{d}_t^\omega \subseteq M_\lambda$ realizes $\text{tp}(\bar{c}_\lambda^\omega \hat{\ } \bar{d}_\lambda^\omega, M_t)$ for $t \in \mathcal{U}$,
- (g) $\langle \bar{c}_t^\omega \hat{\ } \bar{d}_t^\omega : t \in \mathcal{U} \rangle$ is fully indiscernible (in M_λ) over B and even over $B \cup \bar{c}_\lambda^\omega \cup \bar{d}_\lambda^\omega$ (see Definition 3.4(2)),
- (h)₁ if $t_0 <_I \dots <_I t_m <_I \dots <_I t_n$ belongs to \mathcal{U} , so $m < n$ and possibly $m + 1 = n$; moreover possibly $0 = m$ then $\text{tp}(\bar{d}_{t_{m+1}}^\omega, \bar{c}_{t_{m+1}}^\omega + \bar{d}_{t_m}^\omega) \vdash \text{tp}(\bar{d}_{t_{m+1}}^\omega, \bar{c}_{t_0}^\omega + \dots + \bar{c}_{t_{m+1}}^\omega + \dots + \bar{c}_{t_n}^\omega + \bar{d}_{t_0}^\omega + \dots + \bar{d}_{t_m}^\omega + M_{t_0})$,
- (h)₂ if $s <_I t$ are from \mathcal{U} then $\bar{c}_s^\omega \hat{\ } \bar{d}_s^\omega$ is from M_t , (actually follows from clause (g)),
- (h)₃ if $t_0 <_I \dots <_I t_n$ are from \mathcal{U} then $\text{tp}(\bar{d}_{t_1}^\omega, \bar{d}_{t_0}^\omega + \bar{c}_{t_1}^\omega) \vdash \text{tp}(\bar{d}_{t_1}^\omega, M_{t_0} + \bar{d}_{t_0}^\omega + \bar{c}_{t_0}^\omega + \bar{c}_{t_1}^\omega + \dots + \bar{c}_{t_n}^\omega)$ (actually this is the case $m = 0$ in (h)₁).

Remark 3.9. We easily can add:

- (i) \mathbf{x} is normal, i.e. $\text{Rang}(\bar{c}_\omega) \subseteq \text{Rang}(\bar{d}_\omega)$.

Proof. First by induction on n we choose $\bar{d}_n, \bar{c}_n, B_n, \langle (\bar{c}_t^n, \bar{d}_t^n) : t \in \mathcal{U}_n \rangle$ and if $n > 0$ also \bar{e}_n, \bar{e}_t^n (for $t \in \mathcal{U}_n$) such that:

- ⊗_n (a) $\bar{d}_n \in \gamma(*) + \theta \cdot n \mathfrak{C}$ and $\bar{d}_0 = \bar{d}$ and $\bar{e}_n \in \theta \mathfrak{C}$,
- (b) $\mathbf{x}_n = (M_\lambda, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa, \theta}$ is normal,
- (c) $\bar{d}_m \triangleleft \bar{d}_n$ if $m < n$,
- (d) $\bar{c}_m = \bar{c}_n \upharpoonright I_{\mathbf{x}_m}$ if $m < n$,
- (e) $\bar{c}_t^n \hat{\ } \bar{d}_t^n$ is from M_λ and realizes $\text{tp}(\bar{c}_n \hat{\ } \bar{d}_n, M_t)$ for $t \in \mathcal{U}_n$,
- (f) $\bar{d}_n = \bar{d}_m \hat{\ } \bar{e}_n$ and $lg(\bar{e}_n) = \theta$ and $\mathbf{x}_m \leq_1 \mathbf{x}_n$ if $n = m + 1$,
- (g) $\langle \bar{c}_t^n \hat{\ } \bar{d}_t^n : t \in \mathcal{U}_n \rangle$ is fully indiscernible over $B_n + \bar{c}_n + \bar{d}_n$,
- (h) $\mathcal{U}_n \in \mathcal{D}$ decrease with n and is $\subseteq \mathcal{U}_*$ from Observation 3.7(1),
- (i) if $s < t$ are from \mathcal{U}_n then $M_s + \bar{c}_s^n + \bar{d}_s^n \subseteq M_t$ (follows by (g)),
- (j) $\text{tp}(\bar{d}_m, \bar{e}_t^n + \bar{c}_m) \vdash \text{tp}(\bar{d}_m, M_t + \bar{c}_m + \bar{c}_t^m + \bar{d}_t^m)$ if $n = m + 1, t \in \mathcal{U}_n$,
- (k) $(M_t, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa_I(t), \theta}$ for $t \in \mathcal{U}_n$,
- (l) if $n = m + 1, k < \omega$ and $t_0 <_I \dots <_I t_k$ are from \mathcal{U}_n then $\text{tp}(\bar{d}_{t_1}^m, \bar{c}_{t_1}^m + \bar{e}_{t_0}^n) \vdash \text{tp}(\bar{d}_{t_1}^m, \sum_{\ell=0}^k \bar{c}_{t_\ell}^m + \bar{d}_{t_0}^m + \bar{e}_{t_0}^n + M_{t_0})$.

Case 1: $n = 0$. First, let $\bar{d}_n = \bar{d}$. Recalling Claim 2.7(4) clearly $\mathbf{y}_n = (M_\lambda, \emptyset, \langle \rangle, \bar{d}_n) \in K_{\kappa, \theta}$.

Second, by Theorem 2.12 we can find B_n, \bar{c}_n such that $\mathbf{y}_n \leq_1 \mathbf{x}_n := (M_\lambda, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa, \theta}$; we can add \mathbf{x}_n is normal by Claim 2.7(6) but then $\ell_g(\bar{d}_n) = \gamma(*) + \theta$.

Third, for $t \in I$ we can choose $\bar{c}_t^n \wedge \bar{d}_t^n$ from M_λ which realizes $\text{tp}(\bar{c}_n \wedge \bar{d}_n, M_t)$.

Fourth, by Fact 3.6 we choose $\mathcal{U}_n \in \mathcal{D}$ such that $\langle \bar{c}_t^n \wedge \bar{d}_t^n : t \in \mathcal{U}_n \rangle$ is a fully indiscernible sequence over $B_n + \bar{d}_n + \bar{c}_n$ and (by the normality of the filter \mathcal{D}) in particular $\bar{c}_s^n \wedge \bar{d}_s^n \subseteq M_t$ when $s < t \in \mathcal{U}_n$ are from I and \mathcal{U}_n is $\subseteq \mathcal{U}_*$ from Claim 3.7. It is easy to check that all the demands hold, recalling \bar{c}_n for $n = 0$ is not required in \otimes_n . In particular for clause (k) use the reflective property of D or Claim 2.10.

Case 2: $n = m + 1$. First, by clause (k) for m and Conclusion 2.21 for each $t \in \mathcal{U}_m$ recalling $M_t + \bar{c}_t^m + \bar{d}_t^m$ is $\subseteq M_\lambda$ and is of cardinality $< \kappa$ and M_λ is κ -saturated, there is $\bar{e}_t^{n,*} \in {}^\theta(M_\lambda)$ such that:

$$(*)_1 \text{ tp}(\bar{d}_m, \bar{c}_m + \bar{e}_t^{n,*}) \vdash \text{tp}(\bar{d}_m, M_t + \bar{c}_m + \bar{c}_t^m + \bar{d}_t^m).$$

Second, by Fact 3.6 choose $\mathcal{U}'_n \subseteq \mathcal{U}_m$ which belongs to \mathcal{D} such that $\langle \bar{c}_t^m \wedge \bar{d}_t^m \wedge \bar{e}_t^{n,*} : t \in \mathcal{U}'_n \rangle$ is fully indiscernible over $B_m + \bar{c}_m + \bar{d}_m$.

We shall now prove

$$\odot_1 \text{ if } t_0 < \dots < t_k \text{ are from } \mathcal{U}'_n \text{ then } \text{tp}(\bar{d}_{t_1}^m, \bar{c}_{t_1}^m + \bar{e}_{t_0}^{n,*}) \vdash \text{tp}(\bar{d}_{t_1}^m, \bar{c}_{t_1}^m + \dots + \bar{c}_{t_k}^m + M_{t_0}).$$

Toward this, by clause (e) of \otimes_m we have

$$(*)_2 \text{ tp}(\bar{c}_{t_1}^m \wedge \bar{d}_{t_1}^m, M_{t_1}) = \text{tp}(\bar{c}_m \wedge \bar{d}_m, M_{t_1}).$$

By $(*)_2$ there is an elementary mapping f mapping $\bar{c}_m \wedge \bar{d}_m$ to $\bar{c}_{t_1}^m \wedge \bar{d}_{t_1}^m$ which is the identity on M_{t_1} . But $M_{t_0} + \bar{c}_{t_0}^m + \bar{d}_{t_0}^m \subseteq M_{t_1}$ by \otimes_m (i) and $\bar{e}_{t_0}^{n,*} \subseteq M_{t_1}$ by the full indiscernibility, i.e. by the choice of \mathcal{U}'_n above, hence by applying f on $(*)_1$ for t_0 we get

$$(*)_3 \text{ tp}(\bar{d}_{t_1}^m, \bar{c}_{t_1}^m + \bar{e}_{t_0}^{n,*}) \vdash \text{tp}(\bar{d}_{t_1}^m, M_{t_0} + \bar{c}_{t_1}^m + \bar{c}_{t_0}^m + \bar{d}_{t_0}^m).$$

Now by clause (k) of \otimes_m

$$(*)_4 (M_{t_0}, B_m, \bar{c}_m, \bar{d}_m) \in \text{mxK}_{\kappa_I(t_0), \theta}.$$

But by clause (e) of \otimes_m , recalling $t_0 < t_1$ hence $M_{t_0} \subseteq M_{t_1}$

$$(*)_5 \text{ tp}(\bar{c}_{t_1}^m \wedge \bar{d}_{t_1}^m, M_{t_0}) = \text{tp}(\bar{c}_m \wedge \bar{d}_m, M_{t_0}).$$

By $(*)_4 + (*)_5$

$$(*)_6 (M_{t_0}, B_m, \bar{c}_{t_1}^m, \bar{d}_{t_1}^m) \in \text{mxK}_{\kappa_I(t_0), \theta}.$$

Also easily by clauses (k) + (i) of \otimes_m applied to $t = t_2, t_3, \dots, t_k$ (recalling Observation 1.4(3))

$$(*)_7 \text{ tp}(\bar{c}_{t_2}^m \wedge \dots \wedge \bar{c}_{t_k}^m, M_{t_0} + \bar{c}_{t_1}^m + \bar{d}_{t_1}^m) \text{ does not split over } B_m.$$

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By $(*)_6 + (*)_7$ and the weak orthogonality Claim 2.14(1) we have

$$(*)_8 \text{tp}(\bar{d}_{t_1}^m, M_{t_0} + \bar{c}_{t_1}^m) \vdash \text{tp}(\bar{d}_{t_1}^m, M_{t_0} + \bar{c}_{t_1}^m + \cdots + \bar{c}_{t_k}^m).$$

By $(*)_3 + (*)_8$

$$(*)_9 \text{tp}(\bar{d}_{t_1}^m, \bar{c}_{t_1}^m + \bar{c}_{t_0}^{n,*}) \vdash \text{tp}(d_{t_1}^m, M_{t_0} + \bar{c}_{t_1}^m + \cdots + \bar{c}_{t_k}^m)$$

as promised in \odot_1 .

Now we continue to deal with Case 2, choose $F_n : \mathcal{U}'_n \rightarrow \mathcal{U}'_n$ such that $s \in \mathcal{U}'_n \Rightarrow s <_I F_n(s) \in \mathcal{U}'_n$ and for $t \in \mathcal{U}'_n$ we let $\bar{e}_t^n := \bar{e}_{F_n(t)}^{n,*}$. Let $\bar{e}_n \in {}^\theta \mathfrak{C}$ be such that $\bar{c}_m \hat{\ } \bar{d}_m \hat{\ } \bar{e}_n$ realizes $\text{Av}(\langle \bar{c}_t^m \hat{\ } \bar{d}_t^m \hat{\ } \bar{e}_t^n : t \in \mathcal{U}'_n \rangle / \mathcal{D}, M_\lambda)$. Let $\mathcal{U}''_n \in \mathcal{D}$ be $\subseteq \mathcal{U}'_n$ and such that $s \in \mathcal{U}''_n \wedge t \in \mathcal{U}''_n \wedge s <_I t \Rightarrow F_n(s) <_I t$ and the sequence $\langle \bar{c}_t^m \hat{\ } \bar{d}_t^m \hat{\ } \bar{e}_{F(t)}^{n,*} : t \in \mathcal{U}''_n \rangle$ is fully indiscernible over $\bar{c}_m \hat{\ } \bar{d}_m \hat{\ } \bar{e}_n$.

Let $\bar{d}_n = \bar{d}_m \hat{\ } \bar{e}_n$ and $\bar{d}_t^n = \bar{d}_t^m \hat{\ } \bar{e}_t^n$ for $t \in \mathcal{U}''_n$.

Let $\mathbf{y}_n := (M, B_n, \bar{c}_n, \bar{d}_n) \in K_{\kappa, \theta}$ so clearly $\mathbf{x}_m \leq_1 \mathbf{y}_n$ hence by Theorem 2.12 there is $\mathbf{x}_n = (M_\lambda, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa, \theta}$ such that $\mathbf{y}_n \leq_2 \mathbf{x}_n$ so \bar{c}_n and B_n are well defined^v and $\bar{c}_m = \bar{c}_n \upharpoonright I_{\mathbf{x}_m}$; can add \mathbf{x}_n is normal by Claim 2.7(6). For $t \in \mathcal{U}''_n$, let \bar{c}_t^n be a sequence from M_λ such that $\bar{c}_t^m = \bar{c}_t^n \upharpoonright \text{Dom}(\bar{c}_m)$ and $\text{tp}(\bar{c}_t^n \hat{\ } \bar{d}_t^n, M_t) = \text{tp}(\bar{c}_n \hat{\ } \bar{d}_n, M_t)$, this is possible as M is κ -saturated, $|M_t + \bar{c}_t^n + \bar{d}_t^n| < \kappa$, $|\text{Rang}(\bar{c}_n)| \leq \theta < \kappa$ and $\bar{c}_t^n \hat{\ } \bar{d}_t^n$ realizes the type $\text{tp}(\bar{c}_m \hat{\ } \bar{d}_n, M_t)$.

Lastly, let \mathcal{U}_n be a subset of \mathcal{U}''_n which belongs to \mathcal{D} such that:

$$\odot_2 \langle \bar{c}_t^n \hat{\ } \bar{d}_t^n : t \in \mathcal{U}_n \rangle \text{ is fully indiscernible over } B_n + \bar{c}_n + \bar{d}_n;$$

$$\odot_3 (M_t, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa, \theta} \text{ for every } t \in \mathcal{U}_n.$$

[Why \mathcal{U}_n exists? By Fact 3.6 and Claim 2.10.]

It is easy to check that $\mathbf{x}_n, \bar{c}_n, \bar{d}_n, \langle (\bar{c}_t^n, \bar{d}_t^n, \bar{e}_t^n) : t \in \mathcal{U}_n \rangle$ are as required. For example, clause (f) holds as $lg(\bar{d}_0) = lg(\bar{d}) = \gamma(*)$ and $lg(\bar{d}_m) = \gamma(*) + \theta \cdot m$ by clause (a) of θ_n and $lg(\bar{e}_n) = \theta$ by $\otimes_n(f)$ we can prove that $lg(\bar{d}_n) = lg(\bar{d}_m) + \theta = \gamma(*) + \theta \cdot m + \theta = \gamma(*) + \theta \cdot n$, so we clearly are done. For clause (e) note \odot_1 and the choices of F and $\mathcal{U}''_n, \mathcal{U}_n$. For clause $\otimes_n(\ell)$ note that $s \in \mathcal{U}'_n \Rightarrow s <_I F_n(s) \in \mathcal{U}'_n$ so $\bar{c}_s^m \subseteq M_{F_n(s)}$.

So we have carried the induction. Second, let $\bar{c}_\omega = \bar{c}_\lambda^\omega = \bigcup_{n < \omega} \bar{c}_n, \bar{d}_\omega = \bar{d}_\lambda^\omega = \bigcup_{n < \omega} \bar{d}_t^n$ and $B = \bigcup \{B_n : n < \omega\}$ and $\mathcal{U} = \bigcap \{\mathcal{U}_n : n < \omega\}$ and for $t \in \mathcal{U}$ let $\bar{c}_t = \bigcup_n \bar{c}_t^n, \bar{d}_t = \bigcup_n \bar{d}_t^n$.

Let us check that \boxtimes from the theorem holds indeed.

Clause (a): $\mathcal{U} \in \mathcal{D}$ as each $\mathcal{U}_n \in \mathcal{D}$ by $\otimes_n(h)$ and \mathcal{D} is κ -complete and $\kappa > \aleph_0$ recalling $\mathcal{U} = \bigcap \{\mathcal{U}_n : n < \omega\}$.

Clause (b): $B \in [M_\lambda]^{< \kappa}$ as $B_n \subseteq M_\lambda, |B_n| < \kappa$ by $\otimes_n(b)$ for $n < \omega$ and κ is regular uncountable recalling and $B := \bigcup \{B_n : n < \omega\}$.

Clause (c): By $\otimes_n(a) + (c)$ for $n < \omega$.

^vIn fact, we can demand that $\text{tp}(\bar{c}_n \upharpoonright (I_{\mathbf{x}_n} \setminus I_{\mathbf{x}_m}), M + \bar{c}_m)$ does not split over $B_{\mathbf{x}_n}$.

Clause (d): $\bar{d} = \bar{d}_0 = \bar{d}_\omega \upharpoonright \gamma(*)$ is proved as in clause (c).

Clause (e): As $\mathbf{x}_n = (M, B_n, \bar{c}_n, \bar{d}_n) \in \text{mxK}_{\kappa, \theta}$ by clause (b) of \otimes_n and $\mathbf{x}_n \leq_1 \mathbf{x}_{n+1}$ by $\otimes_{n+1}(f)$ and $(M, B, \bar{c}_\omega, \bar{d}_\omega) = \cup\{\mathbf{x}_n : n < \omega\}$, clearly by Claim 2.14(2) we are done.

Clause (f): By clause (e) of \otimes_n (and the choice of $\bar{c}_\omega, \bar{d}_\omega, \bar{c}_t^\omega, \bar{d}_t^\omega$, etc.).

Clause (g): Similarly by clause (g) of \otimes_n .

Clause (h)₁: By clauses (h)₂ + (h)₃ proved below.

Clause (h)₂: By clause (i) of \otimes_n .

Clause (h)₃: Holds by clause (ℓ) of \otimes_n . □_{3.8}

Theorem 3.10. (1) If M is κ -saturated of cardinality $\leq \lambda$, $\bar{d} \in \theta^+ > \mathfrak{C}$ then we can find a strict (κ, θ) -decomposition $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ over M such that $\bar{d}_0 \triangleleft \bar{d}$.

(2) Instead Hypothesis 3.1 it is enough to demand: if M is κ -saturated and $p = p(\langle x_i : i < \theta \rangle)$ is a type with parameters from $M \cup C$, $|C| \leq \theta$ which is $(< \kappa)$ -satisfiable in M , i.e. every subset of p of cardinality $< \kappa$ is realized in M then p can be extended to $p^+ \in \mathbf{S}^\theta(M \cup C)$ which is $(< \kappa)$ -satisfiable in M .

Proof. (1) We can choose $\bar{M}' = \langle M'_t : t \in I \rangle$ such that (M, \bar{M}') satisfies the demands on $(M_\lambda, \langle M_t : t \in I \rangle)$ in Hypothesis 3.1(1) or 3.1(2) and apply Theorem 3.8 (as assuming $\text{lg}(\bar{d}) = \theta$ or $\text{lg}(\theta) < \theta^+$ does not matter).

(2) The idea is to repeat the proof of Theorem 3.8, but as of unclear value we leave it to the reader. □_{3.10}

Corollary 3.11. Assume $\kappa = \lambda > |T|$ is weakly compact, $M_\alpha \in \text{EC}_{< \lambda}(T)$ is \prec -increasing continuous, $M = \cup\{M_\alpha : \alpha < \lambda\}$ is saturated. Then Theorem 3.8 and Claim 3.12(2) hold.

Proof. Revise the proof of Theorem 3.8, but in \otimes_n weaken clauses (g), (h) to (g)⁻, (h)⁻ and use the proof of Claim 4.14 in the end where

(g)⁻ $\mathcal{U}_n \in [\kappa]^\kappa$ decreasing with n and $\langle \bar{c}_t^n \hat{\wedge} \bar{d}_t^n : t \in \mathcal{U}_n \rangle$ is just fully n -indiscernible,
 (h)⁻ \mathcal{U}_n does not belong to the weakly compact ideal.

We leave the details to the reader or see [20]. □_{3.11}

Claim 3.12. (1) (even not assuming Hypothesis 3.1) Assume M is κ -saturated, $|T| \leq \theta < \kappa$, $\text{tp}(\bar{c}_n, M + \bar{c}_0 + \dots + \bar{c}_{n-1})$ does not split over B where $|B| < \kappa$, $B \subseteq M$ and $\text{lg}(\bar{c}_n) < \theta^+$ and $\bar{d}_n = \bar{c}_n \upharpoonright u$ for $n < \omega$ (so $u \subseteq \text{Dom}(\bar{c}_n)$) then $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ is a strict (κ, θ) -decomposition over (M, B) .

(2) (Assuming Hypothesis 3.1!) Assume $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ is a strict (κ, θ) -decomposition over (M_λ, B) . For any $\bar{d} \in \theta^+ > \mathfrak{C}$ such that $\langle \bar{c}_{1+n} \hat{\wedge} \bar{d}_{1+n} : n < \omega \rangle$ is an indiscernible sequence over $M + \bar{c}_0 + \bar{d}_0 + \bar{d}$, we can find a strict (κ, θ) -decomposition

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$\langle \langle \bar{c}_n^+, \bar{d}_n^+ \rangle : n < \omega \rangle$ over (M_λ, B) such that $\bar{c}_n \sqsubseteq \bar{c}_n^+, \bar{d}_n \sqsubseteq \bar{d}_n^+$ for $n < \omega$ and $\bar{d}_0 \hat{\wedge} \bar{d} \sqsubseteq \bar{d}_0^+$.

Proof. (1) Easy.

(2) Repeat the proof of Theorem 3.8 starting with $\mathbf{x}_0 = (M_\lambda, B, \bar{c}_0, \bar{d}_0)$ and $\langle \bar{c}_t \hat{\wedge} \bar{d}_t : t \in I \rangle$ and $\bar{c}_t \hat{\wedge} d_t \in {}^{\ell g(\bar{c}_t) + \ell g(\bar{d}_t)} M_\lambda$ realizing $\text{tp}(\bar{c}_0 \hat{\wedge} \bar{d}_0, M_t + \sum_{n < \omega} \bar{c}_{1+n} \hat{\wedge} \bar{d}_{1+n})$.

□_{3.12}

Claim 3.13. The sequence $\langle \langle \bar{c}_t^2, \bar{d}_t^2 \rangle : t \in I_2 \rangle$ is a strict₀ $(0, \theta)$ -decomposition over (B_2, B_1) when:

- (a) $\langle \langle \bar{c}_s^1, \bar{d}_s^1 \rangle : s \in I_1 \rangle$ is a strict₀ $(0, \theta)$ -decomposition over (B_2, B_1) (and $\ell g(\bar{c}_s^1) = \ell g(\bar{c}_t^2), \ell g(\bar{d}_s^1) = \ell g(\bar{d}_t^2) = \ell g(\bar{d}_t^1)$, of course),
- (b) for any n if $I_\ell \models "t_{\ell,0} < \dots < t''_{\ell,n-1}$ for $\ell = 1, 2$ then $\text{tp}(\bar{c}_{t_{1,0}}^1 \hat{\wedge} \bar{d}_{t_{1,0}}^1 \hat{\wedge} \dots \hat{\wedge} \bar{c}_{t_{1,n-1}}^1 \hat{\wedge} \bar{d}_{t_{1,n-1}}^1, B_2) = \text{tp}(\bar{c}_{t_{2,0}}^2 \hat{\wedge} \bar{d}_{t_{2,0}}^2 \hat{\wedge} \dots \hat{\wedge} \bar{c}_{t_{2,n-1}}^2 \hat{\wedge} \bar{d}_{t_{2,n-1}}^2, B_2)$.

Proof. Should be clear.

□_{3.13}

4. Consequences of Strict Decomposition

Here we look again at the generic pair conjecture (see [22, 0.2]). The non-structure side (in a strong version) is proved there for $\lambda = \lambda^{<\lambda}$ non-strong limit and in [18] for $\lambda = \lambda^{<\lambda}$ strong limit (i.e. strongly inaccessible) as follows.

The conjecture is (the instances of G.C.H. are used to make the conjecture transparent) as follows.

Conjecture 4.1 (The Generic Pair Conjecture). Assume^w $\lambda = \lambda^{<\lambda} > |T|, 2^\lambda = \lambda^+, M_\alpha \in \text{EC}_\lambda(T)$ is \prec -increasing continuous for $\alpha < \lambda^+$ with $\cup\{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda^+}(T)$ being saturated.

- (1) The λ -generic pair conjecture says that: T is dependent if and only if for some club E of λ^+ for all pairs $\alpha < \beta < \lambda^+$ from E of cofinality λ^+ , (M_β, M_α) has the same isomorphism type.
- (2) For $\zeta < \lambda$ the λ -generic ζ -tuple conjecture says that: T is dependent if and only if for some club E of λ^+ for all increasing sequences $\langle \alpha_\varepsilon : \varepsilon \leq \zeta \rangle$ of members of E of cofinality λ , the structure $(M_{\alpha_\zeta}, M_{\alpha_\varepsilon})_{\varepsilon < \zeta}$ has the same isomorphism type, (equivalently, if $\langle \alpha_{\ell,\varepsilon} : \varepsilon \leq \zeta \rangle$ is as above for $\ell = 1, 2$ then there is an isomorphism from $M_{\alpha_{1,\zeta}}$ onto $M_{\alpha_{2,\zeta}}$ mapping $M_{\alpha_{1,\varepsilon}}$ onto $M_{\alpha_{2,\varepsilon}}$ for $\varepsilon < \zeta$).

We concentrate on the pair. Note that if $\kappa = \text{cf}(\kappa) < \lambda$, then the λ -generic κ -tuple conjecture implies that for dependent T there is a medium (λ, κ) -limit model, see [22], but we do not succeed to deal with it here.

^wThe " $2^\lambda = \lambda^+$ " is just for making the formulation more transparent, and by absoluteness is equivalent to the formulation not assuming $2^\lambda = \lambda^+$.

Here we prove the “structure” side when λ is measurable. It seemed natural to assume that the first-order theories of such pair is complicated if T is independent and “understandable” for dependent of T .

In fact, it may be better to ask the following questions.

Problem 4.2. (1) Assume $|T| < \theta \leq \kappa \leq \lambda = \lambda^{<\kappa} < \kappa_2 \leq \mu = \mu^{<\aleph_2}$ and $M_1 \prec M_2 \prec \mathfrak{C}$, M_1 is κ_1 -saturated of cardinality λ , M_2 is κ_2 -saturated of cardinality μ . What can we say on $\text{Th}(M_2, M_1)$? On $\text{Th}_{\mathbb{L}_{\infty, \theta}(\tau(T))}(M_2, M_1)$?

More generally, we have the following.

Problem 4.3. (1) Assume $n < \omega$, $|T| < \theta$, $\sigma < \theta < \kappa_0$, $\lambda_\ell = \lambda_\ell^{<\kappa_\ell}$ for $\ell \leq n$, $\lambda_\ell < \kappa_{\ell+1}$ for $\ell < n$. Let M_ℓ be κ_ℓ -saturated of cardinality λ_ℓ for $\ell \leq n$ and $M_\ell \prec M_{\ell+1}$ for $\ell < n$. What can we say on $M^+ = \text{Th}(M_n, \dots, M_1, M_0)$, i.e. M_n expanded by unary predicates for M_ℓ for $\ell < n$? When can we interpret (with first-order formulas with parameters) second-order logic on θ ? i.e. classify T by this.

(2) Similarly for $\mathbb{L}_{\sigma, \sigma}(\tau_{M^+})$.

(3) Similarly allowing n to be $< \theta$.

The proof here, if, e.g., $\kappa = \lambda$ is measurable says that even the $\mathbb{L}_{\infty, \kappa}$ -theory of the pair is constant, but does not say much even on the first-order theory (see [7]). It is known that for many “complicated enough” theories T , for M_2, M_1 as in Problem 4.2, in $\text{Th}(M_2, M_1)$ we can interpret second-order logic on $(\lambda, =)$. This holds, e.g., for $T =$ Peano arithmetic.

On n -independent theories see [23, §2]. Note that we have the following.

Claim 4.4. Assume T is the model completion of T_0 , defined below so seems “the simplest” 2-independent theory. If $M_0 \in \text{EC}_\lambda(T)$ and M_1 is a λ^+ -saturated \prec -extension of M_0 then in (M_1, M_0) we can interpret second-order logic on M_0 , (i.e. quantification on two-place relations, when:

(*) τ_{T_0} consists of P_0, P_1, P_2 (unary predicates) and R (a ternary predicate) and a τ_{T_0} -model M is a model of T_0 if and only if $\langle P_0^M, P_1^M, P_2^M \rangle$ is a partition of $|M|$ and $R^M \subseteq P_0^M \times P_1^M \times P_2^M$.

Proof. Obvious. □_{4.4}

Also for $T =$ theory of Boolean algebras (which is n -independent for every n) the theory is complicated. Of course, it would be better to eliminate the measurable assumption.

Explanation 4.5. Why $K_{\lambda, \kappa, \theta}$ and $\text{mx}K_{\lambda, \kappa, \theta}$ from Sec. 2 does not suffice for us so that in Sec. 3 we deal also with the more complicated $\langle (\bar{c}_t, \bar{d}_t) : t \in J \rangle$ from Definition 3.5? This is motivated by the proof of the generic pair conjecture.

To understand it maybe better consider the class

(*) $\mathbf{N}_\kappa^2 = \{(N, M) : M \prec N \prec \mathfrak{C}_T, M \text{ is } \kappa\text{-saturated and } N \text{ is } \|M\|^+\text{-saturated}\}$.

Proving the generic pair conjecture for κ we consider $\bar{M} = \langle M_\alpha : \alpha < \kappa^+ \rangle$, which is \prec -increasing continuous, $M = \bigcup_\alpha M_\alpha \prec \mathfrak{C}_T$ is saturated of cardinality κ^+ .

Assuming T is dependent we should choose a thin enough club E of κ^+ and consider $\{(M_\beta, M_\alpha) : \alpha < \beta \text{ and } \{\alpha, \beta\} \subseteq E \cap S_\kappa^{\kappa^+}\}$. Now the club E will be chosen such that all relevant pairs (M_β, M_α) are similar enough to those of pairs from \mathbf{N}_κ^2 .

So a sufficient condition for the conjecture is:

- ⊞ assume $(N_1, M_1), (N_2, M_2) \in \mathbf{N}_\kappa^2$, then we can find $\bar{f} = \langle f_s : s \in \mathcal{Y} \rangle$ such that:
- (a) \mathcal{Y} is a partial order,
 - (b) \mathcal{Y} is $(< \kappa)$ -complete, that is, any increasing chain (for $<_{\mathcal{Y}}$) of length $< \kappa$ has an upper bound,
 - (c) f_s is an (N_1, N_2) -elementary mapping,
 - (d) $\text{Dom}(f_s)$ has cardinality $< \kappa$,
 - (e) f_s maps $\text{Dom}(f_s) \cap M_1$ onto $\text{Rang}(f_s) \cap M_2$,
 - (f)₁ if $s \in \mathcal{Y}, A \in [N_1]^{< \kappa}$ then for some $t \in \mathcal{Y}$ we have $s <_{\mathcal{Y}} t \wedge A \subseteq \text{Dom}(f_t)$,
 - (f)₂ if $s \in \mathcal{Y}$ and $A \in [N_2]^{< \kappa}$ then for some $t \in \mathcal{Y}$ we have $s <_{\mathcal{Y}} t \wedge A \subseteq \text{Rang}(f_t)$.

Now the approximation will consist of $B_\ell \in [M_\ell]^{< \kappa}$ and $\langle (\bar{c}_n^\ell, \bar{d}_n^\ell) : n < \omega \rangle$, with $\bar{c}_0^\ell, \bar{d}_0^\ell$ from N_ℓ , which form a strict $(\kappa, < \kappa)$ -decomposition over (M_ℓ, B_ℓ) for $\ell = 1, 2$ and an elementary mapping h mapping B_1 onto B_2 and $(\bar{c}_n^1, \bar{d}_n^1)$ to $(\bar{c}_n^2, \bar{d}_n^2)$. So using $\bar{c}_n^\ell, \bar{d}_n^\ell$ for $n > 0$ is to give us a condition with which we can continue in a good induction hypothesis.

Now it should be clear what we like to have from strict decomposition; however, the decompositions from Sec. 2 are not enough.

We first connect decomposition (i.e. the results of Theorem 3.8) and sufficient conditions for being an indiscernible sequence.

Claim 4.6. *The sequence $\langle \bar{a}_\alpha : \alpha < \alpha^* \rangle$ is an indiscernible sequence over B when for some p, B we have:*

- (a) \bar{a}_α is a sequence of members of \mathfrak{C} and $\bar{a}_\alpha = \bar{c}_\alpha \hat{\ } \bar{d}_\alpha$ where $lg(\bar{c}_\alpha) = lg(\bar{c}_0)$, $lg(\bar{d}_\alpha) = lg(\bar{d}_0)$ (not necessarily finite),
- (b) $\text{tp}(\bar{a}_\alpha \hat{\ } \bar{a}_\beta, B) = p$ for $\alpha < \beta < \alpha^*$.
- (c) if $\alpha < \beta < \alpha^*$ then $\text{tp}(\bar{d}_\beta, \bar{c}_\beta + \bar{d}_\alpha + B) \vdash \text{tp}(\bar{d}_\beta, \bar{c}_\beta \cup \{\bar{a}_\gamma : \gamma \leq \alpha\} \cup B)$,
- (d) $\text{tp}(\bar{c}_\alpha, \cup\{\bar{a}_\beta : \beta < \alpha\} \cup B)$ is increasing with α ,
- (e) $\text{tp}(\bar{c}_\alpha, \cup\{\bar{a}_\beta : \beta < \alpha\} \cup B)$ does not split over B .

Remark 4.7. Concerning Sec. 3, it is natural to omit the first occurrence of B in clause (c).

Proof. For $u, v \subseteq \alpha^*$ let $B_{u,v} = \cup\{\bar{a}_\alpha : \alpha \in u\} \cup \cup\{\bar{c}_\alpha : \alpha \in v\} \cup B$. For $u, v \subseteq \alpha^*$ and increasing functions h_1 from u to α^* and h_2 from v to α_* such that

$h_1 \upharpoonright (u \cap v) = h_2 \upharpoonright (u \cap v)$ and even $h = h_1 \cup h_2$ is an increasing function from $u \cup v$ to α^* , let $f = f_{h_1, h_2}$ be defined as follows:

- ⊗ (a) $\text{Dom}(f) = B_{u, v}$,
- (b) $f \upharpoonright B = \text{id}_B$,
- (c) f maps \bar{a}_α to $\bar{a}_{h(\alpha)}$ for $\alpha \in u$,
- (d) f maps \bar{c}_α to $\bar{c}_{h(\alpha)}$ for $\alpha \in v$.

Is $f_{h \upharpoonright u, h \upharpoonright v}$ a well-defined function and even one to one? For this it suffices to check the following three demands, which follows by clause (b) of the assumption

- (*)₁ (α) if $\alpha, \beta < \alpha^*$, $b \in B$ and $i < \ell g(\bar{a}_\alpha)$ then $(\bar{a}_\alpha)_i = b \Leftrightarrow (\bar{a}_\beta)_i = b$,
- (β) if $\alpha, \beta < \alpha^*$ and $i, j < \ell g(\bar{a}_\alpha)$ then $(\bar{a}_\alpha)_i = (\bar{a}_\alpha)_j \Leftrightarrow (\bar{a}_\beta)_i = (\bar{a}_\beta)_j$,
- (γ) if $\alpha_1 < \alpha_2 < \alpha^*$ and $\beta_1 < \beta_2 < \alpha^*$ and $i, j < \ell g(\bar{a}_\alpha)$ then $(\bar{a}_{\alpha_1})_i = (\bar{a}_{\alpha_2})_j \Leftrightarrow (\bar{a}_{\beta_1})_i = (\bar{a}_{\beta_2})_j$.

We prove by induction on n that $(\bar{a}_\alpha : \alpha < \alpha^*)$ is an n -indiscernible sequence over B (when $n < \alpha^*$). For $n \leq 2$ this is trivial by clause (b) of the assumption. So assume $n = m + 1 > 2$ and we have proved it up to m . So let $\alpha_0 < \dots < \alpha_m < \alpha^*$, $\beta_0 < \dots < \beta_m < \alpha^*$ and we shall prove that $\bar{a}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{a}_{\alpha_m}, \bar{a}_{\beta_0} \hat{\ } \dots \hat{\ } \bar{a}_{\beta_m}$ realize the same type over B , this suffices.

Now by symmetry without loss of generality $\alpha_m \leq \beta_m$ let $h_0 = \{(\alpha_\ell, \beta_\ell) : \ell < m\}$, $h_1 = \{(\alpha_\ell, \alpha_\ell) : \ell < m\}$, $h_2 = h_1 \cup \{(\alpha_m, \beta_m)\}$ and $h_3 = h_0 \cup \{(\beta_m, \beta_m)\}$ and $h_4 = h_0 \cup \{(\alpha_m, \beta_m)\}$.

Let f_0 be the mapping f_{h_0, h_0} . By the induction hypothesis

- (*)₂ f_0 is an elementary mapping.

Let f_1 be the mapping f_{h_1, h_2} , now by clause (d) of the assumption

- (*)₃ f_1 is an elementary mapping.

By (*)₂ we know that $\bar{a}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{a}_{\alpha_{m-1}}$ and $f_0(\bar{a}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{a}_{\alpha_{m-1}})$ realize the same type over B and they are included in $B_{\alpha_m, \alpha_m}, B_{\beta_m, \beta_m}$ respectively. But $\alpha_m \leq \beta_m$ so both sequences are from B_{β_m} hence by clause (e) of the assumption, i.e. as $\text{tp}(\bar{c}_{\beta_m}, B_{\beta_m, \beta_m})$ does not split over B , recalling (*)₂ we have

- (*)₄ $f_2 := f_{h_0, h_3}$ is an elementary mapping.

By (*)₃ + (*)₄, comparing we have

- (*)₅ $f_3 := f_{h_0, h_4}$ being $f_2 \circ f_1$ is an elementary mapping.

Note that

- (*)₆ $f_3(\bar{c}_{\alpha_m} \hat{\ } \bar{d}_{\alpha_{m-1}}) = \bar{c}_{\beta_m} \hat{\ } \bar{d}_{\beta_{m-1}}$.

By clause (b) of the assumption and (*)₅ + (*)₆ clearly

- (*)₇ $f_3(\text{tp}(\bar{d}_{\alpha_m}, \bar{c}_{\alpha_m} + \bar{d}_{\alpha_{m-1}} + B)) = \text{tp}(\bar{d}_{\beta_m}, \bar{c}_{\beta_m} + \bar{d}_{\beta_{m-1}} + B)$.

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By clause (c) of the assumption

$$\begin{aligned}
 (*)_8 \quad & \text{tp}(\bar{d}_{\alpha_m}, \bar{c}_{\alpha_m} + \bar{d}_{\alpha_{m-1}} + B) \vdash \text{tp}(\bar{d}_{\alpha_m}, \bar{c}_{\alpha_m} + \bar{a}_{\alpha_0} + \cdots + \bar{a}_{\alpha_{m-1}} + B), \\
 (*)_9 \quad & \text{tp}(\bar{d}_{\beta_m}, \bar{c}_{\beta_m} + \bar{d}_{\beta_{m-1}} + B) \vdash \text{tp}(\bar{d}_{\beta_m}, \bar{c}_{\beta_m} + \bar{a}_{\beta_0} + \cdots + \bar{a}_{\beta_{m-1}} + B).
 \end{aligned}$$

Together $f_4 = f_{h_4, h_4}$ is an elementary mapping and it maps \bar{a}_{α_ℓ} to \bar{a}_{β_ℓ} for $\ell \leq m$ (and extend id_B) so we are done. □_{4.6}

Observation 4.8. The sequence $\langle (\bar{c}_t, \bar{d}_t) : t \in I \rangle$ is a strict₀ $(0, \theta)$ -decomposition over (B^0, B) when for some $\langle I_\ell, B, B^\ell : \ell < n \rangle$ we have:

- (a) $n < \omega$ and $n \geq 2$,
- (b) the linear order I is $I_0 + \cdots + I_n$ where I_ℓ is infinite for $\ell = 1, \dots, n-1$,
- (c) $\langle (\bar{c}_t, \bar{d}_t) : t \in I_\ell \rangle$ is a strict₀ $(0, \theta)$ -decomposition over (B^ℓ, B) for $\ell \leq n$,
- (d) $\langle \bar{c}_t \hat{\ } \bar{d}_t : t \in I_\ell + I_{\ell+1} \rangle$ is indiscernible over B^ℓ and $\ell g(\bar{c}_t)$ for $t \in I_\ell \cup I_{\ell+1}$ is constant for $\ell < n$,
- (e) $B^{\ell+1} \supseteq \{ \bar{c}_t \hat{\ } \bar{d}_t : t \in I_\ell \} \cup B^\ell$,
- (f) $B \subseteq B^0$.

Proof. In Definition 3.5, clause (a) holds trivially as $B^0 \supseteq B$ by clause (f) here (recalling that κ there stands for 0 here). For clause (b) of Definition 3.5 the sequence $\langle \ell g(\bar{c}_t) : t \in I \rangle$ is constant as for each $\ell < n$ the sequence $\langle \ell g(\bar{c}_t) : t \in I_\ell \cup I_{\ell+1} \rangle$ is constant (by clause (c) here and (b) in Definition 3.5) and use transitivity of equality and I_ℓ for $\ell = 1, \dots, n-1$ being non-empty by clause (b) here. Similarly $\langle \ell g(\bar{d}_t) : t \in I \rangle$ is constant, so Definition 3.5(b) indeed holds.

Similarly, clause (c) of Definition 3.5 follows from Definition 3.5(d) proved below and also clause (c) here for $\ell = 1$ (recalling I_1 is infinite). Clause (e)⁻ of Definition 3.5 follows from Definition 3.5(3) similarly using $B' \supseteq B^0$ by clause (e) here and (e)⁻ there and clause (d) proved below.

So we are left with clause (d) of Definition 3.5, that is $\langle \bar{c}_t \hat{\ } \bar{d}_t : t \in I \rangle$ is an indiscernible sequence over B^0 . For this we prove by induction on $k \leq n$ that $\langle \bar{c}_t \hat{\ } \bar{d}_t : t \in \cup \{ I_\ell : \ell \in [n-k, n] \} \rangle$ is an indiscernible sequence over B^{n-k} . For $k = 0, 1$ this holds by clause (d), for $k \geq 2$, let $s_0 <_I \cdots <_I s_{m-1}, t_0 <_I \cdots <_I t_{m-1}$ be from $\cup \{ I_\ell : \ell \in [n-k, n] \}$ be given. Choose s'_i, t'_i from $I_{n-k} \cup I_{n-k+1}$ for $i < m$ such that $s'_0 <_I \cdots <_I s'_{m-1}, t'_0 <_I \cdots <_I t'_{m-1}$ and $s_\ell \in I_{n-k} \Rightarrow s'_\ell = s_\ell, s_\ell \notin I_{n-k} \Rightarrow s'_\ell \in I_{n-k+1}$ and $t_\ell \in I_{n-k} \Rightarrow t'_\ell = t_\ell, t_\ell \notin I_{n-k} \Rightarrow t'_\ell \in I_{n-k+1}$. This is possible as I_{n-k+1} is infinite because $k \in [2, n]$.

Let $j \leq m$ be such that $\ell < m \Rightarrow (s_\ell \in I_{n-k} \Leftrightarrow \ell < j)$; so by the induction hypothesis we have $\text{tp}(\bar{a}_{s_j} \hat{\ } \cdots \hat{\ } \bar{a}_{s_{m-1}}, B^{n-k+1}) = \text{tp}(\bar{a}_{s'_j} \hat{\ } \cdots \hat{\ } \bar{a}_{s'_{m-1}}, B^{n-k+1})$. As $\ell < j \Rightarrow \bar{a}_{s_\ell} = \bar{a}_{s'_\ell} \subseteq B^{n-k+1}$ and $B^{n-k} \subseteq B^{n-k+1}$ it follows that $\text{tp}(\bar{a}_{s_0} \hat{\ } \cdots \hat{\ } \bar{a}_{s_{m-1}}, B^{n-k})$ is equal to $\text{tp}(\bar{a}_{s'_0} \hat{\ } \cdots \hat{\ } \bar{a}_{s'_{m-1}}, B^{n-k})$. This type by clause (d) is equal to $\text{tp}(\bar{a}_{t'_0} \hat{\ } \cdots \hat{\ } \bar{a}_{t'_{m-1}}, B^{n-k})$. Similarly to the proof in the beginning of the paragraph, this type by the induction hypothesis is equal to $\text{tp}(\bar{a}_{t_0} \hat{\ } \cdots \hat{\ } \bar{a}_{t_{m-1}}, B^{n-k})$, so together we are done. □_{4.8}

The following is a local version of Claim 4.6 (see Remark 4.10)

Claim 4.9. Assume $(n(*) < \omega$ and $\alpha(*) > n(*)$ for each $k < n(*)$)

- (a) $\bar{c}_\alpha^k \in \gamma^{(k,1)}\mathfrak{C}$ for $k \leq n(*), \alpha < \alpha(*)$; this means that the length of \bar{c}_α^k may depend on k but not on α and may be infinite,
- (b) $\bar{d}_\alpha = \bar{d}_\alpha^0 \in \gamma^{(k,0)}\mathfrak{C}$ for $\alpha < \alpha(*)$,
- (c) $\bar{c}_\alpha^k \triangleleft \bar{c}_\alpha^{k+1}$ for $\alpha < \alpha(*), k < n(*)$,
- (d) $\bar{e}_\alpha^k \in \gamma^{(m,2)}\mathfrak{C}$ for non-zero $k \leq n(*), \alpha < \alpha(*)$,
- (e) for all $\alpha < \beta < \alpha(*)$ the type $\text{tp}(\bar{c}_\beta^{k+1} \hat{\ } \bar{e}_\alpha^{k+1} \hat{\ } \bar{d}_\beta \hat{\ } \bar{e}_\beta^k, B)$ is the same,
- (f) $\text{tp}(\bar{d}_\beta \hat{\ } \bar{e}_\beta^k, \bar{c}_\beta^k + \bar{e}_\alpha^{k+1} + B) \vdash \text{tp}(\bar{d}_\beta^k \hat{\ } \bar{e}_\beta^k, \bar{c}_\beta^k + \sum_{\gamma < \beta} \bar{c}_\gamma^k \hat{\ } \bar{d}_\gamma + B)$,
- (g) $\text{tp}(\bar{c}_\alpha^k, \cup\{\bar{c}_\beta^{k+1} \hat{\ } \bar{d}_\beta \hat{\ } \bar{e}_\beta^{k+1} : \beta < \alpha\} \cup B)$ is increasing with α ,
- (h) $\text{tp}(\bar{c}_\alpha^k, \cup\{\bar{c}_\beta^{k+1} \hat{\ } \bar{d}_\beta \hat{\ } \bar{e}_\beta^{k+1} : \beta < \alpha\} \cup B)$ does not split over B ,
- (i) $\text{tp}(\bar{c}_\alpha^{n(*)} \hat{\ } \bar{d}_\alpha \hat{\ } \bar{e}_\alpha^k, B)$ is the same for all $\alpha < \alpha(*)$.

Then $\langle \bar{c}_\alpha^0 \hat{\ } \bar{d}_\alpha : \alpha < \alpha(*) \rangle$ is an $n(*)$ -indiscernible sequence over B .

Remark 4.10. (1) In what sense is Claim 4.9 a local version of Claim 4.6? In the second we get only $n(*)$ -indiscernibility. Note that the role of \bar{d}_α there is played by $\bar{d}_\alpha, \bar{e}_\alpha^m (m < n(*))$ here.

(2) The claim is not used in the rest of the section.

Proof. We prove by induction on $n < n(*)$ that

- \odot_1 if $k \leq n(*) - n$ and $\alpha_0 < \dots < \alpha_n$ and $\beta_0 < \beta_1 < \dots < \beta_n$ then $\bar{c}_{\alpha_0}^k \hat{\ } \bar{d}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha_{n-1}}^k \hat{\ } \bar{d}_{\alpha_{n-1}} \hat{\ } \bar{c}_{\alpha_n}^k \hat{\ } \bar{d}_{\alpha_n} \hat{\ } \bar{e}_{\alpha_n}^k$ and $\bar{c}_{\beta_0}^k \hat{\ } \bar{d}_{\beta_0} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha_{n-1}}^k \hat{\ } \bar{d}_{\alpha_{n-1}} \hat{\ } \bar{c}_{\beta_n}^k \hat{\ } \bar{d}_{\beta_n} \hat{\ } \bar{e}_{\beta_n}^k$ realize the same type over B .

For $n = n(*) - 1, k = 0$ we get the desired conclusion.

For $n = 0$ this holds by clause (i) of the assumption. So assume $n = m + 1$ and $k \leq n(*) - n$ and we have proved this for m . Note that $k + 1 \leq n(*) - m$. So let $\alpha_0 < \dots < \alpha_{m+1} < \alpha^*, \beta_0 < \dots < \beta_{m+1} < \alpha^*$ be given and without loss of generality $\alpha_n \leq \beta_n$ and we shall prove the equality of types from \odot_1 in this case, this suffices. Now

- (*)₁ $\bar{c}_{\alpha_0}^{k+1} \hat{\ } \bar{d}_{\alpha_0} \hat{\ } \dots \hat{\ } \bar{c}_{\alpha_{m-1}}^{k+1} \hat{\ } \bar{d}_{\alpha_{m-1}} \hat{\ } \bar{c}_{\alpha_m}^{k+1} \hat{\ } \bar{d}_{\alpha_m} \hat{\ } \bar{e}_{\alpha_m}^{k+1}$ and $\bar{c}_{\beta_0}^{k+1} \hat{\ } \bar{d}_{\beta_0} \hat{\ } \dots \hat{\ } \bar{c}_{\beta_{m-1}}^{k+1} \hat{\ } \bar{d}_{\alpha_{m-1}} \hat{\ } \bar{c}_{\alpha_m}^{k+1} \hat{\ } \bar{d}_{\alpha_m} \hat{\ } \bar{e}_{\alpha_m}^{k+1}$ realize the same type over B .

[Why? By the induction hypothesis.]

- (*)₂ $\text{tp}(\bar{c}_{\beta_n}^k, \cup\{\bar{c}_\gamma^{k+1} \hat{\ } \bar{d}_\gamma \hat{\ } \bar{e}_\gamma^{k+1} : \gamma < \beta_n\} \cup B)$ extends $\text{tp}(\bar{c}_{\alpha_n}^k, \cup\{\bar{c}_\gamma^{k+1} \hat{\ } \bar{d}_\gamma \hat{\ } \bar{e}_\gamma^{k+1} : \gamma < \alpha_n\} \cup B)$.

[Why? By clause (g) of the assumption.]

- (*)₃ $\text{tp}(\bar{c}_{\beta_n}^k, \cup\{\bar{c}_\gamma^{k+1} \hat{\ } \bar{d}_\gamma \hat{\ } \bar{e}_\gamma^{k+1} : \gamma < \beta_n\} \cup B)$ does not split over B .

[Why? By clause (h) of the assumption.]

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(*)₄ $\bar{c}_{\alpha_0}^{k+1} \hat{\bar{d}}_{\alpha_0} \hat{\dots} \hat{\bar{c}}_{\alpha_{m-1}}^{k+1} \hat{\bar{d}}_{\alpha_{m-1}} \hat{\bar{c}}_{\alpha_m}^{k+1} \hat{\bar{d}}_{\alpha_m} \hat{\bar{e}}_{\alpha_m}^{k+1} \hat{\bar{c}}_{\beta_{m+1}}^k$ and $\bar{c}_{\beta_0}^{k+1} \hat{\bar{d}}_{\beta_0} \hat{\dots} \hat{\bar{c}}_{\beta_{m-1}}^{k+1} \hat{\bar{d}}_{\beta_{m-1}} \hat{\bar{c}}_{\beta_m}^{k+1} \hat{\bar{d}}_{\beta_m} \hat{\bar{e}}_{\beta_m}^{k+1} \hat{\bar{c}}_{\beta_{m+1}}^k$ realize the same type over B .

[Why? By (*)₁ + (*)₃.]

(*)₅ in (*)₄ we can replace $\bar{c}_{\beta_{m+1}}^k$ in the first sequence by $\bar{c}_{\alpha_{m+1}}^k$.

[Why? By (*)₄ + (*)₂.]

But

(*)₆ $(\bar{c}_{\alpha_{m+1}}^{k+1} \hat{\bar{e}}_{\alpha_m}^{k+1}) \hat{(\bar{d}}_{\alpha_{m+1}} \hat{\bar{e}}_{\alpha_{m+1}}^k)$ and $(\bar{c}_{\beta_{m+1}}^{k+1} \hat{\bar{e}}_{\beta_m}^{k+1}) \hat{(\bar{d}}_{\beta_{m+1}} \hat{\bar{e}}_{\beta_{m+1}}^k)$ realize the same type over B .

[Why? By clause (e) of the assumption.]

(*)₇ in (*)₄, (*)₅ we can replace $\bar{c}_{\alpha_\ell}^{k+1} (\ell \leq m)$ by $\bar{c}_{\alpha_\ell}^k$.

[Why? As $\bar{c}_\gamma^k \triangleleft \bar{c}_\gamma^{k+1}$ by clause (c) of the assumption.]

(*)₈ $\bar{c}_{\alpha_0}^k \hat{\bar{d}}_{\alpha_0} \hat{\dots} \hat{\bar{c}}_{\alpha_{m-1}}^k \hat{\bar{d}}_{\alpha_{m-1}} \hat{\bar{c}}_{\alpha_m}^k \hat{\bar{d}}_{\alpha_m} \hat{\bar{c}}_{\alpha_{m+1}}^k \hat{\bar{d}}_{\alpha_{m+1}}^m \hat{\bar{e}}_{\beta_{m+1}}^k$ and $\bar{c}_{\beta_0}^k \hat{\bar{d}}_{\beta_0} \hat{\dots} \hat{\bar{c}}_{\beta_{m-1}}^k \hat{\bar{d}}_{\beta_{m-1}} \hat{\bar{c}}_{\beta_m}^k \hat{\bar{d}}_{\beta_m} \hat{\bar{c}}_{\beta_{m+1}}^k \hat{\bar{d}}_{\beta_{m+1}} \hat{\bar{e}}_{\beta_{m+1}}^k$ realize the same type over B .

[Why? By (*)₇ + (*)₆ and clause (f) of the assumption.]

We finish the induction step. Hence we get the desired statement. □_{4.9}

We now return to the generic pair conjecture. Central here is the following definition; the best case is $\lambda = \kappa = \theta$ is a measurable cardinal.

Definition 4.11. We say that the triple $(\lambda, \kappa, < \theta)$ is good or T -good when:

- (A) $|T| < \theta = \text{cf}(\theta) \leq \kappa = \text{cf}(\kappa) \leq \lambda = \lambda^{< \kappa}$,
- (B) T is dependent,
- (C) if M is κ -saturated of cardinality $\leq \lambda$ and $\bar{d} \in \theta^{> \mathfrak{C}}$ then we can find $B \subseteq M$ of cardinality $< \kappa$ and a strict $(\kappa, < \theta)$ -decomposition $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ over (M, B) such that $\bar{d} \triangleleft \bar{d}_0$,
- (D) if M is κ -saturated of cardinality $\leq \lambda$, $B \subseteq M$ has cardinality $< \kappa$, $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ is a strict $(\kappa, < \theta)$ -decomposition over (M, B) and $\bar{d} \in \theta^{> \mathfrak{C}}$ such that $\langle \bar{c}_{1+n} \hat{\bar{d}}_{1+n} : n < \omega \rangle$ is an indiscernible sequence over $M + \bar{c}_0 + \bar{d}_0 + \bar{d}$ then there is a strict $(\kappa, < \theta)$ -decomposition $\langle (\bar{c}_n^+, \bar{d}_n^+) : n < \omega \rangle$ over (M, B) such that $\bar{c}_n \trianglelefteq \bar{c}_n^+$, $\bar{d}_n \triangleleft \bar{d}_n^+$ for $n < \omega$ and $\bar{d}_0 \hat{\bar{d}} \trianglelefteq \bar{d}_0^+$.

So to begin our analysis we need the following observation.

Observation 4.12. (1) If T is a dependent and $\lambda > |T|$ is a measurable cardinal then $(\lambda, \lambda, < \lambda)$ is T -good.

(2) If T is dependent, κ a supercompact cardinal $> |T|$ and $\lambda = \lambda^{< \lambda} \geq \kappa$, then $(\lambda, \kappa, < \kappa)$ is T -good.

Proof. Immediate by Claim 3.12(2) and Observation 3.7(2), you may use Theorem 3.8, 3.12(2) too. $\square_{4.12}$

For the rest of this section we assume, till but not including the end that is, Conclusion 4.23.

Hypothesis 4.13. (1) T is dependent.

(2) $|T| < \theta = \text{cf}(\theta) \leq \text{cf}(\kappa) = \kappa \leq \lambda$ and $\lambda = \lambda^{<\lambda} > |T|$ and $(\lambda, \kappa, < \theta)$ is T -good.

Claim 4.14. Assume

- (a) δ is a limit ordinal $< \theta = \text{cf}(\theta)$,
- (b) $\langle (\bar{c}_n^\alpha, \bar{d}_n^\alpha) : n < \omega \rangle$ is a strict $(\kappa, < \theta)$ -decomposition over (M, B_α) for each $\alpha < \delta$,
- (c) $\bar{c}_n^\alpha \sqsubseteq \bar{c}_n^\beta \wedge \bar{d}_n^\alpha \sqsubseteq \bar{d}_n^\beta$ for $\alpha < \beta < \delta, n < \omega$,
- (d) $B_\alpha \subseteq B_\beta$ for $\alpha < \beta < \delta$,
- (e) we define $\bar{c}_n^\delta = \cup \{\bar{c}_n^\alpha : \alpha < \delta\}$, $\bar{d}_n^\delta = \cup \{\bar{d}_n^\alpha : \alpha < \delta\}$, $B_\delta = \cup \{B_\alpha : \alpha < \delta\}$.

Then $\langle (\bar{c}_n^\delta, \bar{d}_n^\delta) : n < \omega \rangle$ is a strict $(\kappa, < \theta)$ -decomposition over (M, B_δ) .

Remark 4.15. On another approach see [20, 3.23 = c70].

Proof. We have to check clauses (a)–(f) of Definition 3.5(1). Clause (a) is trivial by assumption (b) of Claim 4.14 recalling $\delta < \theta \leq \text{cf}(\kappa)$. Clause (b) holds as $\delta < \theta = \text{cf}(\theta)$ by assumption (a) of Claim 4.14 and $\langle \ell g(\bar{c}_n^\delta) : n < \omega \rangle$ is constant by assumptions (b) and (c) and similarly $\langle \ell g(\bar{d}_n^\delta) : n < \omega \rangle$ is constant. Next clauses (c)–(e) hold by their local character and assumptions (b) + (c) of Claim 4.14.

Lastly, proving clause (f) is the main point, it means to show:

- \odot_1 if $B_\delta \subseteq A \subseteq M$ and $|A| < \kappa$ then for some pair (\bar{c}, \bar{d}) of sequences from M we have $\langle \bar{c} \hat{\wedge} \bar{d} \rangle \hat{\wedge} \langle \bar{c}_n^\delta \hat{\wedge} \bar{d}_n^\delta : n < \omega \rangle$ is an indiscernible sequence over A .

Toward this by induction on $\alpha < \delta$ we choose a pair $(\bar{c}_\alpha^*, \bar{d}_\alpha^*)$ from M such that

- \otimes_1 $\langle \bar{c}_\alpha^* \hat{\wedge} \bar{d}_\alpha^* \rangle \hat{\wedge} \langle \bar{c}_n^\alpha \hat{\wedge} \bar{d}_n^\alpha : n < \omega \rangle$ is an indiscernible sequence over $A \cup \cup \{\bar{c}_\beta^* \hat{\wedge} \bar{d}_\beta^* : \beta < \alpha\}$.

[Why possible? We can choose $(\bar{c}_\alpha^*, \bar{d}_\alpha^*)$ because $\langle (\bar{c}_n^\alpha, \bar{d}_n^\alpha) : n < \omega \rangle$ being a strict $(\kappa, < \theta)$ -decomposition over B_α , we can apply clause (f) of Definition 3.5 recalling B_α being a subset of $A \cup \cup \{(\bar{c}_\beta^* \hat{\wedge} \bar{d}_\beta^* : \beta < \alpha\}$ and the latter being $\subseteq M$ and of cardinality $< \kappa$ as κ is regular $\geq \theta > \delta$.]

We can find $\langle (\bar{c}_\alpha, \bar{d}_\alpha) : \alpha < \delta \rangle$ though not necessarily in M such that

- \otimes_2 (a) $\langle \bar{c}_\alpha \hat{\wedge} \bar{d}_\alpha : \alpha < \delta \rangle \hat{\wedge} \langle \bar{c}_n^\delta \hat{\wedge} \bar{d}_n^\delta : n < \omega \rangle$ is an indiscernible sequence over A ,
- (b) $\bar{c}_\alpha^* \sqsubseteq \bar{c}_\alpha$ and $\bar{d}_\alpha^* \sqsubseteq \bar{d}_\alpha$ for $\alpha < \delta$ and, of course, $\ell g(\bar{c}_\alpha) = \ell g(\bar{c}_n^\alpha)$, $\ell g(\bar{d}_\alpha) = \ell g(\bar{d}_n^\alpha)$.

[Why? For this by using the saturation of \mathfrak{C} , it is enough to prove that: if $n < m < \omega, \alpha_0 < \dots < \alpha_{n-1} < \alpha_n < \delta$ then the sequences $\bar{c}_{\alpha_0}^* \hat{\wedge} \bar{d}_{\alpha_0}^* \hat{\wedge} \dots \hat{\wedge} \bar{c}_{\alpha_{n-1}}^* \hat{\wedge} \bar{d}_{\alpha_{n-1}}^*$ and $\bar{c}_0^{\alpha_0} \hat{\wedge} \bar{d}_0^{\alpha_0} \hat{\wedge} \bar{c}_1^{\alpha_1} \hat{\wedge} \bar{d}_1^{\alpha_1} \hat{\wedge} \dots \hat{\wedge} \bar{c}_{n-1}^{\alpha_{n-1}} \hat{\wedge} \bar{d}_{n-1}^{\alpha_{n-1}}$ realize the same type over $A \cup \{\bar{c}_k^\delta \hat{\wedge} \bar{d}_k^\delta : k \in [n, m]\}$. This is proved by induction on $n < \omega$. For $n = 0$ this is trivial. For $n + 1$, so assuming $n + 1 < m$:

First: $\bar{c}_{\alpha_n}^* \hat{\wedge} \bar{d}_{\alpha_n}^*$ and $\bar{c}_n^{\alpha_n} \hat{\wedge} \bar{d}_n^{\alpha_n}$ realize the same type over $A \cup \{\bar{c}_{\alpha_\ell}^* \hat{\wedge} \bar{d}_{\alpha_\ell}^* : \ell < n\} \cup \{\bar{c}_k^\delta : k \in [n + 1, m]\}$ by the choice of $(\bar{c}_{\alpha_n}^*, \bar{d}_{\alpha_n}^*)$.

Second: $\bar{c}_{\alpha_0}^* \hat{\wedge} \bar{d}_{\alpha_0}^* \hat{\wedge} \bar{c}_{\alpha_1}^* \hat{\wedge} \bar{d}_{\alpha_1}^* \hat{\wedge} \dots \hat{\wedge} \bar{c}_{\alpha_{n-1}}^* \hat{\wedge} \bar{d}_{\alpha_{n-1}}^*$ and $\bar{c}_0^{\alpha_0} \hat{\wedge} \bar{d}_0^{\alpha_0} \hat{\wedge} \bar{c}_1^{\alpha_1} \hat{\wedge} \bar{d}_1^{\alpha_1} \hat{\wedge} \dots \hat{\wedge} \bar{c}_{n-1}^{\alpha_{n-1}} \hat{\wedge} \bar{d}_{n-1}^{\alpha_{n-1}}$ realize the same type over $A \cup \{\bar{c}_k^\delta \hat{\wedge} \bar{d}_k^\delta : k \in [n, m]\}$ by the induction hypothesis hence also over $A \cup (\bar{c}_n^{\alpha_n} \hat{\wedge} \bar{d}_n^{\alpha_n}) \cup \{\bar{c}_k^\delta \hat{\wedge} \bar{d}_k^\delta : k \in [n + 1, m]\}$ by monotonicity. By the last two sentences we are done.]

Hence we can find a pair (\bar{c}', \bar{d}') such that:

$$\textcircled{*}_3 \langle \bar{c}' \hat{\wedge} \bar{d}' \rangle \hat{\wedge} \langle \bar{c}_\alpha \hat{\wedge} \bar{d}_\alpha : \alpha < \delta \rangle \hat{\wedge} \langle \bar{c}_n^\delta \hat{\wedge} \bar{d}_n^\delta : n < \omega \rangle \text{ is an indiscernible sequence over } A.$$

Lastly, we choose (\bar{c}'', \bar{d}'') such that

$$\textcircled{*}_4 (\bar{c}'', \bar{d}'') \text{ is a pair of sequences from } M \text{ such that } \bar{c}'' \hat{\wedge} \bar{d}'' \text{ realizes } \text{tp}(\bar{c}' \hat{\wedge} \bar{d}', A \cup \bigcup \{\bar{c}_\alpha^* \hat{\wedge} \bar{d}_\alpha^* : \alpha < \delta\}), \text{ of course with } \text{lg}(\bar{c}'') = \text{lg}(\bar{c}_0^\delta), \text{lg}(\bar{d}'') = \text{lg}(\bar{d}_0^\delta); \text{ equivalently there is an automorphism of } \mathfrak{C} \text{ which is the identity on } A \cup \bigcup \{\bar{c}_\alpha^* \hat{\wedge} \bar{d}_\alpha^* : \alpha < \delta\} \text{ mapping } \bar{c}' \hat{\wedge} \bar{d}' \text{ to } \bar{c}'' \hat{\wedge} \bar{d}''.$$

We shall prove that (\bar{c}'', \bar{d}'') is as required. Now to prove that (\bar{c}'', \bar{d}'') is as required in $\textcircled{*}_1$ it suffices to prove, for each $\alpha < \delta$ that

$$\textcircled{*}_2 \langle (\bar{c}'' \upharpoonright \text{lg}(\bar{c}_0^\alpha)) \hat{\wedge} (\bar{d}'' \upharpoonright \text{lg}(\bar{d}_0^\alpha)) \rangle \hat{\wedge} \langle \bar{c}_n^\alpha \hat{\wedge} \bar{d}_n^\alpha : n < \omega \rangle \text{ is an indiscernible sequence over } A.$$

[Why? As $\textcircled{*}_1$ is a “local” demand, i.e. it says that $\bar{c}'' \hat{\wedge} \bar{d}''$ is a sequence realizing an appropriate type q (and is from M) and for this it suffices to check every finite subtype so $\textcircled{*}_2$ suffices.]

Now $\textcircled{*}_2$ follows by $\textcircled{*}_6$ below. Let $(\bar{c}_{\beta,\gamma}^*, \bar{d}_{\beta,\gamma}^*) = (\bar{c}_\beta^* \upharpoonright \text{lg}(\bar{c}_0^\gamma), \bar{d}_\beta^* \upharpoonright \text{lg}(\bar{d}_0^\gamma))$ and $(\bar{c}_\gamma'', \bar{d}_\gamma'') = (\bar{c}'' \upharpoonright \text{lg}(\bar{c}_0^\gamma), \bar{d}'' \upharpoonright \text{lg}(\bar{d}_0^\gamma))$ for $\gamma < \delta$ and $\beta \in [\gamma, \delta)$.

Now

$$\textcircled{*}_5 \langle \bar{c}_\alpha'' \hat{\wedge} \bar{d}_\alpha'' \rangle \hat{\wedge} \langle \bar{c}_{\beta,\alpha}^* \hat{\wedge} \bar{d}_{\beta,\alpha}^* : \beta \in [\alpha, \delta) \rangle \text{ is a strict}_0 \text{ decomposition over } (A, A).$$

[Why? By $\textcircled{*}_3$, this holds for $\langle (\bar{c}'' \upharpoonright \text{lg}(\bar{c}_0^\alpha)) \hat{\wedge} (\bar{d}'' \upharpoonright \text{lg}(\bar{d}_0^\alpha)) \rangle \hat{\wedge} \langle \bar{c}_{\beta,\alpha}^* \hat{\wedge} \bar{d}_{\beta,\alpha}^* : \beta \in [\alpha, \delta) \rangle$ and we use preservation by automorphism of \mathfrak{C} , i.e. use $\textcircled{*}_4$.]

$$\textcircled{*}_6 \text{ For } \alpha < \delta \text{ and } i \leq \omega \text{ the sequence } \langle \bar{c}_\alpha'' \hat{\wedge} \bar{d}_\alpha'' \rangle \hat{\wedge} \langle \bar{c}_{\beta,\alpha}^* \hat{\wedge} \bar{d}_{\beta,\alpha}^* : \beta \in [\alpha, \delta) \rangle \hat{\wedge} \langle \bar{c}_n^\alpha \hat{\wedge} \bar{d}_n^\alpha : n < i \rangle \text{ is a strict}_0 \text{ decomposition over } (A, A).$$

[Why? We prove this by induction on i , noticing that for $i = \omega$ we get the desired conclusion; also for $i = \omega$ the inductive step is trivial and for $i = 0$ use $\textcircled{*}_5$.

So assume $i = n + 1$, let

- (*)₁ • $A_1 = A \cup \text{Rang}(\bar{c}''_\alpha) \cup \text{Rang}(\bar{d}''_\alpha)$,
- $A_2 = A_{2,\delta}$ where
- for $\gamma \in [\alpha, \delta]$ we let $A_{2,\gamma} = \cup\{\text{Rang}(\bar{c}^*_{\beta,\alpha}) \cup \text{Rang}(\bar{d}^*_{\beta,\alpha}) : \beta \in [\alpha, \gamma]\}$,
- $A_3 = \cup\{\bar{c}^\alpha_\ell \wedge \bar{d}^\alpha_\ell : \ell < n\}$.

Clearly

- (*)₂ (a) $A_1, A_{2,\delta}$ are $\subseteq M$ and $A_3 \subseteq \mathfrak{C}$,
- (b) $A_{2,\gamma}$ is \subseteq -increasing for $\gamma \in [\alpha, \delta]$.

Let $\gamma \in (\alpha, \delta)$ be a successor ordinal and we shall prove

- (*)₃ $(c^*_{\gamma,\alpha} \wedge \bar{d}^*_{\gamma,\alpha}) \wedge \dots \wedge (\bar{c}^*_{\gamma+n,\alpha} \wedge \bar{d}^*_{\gamma+n,\alpha})$ and $(\bar{c}^\alpha_0 \wedge \bar{d}^\alpha_0) \wedge \dots \wedge (\bar{c}^\alpha_n \wedge \bar{d}^\alpha_n)$ realize the same type over $A_1 \cup A_{2,\gamma}$.

As δ is a limit ordinal this suffices by Observation 4.8 (with $n = 2, I_0$ a singleton, I_1 isomorphic to δ, I_2 isomorphic to ω). Now $(c^*_{\gamma,\alpha} \wedge \bar{d}^*_{\gamma,\alpha}) \wedge \dots \wedge (\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha})$ and $(\bar{c}^\alpha_0 \wedge \bar{d}^\alpha_0) \wedge \dots \wedge (\bar{c}^\alpha_{n-1} \wedge \bar{d}^\alpha_{n-1})$ realize the same type over $A_1 \cup A_{2,\gamma}$ by the induction hypothesis.

Next by clause (b) of the assumption $\text{tp}(\bar{c}^\alpha_n, M \cup \cup\{\bar{c}^\alpha_m \wedge \bar{d}^\alpha_m : m < n\})$ does not split over B_α hence $\text{tp}(\bar{c}^\alpha_n, A \cup A_1 \cup A_2 \cup A_3)$ does not split over $B_\alpha \subseteq A$ by (*₁).

Hence by the induction hypothesis $\langle \bar{c}^\alpha_\ell \wedge \bar{d}^\alpha_\ell \rangle \langle \bar{c}^*_{\beta,\alpha} \wedge \bar{d}^*_{\beta,\alpha} : \beta \in [\alpha, \delta] \rangle \langle (\bar{c}^\alpha_k \wedge \bar{d}^\alpha_k) : k < n \rangle$ is an indiscernible sequence over $A \cup \bar{c}^\alpha_n$, hence

- $(\bar{c}^*_{\gamma,\alpha} \wedge \bar{d}^*_{\gamma,\alpha}) \wedge \dots \wedge (\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \wedge \bar{c}^\alpha_n$ and $(\bar{c}^\alpha_0 \wedge \bar{d}^\alpha_0) \wedge \dots \wedge (\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \wedge \bar{c}^\alpha_n$ realize the same type over $A_1 \cup A_{2,\gamma}$.

But by \otimes_1 clearly $\bar{c}^\alpha_n, \bar{c}^*_{\gamma+n,\alpha}$ realize the same type over $A_1 \cup A_{2,\gamma}$ hence

- $(c^*_{\gamma,\alpha} \wedge \bar{d}^*_{\gamma,\alpha}) \wedge \dots \wedge (\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \wedge \bar{c}^*_{\gamma+n,\alpha}$ and $(\bar{c}^\alpha_0 \wedge \bar{d}^\alpha_0) \wedge \dots \wedge (\bar{c}^\alpha_{n-1} \wedge \bar{d}^\alpha_{n-1}) \wedge \bar{c}^\alpha_n$ realize the same type over $A_1 \cup A_{2,\gamma}$.

We can choose \bar{d}' in \mathfrak{C} such that $(\bar{c}^*_{\gamma,\alpha} \wedge \bar{d}^*_{\gamma,\alpha}) \wedge \dots \wedge (\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \wedge (\bar{c}^*_{\gamma+n,\alpha} \wedge \bar{d}')$ and $(\bar{c}^\alpha_0 \wedge \bar{d}^\alpha_0) \wedge \dots \wedge (\bar{c}^\alpha_{n-1} \wedge \bar{d}^\alpha_{n-1}) \wedge (\bar{c}^\alpha_n \wedge \bar{d}^\alpha_n)$ realize the same type over $A_1 \cup A_{2,\gamma}$ so (to prove (*₃) hence \otimes_6) it suffices to prove that $\bar{d}^*_{\gamma+n,\alpha}, \bar{d}'$ realize the same type over $A_1 \cup A_{2,\gamma+n} \cup \bar{c}^*_{\gamma+n,\alpha}$.

Recall that $\langle \bar{c}^*_{\beta,\alpha} \wedge \bar{d}^*_{\beta,\alpha} : \beta \in [\alpha, \delta] \rangle \langle \bar{c}^\alpha_n \wedge \bar{d}^\alpha_n : n < \omega \rangle$ is an indiscernible sequence, hence the sequences $\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha} \wedge \bar{c}^*_{\gamma+n,\alpha} \wedge \bar{d}^*_{\gamma+n,\alpha}$ and $\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha} \wedge \bar{c}^\alpha_n \wedge \bar{d}^\alpha_n$ realize the same type.

By the two previous sentences and the transitivity of the equality of types $\bar{d}^*_{\gamma+n,\alpha}, \bar{d}'$ realize the same type over $(\bar{c}^*_{\gamma+n-1,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \wedge \bar{c}^*_{\gamma+n,\alpha}$, but by clause (e)⁻ of Definition 3.5 which applies by clause (b) of the assumption and \otimes_5 above we have $\text{tp}(\bar{d}^*_{\gamma+n,\alpha}, \bar{c}^*_{\gamma+n,\alpha} \wedge \bar{d}^*_{\gamma+n-1,\alpha}) \vdash \text{tp}(\bar{d}^*_{\gamma+n,\alpha}, A_1 + A_{2,\gamma+n} + \bar{c}^*_{\gamma+n,\alpha})$ so we are done. □_{4.14}

S. Shelah

- Definition 4.16.** (1) We say that the strict $(\kappa, < \theta)$ -decompositions $\langle (\bar{c}'_\varepsilon, \bar{d}'_\varepsilon) : \varepsilon < \delta \rangle, \langle (\bar{c}''_\varepsilon, \bar{d}''_\varepsilon) : \varepsilon < \delta \rangle$ over M are equivalent over $B \in [M]^{< \lambda}$ when for some automorphism f of M over B for every n and $\varepsilon_0 < \dots < \varepsilon_{n-1} < \delta, f$ maps the type $\text{tp}((\bar{c}'_{\varepsilon_\delta} \hat{\ } \bar{d}'_{\varepsilon_\delta}) \hat{\ } \dots \hat{\ } (\bar{c}'_{\varepsilon_{n-1}} \hat{\ } \bar{d}'_{\varepsilon_{n-1}}), M)$ to the type $\text{tp}((\bar{c}''_{\varepsilon_0} \hat{\ } \bar{d}''_{\varepsilon_0}) \hat{\ } \dots \hat{\ } (\bar{c}''_{\varepsilon_{n-1}} \hat{\ } \bar{d}''_{\varepsilon_{n-1}}), M)$ and $\text{lg}(\bar{c}'_{\varepsilon_0}) = \text{lg}(\bar{c}''_{\varepsilon_0}), \text{lg}(\bar{d}'_{\varepsilon_0}) = \text{lg}(\bar{d}''_{\varepsilon_0})$.
- (2) In part (1) we say “weakly equivalent over B ” when for every $\zeta < \kappa$ and $\bar{b}' \in {}^\zeta M$ there is $\bar{b}'' \in {}^\zeta M$ and vice versa and elementary mapping f such that: $f \supseteq \text{id}_B, f(\bar{b}') = \bar{b}''$ and $f(\bar{c}'_\varepsilon) = f(\bar{c}''_\varepsilon), f(\bar{d}'_\varepsilon) = f(\bar{d}''_\varepsilon)$ for $\varepsilon < \delta$.
- (3) If $B = \emptyset$ then we may omit it.

Claim 4.17. (1) If M is κ -saturated of cardinality λ and $B \in [M]^{< \kappa}$ then the number of strict $(\lambda, \kappa, < \theta)$ -decompositions $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ over (M, B) such that $\text{lg}(\bar{d}_n) > 0$ up to weak equivalence or when $\text{lg}(\bar{d}_n) = 0$ up to equivalence over B is $\leq 2^{< \kappa}$, see Definition 4.16.

- (2) For M, B as above, two strict $(\kappa, < \theta)$ -decompositions are equivalent when they are weakly equivalent over B and $\lambda = \kappa$.
- (3) If $\langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ is a $(\kappa, < \theta)$ -decomposition over (M, B) so $B \in [M]^{< \kappa}$ and $C = \cup\{\text{rang}(\bar{c}_n \hat{\ } \bar{d}_n) : n < \omega\} \cup B$ then $M_{[C]}$ is a κ -sequence homogeneous model (see [14, 2.1–2.4]).
- (4) Assume

- (a) $\langle (\bar{c}^\iota_{\alpha,n}, \bar{d}^\iota_{\alpha,n}) : n < \omega \rangle$ is a strict $(\kappa, < \theta)$ -decomposition of (M, B_ι) for $\alpha \leq \delta$ such that $\iota = 1, 2,$
- (b) $\alpha < \beta \leq \delta \Rightarrow \bar{c}^\iota_{\alpha,n} \leq \bar{c}^\iota_{\beta,n} \wedge \bar{d}^\iota_{\alpha,n} \leq \bar{d}^\iota_{\beta,n}$ and $\bar{c}^\iota_{\delta,n} = \bigcup_{\alpha < \delta} \bar{c}^\iota_{\alpha,n}, \bar{d}^\iota_{\delta,n} = \bigcup_{\alpha < \delta} \bar{d}^\iota_{\alpha,n},$
- (c) $\langle (\bar{c}^1_{\alpha,n}, \bar{d}^1_{\alpha,n}) : n < \omega \rangle$ is weakly equivalent to $\langle (\bar{c}^2_{\alpha,n}, \bar{d}^2_{\alpha,n}) : n < \omega \rangle$ over B for $\alpha < \delta$.

Then clause (c) holds also for $\alpha = \delta$.

Proof. (1) First, if $\text{lg}(\bar{d}_n) = 0$, by Fact 1.2(4), that is [17, 5.26], however in the present case κ is measurable hence strongly inaccessible so 4.17(1) is easy. That is, fixing B , also the number of $\alpha < \lambda$ and $p(\bar{x}_\alpha) \in \mathbf{S}(M)$ not splitting over B is $\leq \lambda$ by Fact 1.2. The case for weakly equivalent also is easy.

(2) Let $\langle (\bar{c}'_n, \bar{d}'_n) : n < \omega \rangle, \langle (\bar{c}''_n, \bar{d}''_n) : n < \omega \rangle$ be two strict $(\kappa, < \theta)$ -decompositions over M . As a λ -sequence-homogeneous model of cardinality λ is determined up to isomorphisms by the set of complete types of finite tuples in it (by [5] or see [14, Sec. 2]) by part (3) it suffices to show:

- (*)₁ for every $\bar{b}' \in {}^\omega M$ there are $\bar{b}'' \in {}^\omega M$ and an elementary mapping f such that $f(\bar{b}') = \bar{b}'', f \supseteq \text{id}_B$ and $\varepsilon < \delta \Rightarrow f(\bar{c}'_\varepsilon) = \bar{c}''_\varepsilon \wedge f(\bar{d}'_\varepsilon) = \bar{d}''_\varepsilon,$
- (*)₂ for every $\bar{b}'' \in {}^\omega M$ there are $\bar{b}' \in {}^\omega M$ and f as above.

But this follows from weakly equivalent.

- (3) Let g be the identity mapping on $\cup\{\text{Rang}(\bar{c}_n \hat{\ } \bar{d}_n) : n < \omega\} \cup B$.

Let

- (*)₁ \mathcal{F} be the set of f such that:
- f is an elementary mapping,
 - $\text{Dom}(f) \subseteq M$ has cardinality $< \kappa$,
 - $\text{Rang}(f) \subseteq M$,
 - $f \cup g$ is an elementary mapping.

It suffices to prove (*)₁–(*)₄ below:

- (*)₁ $\mathcal{F} \neq \emptyset$.

[Why? As $f = \text{id}_B$ belongs to \mathcal{F} .]

- (*)₂ $f \in \mathcal{F}$ if and only if $f^{-1} \in \mathcal{F}$

- (*)₃ \mathcal{F} is closed under union of increasing chains of length $< \kappa$.

[Why? Just check.]

- (*)₄ if $f \in \mathcal{F}$, $\ell \in \{1, 2\}$ and $a_\ell \in M$ then for some $a_{3-\ell} \in M$ we have $f \cup \{(a_1, a_2)\} \in \mathcal{F}$.

Why? By symmetry, without loss of generality $\ell = 1$. Let δ be a limit ordinal $< \kappa$. We choose $\bar{c}'_\varepsilon, \bar{d}'_\varepsilon$ sequences from M of length $\text{lg}(\bar{c}_0), \text{lg}(\bar{d}_0)$ respectively by induction on $\varepsilon < \delta$, by applying Definition 3.5(f) to $M, \langle (\bar{c}_n, \bar{d}_n) : n < \omega \rangle$ and $B_\varepsilon = B \cup \text{Dom}(f) \cup \{a_1\} \cup \text{Rang}(f) \cup \bigcup \{\bar{c}'_\zeta \hat{\ } \bar{d}'_\zeta : \zeta < \varepsilon\}$ and get $\bar{c}'_\varepsilon, \bar{d}'_\varepsilon$ from M as there.

Clearly

- (*)_{4.1} $\langle \bar{c}'_\varepsilon \hat{\ } \bar{d}'_\varepsilon : \varepsilon < \delta \rangle \hat{\ } \langle (\bar{c}_n \hat{\ } \bar{d}_n) : n < \omega \rangle$ is an indiscernible sequence over B_0 .

This implies that, letting $B_* = \bigcup \{\bar{c}'_\varepsilon \hat{\ } \bar{d}'_\varepsilon : \varepsilon < \delta\} \cup B$:

- (*)_{4.2} $f \cup g \cup \text{id}_{B_*}$ is an elementary mapping.

As M is κ -saturated

- (*)_{4.3} there is $a_2 \in M$ such that $f \cup \text{id}_{B_*} \cup \{(a_1, a_2)\}$ is an elementary mapping.

So we can prove as in the proof of Claim 4.14

- (*)_{4.4} $f \cup \{(a_1, a_2)\} \in \mathcal{F}$.

So we are done.

- (4) Repeat the proof of Claim 4.14. □_{4.17}

Clearly Claim 4.14 is a step forward. Now we prove the generic pair conjecture; instead of assuming that the cardinality λ is measurable we can restrict T .

Toward this (the λ^+ is an overkill, λ and even θ suffice).

Definition 4.18. We say that the triple $\mathbf{m} = (M, N, \mathcal{A}) = (M_{\mathbf{m}}, N_{\mathbf{m}}, A_{\mathbf{m}})$ is a $(\lambda, \kappa, < \theta)$ -system when $(\lambda = \lambda^{< \lambda} \geq \text{cf}(\kappa) = \kappa \geq \theta = \text{cf}(\theta) > |T|)$ and it satisfies clauses (a)–(d) below, and say the triple is a full $(\lambda, \kappa, < \theta)$ -system when it satisfies clauses (a)–(i) below where:

- ⊞ (a) M is κ -saturated of cardinality λ , even strongly κ -saturated^x (just to simplify clause (i)),
- (b) $M \prec N \prec \mathfrak{C}$ and N has cardinality λ ,
- (c) $\mathcal{A}_{\mathbf{m}}$ is a set of cardinality λ of objects \mathbf{p} such that:
 - (α) \mathbf{p} is of the form $\langle (\bar{c}_{\varepsilon}, \bar{d}_{\varepsilon}) : \varepsilon < \lambda^+ \rangle = \langle (\bar{c}_{\varepsilon}[\mathbf{p}], \bar{d}_{\varepsilon}[\mathbf{p}]) : \varepsilon < \lambda^+ \rangle$,
 - (β) $\bar{c}_0 \hat{\wedge} \bar{d}_0 \subseteq N$,
 - (γ) $\langle (\bar{c}_{\varepsilon}, \bar{d}_{\varepsilon}) : \varepsilon < \lambda^+ \rangle$ is a strict $(\kappa, < \theta)$ -decomposition over M ,
- (d) $\mathcal{A}_{\mathbf{m}}$ is partially ordered by: $\mathbf{p} \leq_{\mathbf{m}} \mathbf{q}$ if and only if
 - $\bar{c}_0[\mathbf{p}] \trianglelefteq \bar{c}_0[\mathbf{q}]$ and $\bar{d}_0[\mathbf{p}] \trianglelefteq \bar{d}_0[\mathbf{q}]$,
 - for every large enough $\varepsilon < \lambda^+$, $\bar{c}_{\varepsilon}[\mathbf{p}] \trianglelefteq \bar{c}_{\varepsilon}[\mathbf{q}]$, $\bar{d}_{\varepsilon}[\mathbf{p}] \trianglelefteq \bar{d}_{\varepsilon}[\mathbf{q}]$,
- (e) $\mathcal{A}_{\mathbf{m}}$ is closed under union of increasing chains of length $< \theta$ defined naturally,
- (f) if $\mathbf{p} \in \mathcal{A}_{\mathbf{m}}$ and $\bar{d} \in {}^{\theta}N$ then for some $\mathbf{q} \in \mathcal{A}$ above \mathbf{p} we have $\text{Rang}(\bar{d}) \subseteq \text{Rang}(\bar{d}_0[\mathbf{q}])$,
- (g) N is \aleph_0 -saturated,
- (h) up to really equivalence every possible \mathbf{p} in clause (c) occurs (i.e. is represented in \mathcal{A}), where \mathbf{p}, \mathbf{q} are really equivalent when they are as in (c) and $(\bar{c}_{\mathbf{p},0}, \bar{d}_{\mathbf{p},0}) = (\bar{c}_{\mathbf{q},0}, \bar{d}_{\mathbf{q},0})$ and are equivalent,
- (i) there is $\mathbf{q}_2 \in \mathcal{A}_{\mathbf{m}}$ such that $\mathbf{p}_2 \leq_{\mathbf{m}} \mathbf{q}_2$ and $\mathbf{q}_2, \mathbf{q}_1$ are equivalent as witnessed by f , when:
 - (α) $\mathbf{p}_1 \leq_{\mathbf{m}} \mathbf{q}_1$ so both are from $\mathcal{A}_{\mathbf{m}}$,
 - (β) $\mathbf{p}_2 \in \mathcal{A}_{\mathbf{m}}$ is really equivalent to \mathbf{p}_1 ,
 - (γ) $f \in \text{aut}(M)$ maps \mathbf{p}_1 to \mathbf{p}_2 , yes the same f .

- Definition 4.19.** (1) Let $\text{BP} = \text{BP}_{\lambda, \kappa, < \theta}$ be the set of $(\lambda, \kappa, < \theta)$ -systems.
- (2) If $(M_{\ell}, N_{\ell}, \mathcal{A}_{\ell})$ is a $(\lambda, \kappa, < \theta)$ -system for $\ell = 1, 2$ we say $(M_2, N_2, \mathcal{A}_2)$ is above $(M_1, N_1, \mathcal{A}_1)$ or $(M_1, N_1, \mathcal{A}_1) \leq_{\text{BP}_{\lambda, \kappa, < \theta}} (M_2, N_2, \mathcal{A}_2)$ when $M_1 = M_2$, $N_1 \prec N_2$ and $\mathcal{A}_1 \subseteq \mathcal{A}_2$.
- (3) We may write just $\text{BP}, \leq_{\text{BP}}$ when $(\lambda, \kappa, < \theta)$ is clear from the context.
- (4) If $\mathbf{m} \in \text{BP}$ and $\mathbf{p} \in \mathcal{A}_{\mathbf{m}}$, we say that B is a base of \mathbf{p} when $B \subseteq M_{\mathbf{m}}$ has cardinality $< \kappa$ and \mathbf{p} is a strict $(\kappa, < \theta)$ -decomposition over (M, B) .

Claim 4.20. Assume $\kappa = \lambda = \lambda^{< \lambda}$ (see Definition 4.11).

- (1) If $M \prec \mathfrak{C}$ is strongly κ -saturated of cardinality λ then there is a pair (N, \mathcal{A}) such that (M, N, \mathcal{A}) is a $(\lambda, \kappa, < \theta)$ -system.

^xThis means that for every (M, M) -elementary mapping of cardinality $< \kappa$ can be extended to an automorphism of M .

- (2) If (M, N_1, \mathcal{A}_1) is a $(\lambda, \kappa, < \theta)$ -system and $N_2 \prec \mathfrak{C}$ and $\|N_2\| = \lambda$ then there is a pair (N_3, \mathcal{A}_3) such that (M, N_3, \mathcal{A}_3) is a $(\lambda, \kappa, < \theta)$ -system above (M, N_1, \mathcal{A}_1) and $N_2 \prec N_3$.
- (3) If $\langle (M, N_\varepsilon, \mathcal{A}_\varepsilon) : \varepsilon < \delta \rangle$ is an increasing sequence of $(\lambda, \kappa, < \theta)$ -systems and δ is a limit ordinal $< \lambda^+$ then the union $(M, \bigcup_{\alpha < \delta} N_\alpha, \bigcup_{\alpha < \delta} \mathcal{A}_\alpha)$ is a $(\lambda, \kappa, < \theta)$ -system, which is a least upper bound of $\{(M, N_\varepsilon, \mathcal{A}_\varepsilon) : \varepsilon < \delta\}$.
- (4) If in part (3) we have $\text{cf}(\delta) = \lambda$ or just $\text{cf}(\delta) \geq \theta$ and each is full, then so is the union.

Proof. (1) Let $\mathbf{m} = (M, M, \emptyset)$ and check.

- (2) Let $N_3 \prec \mathfrak{C}$ be such that $N_1 \cup N_2 \prec N_3$, N_3 is κ -saturated of cardinality λ and use (M, N_3, \mathcal{A}_1) .
- (3) Easy.
- (4) Easy. □4.20

Claim 4.21. (1) We have:

- (a) \leq_{BP} is a partial order on BP,
 - (b) any \leq_{BP} -increasing sequence of length $< \lambda^+$ has an upper bound,
 - (b)⁺ moreover if $\langle (M, N_\alpha, \mathcal{A}_\alpha) : \alpha < \delta \rangle$ is \leq_{BP} -increasing then $(M, \bigcup\{N_\alpha : \alpha < \delta\}, \bigcup\{\mathcal{A}_\alpha : \alpha < \delta\})$ is a least \leq_{BP} -upper bound,
 - (c) BP is not empty, moreover for every $M \in \text{EC}_{\lambda, \lambda}(T)$ there is $\mathbf{m} \in \text{BP}$ such that $M_{\mathbf{m}} = M = N_{\mathbf{m}}, \mathcal{A}_{\mathbf{m}} = \emptyset$.
- (2) If \mathbf{m} is a $(\lambda, \kappa, < \theta)$ -system then:
- (a) $\leq_{\mathbf{m}}$ is a partial order of $\mathcal{A}_{\mathbf{m}}$,
 - (b) if also \mathbf{n} is a $(\lambda, \kappa, < \theta)$ -system and $\mathbf{m} \leq_{\text{BP}} \mathbf{n}$ then $\leq_{\mathbf{m}} = \leq_{\mathbf{n}} \upharpoonright \mathcal{A}_{\mathbf{m}}$,
 - (c) if δ as a limit ordinal $< \theta$ and $\mathbf{p}_\alpha \in \mathcal{A}_{\mathbf{m}}$ for $\alpha < \delta$ is $\leq_{\mathbf{m}}$ -increasing with α then there is $\mathbf{n} \in \text{BP}$ such that $\mathbf{m} \leq_{\text{BP}} \mathbf{n}$ and $\leq_{\mathbf{n}}$ -upper bound of $\{\mathbf{p}_\alpha : \alpha < \delta\}$,
 - (d) if $\mathbf{p} \in \mathcal{A}_{\mathbf{m}}$ and $\bar{d} \in {}^{\theta}N$ then there are a $(\lambda, \kappa, < \theta)$ -system \mathbf{n} satisfying $\mathbf{m} \leq_{\text{BP}} \mathbf{n}$ and $\mathbf{q} \in \mathcal{A}_{\mathbf{n}}$ such that $\mathbf{p} \leq_{\mathbf{n}} \mathbf{q}$ and $\bar{d} \subseteq \text{Rang}(\bar{d}_{\mathbf{q}}^{\mathbf{q}})$,
 - (e) if $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{A}_{\mathbf{m}}$ are equivalent and $\mathbf{p}_1 \leq_{\mathbf{m}} \mathbf{q}_2$ then for some $\mathbf{n} \in \text{BP}$ we have $\mathbf{m} \leq_{\text{BP}} \mathbf{n}$ and for some $\mathbf{q}_2 \in \mathcal{A}_{\mathbf{n}}$ equivalent to \mathbf{q}_1 we have $\mathbf{p}_1 \leq_{\mathbf{n}} \mathbf{q}_2$.
- (3) If (M, N, \mathcal{A}) is a $(\lambda, \kappa, < \theta)$ -system then there is a full $(\lambda, \kappa, < \theta)$ -system above it.

Proof. (1) Clause (a) holds easily by checking Definition 4.19(2). Clauses (b) and (b)⁺ hold easily by Claim 4.20(3), and for clause (c) use (M, M, \emptyset) .

(2) Why? Clauses (a) and (b) Easy.

Clause (c): For $\alpha < \beta < \delta$ let $\zeta = \zeta(\alpha, \beta) < \lambda^+$ be such that:

- if $\varepsilon \in [\zeta, \lambda^+)$ then $\bar{c}_\varepsilon^{\mathbf{p}_\alpha} \sqsubseteq \bar{c}_\varepsilon^{\mathbf{p}_\beta}, \bar{d}_\varepsilon^{\mathbf{p}_\alpha} \sqsubseteq \bar{d}_\varepsilon^{\mathbf{p}_\beta}$.

Let $\zeta(*) = \sup\{\zeta(\alpha, \beta) + 2 : \alpha < \beta < \delta\}$, so necessarily $\zeta(*) < \lambda^+$. Clearly if $\zeta \in [\zeta(*), \lambda^+) \cup \{0\}$ then $\langle \bar{c}_\zeta^{\mathbf{P}^\alpha} : \alpha < \delta \rangle$ is \triangleleft -increasing and also the sequence $\langle \bar{d}_\zeta^{\mathbf{P}^\alpha} : \alpha < \delta \rangle$ is \triangleleft -increasing.

We choose $\bar{c}_0 = \cup\{\bar{c}_0^{\mathbf{P}^\alpha} : \alpha < \delta\}$, $\bar{d}_0 = \cup\{\bar{d}_0^{\mathbf{P}^\alpha} : \alpha < \delta\}$ and for $\varepsilon < \lambda^+$ let $\bar{c}_{1+\varepsilon} = \cup\{\bar{c}_{\zeta(*)+\varepsilon}^{\mathbf{P}^\alpha} : \alpha < \delta\}$ and $\bar{d}_{1+\varepsilon} = \cup\{\bar{d}_{\zeta(*)+\varepsilon}^{\mathbf{P}^\alpha} : \alpha < \delta\}$.

Now easily

- $\mathbf{p} = \langle \langle \bar{c}_\varepsilon, \bar{d}_\varepsilon \rangle : \varepsilon < \lambda^+ \rangle$ is well defined and satisfies the demand in Definition 4.18(c) $(\alpha) + (\beta) + (\gamma)$.

[Why? By Claim 4.14.]

So we define \mathbf{n} as $(M, N_{\mathbf{m}}, \{\mathbf{p}\} \cup \mathcal{A}_{\mathbf{m}})$, easily

- $\mathbf{n} \in \text{BP}$ and $\mathbf{m} \leq \mathbf{n}$.

Lastly, $\alpha < \delta \Rightarrow \mathbf{p}_\alpha \leq_{\mathbf{n}} \mathbf{p}$ as witnessed by $\zeta(*) \times \omega$. So we are done proving clause (c).

Clause (d): Easy. By clause (D) of Definition 4.11 of “ $(\lambda, \kappa, < \theta)$ is T -good” which holds by Hypothesis 4.13.

Clause (e): By the definition of “equivalence” in Definition 4.16 (and Definition 4.18(h)).

(3) We fix a $(\lambda, \kappa, < \theta)$ -system (M, N_*, \mathcal{A}_*) . We now shall choose $\mathbf{m}_i \in \text{BP}$, \leq_{BP} -increasing by induction on $i \leq \lambda$ and

- \odot_1 for $i = 0$: \mathbf{m}_0 is (M, N_*, \mathcal{A}_*) ,
- \odot_2 for limit $i < \lambda$, \mathbf{m}_i is a \leq_{BP} -upper bound of $\langle \mathbf{m}_j : j < i \rangle$, in fact the union.

[Why? Possible by clause (b)⁺ of part (1).]

By bookkeeping

- \odot_3 if $\langle \mathbf{p}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{m}_j}$ -increasing, δ a limit ordinal $< \theta$ and $j < \lambda$, then for some $i \in (j, \lambda)$ the sequence has an upper bound by $\leq_{\mathbf{m}_{i+1}}$.

[Why? By clause (c) of part (2).]

- \odot_4 if $\mathbf{p} \in \mathcal{A}_{\mathbf{m}_j}$ and $\bar{d} \in {}^\theta N_{\mathbf{m}_j}$ and $j < \lambda$ then for some $i \in (j, \lambda)$, there is $\mathbf{q} \in \mathcal{A}_{\mathbf{m}_{i+1}}$ such that $\mathbf{p} \leq_{\mathbf{m}_{i+1}} \mathbf{q}$ and $\text{Rang}(\bar{d}) \subseteq \text{Rang}(\bar{d}_0^{\mathbf{q}})$.

[Why? By clause (d) of part (2).]

- \odot_5 if $j < \lambda$ and $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, f$ satisfy clauses (α) – (ε) of Definition 4.18(i) with \mathbf{m}_j here standing for \mathbf{m} there then for some $i \in (j, \lambda)$ there is $\mathbf{q}_2 \in \mathcal{A}_{\mathbf{m}_{i+1}}$ such that $\mathbf{p}_2 \leq_{\mathbf{m}_{i+1}} \mathbf{q}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ are equivalent as witnessed by f .

[Why? Use clause (e) of part (2).]

- \odot_6 for every $i < \lambda$ for some $j \in (i, \lambda)$ and N' , the model N' is κ -saturated and even saturated and $N_j \prec N' \prec N_{j+1}$.

[Why: By 4.20(20).]

So we can carry the induction. Now \mathbf{m}_λ is as required. □4.21

Theorem 4.22. *Assume $(\lambda, \kappa, < \theta) = (\lambda, \lambda, < \lambda)$ so is a T -good triple, see Definition 4.11, Hypothesis 4.13.*

- (1) *If $\lambda^+ = 2^\lambda$, $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is an \prec -increasing continuous sequence of members of $\text{EC}_\lambda(T)$, M_α saturated if α is non-limit and $M = \cup\{M_\alpha : \alpha < \lambda^+\}$ is saturated then for some club (= closed unbounded subset) E of λ^+ for any $\alpha < \beta < \delta \in E$ and α, β are non-limit or are from $S_\lambda^{\lambda^+}$, the pairs (M_δ, M_α) and (M_δ, M_β) are isomorphic.*
- (2) *If $\mathbf{m}_\ell = (M_\ell, N_\ell, \mathcal{A}_\ell)$ is a full $(\lambda, \lambda, < \lambda)$ -system for $\ell = 1, 2$ then $(N_1, M_1) \cong (N_2, M_2)$ that is there is an isomorphism f from N_1 onto N_2 mapping M_1 onto M_2 .*

Proof. (1) By part (2), noting that M_α, M_β are saturated and recalling Claim 4.21 and its proof but we elaborate.

For $\alpha < \lambda^+$ let $\Lambda_\alpha = \{\eta : \eta \text{ an increasing continuous sequence of ordinals with supremum } < \alpha \text{ of length } < \lambda \text{ and } \eta(0) \text{ has cofinality } \lambda \text{ and } \eta(i+1) \text{ successor}\}$ and $\Lambda_\alpha^+ = \Lambda_{\alpha+1} \setminus \Lambda_\alpha$. We now by induction on α choose M'_α and $(N_\eta, \mathcal{A}_\eta)$ for $\eta \in \Lambda_{\alpha+1}^+$ such that:

- (*) (a) $M'_\alpha \prec \mathfrak{C}$ has cardinality λ ,
- (b) M'_α is saturated when $\text{cf}(\alpha) \notin [\aleph_0, \lambda)$,
- (c) if $p \in \mathbf{S}(M'_\beta)$ then for some $\gamma \in (\beta, \lambda^+)$, p is realized in M'_γ ,
- (d) if $\alpha = \beta + 1$ and $\eta \in \Lambda_{\alpha+1}^+$ (so a $\text{sup Rang}(\eta) = \alpha$) then
 - (α) $M'_\beta \prec N_{\beta, \eta} \prec M'_\alpha$,
 - (β) $(M'_{\eta(0)}, N_{\beta, \eta}, \mathcal{A}_\eta) \in \text{BP}$,
 - (γ) if β is a limit ordinal then $(M'_{\eta(0)}, N_\eta, \mathcal{A}_\eta) = (M'_{\eta(0)}, \cup\{N_{\eta \upharpoonright i} : i < \text{lg}(\eta)\}, \cup\{\mathcal{A}_{\eta \upharpoonright i} : i < \text{lg}(\eta)\})$ so $N_\eta = M'_\beta$,
 - (δ) if β is a successor ordinal then $(M'_{\eta(0)}, N_\eta, \mathcal{A}_\eta)$ is a full $(\lambda, \lambda, < \lambda)$ -system.

There is no problem to carry the induction. By clause (c), $M = \cup\{M'_\alpha : \alpha < \lambda^+\}$ is saturated so without loss of generality $\cup\{M'_\alpha : \alpha < \lambda^+\} = \cup\{M_\alpha : \alpha < \lambda^+\}$, let $E_* = \{\delta < \lambda^+ : \delta \text{ a limit ordinal divisible by } \lambda \text{ and } M'_\alpha = M_\alpha\}$, clearly a club of λ^+ .

Now if $\alpha < \beta$ are from E and has cofinality λ , let $\eta = \eta_{\alpha, \beta}$ be an increasing continuous sequence of ordinals of length λ with limit β , such that $\eta(0) = \alpha$ and $\eta(i+1)$ a successor for $i < \lambda$. Now $M_\alpha^* = M'_\alpha, M_\beta = M'_\beta = \cup\{N_{\eta \upharpoonright i} : i < \lambda\}$ and let $\mathcal{A}_{\alpha, \beta} = \cup\{\mathcal{A}_{\eta \upharpoonright i} : i < \lambda\}$, so $(M_\alpha, M_\beta, \mathcal{A}_{\alpha, \beta})$ is a full $(\lambda, \lambda, < \lambda)$ -system. By part (2) we can finish.

- (2) We define the set AP of approximation:

(*)₁ AP is the set of triples $\mathbf{h} = (\mathbf{p}_1, B_1, \mathbf{p}_2, B_2, f) = (p_1[\mathbf{h}], B_2[\mathbf{h}], p_2[\mathbf{h}], B_2[\mathbf{h}], f[\mathbf{h}])$ satisfying:

- (a) $\mathbf{p}_\ell \in \mathcal{A}_\ell$ for $\ell = 1, 2$,
- (b) $B_\ell \in [M_\ell]^{<\lambda}$ is a base for \mathbf{p}_ℓ , see Definition 4.19(4),
- (c) f is an elementary mapping which maps B_1 onto B_2 such that $\text{Dom}(f) = B_1$,
- (d) there is an isomorphism f^+ from M_1 onto M_2 extending f such that: if $\alpha_0 < \dots < \alpha_{n-1} < \lambda^+$ then f^+ maps $\text{tp}((\bar{c}_{\mathbf{p}_1, \alpha_1} \hat{\ } \bar{d}_{\mathbf{p}_1, \alpha_1}) \hat{\ } \dots \hat{\ } (\bar{c}_{\mathbf{p}_1, \alpha_{n-1}} \hat{\ } \bar{d}_{\mathbf{p}_1, \alpha_{n-1}}), M_1)$ onto $\text{tp}((\bar{c}_{\mathbf{p}_2, \alpha_0} \hat{\ } \bar{d}_{\mathbf{p}_2, \alpha_0}) \hat{\ } \dots \hat{\ } (\bar{c}_{\mathbf{p}_2, \alpha_{n-1}} \hat{\ } \bar{d}_{\mathbf{p}_2, \alpha_{n-1}}), M_2)$.

(*)₂ we define the two-place relation \leq_{AP} by: $\mathbf{h}_1 \leq_{\text{AP}} \mathbf{h}_2$ if and only if

- (a) both are in AP,
- (b) $\mathbf{p}_\ell[\mathbf{h}_1] \leq_{\mathbf{m}_\ell} \mathbf{p}_\ell[\mathbf{h}_2]$ for $\ell = 1, 2$,
- (c) $B_\ell[\mathbf{h}_1] \subseteq B_\ell[\mathbf{h}_2]$,
- (d) $f_1[\mathbf{h}_1] \subseteq f[\mathbf{h}_2]$.

Obviously

(*)₃ \leq_{AP} partially ordered AP.

Also

(*)₄ if $\delta < \lambda$ is a limit ordinal and $\langle \mathbf{h}_\alpha : \alpha < \delta \rangle$ is \leq_{AP} -increasing, then this sequence has an \leq_{AP} -upper bound.

[Why? Clauses (a)–(c) of (*)₁ hold as in the proof of Claim 4.21(2)(c) and clause (d) of (*)₁ holds by Claims 4.17(4) and 4.17(2).]

(*)₅ if $\mathbf{h} \in \text{AP}$ and $a \in M_1$ then there is $\mathbf{h}' \in \text{AP}$ such that $\mathbf{h} \leq_{\text{AP}} \mathbf{h}'$ and $a \in B_{\mathbf{p}_\ell}[\mathbf{h}'_1]$.

[Why? Let $a_1 = a$ and let $f^+ \supseteq f_{\mathbf{h}}$ be as in (*)₁(d) and let $a_2 \in M_2$ realize $f^+(\text{tp}(a_1, B_{\mathbf{p}_1}[\mathbf{h}]))$.

Let $B'_\ell = (B_{\mathbf{p}_\ell}[\mathbf{h}] \cup \{a_\ell\})$ and let $\mathbf{p}'_\ell \in \mathcal{A}_{\mathbf{m}_\ell}$ be defined by: $(\bar{c}_\varepsilon[\mathbf{p}'_\ell], \bar{d}_\varepsilon[\mathbf{p}'_\ell]) = (\bar{c}_\varepsilon[\mathbf{p}_\ell[\mathbf{h}_1]], \bar{d}_\varepsilon[\mathbf{p}_\ell[\mathbf{h}]])$.

Lastly, let $\mathbf{h}' = (\mathbf{p}'_1, B'_1, \mathbf{p}'_2, B'_2, f_{\mathbf{h}} \cup \{(a_1, a_2)\})$.

(*)₆ If $\mathbf{h} \in \text{AP}$ and $a \in M_2$ then there is $\mathbf{h}' \in \text{AP}$ such that $\mathbf{h} \leq_{\text{AP}} \mathbf{h}'$ and $a \in B_2[\mathbf{h}']$.

[Why? Like (*)₅.]

(*)₇ if $\mathbf{h} \in \text{AP}$ and $d \in N_1$ then for some $\mathbf{h}' \in \text{AP}$ we have $\mathbf{h} \leq_{\text{AP}} \mathbf{h}'$ and $d \in \text{Rang}(\bar{d}_0[\mathbf{p}_1[\mathbf{h}']])$.

[Why? By Claim 4.21(2)(d) there is $\mathbf{q}_1 \in \mathcal{A}_{\mathbf{m}_1}$ such that $\mathbf{p}_1[\mathbf{h}] \leq_{\mathbf{m}_1} \mathbf{q}_1$ and $d \in \bar{d}_0[\mathbf{p}_1[\mathbf{h}']]$.

Let $f' \supseteq f_{\mathbf{h}}$ be as in $(*)_1(d)$ so in particular an isomorphism from M_1 onto M_2 .
 Now by clause (i) of Definition 4.18 there is $\mathbf{q}_2 \in \mathcal{A}_{\mathbf{m}_2}$ such that f' maps \mathbf{q}_1 to \mathbf{q}_2 .

The rest should be clear.]

$(*)_8$ if $\mathbf{h} \in \text{AP}$ and $d \in N_2$ then for some $\mathbf{h}' \in \text{AP}$ we have $\mathbf{h} \leq_{\text{AP}} \mathbf{h}'$ and $d \in \text{Rang}(\bar{d}_0[\mathbf{p}_1(\mathbf{h}')])$.

[Why? Like $(*)_7$.]

Together

\odot there is a sequence $\langle \mathbf{h}_i : i < \lambda \rangle$ such that (for $\ell = 1, 2$)

(a) it is \leq_{AP} -increasing,

(b) $_{\ell}$ if $a \in M_{\ell}$ then $a \in B_{\ell}[\mathbf{h}_i]$ for some i ,

(c) $_{\ell}$ if $d \in N_{\ell}$ then $d \in \text{Rang}(\bar{d}_0[\mathbf{p}_{\ell}[\mathbf{h}_i]])$ for some $i < \lambda$.

From this sequence we can “read” an isomorphism as required, say $g(a_1) = a_2$ if and only if for some i and $\varepsilon < \ell g(\bar{d}_0[\mathbf{p}_1[\mathbf{h}_i]])$ we have $a_1 = (\bar{d}_0[\mathbf{p}_1[\mathbf{h}_i]])_{\varepsilon}$, $a_2 = (\bar{d}_0[\mathbf{p}_2[\mathbf{h}_i]])_{\varepsilon}$. $\square_{4.22}$

Another form, not assuming Hypothesis 4.13, is the following.

Conclusion 4.23. Assume $(\lambda, \lambda, < \lambda)$ is T -good, e.g., $\lambda > |T|$ is a measurable cardinality and $\lambda = \lambda^{<\lambda}$. Then for some \mathbf{F} we have:

(A) (a) \mathbf{F} is as in [14, 3.3], i.e.

(α) \mathbf{F} is a function with domain $\{\bar{M} : \bar{M} \text{ has the form } \langle M_i : i \leq \beta \rangle, \text{ an } \prec\text{-increasing continuous sequence such that } M_i \text{ is models of } T \text{ of cardinality } \lambda \text{ with universe an ordinal } \in [\lambda, \lambda^+) \text{ and if } i \text{ is non-limit then } M_i \text{ is saturated}\}$,

(β) $\mathbf{F}(\bar{M})$ is such that $\bar{M} \hat{\ } \langle \mathbf{F}(\bar{M}) \rangle \in \text{Dom}(\mathbf{F})$;

(B) if $\bar{M} = \langle M_{\alpha} : \alpha < \lambda^+ \rangle$ obeys \mathbf{F} which means that $\lambda^+ = \sup\{\alpha : \mathbf{F}(\bar{M} \upharpoonright (\alpha+1)) \prec M_{\alpha+1}\}$ then for some club E of λ^+ we have:

(a) $\text{cf}(\alpha) = \lambda \Rightarrow M_{\alpha}$ is saturated,

(b) if $M_{\alpha_{\ell}}$ is saturated, $\text{cf}(\delta_{\ell}) = \lambda$ and $\alpha_{\ell} < \delta_{\ell} \in E$ for $\ell = 1, 2$ then $(M_{\delta_1}, M_{\alpha_1}) \cong (M_{\delta_2}, M_{\beta_2})$.

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