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EXISTENCE OF MANY $L_{\infty,\lambda}$-EQUIVALENT, NON-ISOMORPHIC MODELS OF T OF POWER λ

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Suppose T is not superstable, $T \subseteq T_1$ (both first-order theories). If $\lambda > |T_1|$ is regular or strong limit, we construct 2^λ non-isomorphic, pairwise $L_{\infty,\lambda}$-equivalent models of T of power λ, which are reducts of models of T_1. Note, however, that the proof applies to the class of models of T, T (superstable but) with dop or otop and even to appropriate non-elementary classes as well.

0. Introduction

This paper has a place in two lines of research: existence of $L_{\infty,\lambda}$-equivalent non-isomorphic models of power λ and classification theory.

On the history of construction of $L_{\infty,\lambda}$-equivalent, non-isomorphic models of power λ, see e.g., [1], [3], [7], [8].

In the mid-seventies some people became interested in building such models for specific theories and have gotten some results (Nadel, Stavi, Macintyre).

In [8] we have announced the solution for any non-superstable T (and for such pseudo-elementary classes).

Our main result is: (The unexplained notions are defined in 1.2A.)

0.1. Theorem. *Suppose $L \subseteq L_1$ are vocabularies, and for every $I \in K^{\omega}_{\mathrm{tr}}$ (see Definition 1.2) $\mathrm{EM}^1(I, \Phi)$ is an L_1-model, which is the Skolem-hull of $\bigcup_{\eta\in I} \bar{a}_\eta$, $\langle \bar{a}_\eta : \eta \in I\rangle$ is indiscernible in $\mathrm{EM}^1(I, \Phi)$ (see Definition 1.1). Suppose further that $\phi_n(\bar{x}, \bar{y})$ are formulas in the vocabulary L (not necessarily first-order), and for $\eta \in P^I_\omega$, $\nu \in P^I_n$ we have: $\mathrm{EM}^1(I, \Phi) \vDash \phi_{\lg(\eta)}[\bar{a}_\eta, \bar{a}_\nu]$ iff $\nu < \eta$.*

Then for every $\lambda > |L_1|$, λ regular or strong limit, there are 2^λ models of the form $\mathrm{EM}^1(I, \Phi)$, $|I| = \lambda$, $(L_1)_{\infty,\lambda}$-equivalent, but with non-isomorphic L-reducts in pairs.

Remark. Of course, we got many index models I_α of cardinality λ which are quite similar, but $\{\mathrm{EM}(I_\alpha, \Phi) : \alpha\}$ are pairwise non-isomorphic. We can combine the proofs here with those of [6] demanding on the I_α's conditions as there. A consequence is getting many pairwise non-elementarily embeddable such models.

As an example look at the class of separable abelian p-groups. Define

$\mathrm{EM}(I, \Phi)$ (for $I \in K_{\mathrm{tr}}^{\omega}$) as the abelian group freely generated by x_η ($\eta \in P^I_{\lg(\eta)}$, $\lg(\eta) < \omega$) and x^n_η ($\eta \in P^I_\omega$, $n < \omega$), except the relations $x^n_\eta - p x^{n+1}_\eta = x_{\eta \restriction n}$ when $\eta \in P^I_\omega$ (so, informally, $x^n_\eta = \sum_{k \geqslant n} p^{k-n} x_{\eta \restriction k}$). There is no problem to define L_1, Φ.

We can conclude that there are 2^λ, $L_{\infty,\lambda}$-equivalent non-isomorphic separable abelian p-groups of power λ. On this see [6], [9]. [Note that, as mentioned in [9, p. 244] by the general results of [5, Ch. VIII, §2] for $\lambda > \aleph_0$ there are 2^λ non-isomorphic separable abelian p-groups of power λ.)

If we want that no one (group from the family) is embeddable into another (not just not-purely embeddable) see [6, p. 106, Remark 2], and we can combine this with the $L_{\infty,\lambda}$-equivalence.

Note. For the regular case we also use a construction of a linear order (see Appendix) which is a variant of the one from [2, §3] due to Galvin and Laver.

Let us now turn to classification theory. It is reasonable to say that if there are models of T of power λ, $L_{\infty,\lambda}$-equivalent but not isomorphic, then T has no good structure theorem. For the possible invariants distinguishing the two models cannot have a simple definition (see a discussion in [10]), we now can prove

0.2. Main Conclusion. *Let K be the class of models of T which are reducts of models of T_1, T complete in L, $T \subseteq T_1$. Then in K there are 2^λ pairwise non-isomorphic, $L_{\infty,\lambda}$-equivalent models of power λ for every $\lambda > |T_1|$ which is regular or strong limit if at least one of the following holds*:

(a) *T not superstable.*

(b) *$T_1 = T$, and T has the dop* (*see* [11, Definition X 2.1]).

(c) *$T_1 = T$, T countable and T has the otop* (*see* [11, Definition XII 4.1]).

This clearly includes many examples, but more important is that for countable $T_1 = T$ it is best possible: by [11, XIII 1.1] for countable superstable T without the dop nor the otop and $\lambda > 2^{\aleph_0}$ any two $L_{\infty,\lambda}$-equivalent models of T power λ are isomorphic (and we can even weaken the logic).

Proof of 0.1. We apply 2.10 for λ regular, 3.1 for λ strong limit.

Proof of 0.2. We use 0.1. For case (a) its assumption holds by 1.3. For cases (b), (c) we have to replace 1.3 by parallel theorems, and they are [11, X, Fact 2.5A], [11, XII, the proof of 6.1(1)], respectively.

Notation. Bar means a finite sequence, $\bar{s} \in I$ means $\bar{s} = \langle s_0, \ldots \rangle$, $s_0, \ldots \in I$.

$$\mathrm{tp}_\Delta(\bar{a}, B, M) = \{\phi(\bar{x}, \bar{b}) : \bar{b} \in M,\ M \models \phi[\bar{a}, \bar{b}],\ \phi(\bar{x}, \bar{y}) \in \Delta,\ \bar{b} \in B\}.$$

For Δ the set of atomic formulas of $L(M)$ we write 'at'; for Δ the set of basic ($=$ atomic or negation of atomic) formulas of $L(M)$ we write 'bs'.

Let L denote a vocabulary (i.e., a set of predicates and function symbols). So an L-model M is a universe $|M|$ and interpretation of the predicates and function symbols of L as relations and functions on $|M|$.

1. Preliminaries

We review here some of the necessary material we shall use. Recall from [5, Ch. VII, Definitions 2.4, 2.6, p. 393]:

1.1. Definition. We define *generalized EM-models* (EM for Ehrenfeucht–Mostowski).

Let K be a class of models we call the *index models* and we denote its members by I, J. Let $L \subseteq L^1$ be vocabularies.

For $I \in K$ we say that $\langle \bar{a}_s : s \in I \rangle$ is *indiscernible* in M, if, (denoting $\bar{a}_{\langle s_0,\ldots,s_{n-1}\rangle} = \bar{a}_{s_0} {}^\wedge \bar{a}_{s_1} {}^\wedge \cdots {}^\wedge \bar{a}_{s_{n-1}}$) for every $\bar{s}, \bar{t} \in J$ realizing the same atomic type in I, $\bar{a}_s, \bar{a}_t$ realize the same type in M. If L^1 is a vocabulary, $L \subseteq L^1$, Φ a function with domain including $\{\mathrm{tp}_{\mathrm{at}}(\bar{s}, \emptyset, M^1) = \bar{s} \in I\}$, and $I \in K$, we let $M^1 = \mathrm{EM}^1(I, \Phi)$ be an L^1*-model* generated by $\bigcup_{s \in I} \bar{a}_s$ such that $\mathrm{tp}_{\mathrm{at}}(\bar{a}_{\bar{s}}, \emptyset, M^1) = \Phi(\mathrm{tp}_{\mathrm{at}}(\bar{s}, \emptyset, I))$.

We say Φ is *proper* for K if for every $I \in K$, $\mathrm{EM}^1(I, \Phi)$ is well defined.

Let $\mathrm{EM}(I, \Phi)$ be the *L-reduct of* $\mathrm{EM}^1(I, \Phi)$.

Remark. The case we have in mind is T a complete theory in L, T_1 a complete theory in L^1 extending T and having Skolem function, $\mathrm{EM}^1(I, \Phi)$ a model of T_1.

1.2. Definition. Let K_{or} be the class of linear orders.

We shall write K_{tr}^ω for the class of trees with $(\omega + 1)$ levels (see 1.2A).

1.2A. Definition. Let K_{tr}^ω be the class of models isomorphic to some $(A, \triangleleft, P_n, <, h)_{n<\omega}$ where $A \subseteq {}^{\omega \geqslant} I$ for some linear order I and:

(1) A is closed under initial segments.

(2) $\triangleleft$ denotes the initial segment relation; $h(\eta, \nu)$ is the maximal common initial segment of η and ν.

(3) $P_n = \{\eta \in A: \lg(\eta) = \eta\}$.

(4) $<$ is $\bigcup_{\eta \in A} (< \restriction \mathrm{Suc}_A(\eta))$, (i.e., $x < y \rightarrow \exists \eta\, [x,y \in \mathrm{Suc}_A(\eta)]$ and $< \restriction \mathrm{Suc}_A(\eta)$ is a linear order $\eta {}^\wedge \langle x \rangle < \eta {}^\wedge \langle y \rangle$ iff $I \models x < y$, where $\mathrm{Suc}_A(\eta) = \{\nu \in a : \eta \triangleleft \nu \text{ and } \lg(\nu) = \lg(\eta) + 1\}$.

The partial order $<$ extends naturally to the lexicographical order on A. (We will not distinguish strictly between $<$ and the lexicographical order.)

For $\eta \in I \in K_{\mathrm{tr}}^\omega$, $\lg(\eta)$ is the unique $n \leqslant \omega$ such that $I \models P_n[\eta]$, and $\eta \restriction n$ (where $n \leqslant \omega$) is the unique ν, $I \models P_n(\nu) \wedge \nu \triangleleft \eta$.

We identify such $A \subseteq {}^{\omega \geqslant} I$ with the model $(A, \triangleleft, P_n, <, h)_{n \leqslant \omega}$ and call it *standard*.

1.3. Theorem. *Suppose $L \subseteq L^1$, T a complete (first-order) theory in L, T_1 a complete theory in L^1 with Skolem function and $T \subseteq T_1$. Suppose further T is not superstable and $\phi_n(x, \bar{y}_n)$ $(n < \omega)$ witnesses this (see* [5, II, 3.14, p. 52; 3.9, p. 46]).

Then there is a Φ, proper for K_{tr}^{ω} such that for every $I \in K_{\mathrm{tr}}^{\omega}$, $\mathrm{EM}^1(I, \Phi)$ is a model of T_1, and for $\eta \in P_n^I$, $\nu \in P_\omega^I$, $\mathrm{EM}^1(I, \Phi) \Vdash \phi_n(\bar{a}_\nu, \bar{a}_\eta)$ iff $I \vDash \eta < \nu$.

Proof. See [5, VII 3.5(2), VII 3.6(2)].

By a theorem of Karp (see Dickman [1]):

1.4. Theorem. (1) *The L-models M,N are $L_{\infty,\lambda}$-equivalent iff there is a non-empty family $\mathscr{F}$ of functions such that:*

(a) *Each $f \in \mathscr{F}$ is a partial isomorphism from M to N (i.e., f is a one-to-one function from $\mathrm{Dom}(f) \subseteq M$ into N) and for every $\bar{a} \in \mathrm{Dom}(f)$, $\mathrm{tp}_{\mathrm{at}}(\bar{a}, \emptyset, M) = \mathrm{tp}_{\mathrm{at}}(f(\bar{a}), \emptyset, M)$.*

(b) *For every $A \subseteq M$, $|A| < \lambda$ and $f \in \mathscr{F}$ there is $g \in \mathscr{F}$ extending f such that $A \subseteq \mathrm{Dom}(f)$.*

(c) *For every $A \subseteq N$, $|A| < \lambda$ and $f \in \mathscr{F}$ there is $g \in \mathscr{F}$ extending f such that $A \subseteq \mathrm{Rang}(f)$.*

(2) *In such case we say $\mathscr{F}$ exemplifies M,N are $L_{\infty,\lambda}$-equivalent.*

1.5. Lemma. *If I,J are $L_{\infty,\lambda}$-equivalent, then so are $\mathrm{EM}(I, \Phi)$, $\mathrm{EM}(J, \Phi)$.*

Proof. Let $\mathscr{F}$ exemplify I,J are $L_{\infty,\lambda}$-equivalent by 1.4. For each $f \in \mathscr{F}$ we define f^*: it is a function whose domain is $\{\tau(\bar{a}_{s_1}, \dots, \bar{a}_{s_n}) : s_1, \dots, s_n \in \mathrm{Dom}(f)$, τ is an L_1-term$\}$ and $f^*(\tau(\bar{a}_{s_1}, \dots, \bar{a}_{s_n})) = \tau(\bar{a}_{f(s_1)}, \dots, \bar{a}_{f(s_n)})$. It is easy to check that $\{f^* : f \in \mathscr{F}\}$ is as required.

1.6. Definition. $I \subseteq_c J$ for I,J from K_{tr}^{ω} means: I is a submodel of J and for $\eta \in P_\omega^J$, if $(\forall l < \omega)\, \eta \restriction l \in I$, then $\eta \in I$.

1.7. Claim. *Suppose $I \subseteq_c J$ (both in K_{tr}^{ω}, see Definition 2.6(2)), Φ proper for K_{tr}^{ω} and:*

$(*)$ *For every $\eta \in I - P_\omega^I$, and countable $A \subseteq \mathrm{Suc}_I(\eta)$ and distinct $\nu_1, \dots, \nu_k \in \mathrm{Suc}_J(\eta) - A$ there are distinct $\nu_1', \dots, \nu_k' \in \mathrm{Suc}_I(\eta) - A$ such that*

$$(\forall \rho \in A) \bigwedge_{l=1}^{k} [\rho < \nu_l \equiv \rho < \nu_l' \text{ and } \nu_l < \rho \equiv \nu_l' < \rho \text{ and } \bigwedge_{m=1}^{k} \nu_l < \nu_m \equiv \nu_l' < \nu_m'].$$

$(**)$ *For every $\eta \in I - P_\omega^I$, $(\exists \nu \in P_\omega^I)\, [\eta < \nu]$ and $\mathrm{Suc}_I(\eta)$ is infinite.*

Then for every countable set of L_1-formulas $p = \{\psi_i(\bar{x}, \bar{b}_i) : i < \omega\}$ where $\bar{b}_i \in \mathrm{EM}^1(I, \Phi)$, if p is realized in $\mathrm{EM}^1(J, \Phi)$, then p is realized in $\mathrm{EM}^1(I, \Phi)$.

Proof. Easy.

1.8 Claim. *In* 1.7, *we can replace* $(*)$ *by*:

$(*)'$ $\bar{b}_i = \bar{\tau}_i(\bar{\eta}_i)$ *and for every* $\nu \in I - P^I_\omega$, $A = \{\rho \restriction l : l \leq \lg(\rho), \rho \in (\bigcup_{i<\omega} \bar{\eta}_i)\} \cap \mathrm{Suc}_I(\nu)$ *satisfies* $(*)$.

Proof. Easy.

1.9. Definition. Let λ be an uncountable regular cardinal.

(1) $\bar{A}$ is a *λ-representation of A* if $\bar{A} = \langle A_\alpha : \alpha < \lambda \rangle$, A_α increasing continuous in α, $|A_\alpha| < \lambda$ and $A = \bigcup_{\alpha<\lambda} A_\alpha$. So a λ-representation of M is a λ-representation of $|M|$ (the universe of the model) but we write $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$.

(2) A function $\mathbf{F}$ is *D_λ-invariant* (function) (D_λ is the filter of closed unbounded subsets of λ) if its domain is a class of λ-representations and

(a) for every λ-representation $\bar{M}$, $\mathbf{F}(\bar{M})$ is a subset of λ and for λ representations $\bar{M}^1$, $\bar{M}^2$ of M, $\mathbf{F}(\bar{M}^1) \equiv \mathbf{F}(\bar{M}^2) \bmod D_\lambda$, and

(b) if $\bar{M}^1$, $\bar{M}^2$ are representations of M^1, M^2 resp., models of power λ which are isomorphic, then $\mathbf{F}(\bar{M}^1) = \mathbf{F}(\bar{M}^2) \bmod D_\lambda$.

(3) For $\mathbf{F}$ D_λ-invariant, $\mathbf{F}(M)$ is $\mathbf{F}(\bar{M})/D_\lambda$ for every ($\equiv$ some) λ-representation $\bar{M}$ of M.

1.10. Definition. (1) For a λ-representation $\bar{M}$ let (on splitting see below)

$$\mathrm{Sp}(\bar{M}) = \{\delta < \lambda : \delta \text{ limit, and for some } \bar{a} \in \textstyle\bigcup_{\alpha<\lambda} M_\alpha \text{ for every } \beta < \delta \\ \mathrm{tp}(\bar{a}, M_\delta, M) \text{ split over } M_\beta\}.$$

(2) $$\mathrm{Sp}_{\Delta_1,\Delta_2}(\bar{M}) = \{\delta < \lambda : \delta \text{ limit, and for some } \bar{a} \in \textstyle\bigcup_{\alpha<\lambda} M_\alpha \text{ for every } \beta < \delta \\ \text{the type } \mathrm{tp}_{\Delta_1}(\bar{a}, M_\delta, M)\ (\Delta_1, \Delta_2)\text{-splits over } M_\beta\}.$$

1.10A. Remark. (1) Sp is D_λ-invariant (when $\lambda = \mathrm{cf}\,\lambda > \aleph_0$).

(2) On splitting see [5, I Section 2]. We say $\mathrm{tp}_{\Delta_1}(a, B, M)$ (Δ_1, Δ_2)-splits over $A \subseteq M$ (where $\bar{a} \in M, B \subseteq M$) if for some $\bar{b}_1, \bar{b}_2 \in B$, $\mathrm{tp}_{\Delta_2}(\bar{b}_1, A, M) = \mathrm{tp}_{\Delta_2}(\bar{b}_2, A, M)$ but $\mathrm{tp}_{\Delta_1}(\bar{a} \wedge \bar{b}_1, A, M) \neq \mathrm{tp}_{\Delta_1}(\bar{a} \wedge \bar{b}_2, A, M)$.

1.11. Definition. (1) M is *$(\lambda, \Delta_1, \Delta_2)$-nice* if $\mathrm{Sp}_{\Delta_1,\Delta_2}(M) = \emptyset/D_\lambda$.

(2) $I \in K^\omega_{\mathrm{tr}}$ is *locally* $(\lambda, \mathrm{bs}, \mathrm{bs})$*-nice* [*locally* $(<\lambda, \mathrm{bs})$-stable] if for every $\eta \in I - P^I_\omega$, $(\mathrm{Suc}_I(\eta), <)$ is $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice [$(<\lambda, \mathrm{bs})$-stable] and $(**)$ of 1.7 holds.

(3) M is *$(<\lambda, \Delta)$-stable* if for every $A \subseteq |M|$ of power $<\lambda$

$$\lambda > |\{\mathrm{tp}_\Delta(\bar{a}, A, M) : \bar{a} \in |M|\}|.$$

2. Regular cardinals

2.1. Assumption. Let λ be a fixed uncountable regular cardinal (for this section).

Let D_λ be the filter generated by the closed unbounded subsets of λ. Let $L \subseteq L^1$, ϕ_n $(n<\omega)$, Φ be as in the conclusion of 1.3 and assume $\lambda > |L_1|$.

2.2. Definition. For $I \in K^\omega_{\mathrm{tr}}$ of power λ, we define a set $S = S(I) \subseteq \lambda$ modulo D_λ (so formalistically, $S(I)$ belongs to $\mathscr{P}(\lambda)/D_\lambda$).

For any λ-representation $\bar I = \langle I_\alpha : \alpha<\lambda\rangle$ of I let $S(\bar I) = \{\delta<\lambda : \delta$ a limit ordinal and for some $\eta \in P^I_\omega$, $\{\eta \restriction n : n<\omega\} \subseteq I_\delta$ but for no $\alpha<\delta$, $\{\eta \restriction n : n<\omega\} \subseteq I_\alpha\}$.

By the following fact $S(\bar I)$ is determined by I, modulo D_λ.

2.3. Fact. (1) *The function of S is D_λ-invariant.*

(2) *If $I \in K^\omega_{\mathrm{tr}}$ is locally λ-nice, then* $\mathrm{Sp}(I) = S(I)$. *Moreover, for any λ-representation $\bar I$ of I,* $\mathrm{Sp}(\bar I) = S(\bar I)$ *provided that for $\eta \in I - P^I_\omega$, $\eta \in I_\alpha$,* $\mathrm{Sp}(\langle \mathrm{Suc}_I(\eta) \cap I_{\alpha+i} : i<\lambda\rangle) = \emptyset$.

2.3A. Remark. Almost nothing changes if we replace S by S', $S'(I) = \{\delta<\lambda : \delta$ a limit ordinal and for some $\eta \in P^I_\omega$, $\{\eta \restriction n : n<\omega\} \subseteq I_\delta$ but $\eta \notin I_\delta\}$ and restrict ourselves to $(<\lambda, \mathrm{bs})$-stable I.

2.4. Theorem. (1) *If $I, J \in K^\omega_{\mathrm{tr}}$ have power λ, λ regular $>\aleph_0$ and I, J are locally* $(\lambda, \mathrm{bs}, \mathrm{bs})$*-nice,* $(<\lambda, \mathrm{bs})$*-stable and* $\mathrm{EM}(I, \Phi)$, *$\mathrm{EM}(J, \Phi)$ are isomorphic, then* $S(I)/D_\lambda = S(J)/D_\lambda$.

(2) *If $I, J \in K^\omega_{\mathrm{tr}}$ have power λ, are locally* $(\lambda, \mathrm{bs}, \mathrm{bs})$*-nice,* $(<\lambda, \mathrm{bs})$*-stable and there is an embedding of* $\mathrm{EM}(I, \Phi)$ *into* $\mathrm{EM}(J, \Phi)$ *preserving the formulas $\phi_n, \neg\phi_n$ (for $n<\omega$), then* $S(I) \subseteq S(J) \bmod D_\lambda$.

Remark. (1) This is like [5, Ch.V.III 2.1, 2.2].

(2) Really "I,J are $(<\lambda, \mathrm{bs})$-stable" can be weakened to "I,J are locally $(<\lambda, bs)$-stable".

Proof. As $|I| = \lambda > \aleph_0$ and I is locally λ-nice and $(<\lambda, \mathrm{bs})$-stable there is $\bar I = \langle I_\alpha : \alpha<\lambda\rangle$, such that

(i) $\bar I$ is a λ-representation of I.

(ii) If $\eta \in I_\alpha$, $\nu \in I$, $\nu < \eta$, then $\nu \in I_\alpha$.

(iii) If $\eta \in I_\alpha$, $\alpha<\delta<\lambda$, δ limit, $\nu \in \mathrm{Suc}_I(\eta)$, then $\mathrm{tp}_{\mathrm{bs}}(\nu, I_\delta, I)$ does not (bs, bs)-split over I_β for some $\beta<\delta$, i.e., for some β, $\alpha<\beta<\delta$ and one of the following holds:

(a) $(\forall \sigma \in \mathrm{Suc}_{I_\delta}(\eta))[\sigma < \eta \Leftrightarrow (\exists \sigma' \in \mathrm{Suc}_{I_\beta}(\eta))(\sigma < \sigma' < \eta)]$ or

(b) $(\forall \sigma \in \mathrm{Suc}_{I_\delta}(\eta))[\sigma > \eta \Leftrightarrow (\exists \sigma' \in \mathrm{Suc}_{I_\beta}(\eta))(\sigma > \sigma' > \eta)]$.

Similarly for J there is a sequence $\langle J_\alpha : \alpha<\lambda\rangle$. Suppose f is a function from $\mathrm{EM}(J, \Phi)$ into $\mathrm{EM}(J, \Phi)$ preserving the formulas ϕ_n, $\neg\phi_n$ for $n<\omega$.

For a sequence $\bar a = \langle a_0, \dots, a_{n-1}\rangle$ from $\mathrm{EM}(I, \Phi)$ let $f(\bar a) = \langle f(a_0), \dots, f(a_{n-1})\rangle$. For $\eta \in I$ let $f(\bar a_\eta) = \bar\tau_\eta(\bar\nu_\eta)$ (i.e., $\bar\tau_\eta$ is a finite sequence of terms in the vocabulary $\mathrm{EM}^1(J, \Phi)$, $\bar\nu_\eta$ a finite sequence of elements of J).

Let $\bar{v}_n = \langle v^l_\eta : l < \lg(\bar{v}_\eta)\rangle$. Let

$$C_0 = \{\delta < \lambda : \delta \text{ is limit, and } (\forall \eta \in I)(\eta \in I_\delta \Leftrightarrow \bar{v}_\eta \subseteq J_\alpha\},$$
$$C_1 = \{\delta \in C_0 : (\forall \alpha < \delta)(\forall \eta \in I_\alpha)(\forall \rho_1 \in \mathrm{Suc}_I(\eta))(\exists \rho_2 \in \mathrm{Suc}_{I_\delta}(\eta))[v_{\rho_1}, v_{\rho_2} \text{ realize the same atomic type over } J_\alpha]\},$$
$$C = \{\delta \in C_1 : C_1 \cap \delta \text{ has order type } \delta\}.$$

Clearly C_0, C_1, C are closed unbounded subsets of λ. So it suffices to prove

$$(*) \qquad S((I_\alpha : \alpha < \lambda)) \cap C \subseteq S((J_\alpha : \alpha < \lambda)).$$

So suppose $\delta \in S(\langle I_\alpha : \alpha < \lambda\rangle) \cap C - S(\langle J_\alpha : \alpha < \lambda\rangle)$ and we shall eventually derive a contradiction. As $\delta \in S(\langle I_\alpha : \alpha < \lambda\rangle)$ there is $\eta \in I$, $I \vDash P_\omega(\eta)$, and for $n < \omega$, $\eta \restriction n \in I_\delta$ but for no $\alpha < \delta$, $\{\eta \restriction n : n < \omega\} \subseteq I_\alpha$.

Now for each $l < \lg(\bar{v}_\eta)$ there are $\alpha_l < \delta$, $\sigma_l \in I_{\alpha_l+1}$ and $m_l \leqslant \omega$ such that:

(α) $(v^2_\eta) \restriction m_l \in J_{\alpha_l}$.

(β) If $m_l < \lg(v^l_\eta)$, then $(v^l_\eta) \restriction (m+1) \notin J_\delta$.

(γ) If $m_l < \lg(v^l_v)$, then

(a) $(\forall \sigma \in \mathrm{Suc}_{I_\delta}(v^l_\eta \restriction m_l))[\sigma < v^l_\eta \restriction (m+1) \Leftrightarrow (\exists \sigma' \in J_{\alpha_l})(v^l_n \restriction (n+1) > \sigma' > \sigma)$.

or

(b) $(\forall \sigma \in \mathrm{Suc}_{I_\delta}(v^l_\eta \restriction m_l))[\sigma > v^l_\eta \restriction (m+1) \Leftrightarrow (\exists \sigma' \in J_{\alpha_l})(v^l_n \restriction (n+1) < \sigma' < \sigma)$.

Let $\alpha = \mathrm{Max}\{\alpha_l + 1 : l < \lg(\bar{v}_\eta)\}$, so $\alpha < \delta$. As $\delta \in C$, $\delta \cap C_1$ has order type δ, so we can find $\beta, \gamma \in C_1$, $\alpha < \beta < \gamma < \delta$, and

$$\bigwedge_{n<\omega} [n \restriction n \in I_\gamma \Rightarrow \eta \restriction n \in I_\beta].$$

Let $n < \omega$ be maximal such that $\eta \restriction n \in I_\beta$ (exists by the choice of η).

Let $\rho_1 = \eta \restriction (n+1)$. We shall prove:

($*$) There is $\rho_2 \in I_\gamma - I_\beta$, $I \vDash (\rho_1 < \rho_2) \wedge (\rho_1 \restriction n = \rho_2 \restriction n)$, $\lg(\rho_2) = n+1$ such that $\bar{\tau}_{\rho_1} = \bar{\tau}_{\rho_2}$ and (in J) $\bar{v}_{\rho_1}$, $\bar{v}_{\rho_2}$ realize the same atomic type over $\bar{v}_\eta$.

This suffices as then

$$\mathrm{EM}(J, \Phi) \vDash \phi_{n+1}(\bar{\tau}_\eta(\bar{v}_\eta), \bar{\tau}_{\rho_1}(\bar{v}_{\rho_1})) \equiv \phi_{n+1}(\bar{\tau}_\eta(\bar{v}_\eta), \bar{\tau}_{\rho_2}(\bar{v}_{\rho_2}))$$

but

$$\mathrm{EM}(I, \Phi) \vDash \phi_{n+1}(\bar{a}_\eta, \bar{a}_{\rho_1}) \wedge \neg\phi_{n+1}(\bar{a}_\eta, \bar{a}_{\rho_2})$$

so we get contradiction to the property of the function f.

Proof of ($*$). By the choice of α (and the α_l's) and as $\langle J_\alpha : \alpha < \lambda\rangle$ satisfies (iii), it is enough that $\bar{v}_{\rho_1}$, $\bar{v}_{\rho_2}$ satisfy the same atomic type over J_β. This is possible as $\beta, \gamma \in C_1$ (see its definition).

2.5. Remarks. (1) See more (particularly on singular λ) in [6].

(2) From the isomorphism type of $M = \mathrm{EM}(I, \Phi)$ we can reconstruct $S(I)$: using Φ as a parameter: trivially (as $S(J)$ for every J such that $\mathrm{EM}(J, \Phi) \cong M$).

But the use of Φ is not necessary as also: $S(J)$ for every J such that for some Φ' (corresponding to a vocabulary L_1', $|L_1'|<\lambda$) $M\cong \mathrm{EM}(J,\phi')$.

A more direct definition is the minimal S/D_λ, such that there is $\langle E_A : A\subseteq M$ finite$\rangle$ such that each E_A is an equivalence relation on the family of finite sequences from M with $<\lambda$ equivalence classes and

$$S=\{\delta<\lambda:\delta \text{ limit of cofinality } \aleph_0, \text{ and for some } \bar a\in M, \text{ for every finite } A\subseteq M_\delta, \text{ there are } \bar b,\bar c\in M_0 \text{ which are } E_A\text{-equivalent}: \mathrm{tp}(\bar b, A\cup\bar a, M)\neq \mathrm{tp}(\bar c, A\cup \bar a, M)\}.$$

Alternatively: the maximal S such that for some expansion M^1 of M (of vocabulary of power $<\lambda$), M^1 is $(<\lambda, L_1)$-stable and $(\lambda-S, L_1, L)$-nice.

See more on this in 2.11.

(3) In 2.4 we actually prove: if $I\in K_{\mathrm{tr}}^\omega$, then I is $(<\lambda,\mathrm{bs})$-stable and locally $(\lambda,\mathrm{bs},\mathrm{bs})$-nice.

(4) The proof in Section 2 can be made more similar to the one in Section 3, building the J_α's by hence-and-forth argument, but less explicitly.

2.6. Definition. (1) For $I,J\in K_{\mathrm{tr}}^\omega$, $I\subseteq J$ means I is a *submodel of* J hence necessarily $\nu\in J$, $\nu<\eta$, $\eta\in I$ imply $\nu\in I$.

(2) For $I,J\in K_{\mathrm{tr}}^\omega$, $I\subseteq_{\mathrm c} J$ (I a *closed* submodel of J) means $I\subseteq J$ and ($\eta\in P_\omega^J$, $\{\eta\restriction n : n<\omega\}\subseteq I \Rightarrow \eta\in I$).

(3) For $I,J\in K_{\mathrm{tr}}^\omega$, we say f is an *embedding* [*closed* embedding] of I onto J if it is an isomorphism from I onto some $I'\subseteq J$ [$I'\subseteq_{\mathrm c} J$].

2.7. Claim. (1) *K_{tr}^ω has the amalgamation property for closed embeddings.*

(2) *If f_l is a closed embedding of I_0 into I_l for $l=1,2$, then we can find J and closed embeddings g_l of I_l into J (for $l=1,2$) such that $g_1f_1=g_2f_2$. Moreover $\|J\|\leq \|I_0\|+\|I_1\|$, $J=g_1(I_1)\cup g_2(I_2)$ and if f_1 is the identity on I_0 we can choose g_1 as the identity (on I_1).*

Proof. Left to the reader.

2.8. Claim. *Suppose $J,I\in K_{\mathrm{tr}}^\omega$ have power λ, $I\subseteq J$ and there is a function h such that:*

(a) *The domain of h is $P_\omega^J-P_\omega^I$,*

(b) *For every $\eta\in\mathrm{Dom}(h)$, $h(\eta)<\eta$*

(c) *For every $\nu\in J$, the set $\{\eta\in\mathrm{Dom}(h):h(\eta)=\nu\}$ has power $<\lambda$.*

Then $S(I)=S(J)$ (mod D_λ).

Proof. Let $\bar J=\langle J_i : i<\lambda\rangle$ represent J. We define a function g from $J-P_\omega^J$ into λ: $g(\nu)$ is the first ordinal $\alpha<\lambda$ such that $\{\eta\in\mathrm{Dom}(h):h(\eta)=\nu\}\subseteq J_\alpha$.

Now α exists as λ is regular and the set above has power $<\lambda$ by (c) of the claim. Now define $C=\{\delta<\lambda:\delta \text{ limit and for every } \nu\in J_\delta,\ g(\nu)<\delta\}$. As λ is regular and $|J_\alpha| < \lambda$ for $\alpha < \lambda$, and as g is a function into λ, C is a closed unbounded subset of λ. Now we shall prove that

$$C \cap S(\langle J_\alpha : a < \lambda \rangle) = C \cap S(\langle I \cap J_\alpha : \alpha < \lambda \rangle).$$

As $\langle I \cap J_\alpha : \alpha < \lambda \rangle$ is a representation of I this is enough for showing $S(I) = S(J)$. The inclusion $\supseteq$ is trivial. For the other direction suppose $\delta \in C \cap S(\langle J_\alpha : \alpha < \lambda \rangle)$. So for some $\eta \in P^J_\omega$, $\{\eta \restriction n : n < \omega\} \subseteq J_\delta$ but for $\alpha < \delta$, $\{\eta \restriction n : n < \omega\} \not\subseteq J_\alpha$.

If $\eta \notin I$, then $\eta \in \mathrm{Dom}(h)$, now $h(\eta) \in \{\eta \restriction n : n < \omega\}$ (as $h(n) \lessdot \eta$), hence $h(\eta) \in J_\delta$, which implies $g(h(\eta)) < \delta$, but then $\eta \in J_{g(h(\eta))} \subseteq J_\delta$, contradiction. So $\eta \in I$, hence (as $I \subseteq J$), $\{\eta \restriction n : n < \omega\} \subseteq I$, hence $\{\eta \restriction n : n < \omega\} \subseteq I \cap J_\delta$. So η witnesses $\delta \in S(\langle I \cap J_\alpha : \alpha < \lambda \rangle)$, but trivially $\delta \in C$. So we have proved the second inclusion hence the claim.

2.9. Lemma. *Suppose $I_\alpha \in K^\omega_{\mathrm{tr}}$, $|I_\alpha| = \lambda$ for $\alpha < \lambda$. Then we can define $J_\alpha \in K^\omega_{\mathrm{tr}}$ $(\alpha < \lambda)$ such that:*

(a) $|J_\alpha| = \lambda$, $I_\alpha \subseteq_c J_\alpha$.

(b) $S(I_\alpha) = S(J_\alpha)$.

(c) *For* $\alpha, \beta < \lambda$ $J_\alpha \equiv_{\infty,\lambda} J_\beta$.

(d) *If each I_α is* [*locally*] ($<\lambda$, bs)-*stable, then so is each* J_α

(e) *If each I_α is locally* (λ, bs, bs)-*nice, then so is each* J_α.

Notation. For $J \in K^\omega_{\mathrm{tr}}$ let $(J)^{\mathrm{fin}} = |\bigcup_{n<\omega} P^J_n|$.

Proof. Without loss of generality the models I_α have pairwise disjoint universes.

Subfact. *There are a linear order M_0 and functions H_1, H_2 from M_0 onto $\bigcup_{\alpha<\lambda} (I_\alpha)^{\mathrm{fin}}$, $\{\alpha : \alpha \leq \lambda\}$ respectively such that:*

($*$) *If $a, b \in M_0$, then for some automorphism $g = g_{a,b}$ of M_0,*

(A) $g(a) = b$,

(B) $(\forall c \in M_0)[c \neq a \to H_1(c) = H_1(g(c)) \wedge H_2(c) = H_2(g(c))]$,

(C) M_0 *is* ($<\lambda$, bs)-*stable*,

(D) M_0 *is* (λ, bs, bs)-*nice*,

(E) *for every $x \in \bigcup_{\alpha<\lambda} |I_\alpha|^{\mathrm{fin}}$, $\alpha \leq \lambda$, the set $\{c \in M_0 : H_1(c) = x, H_2(c) = \alpha\}$ is a dense subset of M_0.*

Proof of the Subfact. We want to apply A3 of the Appendix. So let $\mu_1 = \mu_2 = \lambda^+$, $f_1 = f_2$ is a function from $\mathrm{Reg}(\lambda^+)$ to λ (see A1 of the Appendix), $f_l(\theta) = \theta$, and $g_1 = g_2$ is a function from λ to $\mathrm{Reg}(\lambda^+)$, $g_1(\alpha)$ is α if $\alpha \in \mathrm{Reg}(\lambda^+)$, $g_1(\alpha) = \aleph_0$ otherwise. Let $\chi_1 = \chi_2 = \aleph_0$, and choose $(M_1, P_\alpha)_{\alpha<\lambda} \in K_{\chi_1,\chi_2}$ (exists by A3 of the Appendix). Let $M_0 = M_1 \restriction \bigcup \{P_\alpha : \alpha < \lambda, \alpha \notin \mathrm{Reg}(\lambda^+))$. Now define H_1, H_2 such that for each $\alpha < \lambda$, $H_1 \restriction P_\alpha$, $H_2 \restriction P_\alpha$ are constant (and (E) holds).

Let $H_3 : M \to \lambda$ be defined by: $H_3(a) = \alpha \Leftrightarrow H_1(\eta) \in I_\alpha$.

Let $M=\{\langle a,\eta\rangle : a\in M_0,\ \eta\in\bigcup_{\alpha<\lambda}(I_\alpha-P^{I_\alpha}_\omega)\}$ be ordered by:

$$\begin{aligned}&\langle a_1,\eta_1\rangle\langle a_2,\eta_2\rangle \quad\text{iff}\quad M_0\vDash a_1<a_2 \text{ or } a_1=a_2,\\ &\eta_1\in I_{\alpha_1},\ \eta_2\in I_{\alpha_2},\ \alpha_1<\alpha_2 \text{ or } a_1=a_2,\ \eta_1\in P^{I_\alpha}_n,\ \eta_2\in P^{I_\alpha}_m,\\ &n<m \text{ or } \eta_1,\eta_2\in P^{I_\alpha}_n,\ a_1=a_2 \text{ and in } I_\alpha,\ \eta_1<_{\mathrm{lx}}\eta_2.\end{aligned}$$

Let $H_0(\langle a,\eta\rangle)=a$, for $l=1,2,3$ $H_l(\langle a,\eta\rangle)=H_l(a)$, $H_4(\langle a,\eta\rangle)=\eta$.

Let for $\alpha<\lambda$, $c^{[\alpha]}\in M_0$ be such that $H_1(c^{[\alpha]})=$ the $<$-minimal element of I_α and $H_2(c^{[\alpha]})=\lambda$ and for $\alpha<\lambda$, $\gamma\leqslant\lambda$, $c^{[\alpha]}_\gamma\in M_0$ be such that $H_1(c^{[\alpha]}_\gamma)=H_1(c^{[\alpha]})$, $H_2(c^{[\alpha]}_\gamma)=\gamma$, and w.l.o.g. $c^{[\alpha]}_\lambda=c^{[\alpha]}$.

Let $\langle I_{\alpha,\xi}:\xi<\lambda\rangle$ be a λ-representation of I_α for each $\alpha<\lambda$. Now we shall define for each α, J_α:

$$\begin{aligned}J_\alpha \stackrel{\mathrm{def}}{=} {}^{\omega>}M\cup\{\eta:\eta\in{}^{\omega}M, &\text{ and for some}\\ &m_\eta<\omega,\ \alpha_\eta\leqslant\lambda,\ \gamma_\eta\leqslant\lambda \text{ and } \rho_\eta\in P^{I_{\alpha(\eta)}}_\omega \text{ (where } \alpha(\eta)=\alpha_\eta):\\ &\text{(i) } (\forall l)[m_\eta\leqslant l<\omega \Rightarrow H_2(\eta(l))=\gamma_\eta],\\ &\text{(ii) } (\forall l)[m_\eta\leqslant l<\omega \Rightarrow H_3(\eta(l))=\alpha_\eta],\\ &\text{(iii) if } \gamma_\eta<\lambda, \text{ then } \rho_\eta\in\bigcup\{I_{\alpha_\eta,\xi}:\xi<\gamma_\eta\},\\ &\text{(iv) if } \gamma_\eta=\lambda, \text{ then } m_\eta=0,\ \alpha_\eta=\alpha,\\ &\text{(v) } (\forall l)[m_\eta\leqslant l<\omega\rightarrow\eta(l)=\langle c^{[\alpha(\eta)]}_{\gamma_\eta},\rho_\eta\restriction(l+1)\rangle]\}.\end{aligned}$$

We shall identify $\eta\in P^{I_\alpha}_n$ $(n\leqslant\omega)$ with $\langle t^l_\eta:l<n\rangle$ where $t^l_n=\langle c^{[\alpha]},\eta\restriction l\rangle$. Note:

$(**)_1$ If $\nu,\eta\in J_\alpha$, $l<\lg(\eta)$, $l\leqslant\lg(\nu)$, then $\langle\nu(m):m<l\rangle{}^\frown\langle\eta(m):l\leqslant m<\lg(\eta)\rangle$ belongs to J_α.

$(**)_2$ For $\alpha,\beta<\lambda$, $J_\alpha\restriction(\bigcup_{n<\omega}P^{J_\alpha}_n)=J_\beta\restriction(\bigcup_{n<\omega}P^{J_\beta}_n)$.

$(**)_3$ If $\eta\in P^{J_\alpha}_\omega$, $\gamma_\eta<\lambda$, $\alpha<\lambda$ and $\beta<\lambda$, then $\eta\in P^{J_\beta}_\omega$.

Now we should prove that J_α $(\alpha<\lambda)$ are as required.

Proof of (a). Clearly $|J_\alpha|\geqslant|M|=\lambda$ and $|J_\alpha|\leqslant\sum_{n<\omega}\|M\|^n+\sum\{|I_\gamma|:\eta\in{}^{\omega>}M,\ \gamma=H_2(\eta\restriction l))\}\leqslant\lambda$ hence $|J_\alpha|=\lambda$.

It is also clear (by the identification after the definition of J_α) that $I_\alpha\subseteq J_\alpha$, and looking more carefully that $I_\alpha\subseteq_{\mathrm{c}} J_\alpha$.

Proof of (b). We define a function h_α with domain $P^{J_\alpha}_\omega-P^{I_\alpha}_\omega$. Now, if $\eta\in P^{J_\alpha}_\omega-P^{I_\alpha}_\omega$, then $\gamma_\eta<\lambda$ (see (iv) in the definition of J_α).

Defined $h_\alpha(\eta)=\eta\restriction(m_\eta+1)$. By the previous sentence for every η_0, $|\{\eta:h_\alpha(\eta)=h_\alpha(\eta_0)\}|\leqslant|J_{\alpha,\gamma_{\eta_0}}|<\lambda$. Hence by 2.8, $S(I_\alpha)=S(J_\alpha)\ (\mathrm{mod}\ D_\lambda)$.

Proof of (c). Let $\alpha<\beta<\lambda$. We define below a family $\mathscr{F}_{\alpha,\beta}$ of partial isomorphisms from J_α into J_β:

$f\in\mathscr{F}_{\alpha,\beta}$ iff (a) f is a partial isomorphism from J_α into J_β, the $<$-minimal element of J_α in its domain,
(b) if $\eta\in\mathrm{Dom}(f)$, $l\leqslant\lg(\eta)$, then $\eta\restriction l\in\mathrm{Dom}(f)$,
(c) if $\eta\in\mathrm{Dom}(f)$, $l<\lg(\eta)$, then $\mathrm{Suc}_{J_\alpha}(\eta\restriction l)\subseteq\mathrm{Dom}(f)$,

(d) $J_\alpha \restriction (\mathrm{Dom} f) \subseteq_c J_\alpha$,
(e) like (b), (c), (d) replacing α, f by β, f^{-1}.

Now, by 1.4, $\mathscr{F}_{\alpha,\beta}$ exemplify that $J_\alpha \equiv_{\infty,\lambda} J_\beta$.

It is nonempty as the function f, $\mathrm{Dom}(f) = P_0^{J_\alpha}$, $\mathrm{Rang}(f) = P_0^{J_\beta}$ (it exists and is unique) belongs to it. Also condition (a) of Theorem 1.4(1) is satisfied. The proofs of conditions (b) and (c) are in fact identical, so we shall prove (b) only.

Let $f \in \mathscr{F}_{\alpha,\beta}$, $A \subseteq J_\alpha$, be such that $|A| < \lambda$, W.l.o.g. A is closed under initial segments and *if* $\nu \in J_\alpha - P_\omega^{J_\alpha}$, and $\{\nu(l) : l < \lg(\nu)\} \subseteq \{\rho(l) : \rho \in A, l < \lg(\rho)\}$, *then* $\nu \in A$. W.l.o.g. A is closed, weakening "$|A| < \lambda$" to "$|A - P_\omega^{J_\alpha}| < \lambda$".

As λ is regular, by the choice of $\langle I_{\alpha,\xi} : \xi < \lambda\rangle$ for some $\gamma < \lambda$:

$$A \cap I_\alpha \subseteq I_{\alpha,\gamma}.$$

Let

$$B \stackrel{\text{def}}{=} \{\eta \restriction l : \eta \in A,\ \eta \notin \mathrm{Dom}(f),\ \eta \restriction l \in \mathrm{Dom}(f),\ \eta \restriction (l+1) \notin \mathrm{Dom}(f)\}$$

and for $\eta \in B$, let

$$A_\eta \stackrel{\text{def}}{=} \{\nu : \nu \in A - \mathrm{Dom}(f),\ (\exists l)\nu \restriction l = \eta\} \cup \{\eta\}.$$

We now define by induction on $k < \omega$, $A_\eta^k \subseteq A_\eta$, increasing in k, such that $[\nu \in A_\eta^k,\ \nu \restriction l \in A_\eta \Rightarrow \nu \restriction l \in A_\eta^k]$.

$$\begin{aligned}
A_\eta^0 &= \{\eta\},\\
A_\eta^{2k+1} &= A_\eta^{2k} \cup \{\nu \in A_\eta : \text{for some } \nu' \in I_\alpha \text{ and } l,\ \lg(\nu) = \lg(\nu'),\\
&\qquad \nu \restriction l \in A_\eta^{2k},\ \nu \restriction (l+1) \notin A_\eta^{2k},\\
&\qquad (\forall m)[l \le m < \lg(\nu) - 1 \rightarrow \nu(l) = \nu'(l)]\},\\
A_\eta^{2k+2} &= A_\eta^{2k+1} \cup \{\nu \in A_\eta : \text{for some } l < \lg(\nu),\ \nu \restriction l \in A_\eta^{2k+1}, \text{ and}\\
&\qquad (\forall m)[l \le m < \lg(\nu) - 1 \rightarrow H_2(\eta(l)) < \lambda]\}.
\end{aligned}$$

Let $A^k = (\mathrm{Dom} f) \cup \{A_\eta^k : \eta \in B\}$. Note that each A^k satisfies:
(α) if $\eta \in A^k$, $l < \lg(\eta)$, then $\mathrm{Suc}_{J_\alpha}(\eta \restriction l) \subseteq A^k$,
(β) $A^k \subseteq_c J_\alpha$,
(γ) $\bigcup_{k<\omega} A^k \subseteq_c J_\alpha$.

We now define by induction on $k < \omega$, f_k such that
(δ) $f_k \in \mathscr{F}_{\alpha,\beta}$, $\mathrm{Dom}(f_k) = A^k$,
(ε) $f_0 = f$, $f_k \subseteq f_{k+1}$.

Suppose f_k is defined. Let

$$B_k \stackrel{\text{def}}{=} \{\eta \in A^k : \mathrm{Suc}_{J_\alpha}(\eta) \cap A^k = \emptyset\}$$

and for $\eta \in B_k$, let

$$A_\eta^* = \{\nu \in A_{k+1} : \eta \trianglelefteq \nu\}.$$

It is enough to define $f_{k+1} \restriction A_\eta^*$ for each $\eta \in B_k$.

Case 1: *k is odd.* We define for $\nu \in A_\eta^*$

$$f_{k+1}(\nu) = f_k(\eta) ^\wedge \langle \nu(m) : \lg(\eta) \leq m < \lg(\nu) \rangle.$$

Case 2: *k is even.* Let

$$A_\eta^{**} = \{\eta\} \cup \{\nu : \nu \in A_\eta^*, \text{ and } \mathrm{Suc}_{J_\alpha}(\nu) \cap A_\eta^* \neq \emptyset\}.$$

Note that $[\nu \in A_\eta^{**}, \lg(\eta) \leq l < \lg(\nu) \Rightarrow \nu \restriction l \in A_\eta^{**}]$. Let $g = g_{c^{[\alpha]}, c_\gamma^{[\alpha]}}$ be as in (*), (A), (B) above. We define $f_{k+1} \restriction A_\eta^{**}$ by

$$f_{k+1}(\nu) = f_k(\eta) ^\wedge \langle \langle g(H_0(\nu(l))), H_4(\nu(l)) \rangle : \lg(\eta) \leq l < \lg(\nu) \rangle.$$

We leave the inspection that $f_{k+1} \in \mathscr{F}_{\alpha,\beta}$ and that $\bigcup_{k<\omega} f_k \in \mathscr{F}_{\alpha,\beta}$ to the reader.

Proof of (d). Assume each I_α is locally $(<\lambda, \mathrm{bs})$-stable.

As M_0 is $(<\lambda, \mathrm{bs})$-stable, (by (c)) and for every $\alpha < \lambda$, $\eta \in I_\alpha$, $(\mathrm{Suc}_{I_\alpha}(\eta), <)$ is $(<\lambda, \mathrm{bs})$-stable (by the previous sentence), clearly M is (λ, bs)-stable. But for $\eta \in J_\alpha$, $\alpha < \lambda$, $(\mathrm{Suc}_{J_\alpha}(\eta), <) \cong M$ hence (for $\alpha < \lambda$) J_α is locally $(<\lambda, \mathrm{bs})$-stable.

Next, suppose each I_α is $(<\lambda, \mathrm{bs})$-stable. By the previous paragraph J_α is locally $(<\lambda, \mathrm{bs})$-stable. Let $\alpha < \lambda$, $A \subseteq J_\alpha$, $|A| < \lambda$ and $m < \omega$, and we want to show that $|S_{\mathrm{bs}}^m(A, M)| < \lambda$. Clearly $|S_{\mathrm{bs}}^m(A, M)| \leq |S_{\mathrm{bs}}^1(A, M)|^m + \aleph_0$, so without loss of generality $m = 1$.

As we can increase A (as long as $A \subseteq J_\alpha \wedge |A| < \lambda$) without loss of generality $(\forall \eta \in A)[\bigwedge_{l<\lg(\eta)} \eta \restriction l \in A]$, and

(*) $\quad [\nu \in \mathrm{Dom}\, h_\alpha \cap A, h_\alpha(\eta) = \nu \Rightarrow \eta \in A]$.

Now I_α is $(<\lambda, \mathrm{bs})$-stable, hence

$$\begin{aligned} |\{\mathrm{tp}_{\mathrm{bs}}(b, A, J_\alpha) : b \in I_\alpha\}| &\leq |\{\mathrm{tp}_{\mathrm{bs}}(b, A \cap I_\alpha, J_\alpha) : b \in I_\alpha\}| \\ &\leq |\{\mathrm{tp}_{\mathrm{bs}}(b, A \cap I_\alpha, I_\alpha) : b \in I_\alpha\}| < \lambda. \end{aligned}$$

On the other hand by (*)

$$|\{\mathrm{tp}_{\mathrm{bs}}(b, A, J_\alpha) : b \in J_\alpha - I_\alpha\}| \leq |A| + \aleph_0 < \lambda.$$

Together we get J_α $(<\lambda, \mathrm{bs})$-stable.

Proof of (e). Suppose I_α is locally $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice. So for $\alpha < \lambda$, $\eta \in (I_\alpha)^{\mathrm{fin}}$, $(\mathrm{Suc}_{I_\alpha}(\eta), <)$ is $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice, as also M_0 is $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice (by (b) clearly M is $(\lambda, \mathrm{bs}, \mathrm{bs})$-stable). So J_α is locally $(\lambda, \mathrm{bs}, \mathrm{bs})$-stable.

2.10. Theorem. *There are 2^λ pairwise non-isomorphic $L_{\infty,\lambda}$-equivalent models of the form* $\mathrm{EM}(I, \Phi)$, $(I \in K_{\mathrm{tr}}^\omega)$. *In fact, we can get that no one is embeddable into another by an embedding preserving ϕ_n, $\neg\phi_n$.*

2.10A. Remark. (1) In fact we have expansions which are pairwise $L_{\infty,\lambda}$-equivalent,

(2) Remember, we are assuming 2.1.

Proof. Let $S_\alpha \subseteq \{\delta < \lambda, \operatorname{cf} \delta = \aleph_0\}$ be pairwise disjoint stationary sets. For $\delta \in \bigcup_{\alpha<\lambda} S_\alpha$, let η_δ be an increasing ω-sequence converging to δ. Let $I_\alpha = ({}^{\omega>}\lambda \cup \{\eta_\delta : \delta \in S_\alpha\})$. Clearly $S(I_\alpha) = S_\alpha/D_\lambda$, $I_\alpha \in K^\omega_{\mathrm{tr}}$, I_α is $(<\lambda, \mathrm{bs})$-stable and I_α is locally $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice.

Apply the previous lemma and get J_α $(\alpha < \lambda)$ such that $J_\alpha \in K^\omega_{\mathrm{tr}}$, $|J_\alpha| = \lambda$; the J_α are pairwise $L_{\infty,\lambda}$-equivalent, $S(J_\alpha) = S(I_\alpha) = S_\alpha/D_\lambda$, each J_α is $(<\lambda, \mathrm{bs})$-stable and locally $(\lambda, \mathrm{bs}, \mathrm{bs})$-nice. So clearly the J_α's are pairwise non-isomorphic. By 1.5 the models $\mathrm{EM}(\Phi, I_\alpha)$ (for $\alpha < \lambda$) are pairwise $L_{\infty,\lambda}$-equivalent and by 2.4 they are pairwise non-isomorphic. But we want 2^λ such models not only λ. So without loss of generality each J_α is standard.

For any set $A \subseteq \lambda$, $|A| = \lambda$, let us define J_A, a standard member of K^ω_{tr}. Its set of elements: $\{\langle\ \rangle\} \cup \{\langle \alpha \rangle {}^\frown \eta,\ \eta \in J_\alpha,\ \alpha \in A\}$. The models J_A $(A \subseteq \lambda, |A| = \lambda)$ are in K^ω_{tr} of power λ. They are pairwise $L_{\infty,\lambda}$-equivalent by the Feferman–Vaught theorem. Clearly $S(J_A)$ is the union in $\mathcal{P}(\lambda)/D_\lambda$ of S_α/D_λ $(\alpha \in A)$. So if $\gamma \in A$, $\gamma \notin B$, then $S_\gamma \cap S(J_B) = \emptyset/D_\lambda$ but $S_\gamma/D_\lambda \subseteq S(J_A)$. As S_γ is stationary $S(J_A) \neq S(J_B)$.

Remark. We made no use of $\lambda = \lambda^{<\lambda}$ though in Theorem 1.4 we speak about 'for every $A \subset J_\alpha$, $|A| < \lambda$', as there is a 'cover' of power λ, i.e., 'of small power' was replaced by 'bounded'.

For singular cardinals the situation is more complicated.

2.11. Discussion. There are, of course, various alternatives to the invariants defined in 2.5(2). Let for simplicity $\lambda > \aleph_1 + |T_1|$ be regular. Let $\Delta = \{\phi_n(\bar{x}, \bar{y}_n) : n < \omega\}$, $m(0) = \lg(\bar{x})$ be as in 1.3.

2.11A. Definition. Let for a λ-representation $\bar{M}$,

$$F^m_{\mathrm{cs},\Delta}(\bar{M}) = \{\delta < \lambda : \operatorname{cf} \delta = \aleph_0, \text{ and for every } \bar{a} \in {}^m|M|, \text{ every countable subset of } \mathrm{tp}_\Delta(\bar{a}, M_\delta) \text{ is realized in } M_\delta\}.$$

Clearly

2.11B. Fact. *$F^m_{\mathrm{cs},\Delta}$ is a D_λ-invariant.*

We would like to have

$$(*) \qquad F^{m(0)}_{\mathrm{cs},\Delta)}(\mathrm{EM}(I, \Phi) = S(I)$$

and even

$$(**) \qquad \bigcup_m F^m_{\mathrm{cs},L}((\mathrm{EM}(I, \Phi)) = S(I).$$

2.11C. Fact. *For $(*)$ and $(**)$ to hold, it is enough to demand*:

(a) $|I| = \lambda$, *I is* $(<\lambda, \mathrm{bs})$-*stable.*

(b) *For every $\eta \in P^I_n$, $n < \omega$, there is $\nu \in P^I_\omega$, $\eta < \nu$.*

(c) *Every interval of* $((\mathrm{Suc}_I(\eta), <)$ *is uncountable.*
(d) $T = \mathrm{Th}(\mathrm{EM}(I, \Phi))$ *is stable.*

For understanding the proof in the direction $((a)+(b)+(c)=(d)) \Rightarrow (**))$, see [5, VII 3.2].

2.11D. Conclusion. If $\lambda > \aleph_1 + |T_1|$ *is regular,* $T \subseteq T_1$, T *(complete) stable but not superstable, then* $\{F^1_{\mathrm{cs},L}(M) : M \in \mathrm{PC}(T_1, T)\}$ *is* $\mathcal{P}(\lambda)/D_\lambda$.

2.11E. Fact. *In* 2.11C *if we are satisfied with* $(*)$ *only, we can weaken* (d) *to*:
(d)′ *For* $n < \omega$, $J \in K^\omega_{\mathrm{tr}}$, $M = \mathrm{EM}(J, \Phi)$ *there is no* A, $S^m_{\{\phi_n(\bar{x},\bar{y})\}}(A, M) > (|A| + |\Phi|)$. *Equivalently*:
(d)′ *For* $n < \omega$, $J \in K^\omega_{\mathrm{tr}}$ *there are no* $\bar{a}_\alpha$, $\bar{b}_\alpha$ $(\alpha < |\Phi|^+)$ *such that* $\mathrm{EM}(I, \Phi) \vDash \phi_\eta[\bar{a}_\alpha, \bar{b}_\beta]$ *iff* $\alpha > \beta$.

2.11F. Remark. Now 2.11D is interesting as, if (d)′ fails, then for some Φ^1 proper for K_{or}

$$\{\mathrm{EM}^1(I, \Phi^1) : J \in K_{\mathrm{or}}\} \subseteq \{\mathrm{EM}^1(I, \Phi) : I \in K^\omega_{\mathrm{tr}}\}$$

and for $t_1, t_2 \in J \in K_{\mathrm{or}}$

$$\mathrm{EM}(J, \Phi^1) \vDash \phi[\bar{a}_{t_1}, \bar{a}_{t_2}] \quad \text{iff} \quad J \vDash t_1 < t_2$$

where $\phi(x_1 y_1, \bar{x}_2 \bar{y}_2) = \phi_n(\bar{x}_1, \bar{y}_2)$ for some n. Now for $\mathrm{EM}(J, \Phi^1)$ we can apply [5, VIII 3.1] (its proof, more exactly) to get a reasonable invariant.

2.11G. Definition. For a λ-representation $\bar{M} = \langle M_i : i < \lambda \rangle$ of an L-Model M, Δ a set of L-formulas, $m < \omega$ let

$$\begin{aligned} F^m_{\mathrm{dc},\Delta}(\bar{M}) = \{\delta < \lambda : & \text{if for } \alpha < \omega_1,\ \bar{a}_\alpha \in {}^m M \text{ and } p = \mathrm{Av}_\Delta(\langle \bar{a}_\alpha, \alpha < \omega_1 \rangle, \\ & M_\delta, M) \text{ is a complete } (\Delta, m)\text{-type, then} \\ & p \restriction M_0 \cup \textstyle\bigcup_{\alpha<\omega_1} \bar{a}_\alpha \text{ is realized in } M\}. \end{aligned}$$

2.11H. Fact. *For a* $\lambda > |T_1|$ *regular,* $T \subseteq T_1$, T *unstable*

$$\left\{ \bigcap_{\phi,m} F^m_{\mathrm{dc},\{\phi\}}(M) : M \in \mathrm{PC}(T_1, T),\ \|M\| = \lambda \right\}$$

is $\mathcal{P}(\lambda)/D_\lambda$ *(we can for* $\lambda > |T_1|^+$ *use* $F^1_{\mathrm{dc},L}$*).*

3. The strong limit case

3.1. Theorem. *Let* λ *be a strong limit singular cardinal of uncountable cofinality. There are* $I_\alpha \in K^\omega_{\mathrm{tr}}$ $(\alpha < 2^\lambda)$ *such that*
(1) $|I_\alpha| = \lambda$.

(2) $I_\alpha \equiv_{\infty,\lambda} I_\beta$.

(3) *If* $L \subseteq L^1$, ϕ_n $(n<\omega)$, Φ *are as in the conclusion of* 1.3, *then the models* $\mathrm{EM}(I_\alpha, \Phi)$, $\alpha < 2^\lambda$, *are pairwise non-isomorphic.*

Remark. We can get "no one embeddable into another (by an embedding preserving ϕ_n, $\neg\phi_n$)" if we act as in [6, Theorem 2.6(1), case B].

Proof. Let $\kappa = \mathrm{cf}(\lambda)$, $\lambda = \sum_{i<\kappa} \lambda_i$, λ_i increasing continuous, $\kappa < \lambda_i$, $2^{\lambda_i} < \lambda_{i+1}$. Let $\lambda_\kappa = \lambda$. We first show

3.2. Claim. *There is a linear order M, such that*

(a) $\|M\| = \lambda$.

(b) *For every* $A \subseteq M$, $|A| < \lambda$, $\mu < \lambda$, *there are automorphisms* f_α $(\alpha < \mu)$ *of M such that* $f''_\alpha(A) \cap f''_\beta(A) = \emptyset$ *for* $\alpha \neq \beta$.

(c) *For* $\mu < \lambda$, *in the set* $\{A : A \subseteq M, |A| \leqslant \mu\}$ *there are* $\leqslant 2^{\mu+\kappa}$*-equivalence classes for E_M where AE_MB iff some automorphism f of M,* $f(A) = B$.

Proof. Let for each $i < \kappa$, N_i be a dense strongly λ_i^+-homogeneous linear order (i.e., if $A, B \subseteq N_i$, $|A|$, $|B| \leqslant \lambda_i$, f an isomorphism from $M_i \restriction A$ onto $N_i \restriction B$, then f can be extended to an automorphism of N_i). We further assume $\|N_i\| = 2^{\lambda_i}$.

Choose $a_i \in N_i$. We shall now define M: its set of elements is

$$\left\{f : f \in \prod_{i<\kappa} N_i, \text{ and for all but finitely many } i,\ f(i) = a_i\right\}.$$

The order is the lexicographic order $f < g \Leftrightarrow (\exists i)[f(i) < g(i) \wedge f \restriction i = g \restriction i]$.

It is easy to check that $|M| \leqslant \sum \{\prod_{i\in w} \lambda_i : \text{a finite subset of } \kappa\} \leqslant \lambda \leqslant \sum_{i<\kappa} \lambda_i \leqslant \|M\| = \lambda$, i.e., (a) holds.

Let us prove (b). Choose i such that $|A| + \mu < \lambda_i$, define $B = \{\eta(i) : \eta \in A\}$. So B is a subset of N_i. It is easy to find automorphisms g_α of N_i for $i < \mu$ such that $\langle g_\alpha(B) : \alpha < \mu \rangle$ are pairwise disjoint (e.g., as we can prove N_i is λ_i^+-saturated and $|B| \leqslant |A| < \lambda_i$).

Now we define the automorphisms f_α of M: for $\eta \in M$, $f_\alpha(\eta)$ is a function with domain κ, defined by

$$f_\alpha(\eta)(j) = \begin{cases} \eta(j) & \text{if } j \neq i, \\ g_\alpha(\eta(j)) & \text{if } j = 1. \end{cases}$$

It is easy to check that f_α is an automorphism of M, and $f_\alpha(A)$ $(\alpha < \mu)$ are pairwise disjoint.

We are left with (c). Again choose $i < \kappa$ such that $\mu + |A| < \lambda_i$. For every $A \subseteq M$ define:

$$A^{(i)} = \{\eta(i) : \eta \in A\} \qquad \text{(which is } \subseteq N_i).$$

It is easy to see that:

($*$) If $A_1, A_2 \subseteq M$, f an order preserving function from A_1 onto A_2, and for each $i < \kappa$ there is an automorphism h_j of N_j, $h_j(a_j) = a_j$ and $(\forall \eta \in A_1)[h_j(\eta(j)) = (f(\eta))(j)]$, then we can extend f to an automorphism of M.

The extension f' is simple

$$(f'(\eta))(i) = h_i(\eta).$$

From ($*$) (c) is clear (as each N_i is strongly λ_i^+-homogeneous, $\|N_i\| = 2^{\lambda_i}$).

3.3. Claim. *For M from* 3.2 *we can find M_i $(i < \kappa)$, M_i increasing continuous, for i non-limit $\|M_i\| = 2^{\lambda_{i+1}}$, and if $A_\alpha \subseteq M$ for $\alpha < \lambda_i$, $|A_\alpha| \leqslant \lambda_{i+1}$, then for some automorphisms f_α of M, $f_\alpha(A_\alpha) \subseteq M_{i+1}$,*

$$[\alpha \neq \beta \Rightarrow f_\alpha(A_\alpha) \cap f_\beta(A_\beta) = \emptyset].$$

Proof. Immediate.

3.4. Definition. $I \in K_{\mathrm{tr}}^\omega$ is called *cl-special* if:

(a) $I = \bigcup_{i<\kappa} I_i$, I_i increasing continuous, $|I_i| \leqslant 2^{\lambda_{i+1}}$, and $I_i \subseteq_{\mathrm{c}} I$.

(b) For every $\eta \in P_\omega^I - P_\omega^I$, $(\mathrm{Suc}_I(\eta), < \restriction \mathrm{Suc}_I(\eta))$ is isomorphic to M (from Claim 3.2) and let f_η^I be an isomorphism from M onto $(\mathrm{Suc}_I(\eta), < \restriction \mathrm{Suc}_I(\eta))$.

(c) If $\eta \in I_j$, then f_η^I maps M_j onto $\mathrm{Suc}_I(\eta) \cap I_j$ (M_j from 3.3).

(d) If $\eta \in I_i - \bigcup_{j<i} I_j$, i successor, $J \subseteq \{\nu : \nu$ a sequence of length $\leqslant\omega$, for $l < \mathrm{Min}(\{\lg(\eta), \lg(\nu)\}$, $\nu(l) = \eta(l)$ and for $\lg(\eta) \leqslant l < \lg(\nu)$, $\nu(l) \in M\}$, J closed under initial segments and has power $\leqslant\lambda_i$, then there is a function $g = g_\eta^J$ from J into I_i, such that

(i) $\lg(g(\nu)) = \lg(\nu)$ for $\nu \in J$,

(ii) g preserve $<$,

(iii) the range of g is a closed subset of I_i,

(iv) $g(\eta) = \eta$,

(v) if $\eta < \nu \in J$, $\lg(\nu) < \omega$, then for some automorphism h of M, for every $\nu \hat{} \langle c \rangle \in J$ (so $c \in M$) $(g(\nu \hat{} \langle c \rangle) = f_{g(\nu)}^I(h(c))$.

3.5. Claim. *If I^0, I^1 are cl-special of power λ, then $I^0 \equiv_{\infty,\lambda} I^1$.*

Proof. Let $I^l = \bigcup_{i<\kappa} I_i^l$ as in the definition. Let $\mathscr{F}$ be the set of functions f such that:

(1) $I^0 \restriction \mathrm{Dom} f \subseteq_{\mathrm{c}} I^0$ (hence $\nu < \eta \in \mathrm{Dom} f \Rightarrow \nu \in \mathrm{Dom} f$).

(2) $I^1 \restriction \mathrm{Rang} f \subseteq_{\mathrm{c}} I^1$.

(3) $f : I^0 \to I^1$ is a partial isomorphism.

(4) If $\eta \in P_n^{I^0}$ and $(\exists \nu \in \mathrm{Dom} f)[\eta < \nu]$, then $\{\nu \in P_{n+1}^{I^0} : \eta < \nu\} \subseteq \mathrm{Dom} f$.

(5) If $\eta \in P_n^{I^1}$, $(\exists \nu \in \mathrm{Rang} f)[\eta < \nu]$, then $\{\nu \in P_{\nu+1}^{I^1} : \eta < \nu\} \subseteq \mathrm{Rang} f$.

(6) The power of $\{\eta : \eta \in \mathrm{Dom} f$, not $<$-maximal in $\mathrm{Dom} f\}$ is $<\lambda$.

Why is $\mathscr{F}$ as required? By the definition of cl-special.

3.6. Construction. We define by induction on $i \leqslant \kappa$, for every function A from λ_i to $\{0, 1\}$ (so really it is a set) a model $I_A \in K^{\omega}_{\mathrm{tr}}$ and functions f^A_η $(\eta \in I_A - P^{I_A}_\omega)$ such that:

(1) I_A has power $\leqslant 2^{\lambda_i}$ (if $A:\lambda_i \to \{0, 1\}$).

(2) For $j<i$, $I_{A \restriction \lambda_j} \subseteq_c I_A$.

(3) f^A_η is an isomorphism from M_i onto $(\mathrm{Suc}_{I_A}(\eta), < \restriction \mathrm{Suc}_{I_A}(\eta))$.

(4) For $j<i$, $f^{A\cap\lambda_j}_\eta \subseteq f^A_\eta$ (if $\eta \in I_{A\cap\lambda_j} - P^{I_{A\cap\lambda_j}}_\omega$, of course).

(5) For i successor, condition (d) of Definition 3.4 is satisfied.

(6) If i is limit of uncountable cofinality, then (when $A:\lambda_i \to \{0, 1\}$)

$$I_A = \bigcup_{j<i} I_{A \restriction \lambda_j}.$$

(7) If i is limit $(A \in {}^{\lambda_i}2)$, then $f^A_\eta = \bigcup \{f^{A\cap\lambda_j}_\eta : j<i,\ \eta \in I_{A\restriction\lambda_j}\}$.

(8) If $i=\delta$ with $\mathrm{cf}(\delta)=\aleph_0$, then: for every Φ, $L \subseteq L^1$, $\phi_n(x, \bar{y}_n)$ $(n<\omega)$ as in 1.3, $|L^1|<\lambda_i$; and $A,B:\lambda_i \to 2$, $A \neq B$ and a subtree J of I_A with splitting in the n-th level being λ^n, $\sum_{n<\omega}\lambda^n=\lambda_i$ and a function F from $\bigcup\{\bar{a}_\eta : \eta \in J\}$ into $\mathrm{EM}(I_B, \Phi)$ then there is an ω-branch η of J such that: $\{\phi_n(x, a_{\eta\restriction n}) : n<\omega\}$ is realized in $\mathrm{EM}(J_A, \Phi)$ iff $\{\phi_n(x, F(\bar{a}_{\eta\restriction n})), n<\omega\}$ is not realized in $\mathrm{EM}(I_B, \Phi)$.

3.7. Why does this guarantee non-isomorphism? Suppose $A,B:\lambda \to 2$, $A \neq B F:\mathrm{EM}(I_A, \Phi) \to \mathrm{EM}(I_B, \Phi)$. Let $h(\eta)=\mathrm{Min}\{i<\kappa : F(\bar{a}_\eta) \in \mathrm{EM}(J_{\beta\cap\lambda_i}, \Phi)\}$. The result follows by [6, 2.4].

Theorem. *If $\lambda=\sum_{i<\kappa}\lambda_i$, $\kappa=\mathrm{cf}\,\lambda<\lambda_i<\lambda_j<\lambda$ for $i<\lambda$, λ_i increasing continuous, $h:{}^{\omega>}\lambda \to \kappa$, then for a club of $\delta<\kappa$, if $\mathrm{cf}\,\delta=\aleph_0$ there are $i_n<\delta$, $\delta=\bigcup_{n<\omega} i_n$, $J \subseteq \bigcup_{m<\omega}\prod_{n<m}\lambda_{i_n}$, the splitting of J in the $(n+1)$th level is λ_{i_n} and $h(\eta)<\delta$ for $\eta \in J$.*

3.8. How to do the construction for demand (8). We list all possible A, B, ϕ_n, Φ, F by a list of length $2^{\lambda_\delta}=\lambda_\delta^{\aleph_0}$. We define by induction on $\xi<\lambda_\delta^{\aleph_0}$ a set Γ_ξ of obligations of power $\leqslant|\xi|+\aleph_0$ each of the form:

For some $A:\lambda_\delta \to 2$ and ω-branch $\eta=\langle \eta_l : l<\omega\rangle$ of $\bigcup_{j<\delta} I_{A\restriction j}$ (i.e., $\eta_l \in P_l$) $\langle \eta_l : l<\omega\rangle$ has [or does not have] a $<$-upper bound in I_A where $(\forall j<\delta)$ $(\exists l)(\eta_l \notin I_{A\restriction j})$.

The definition of $\Gamma_{\xi+1}$ takes care of (8) for A^ξ, B^ξ, L^ξ, ϕ^ξ_n, $(L^1)^\xi$, Φ^ξ, F^ξ, J^ξ, and J^ξ has $\lambda_\delta^{\aleph_0}$ ω-branches, so one of them $\langle \eta^l_\xi : l<\omega\rangle$ was not mentioned in Γ^ξ and $(\forall j<\delta)(\exists l)[\eta^l_\xi \notin J_{A\restriction\lambda_j}]$.

If for some definition of I_{B^ξ} compatible with Γ^ξ, $\{\phi_l(x, F(a_{\eta^l_\xi})) : l<\omega\}$ is realized, by adding finitely many positive obligations of I_{B^ξ} we can guarantee this, and by adding "$\langle \eta^l_\xi : l<\omega\rangle$ has no $<$-bound in I_{A^ξ}" we guarantee "$\{\phi_l(x, a_{\eta^l_\xi}) : l<\omega\}$ is not realized in $\mathrm{EM}(I_{A^\xi}, \Phi)$".

If there is no such J_B, let $\Gamma_{\xi+1}=\Gamma_\xi \cup \{\langle \eta^l_\xi : l<\omega\rangle$ has a $<$-bound in $I_{A_\xi}\}$. This guarantees (8).

3.9. *There are no problems in other definitions.*

Appendix: on unique linear orders

A1. Context. λ, μ_1, μ_2 are cardinals such that $\lambda^+ = \text{Max}\{\mu_1, \mu_2\}$. For $l = 1, 2$, f_l is a function from $\text{Reg}(\mu_l) = \{\kappa : \kappa < \mu_l$ a regular (infinite cardinal)$\}$ into λ, g_l is a function from λ onto $\text{Reg}(\mu_l)$ such that $g_l(f_l(\kappa)) = \kappa$ for $\kappa \in \text{Reg}(\mu_l)$ and $[\alpha \neq \beta, g_l(\alpha) = g_l(\beta) \Rightarrow g_l(\alpha) = \aleph_0]$.

Let for a linear order $M = (A, <)$, $M^* = (A, >)$, i.e., its inverse.

A2. Definition. $K = K(\lambda, \mu_1, \mu_2, f_1, f_2, g_1, g_2)$ is the family of models $(M, P_\alpha)_{\alpha<\lambda}$ such that:

(i) M is a linear order.

(ii) M is the union of $\aleph_0$ scattered subsets.

(iii) Each P_α is a dense subset of M.

(iv) $\langle P_\alpha : \alpha < \lambda \rangle$ is a partition of M.

(v) Every increasing sequence in M has length $<\mu_1$, but in every open interval there are increasing sequences of any length $<\mu_1$.

(vi) Every decreasing sequence in M has length $<\mu_2$ but in every open interval there are decreasing sequences of any length $<\mu_2$.

(vii) If $\langle a_i : i < \kappa \rangle$ is an increasing bounded sequence in M, $\aleph_0 < \kappa \in \text{Reg}(\mu_1)$, then for some club C of κ, for $\delta \in C \cup \{\kappa\}$, $\{a_i : i < \delta\}$ has a least upper bound which belongs to $P_{f_1(\text{cf}\,\delta)}$.

(viii) If $\langle a_i : i < \kappa \rangle$ is a decreasing bounded sequence in M, $\aleph_0 < \kappa \in \text{Reg}(\mu_2)$, then for some club C of κ, for $\delta \in C \cup \{\kappa\}$, $\{a_i : i < \delta\}$ has a least upper bound which belongs to $P_{f_2(\text{cf}\,\delta)}$.

(ix) If $x \in P_\alpha$, then $\text{cf}(\{y \in M : y < x\}, <) = g_1(\alpha)$ and $\text{cf}(\{y \in M : y > x\}, >) = g_2(\alpha)$.

A2′. Definition. For χ_1, χ_2 (infinite) cardinals $\leq\lambda$, $K_{\chi_1,\chi_2} = K(\lambda, \mu_1, \mu_1, f_1, f_2, g_1, g_2, \chi_1, \chi_2)$ is the family of $(M, P_\alpha)_{\alpha<\lambda} \in K$, $\text{cf}(M) = \chi_1$, $\text{cf}(M^*) = \chi_2$.

A3. Claim. *For every regular* $\chi_l < \mu_l$, $K_{\chi_1,\chi_1} \neq \emptyset$.

Proof. We define by induction on $n < \omega$, $(M^n, P^n_\alpha)_{\alpha<\lambda}$ such that:

(i) $(M^n, P^n)_{\alpha<\lambda}$ is a submodel of $(M^{n+1}, P^{n+1}_\alpha)_{\alpha<\lambda}$.

(ii) M^n is scattered, and every interval contains a jump.

(iii) $(M^n, P^n_\alpha)_{\alpha<\lambda}$ satisfies from Definition 2: (i), (ii), (iv), first half of (v), first half of (vi), and (vii), (viii).

(iv) If $x \in P^n_\alpha$ has no immediate predecessor, then $\text{cf}(M^n \restriction \{y \in M : y < x\}) = g_1(\alpha)$.

(v) If $x \in P^n_\alpha$ has no immediate successor, then $\text{cf}((M^n \restriction \{y \in M : y > x\})^*) = g_2(\alpha)$.

(vi) $\mathrm{cf}(M^n) = \chi_1$, $\mathrm{cf}((M^n)^*) = \chi_2$.

(vii) If $x \in M^{n+1} - M^n$, then for some $y, z \in M^n : y < x < z$, $\neg(\exists t \in M^n)\, y < t < z$.

(viii) For every $y < z$ in $M^n : \bigwedge_\alpha P_\alpha^{n+2} \cap (y, z)^{M^{n+2}} \neq \emptyset$, in $(y, z)^{M^{n+2}}$ there are increasing sequences of any length $<\mu_1$, in $(y, z)^{M^{n+2}}$ there are decreasing sequences of any length $<\mu_2$.

(ix) $\mathrm{cf}(M^0) = \chi_1$, $\mathrm{cf}((M^0)^*) = \chi_2$ (note that $\mu_1 = \lambda^+$ or $\mu_2 = \lambda^+$).

There is no problem in this and $(\bigcup_n M^n, \bigcup_n P_\alpha^n)_{\alpha<\lambda}$ is as required.

Remark. Really $\chi_l \leq \mu_l^+$ is o.k. if in (v), (vi) of A2 we speak about sequences in some interval, and allow $\kappa = \mu_l$ in (vii), (viii). We can complicate f_l and (vii), (viii).

A4. Claim. *Every two members of K_{χ_1,χ_2} are isomorphic.*

Proof. Like [2, 3.3].

A5. Claim. Every *$M \in K$ is* $(\lambda$, bs, bs)*-nice and* $(<\lambda$, bs)*-stable.*

Proof. As in [12, §6, mainly 'crucial fact' of p. 217].

A6. Claim. (1) *If* $(A, <, P_\alpha)_{\alpha<\lambda} \in K$, $S \subseteq \lambda$, $N = (\bigcup_{\alpha\in S} P_\alpha, < \restriction (\bigcup_{\alpha\in S} P_\alpha), P_\alpha)_{\alpha\in S}$, *then N is* $(<\lambda$, bs)*-stable and* $(\lambda$, bs, bs)*-nice.*

(2) *If* $(M, P_\alpha)_{\alpha<\lambda} \in K$, $x, y \in M$, $g_1(x) = g_2(y)$, $g_2(x) = g_2(y)$, *then there is an automorphism F of M,* $F(x) = y$, $(\forall z \in M)[z \neq x \to \bigwedge_\alpha P_\alpha(z) = P_\alpha(F(z))]$.

Proof. Check.

A7. Remark. In A1 we can change f_l as follows: $\mathrm{Dom}\, f_l = \lambda$, $f_l(\alpha)$ is a function from $g_l(\alpha)$ to λ when $g_l(\alpha) > \aleph_0$ (undefined otherwise) such that

$(*)$ if $\alpha, \beta < \lambda$, $\omega \leq \delta < g_l(\beta)$, $[f_l(\beta)](\delta) = \alpha$, then $g_l(\alpha) = \mathrm{cf}(\delta)$; moreover, if in addition $\mathrm{cf}(\delta) > \aleph_0$, then for some increasing continuous $h : g_l(\alpha) \to \delta$,

$$\delta = \sup \operatorname{Rang} h \quad \text{and} \quad \{\gamma < g_l(\alpha) : [f_l(\alpha)](\gamma) = [f_l(\beta)](h(\gamma))\} \in D_{g_l(\alpha)}.$$

Then in Definition A2, we replace (vii) by (vii)′: if $\langle a_i : i < \kappa \rangle$ is an increasing sequence in M, which is bounded, then for some club C of κ, for every $\delta \in C \cup \{\kappa\}$, $\{a_i : i < \delta\}$ has a least upper bound b_δ and for $\delta < \kappa$ it belongs to $P_{[f_l(\kappa)](\delta)}$.

Similarly we can change (viii).

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