THE JOURNAL OF SYMBOLIC LOGIC
Volume 66, Number 2, June 2001

REGULAR SUBALGEBRAS OF COMPLETE BOOLEAN ALGEBRAS

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Abstract. It is proved that the following conditions are equivalent:

(a) there exists a complete, atomless, σ–centered Boolean algebra, which does not contain any regular, atomless, countable subalgebra,

(b) there exists a nowhere dense ultrafilter on ω.

Therefore, the existence of such algebras is undecidable in ZFC. In "forcing language" condition (a) says that there exists a non–trivial σ–centered forcing not adding Cohen reals.

A subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{A}$ is called regular whenever for every $X \subseteq \mathbb{B}$, $\sup_{\mathbb{B}} X = \mathit{1}$ implies $\sup_{\mathbb{A}} X = \mathit{1}$; see e.g., Heindorf and Shapiro [6]. Clearly, every dense subalgebra is regular. Although every complete Boolean algebra contains a free Boolean algebra of the same size (see the Balcar-Franek Theorem [1]), not always such an embedding is regular. For instance, if $\mathbb{B}$ is a measure algebra, then it contains a free subalgebra of the same cardinality as $\mathbb{B}$, but $\mathbb{B}$ cannot contain any infinite free Boolean algebra as a regular subalgebra. Indeed, measure algebras are weakly σ-distributive but free Boolean algebras are not, and a regular subalgebra of a weakly σ-distributive one is again weakly σ-distributive. Thus $\mathbb{B}$ does not contain any free Boolean algebra. On the other hand, measure algebras are not σ-centered. So, a natural question arises whether there exists a σ-centered, complete, atomless Boolean algebra $\mathbb{B}$ without regular free subalgebras. Since countable atomless Boolean algebras are free and every free Boolean algebra contains a countable regular free subalgebra, it is enough to ask whether $\mathbb{B}$ contains a countable atomless regular subalgebra. In the paper we prove that such an algebra exists iff there exists a nowhere dense ultrafilter.

DEFINITION 1 (Baumgartner [2]). A filter D on ω is called nowhere dense if for every function f from ω to the Cantor set ${}^{\omega}2$ there exists a set $A \in D$ such that $f(A)$ is nowhere dense in ${}^{\omega}2$.

In the sequel we will rather interested in nowhere dense ultrafilters. Observe that every P-ultrafilter (i.e., every P-point in ω^*) is a nowhere dense ultrafilter.

THEOREM 1. *There exists an atomless, complete, σ-centered Boolean algebra without any countable atomless regular subalgebras iff there exists a nowhere dense ultrafilter.*

Received March 24, 1998.

†The research was partially supported by the Basic Research Foundation of the Israel Academy of Sciences and Humanities. This publication has Number 640 in S. Shelah's list.

0022-4812/01/6602-0021/$1.90

By a recent result of Saharon Shelah [7] there exists a model of ZFC in which there are no nowhere dense ultrafilters. So it is consistent with ZFC that there are no atomless, complete, σ-centered Boolean algebras without any countable regular subalgebras.

In the first part of the paper, forcing methods are used to show that nowhere dense ultrafilters exist whenever there exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real. The forcing constructed here uses some ideas from Gitik and Shelah [5]. They have shown that if $\mathbb{P}$ is a σ-centered forcing notion, $\{A_n: n < \omega\}$ are subsets of $\mathbb{P}$ witnessing this, and both $\mathbb{P}$ and A_n's are Borel, then $\mathbb{P}$ adds a Cohen real. On the other hand it is known that a forcing $\mathbb{P}$ adds a Cohen real iff the complete Boolean algebra $\mathbb{B} = RO(\mathbb{P})$ contains an element u such that the reduced Boolean algebra $\mathbb{B}|u$ has a regular infinite free Boolean subalgebra. Thus, to prove the Theorem 1 we need to show in particular the following:

THEOREM 2. *If there exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real then there exists a nowhere dense ultrafilter on ω.*

We shall proceed with the proof by some definitions and a lemma.

DEFINITION 2. (a) A forcing $\mathbb{P}$ is called σ-centered if $\mathbb{P} = \bigcup\{A_n: n < \omega\}$ where each A_n is directed, i.e., for every $p, q \in A_n$ there exists $r \in A_n$ such that $p \leqslant r$ and $q \leqslant r$.

(b) A forcing $\mathbb{P}$ adds a Cohen real if there exists a $\mathbb{P}$–name $\underset{\sim}{r} \in {}^{\omega} 2$ such that for every open dense set $\mathscr{D} \subset {}^{\omega}2$ we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{r} \in \mathscr{D}^*$", where $\mathscr{D}^*$ denotes the encoding of $\mathscr{D}$ in the Boolean universe.

REMARKS. (a) The order of forcing in this notation is inverse of the one in the Boolean algebra.

(b) We can just assume that there is a member p of $\mathbb{P}$ such that if q is above p then there are r_1 and r_2 above q which are incompatible in $\mathbb{P}$.

DEFINITION 3. A set $X \subseteq {}^{\omega>}2$ is somewhere dense if there exists an $\eta \in {}^{\omega>}2$ such that for every $\nu \in {}^{\omega>}2$ there is $\varrho \in X$ with $\eta^\frown\nu \trianglelefteq \varrho$, where $\eta^\frown\nu$ stands for the concatenation of η and ν and the relation $\trianglelefteq$ means that ϱ is an extension of the sequence $\eta^\frown\nu$.

LEMMA 3. *A filter D on ω is not nowhere dense iff it is a so-called well behaved filter, i.e., there is a function $f: \omega \to {}^{\omega>}2$ such that for every $B \in D$ the range of f restricted to B is somewhere-dense.*

PROOF. Suppose $f: \omega \to {}^{\omega}2$ be such that for every $B \in D$ the image of B is not nowhere dense. Without loss of generality we can assume that the range of f is dense in itself. Since every closed and dense in itself subset of the Cantor cube ${}^{\omega}2$ is homeomorphic to the whole ${}^{\omega}2$ we can assume also that the range of f is dense in ${}^{\omega}2$. Moreover, since it is countable it can be identified with a subset of the set ${}^{\omega>}2$ of all rational points of the Cantor set. Thus without loss of generality we can assume that f maps ω into ${}^{\omega>}2$. On the other hand a set $X \subseteq {}^{\omega>}2$ is nowhere dense whenever for every $\eta \in {}^{\omega>}2$ there exists some $\nu \in {}^{\omega>}2$ such that the set of all sequences extending $\eta^\frown\nu$ is disjoint from X. Therefore, since the image of B under

f is not nowhere dense in ${}^{\omega>}2$, it can be identified with a somewhere dense subset of ${}^{\omega>}2$. This in fact completes the proof of the lemma. $\dashv$

REMARK. If D is a filter on ω and $\mathscr{P}(\omega)/\mathrm{D}$ is infinite then D is not nowhere dense. Indeed, if $\langle A_n : n < \omega \rangle$ is a partition of ω such that $\omega \setminus A_n \notin D$ for all $n < \omega$ and $\langle e_n : n < \omega \rangle$ list the set ${}^{\omega>}2$ then the map $f : \omega \to {}^{\omega>}2$ defined by the formula

$$f(e) = e_n \quad \text{iff} \quad e \in A_n$$

witness "D is well behaved".

PROOF OF THEOREM 2. Assume that there are no nowhere dense ultrafilters. Further assume that $\mathbb{P}$ is a forcing in which above each element there are two incompatible ones and $\mathbb{P} = \bigcup\{A_n : n < \omega\}$ where each A_n is directed. We start with the following known fact which we prove here for the sake of completeness: $\dashv$

FACT 4. Every forcing $\mathbb{Q}$ with Knaster condition such that above every element of $\mathbb{Q}$ there are two incompatible ones, adds a real.

In fact, by assumption, forcing with $\mathbb{Q}$ adds a new subset to $\mathbb{Q}$, hence a new subset to some ordinal. In the set

$$\mathscr{K} = \{(\alpha, p, \underset{\sim}{\tau}) : p \in \mathbb{Q}, \alpha \text{ an ordinal and } \underset{\sim}{\tau}$$
$$\text{a } \mathbb{Q} - \text{name of a subset of } \alpha \text{ such that } p \Vdash \text{“}\underset{\sim}{\tau} \notin V\text{”}\}$$

we choose $(\alpha, p, \underset{\sim}{\tau})$ with α being minimal. So necessarily α is a cardinal and

$$p \Vdash \text{“the tree } ({}^{\alpha>}2, \trianglelefteq) \text{ has a new } \alpha\text{-branch in } V^{\mathbb{Q}}\text{”}$$

So, as $\mathbb{Q}$ satisfies the Knaster condition (which follows from σ-centered), necessarily $\mathrm{cf}(\alpha) = \aleph_0$ and letting $\alpha = \bigcup_{n<\omega} \alpha_n$, where $\alpha_n < \alpha_{n+1}$ for some countable $w \subseteq {}^{\alpha>}2$ we get

$$p \Vdash \text{“}(\forall n < \omega)(\underset{\sim}{\tau} \restriction \alpha_n \in w)\text{”},$$

so $p \Vdash$ “we add a new subset to w, $|w| = \aleph_0$”.

We have shown that $I = \{p \in \mathbb{Q} : p \Vdash \text{“}\underset{=}{r} \in {}^{\omega}2 \text{ is new ”}$ for some $\mathbb{Q}$ – name $\underset{=}{r}\}$ is a dense subset of $\mathbb{Q}$. So let $\{p_i : i < \omega\} \subseteq I$ be a maximal antichain and let $\underset{=}{r}_i$ be such that $p_i \Vdash$ “$\underset{=}{r}_i$ is new ”. By density of I we can define the $\mathbb{Q}$-name $\underset{=}{r}$ as follows: $\underset{=}{r} = \underset{=}{r}_i$ if $p_i \in G_{\mathbb{Q}}$. This completes the proof of Fact 4.

Now we fix a $\mathbb{P}$-name of a new real $\underset{=}{r} \in {}^{\omega}2$ added by $\mathbb{P}$. For every $p \in \mathbb{P}$ we set $T_p = \{\eta \in {}^{\omega>}2 : \neg(p \Vdash \neg(\text{“}\eta \trianglelefteq \underset{=}{r}\text{”}))\}$, i.e., $\eta \in T_p$ iff there exists $q \in \mathbb{P}$ such that $p \leqslant q$ and $q \Vdash \text{“}\eta = \underset{=}{r} \restriction \lg \eta\text{”}$, where $\lg \eta$ denotes the length of the sequence η.

FACT 5. For every $p \in \mathbb{P}$, T_p is a subtree of ${}^{\omega>}2$, i.e $\eta \trianglelefteq \nu$ and $\nu \in T_p$ implies $\eta \in T_p$ and $\langle\rangle \in T_p$, where $\langle\rangle$ denotes the empty sequence.

Indeed, if $\eta \trianglelefteq \nu$ and $\nu = \underset{=}{r} \restriction \lg \nu$, then $\eta = \underset{=}{r} \restriction \lg \eta$.

FACT 6. The tree T_p has no maximal elements.

To prove the Fact 6 we fix $\eta \in T_p$. Then there is $q \in \mathbb{P}$ such that $p \leqslant q$ and

$$q \Vdash \text{“}\underset{\sim}{r} \restriction \lg \eta = \eta\text{”}.$$

Let $k = \lg(\eta)$, so $I = \{r \in \mathbb{P}\colon r \text{ forces a value to } \underset{\sim}{r} \restriction (k+1)\}$ is a dense and open subset of $\mathbb{P}$, hence there is $q' \in \mathbb{P}$ such that $q \leqslant q'$ and q' forces a value to $\underset{\sim}{r} \restriction (k+1)$, say ϑ. So q' also forces $\underset{\sim}{r} \restriction k = \vartheta \restriction k$, but $q \leqslant q'$ and $q \Vdash$ “$\underset{\sim}{r} \restriction k = \eta$ hence $\vartheta \restriction k = \eta$”. As q' witnesses $\vartheta \in T_p$ and $\vartheta \in {}^{k+1}2$ and $\eta \in {}^{k}2$, $\eta \trianglelefteq \vartheta$, this completes the proof of Fact 6.

Fact 7. The set $\lim T_p$ of all ω-branches through T_p is closed, i.e., if $\eta \in {}^{\omega}2 \setminus \lim T_p$ then there exists $\nu \in {}^{\omega>}2$ such that $\nu \trianglelefteq \eta$ and the set of all ω-branches extending ν is disjoint from $\lim T_p$.

Indeed, if $\eta \in {}^{\omega}2 \setminus \lim T_p$ then there exists $n \in \omega$ such that $n \leqslant m < \omega$ implies $\eta \restriction m \notin T_p$. By Fact 5 it is clear that every ω-branch extending $\nu = \eta \restriction n$ does not belong to T_p, which proves the Fact 7.

Now let us observe that the family $\{T_p\colon p \in A_n\}$ is directed under inclusion, i.e., if $p, q \in A_n$ and $r \in \mathbb{P}$ is such that $p \leqslant r$ and $q \leqslant r$ then $T_r \subseteq T_p \cap T_q$. Indeed, if $\eta \in {}^{\omega>}2$ and there exists $s \geqslant r$ such that $s \Vdash$ “$\eta = \underset{\sim}{r} \restriction \lg \eta$” then of course $s \geqslant p$ and $s \geqslant q$ and thus η belongs to T_p and T_q.

So by compactness of ${}^{\omega}2$ and Facts 5–7 we get the following:

Fact 8. The set $T_n = \bigcap\{T_p\colon p \in A_n\}$ is a subtree of ${}^{\omega>}2$ and the set of ω-branches of T_n is non-empty.

Now we make a choice:

(1) $$\eta_n^* \text{ is an } \omega\text{ - branch of } T_n.$$

Subsequently for every $n < \omega$ and every $p \in A_n$ we define

$$B_p^n = \{k < \omega\colon (\exists q \in \mathbb{P})(p \leqslant q \wedge q \Vdash \text{“}\underset{\sim}{r} \restriction k = \eta_n^* \restriction k \;\;\&\;\; \underset{\sim}{r}(k) \neq \eta_n^*(k)\text{”})\}$$

We have the following:

Fact 9. For every $n < \omega$ and every $p \in A_n$ the set B_p^n is infinite.

Indeed, since $p \in A_n$ and T_n is a subtree of T_p, η_n^* is an ω-branch of T_p. Let us fix $m < \omega$. Then, by the definition of T_p, there exists $r \in \mathbb{P}$ such that $r \geqslant p$ and

$$r \Vdash \text{“}\eta_n^* \restriction m = \underset{\sim}{r} \restriction m\text{”}.$$

On the other hand

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{r} \neq \eta_n^*\text{”},$$

because $\underset{\sim}{r}$ is a new real. Thus for some $q \in \mathbb{P}$, $q \geqslant r$ and $k < \omega$ we get

$$q \Vdash \text{“}\underset{\sim}{r} \restriction k \neq \eta_n^* \restriction k\text{”}.$$

We can assume that k is minimal with such a property. Since $r \leqslant q$, it must be $k > m$. But $q \geqslant p$ and thus, by minimality of k, we have $k - 1 \in B_p^n$, which proves the Fact 9.

Now we establish for every $n < \omega$ the following definition:

$$\mathscr{D}_n^0 = \{B \subseteq \omega\colon (\exists p \in A_n)(|B_p^n \setminus B| < \omega)\}.$$

Fact 10. For every $n < \omega$, $\mathscr{D}_n^0$ is a filter.

Indeed, let $B_1, B_2 \in \mathscr{D}_n^0$. Then there exist $p_1, p_2 \in A_n$ such that both $B_{p_1}^n \setminus B_1$ and $B_{p_2}^n \setminus B_2$ are finite. Since A_n is directed we can choose $r \in A_n$ such that $p_1 \leqslant r$ and $p_2 \leqslant r$. On the other hand, from the definition of B_p^n it easily follows that

$$p \leqslant q \quad \text{implies} \quad B_q^n \subseteq B_p^n.$$

Thus $B_r^n \subseteq B_{p_1}^n \cap B_{p_2}^n$ and therefore

$$B_r^n \setminus (B_1 \cap B_2) \subseteq (B_{p_1}^n \setminus B_1) \cup (B_{p_2}^n \setminus B_2)$$

is finite. Clearly, every superset of an element of $\mathscr{D}_n^0$ also belongs to $\mathscr{D}_n^0$ and, by the Fact 9, $\mathscr{D}_n^0$ does not contain the empty set, which completes the proof of Fact 10.

Now by Fact 9 and Fact 10, we can make the following choice: for $n < \omega$

(2) $\qquad \mathscr{D}_n$ is a non-principal ultrafilter containing $\mathscr{D}_n^0$

By our hypothesis the ultrafilters $\mathscr{D}_n$ are not nowhere dense and so by Lemma 3 for every $n < \omega$ we can choose a function $f_n \colon \omega \to {}^{\omega>}2$ such that

(3) $$(\forall B \in \mathscr{D}_n)(\exists u \in {}^{\omega>}2)(\forall v \in {}^{\omega>}2)(\exists k \in B)(u^\frown v \trianglelefteq f_n(k)).$$

Without loss of generality we may assume that the empty sequence does not belong to the range of f_n.

Now we have to come back to the sequence $\{\eta_n^* \colon n < \omega\}$ of ω-branches of the trees T_n. Since it can happen that the sequence is not one-to-one we consider the set

$$Y = \{n < \omega \colon \eta_n^* \notin \{\eta_m^* \colon m < n\}\}.$$

Then for $n, m \in Y$ we have $\eta_n^* \neq \eta_m^*$ whenever $n \neq m$.

In the sequel we shall need the following:

CLAIM. If $\langle \eta_n \colon n < \omega \rangle \subseteq {}^{\omega}2$ is a sequence of distinct ω-branches of a tree $T \subseteq {}^{\omega>}2$ there exists an increasing sequence $\langle e_n \colon n < \omega \rangle \subseteq \omega$ such that for all $n < m < \omega$ we have

$(*)$ $$\{\eta_n \restriction l \colon e_n < l < \omega\} \cap \{\eta_m \restriction l \colon e_m < l < \omega\} = \emptyset.$$

To prove the claim observe that $\eta_n \restriction l \neq \eta_m \restriction l$ and $k > l$ implies $\eta_n \restriction k \neq \eta_m \restriction k$. Now assume that $e_0, \dots, e_n$ are defined so that the condition $(*)$ holds true. Since $\eta_{n+1} \notin \{\eta_0, \dots, \eta_n\}$ there exists $k < \omega$ such that $\eta_0 \restriction k, \dots, \eta_n \restriction k, \eta_{n+1} \restriction k$ are pairwise different. We can assume that $k > e_n$ and e_{n+1} to be the first such k. This completes the proof of the claim.

Now using the claim we can choose an increasing sequence $\langle e_n \colon n < \omega \rangle \subseteq \omega$ in such a way that, letting

$$C_n = \{\eta_n^* \restriction l \colon e_n \leqslant l < \omega\},$$

the sequence $\langle C_n \colon n \in Y \rangle$ consists of pairwise disjoint sets, and so that we have

$$\eta_n^* = \eta_m^* \Leftrightarrow e_n = e_m \Leftrightarrow C_n = C_m.$$

Finally, for $\eta \in {}^{\omega}2$ we define

$$u(\eta) = \{n \in Y : (\exists l < \omega)(\eta \restriction l = \eta_n^* \restriction l \wedge (\forall m < n)(\eta \restriction l \neq \eta_m^* \restriction l))\},$$

$n_k(\eta) =$ the k–th member of $u(\eta)$,

$$m_k(\eta) = \min\{m < \omega : e_{n_k(\eta)} < m \wedge \eta \restriction (m+1) \ntrianglelefteq \eta_{n_k(\eta)}^*\},$$

i.e., $m_k(\eta)$ is the smallest $m > e_{n_k(\eta)}$ such that $\eta \restriction (m+1) \neq \eta^*_{n_k(\eta)} \restriction (m+1)$. By definition of $m_k(\eta)$, we have $e_{n_k(\eta)} < m_k(\eta)$. Clearly we also have:

(i) $u(\eta)$ is well-defined,

(ii) $n_k(\eta)$ is well–defined if $k < |u(\eta)|$,

(iii) $m_k(\eta)$ is well–defined if $k < |u(\eta)|$ and $\eta \neq \eta^*_{n_k}$.

Now we can define a function $\tau : {}^{\omega}2 \setminus \{\eta^*_n : n < \omega\} \to {}^{\omega\geq}2$ by the formula:

$$\tau(\eta) = f_{n_0(\eta)}(m_0(\eta))^\frown f_{n_1(\eta)}(m_1(\eta))^\frown \cdots,$$

where, for $n < \omega$, f_n is the function from the condition (3). From the formula it follows easily that $\tau(\eta) \in {}^{\omega\geq}2$ and it is well defined if $\eta \notin \{\eta^*_n : n < \omega\}$ and moreover $\tau(\eta)$ is infinite whenever $u(\eta)$ is infinite, as $\langle\rangle \notin$ Range (f_n).

To complete the proof of the theorem it remains to show:

FACT 11. $\Vdash_{\mathbb{P}}$ "$\tau(\underline{r})$ is Cohen over V".

PROOF. To prove this fact we fix an open dense set $I \subseteq {}^{\omega>}2$ and a $p \in \mathbb{P}$ and we show that there is a $q \in \mathbb{P}$ with $p \leq q$ such that $q \Vdash_{\mathbb{P}}$ "$\tau(\underline{r}) \in [I]$", where $[I]$ is the name of $\{\eta \in {}^{\omega}2 : t \trianglelefteq \eta$ for some $t \in I\}$ in the generic extension. Let $n < \omega$ be such that $p \in A_n$ and let $n^{\otimes} = \min\{m < \omega : \eta^*_m = \eta^*_n\}$. Clearly $n^{\otimes} \leq n$ and $n^{\otimes} \in Y$. Then $u(\eta^*_n)$ is well defined and $n^{\otimes} \in u(\eta^*_n)$; in fact $n^{\otimes}$ is the last member of $u(\eta^*_n)$. Let $k = |u(\eta^*_n)| - 1$, so $n_k(\eta^*_n) = n^{\otimes}$. Also $m_i(\eta^*_n)$ is well defined and finite for $i < k$. Then we set

$$\nu^{\otimes} = f_{n_0(\eta^*_n)}(m_0(\eta^*_n))^\frown \cdots ^\frown f_{n_{k-1}(\eta^*_n)}(m_{k-1}(\eta^*_n)),$$

so if $k = 0$, i.e., if $u(\eta^*_n)$ is a singleton, then $\nu^{\otimes}$ is the empty sequence.

Clearly $\nu^{\otimes} \in {}^{\omega>}2$. Also we have $p \nVdash_{\mathbb{P}}$ "$\underline{r} \restriction (e_n+1) \ntrianglelefteq \eta^*_n$". Hence $p \nVdash_{\mathbb{P}}$ "$\neg\varphi$", where φ is the formula asserting $u(\eta^*_n)$ is an initial segment of $u(\underline{r})$. Note that φ implies $(\forall i < k)(n_i(\underline{r}) = n_i(\eta^*_n)) \wedge m_i(\underline{r}) = m_i(\eta^*_n)$. Since $p \Vdash_{\mathbb{P}}$ "$\underline{r} \neq \eta^*_{n^{\otimes}}$", it follows that $p \Vdash_{\mathbb{P}}$ "$\varphi \to m_k(\underline{r})$ is well–defined". Let $Z = \{\varrho \in {}^{\omega>}2 : p \nVdash_{\mathbb{P}}$ "$\neg(\varphi \wedge f_{n_k(\underline{r})}(m_k(\underline{r})) = \varrho)$"$\}$.

It is enough to show that Z is a somewhere dense subset of ${}^{\omega>}2$. [Suppose that Z is a somewhere dense subset of ${}^{\omega>}2$. Then there is $\varrho_0 \in {}^{\omega>}2$ such that for any $\nu \in {}^{\omega>}2$ there is $\varrho \in Z$ with $\varrho_0{}^\frown\nu \trianglelefteq \varrho$. Let $\tilde{\varrho}_0 = \nu^{\otimes\frown}\varrho_0$ and let $\nu \in {}^{\omega>}2$ be such that $\tilde{\varrho}_0{}^\frown\nu \in I$. Then there is $\varrho \in Z$ such that $\tilde{\varrho}_0{}^\frown\nu \trianglelefteq \varrho$. Let $q \geq p$ be such that $q \Vdash_{\mathbb{P}}$ "$\varphi \wedge f_{n_k}(\underline{r}) = \varrho$". Then $q \Vdash_{\mathbb{P}}$ "$\tilde{\varrho}_0{}^\frown\nu \trianglelefteq \tau(\underline{r})$". And hence we can conclude that $q \Vdash_{\mathbb{P}}$ "$\tau(\underline{r}) \in [I]$".]

Now, we have $p \nVdash_{\mathbb{P}}$ "$\neg(n_k(\underline{r}) = n^{\otimes} \vee \neg\varphi)$". Hence

$$Z = \{\varrho \in {}^{\omega>}2 : p \nVdash_{\mathbb{P}} \text{“}\neg(f_{n^{\otimes}}(m_k(\underline{r})) = \varrho \wedge \varphi)\text{”}\}.$$

Thus, by the choice of $f_{n^{\otimes}}$, it is enough to prove:

$$B_0 = \{m < \omega : p \nVdash_{\mathbb{P}} \text{“}m_k(\underline{r}) \neq m \vee \neg\varphi\text{”}\} \in \mathscr{D}_{n^{\otimes}}.$$

[Suppose that $B_0 \in \mathscr{D}_{n^{\otimes}}$. Then, by (3), there is $\varrho \in {}^{\omega>}2$ such that $(\forall \nu \in {}^{\omega>}2)(\exists k \in B_0)(\varrho^\frown\nu \trianglelefteq f_{n^{\otimes}}(k))$.]

We have $\mathscr{D}_{n^{\otimes}} = \mathscr{D}_n$. Hence it is enough to show $B_0 \in \mathscr{D}_n$. By definition of $m_k(\underline{r})$ and since $\varphi \to n_k(\underline{r}) = n^{\otimes}$, this is equivalent to:

$$\{m < \omega : p \nVdash_{\mathbb{P}} \text{“}\underline{r} \restriction m \neq \eta^*_{n^{\otimes}} \restriction m \vee \underline{r}(m+1) = \eta^*_{n^{\otimes}}(m+1) \vee \neg\varphi\text{”}\} \in \mathscr{D}_n.$$

But $\eta^*_{n\otimes} = \eta^*_n$ and $p \in A_n$. Hence, by definition of $\mathscr{D}^0_n$, the set above does belong to $\mathscr{D}^0_n \subseteq \mathscr{D}_n$. ⊣

Finally we prove that the converse to Theorem 2 is also true, i.e., we shall show that whenever there exists a nowhere dense ultrafilter there exists a σ-centered forcing $\mathbb{P}$ with the property that above each element there are two incompatible ones and moreover $\mathbb{P}$ does not add a Cohen real. To prove this fact we shall use some topological methods, but we can also write it using forcing.

Recall that a subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{A}$ is *regular* whenever $\sup_{\mathbb{A}} X = 1$ for every $X \subseteq \mathbb{B}$ such that $\sup_{\mathbb{B}} X = 1$. The subalgebra $\mathbb{B}$ is regular iff the corresponding map of the Stone spaces is semi-open, i.e., the image of every non-empty clopen set has non-empty interior. Using nowhere dense ultrafilters we construct a dense in itself, separable, extremally disconnected compact space (=Stone space of an atomless, σ-centered, complete Boolean algebra) which has no semi-open continuous maps onto the Cantor set.

We use a topology on the set ${}^{\omega>}\omega = \bigcup\{{}^n\omega : n < \omega\}$. If $s \in {}^{\omega>}\omega$ is a sequence of length n and $k \in \omega$, then $s^\frown k$ denotes the sequence of length $n+1$ extending s in such a way that the n-th term is k. For a set $A \subseteq \omega$ we set $s^\frown A = \{s^\frown k : k \in A\}$. For a given ultrafilter $p \subseteq \mathscr{P}(\omega)$ we consider a topology $\mathscr{T}_p$ on ${}^{\omega>}\omega$ given by the formula:

$$U \in \mathscr{T}_p \text{ iff for every } s \in U \text{ there exists } A \in p \text{ such that } s^\frown A \subseteq U.$$

The set ${}^{\omega>}\omega$ equipped with the topology $\mathscr{T}_p$ we denote G_p. The space G_p is known to be Hausdorff and extremally disconnected; see e.g., Dow, Gubbi and Szymanski, ([4]). Hence the Čech-Stone extension βG_p is extremally disconnected, compact, separable, and dense in itself.

Under a much stronger assumption that there exists a P–point the next theorem was proved by A. Blass [3].

THEOREM 12. *If there exists a nowhere dense ultrafilter then there exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real.*

PROOF. By virtue of a theorem of Silver, it is enough to show that there exists a σ-centered, complete, atomless Boolean algebra $\mathbb{B}$ such that $\mathbb{B}$ does not contain any regular free subalgebra. For this goal we shall use the topological space G_p described above. It remains to show that whenever p is a nowhere dense ultrafilter and $f : \beta G_p \to {}^{\omega}\{0,1\}$ is continuous, then there exists a non-empty clopen set $H \subseteq \beta G_p$ such that $\operatorname{int} f(H) = \emptyset$.

First of all we notice that since p is a nowhere dense ultrafilter, for every $s \in {}^{\omega>}\omega$ there exists $A_s \in p$ such that

$$\operatorname{int}\operatorname{cl} f(s^\frown A_s) = \emptyset. \tag{4}$$

In the sequel L_n will denote the set of all sequences of length n, i.e., L_n is the n-th level of the tree ${}^{\omega>}\omega$. In particular, $L_0 = \{s_0\}$ is the empty sequence. By induction we define a sequence of sets $\{U_n : n < \omega\}$ such that $U_n \subseteq L_n$ for every $n < \omega$ and, moreover

$$\operatorname{int}\operatorname{cl} f(U_n) = \emptyset, \tag{5}$$

(6) for every $s \in U_n$ there exists $A \in p$ such that $s^\frown A \subseteq U_{n+1}$.

We set $U_0 = \{s_0\}$ and $U_1 = s_0{}^\frown A_{s_0}$. Assume U_n is defined, say $U_n = \{s_k : k < \omega\}$. Then by continuity of f and the condition (4) we can choose $A_k \in p$ in such a way that $\operatorname{int} \operatorname{cl} f(s_k{}^\frown A_k) = \emptyset$ and moreover, the diameter of $\operatorname{cl} f(s_k{}^\frown A_k)$ is not greater than $\frac{1}{k}$. Clearly, s_k is an accumulation point of $s_k{}^\frown A_k$, because $A_k \in p$. Hence, for every $k < \omega$ we get

$$\operatorname{cl} f(s_k{}^\frown A_k) \cap \operatorname{cl} f(U_n) \neq \emptyset.$$

Therefore, since diameters of the sets $\operatorname{cl} f(s_k{}^\frown A_k)$ tend to zero, the set of accumulation points of the set $\bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) : k < \omega\}$ is contained in $\operatorname{cl} f(U_n)$. Indeed, every ε-neighborhood of the set $\operatorname{cl} f(U_n)$ has to contain all but finitely many sets of the form $\operatorname{cl} f(s_k{}^\frown A_k)$. So the set $\operatorname{cl} f(U_n) \cup \bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) : k < \omega\}$ is closed. It is also nowhere dense as it is a countable union of nowhere dense sets and is closed. Now we set

$$U_{n+1} = \bigcup\{s_k{}^\frown A_k : k < \omega\}$$

and observe that

$$\operatorname{cl} f(U_{n+1}) \subseteq \operatorname{cl} f(U_n) \cup \bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) : k < \omega\}.$$

Thus the set $f(U_{n+1})$ is nowhere dense, which completes the construction of U_n's.

By the condition (5) , there exists a dense set

$$\{x_n : n < \omega\} \subseteq {}^{\omega}\{0,1\} \setminus \bigcup\{\operatorname{cl} f(U_n) : n < \omega\}.$$

In particular, for every $n, k < \omega$ we have $f^{-1}(\{x_n\}) \cap \operatorname{cl} U_k = \emptyset$, where "cl" denotes here the closure in βG_p. Now, for every $n < \omega$ we choose a clopen set $V_n \subseteq \beta G_p$ such that

(7) $$f^{-1}(\{x_n\}) \subseteq V_n \subseteq \beta G_p \setminus \operatorname{cl}(U_0 \cup \cdots \cup U_n).$$

By induction we construct a sequence $\{W_n : n < \omega\}$ such that the following conditions hold:

(8) $$W_n \subseteq U_n \text{ for } n < \omega \text{ and } W_0 = U_0$$

for every $s \in W_n$ there exists $B_s \in p$ such that

(9) $$s^\frown B_s \subseteq U_{n+1} \setminus (V_0 \cup \cdots \cup V_n),$$

(10) $$W_{n+1} = \bigcup\{s^\frown B_s : s \in W_n\}.$$

Assume the sets $W_0, \ldots, W_n$ are defined in such a way that (8), (9) and (10) are satisfied. Then we have in particular

$$W_n \subseteq U_n \setminus (V_0 \cup \cdots \cup V_{n-1});$$

by the condition (7) we also have

$$U_n \subseteq \beta G_p \setminus V_n.$$

Hence we get $W_n \subseteq \bigcup_{n=0}^{\infty} U_n \setminus (V_0 \cup \cdots \cup V_n)$. Since the set $\bigcup_{n=0}^{\infty} U_n \setminus (V_0 \cup \cdots \cup V_n)$ is open, for every $s \in W_n$ we can choose $B_s \in p$ such that $s^\frown B_s \subseteq U_{n+1} \setminus (V_0 \cup \cdots \cup V_n)$. Then it is enough to set $W_{n+1} = \bigcup\{s^\frown B_s : s \in W_n\}$.

Clearly the set $W = \bigcup\{W_n : n < \omega\}$ is open in G_p and $W \cap V_n = \emptyset$ for every $n < \omega$. Indeed, if $m > n$, then $W_m \cap V_n = \emptyset$ by the conditions (9) and (10), whereas for $m \leqslant n$, $W_m \cap V_n = \emptyset$ because $W_m \subseteq U_m$ and $U_m \cap V_n = \emptyset$ by the condition (7). Since V_n is a clopen set in βG_p we also have

$$\operatorname{cl} W \cap V_n = \emptyset$$

for every $n < \omega$. Since βG_p is extremally disconnected, cl W is clopen subset of βG_p and, by the last equality and condition (7) we get

$$f(\operatorname{cl} W) \cap \{x_n : n < \omega\} = \emptyset.$$

Therefore $f(\operatorname{cl} W)$ is nowhere dense, because $\{x_n : n < \omega\}$ is dense in ${}^{\omega}\{0,1\}$, which completes the proof. ⊣

Acknowledgements. The authors are very indebted to the referee and Heike Mildeberger for simplifying the proof of Theorem 2 as well as for information that Jörg Brendle obtained similar result independently.

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