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SOUSLIN TREES AND SUCCESSORS OF SINGULAR CARDINALS

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Introduction

The questions concerning existence of Aronszajn and Souslin trees are of the oldest and most dealt-with in modern set theory.

There are many results about existence of λ^+-Aronszajn trees for regular cardinals λ. For these cases the answer is quite complete. (See Jech [6] and Kanamory & Magidor [8] for details.)

The situation is quite different when λ is a singular cardinal. There are very few results of which the most important (if not the only) are Jensen's: $V = L$ implies κ-Aronszajn, κ-Souslin and special κ-Aronszajn trees exist iff κ is not weakly-compact [7]. On the other hand, if GCH holds and there are no λ^+-Souslin trees for a singular λ, then it follows (combining results of Dodd–Jensen, Mitchel and Shelah) that there is an inner model (of ZFC) with many measurable cardinals.

In 1978 Shelah found a crack in this very stubborn problem by showing that if λ is a singular cardinal and κ is a super-compact one s.t. $\operatorname{cof}\lambda < \kappa < \lambda$, then a weak version of $\square_\lambda^*$ fails.[1] The relevance of this result to our problem was found by Donder through a remark of Jensen in [7, pp. 283] stating that if $2^\lambda = \lambda^+$, then $\square_\lambda^*$ is equivalent to the existence of a special Aronszajn tree. Shelah, in [10], had shown that the situation can be collapsed down to $\aleph_{\omega+1}$ so we get Cons(ZFC + there is a super-compact cardinal) implies Cons(ZFC + there are no $\aleph_{\omega+1}$-special Aronszajn trees). In the $\aleph_2$ case (of Silver and Mitchel) the nonexistence consistency result for Aronszajn trees followed the result for special Aronszajn trees and used similar methods in the proof.

A natural hope was that the same scheme will work for the Shelah result. Maybe for a singular λ above a large enough cardinal, there can be no λ^+-Aronszajn trees. If this is true, then maybe by collapsing the large cardinal we could get the consistency of : "there are no $\aleph_{\omega+1}$-Aronszajn trees". In this paper we show that this is not the case.

Theorem 1 shows the consistency with the existence of a super-compact

[1] Ben-David noticed that a strongly compact κ suffices for the result.

cardinal of existence of Souslin trees for any successor of a singular cardinal. The proof is via forcing.

Theorem 2 shows that the situation of Theorem 1 can be 'brought' down to obtain the (rather surprising) consistency of "there are no $\aleph_{\omega+1}$-special Aronszajn trees and yet there are $\aleph_{\omega+1}$-Souslin trees" (relative to Cons(ZFC + $\exists$ super-compact cardinal) of course).

Theorem 3 takes a somewhat different approach; imitating Jensen's construction of Souslin trees in L we define $\Box_{\lambda\mu}$, a weaker version of $\Box_\lambda$, and show it implies the existence of a λ^+-Souslin tree for a strong limit singular λ s.t. $2^\lambda = \lambda^+$. The nice property $\Box_{\lambda\mu}$ is that it is consistent with the existence of a super-compact ordinal between cof(λ) and λ ($\Box_\lambda$ fails there) so we get another proof to Theorem 1.

Theorem 4 deals with a situation in which λ is a limit of super-compact cardinals. Now even $\Box_{\lambda\mu}$ fails and we define another principle $\Box^\mu_\lambda$. Theorem 4 states that "$\Box^\mu_\lambda$ + GCH + μ is a λ-compact cardinal, $\mathrm{cof}(\lambda) < \mu < \lambda$" imply the existence of a λ^+-Aronszajn tree. Baumgartner showed that $\Box^\mu_\lambda$ is consistent with the existence of super-compact cardinal κ s.t. $\mu < \kappa \leqslant \lambda$; we show the consistency of the full assumption of Theorem 4.

Theorem 5 gives a λ^+-Souslin tree under the assumptions of Theorem 4.

As the first author is a student of the second, we feel it is worth a few lines to clarify who is responsible for what in the paper. Theorem 1 was proved by Shelah for Aronszajn trees and that was the trigger for the paper. Ben-David improved Theorem 1 to get Souslin trees. Shelah conjectured Theorem 3 which was, then, proved by Ben-David. Theorem 2 was noticed independently by both authors. Shelah proved Theorem 4, and Theorem 5 is a joint result.

Note added in proof

After the completion of this paper we have shown that for strong limit singular λ, if $2^\lambda = \lambda^+$, then the existence of a λ^+-special Aronszajn tree implies the existence of a (λ^+, ∞)-distributive tree on λ^+. It follows that if there is a λ^+-Aronszajn tree, then necessarily there is a non-special one [1].

On the other hand, we proved that the existence of a λ^+-special Aronszajn tree does not imply the existence of a non-reflecting stationary set of λ^+ (assuming the consistency, with ZFC of super-compact cardinals) [2].

Definitions and notation

(1) A λ^+-Aronszajn tree is a tree of height λ^+ with no λ^+-branch s.t. the power of each of its levels is less than λ^+.

(2) A λ^+-special Aronszajn tree is a λ^+-Aronszajn tree T s.t. there is a function $f:T\to\lambda$ satisfying: for any two nodes $x, y \in T$, $x <_T y \Rightarrow f(x) \neq f(y)$.

(3) A λ^+-Souslin tree is a tree of height λ^+ with no antichain of power λ^+.

Remark. It is well known (and quite easy to see) that a Souslin tree is always an Aronszajn tree and that a special Aronszajn tree is never a Souslin tree.

(4) $\Box_\lambda$ denotes the existence of a sequence $\langle C_\alpha : \alpha < \lambda^+\rangle$ s.t. for any limit ordinal α, C_α is a closed unbounded subset of α s.t.

(i) The order type of C_α is less than α, and

(ii) If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

Remark. It is quite common to have (i) in the definition replaced by the demand $|C_\alpha| \leq \lambda$. It is not hard to prove by induction on $\alpha < \lambda^+$ that the two definitions are equivalent.

(5) Let us fix some notations. (i) For a regular cardinal λ, D_λ denotes the filter generated by the closed unbounded subsets of λ.

(ii) For any tree T and an ordinal α, we let $(T)_\alpha$ denote the α's level of T (= the set of all nodes in T that have height α). We let $l(x)$ denote the height of a node x(= the order type of $\{y : y <_T x\}$),

Results

Theorem 1. *For any regular cardinal κ and any singular λ above it there is a forcing notion P^κ_λ satisfying:*

(i) *P^κ_λ is κ-directedly-complete.*

(ii) *P^κ_λ adds no sets of cardinality $\leq \lambda$ to the universe.*

(iii) *If $2^\lambda = \lambda^+$, then $|P^\kappa_\lambda| = \lambda^+$ so it collapses no cardinals.*

(iv) *Forcing with P^κ_λ produces a model of* ZFC *with a λ^+-Souslin tree.*

Corollaries. (i) *Recalling Laver's super-compact indestructibility theorem, our theorem shows the consistency of Souslin trees on successors of singulars above a super-compact cardinal.*

(ii) *Iterating the forcing we can get a model of* "ZFC + *there is a super-compact cardinal + for every singular λ there is a λ^+-Souslin tree*".

Proof. We define a forcing notion P^κ_λ as follows: any member of P^κ_λ is of the form $\langle T_p, F_p\rangle$ where T_p is a tree of height $h(T_p) < \lambda^+$, $h(T_p)$ a successor ordinal and F_p is a set of functions $\{f^\alpha_x : x \in T_p, l(x) < \alpha < h(T_p)\}$. Each f^α_x is a function from κ to the α's level of T_p s.t. (i) for every $\delta < \kappa$, $x \leq_T f^\alpha_x(\delta)$ and (ii) for every

$\alpha<\beta<h(T_p)$, $\{\delta : f_x^\alpha(\delta) <_T f_x^\beta(\delta)\} \in D_\kappa$. The order of P_λ^κ is: $\langle T_p, F_o\rangle \leqslant \langle T_q, F_q\rangle$ if T_q is an end extension of T_p (as trees) and $F_p \subseteq F_q$.

Lemma. (i) *For every* $\alpha<\lambda^+$, $B_\alpha=\{p\in P_\lambda : h(T_p)>\alpha\}$ *is a dense subset of* P_λ^κ.

(ii) P_λ^κ *is* $<\kappa$*-complete.*

(iii) *Forcing with* P_λ^κ *adds no sets of cardinality less than* λ^+ *to the universe.*

(iv) *Forcing with* P_λ^κ *adds a* λ^+*-Souslin tree.*

Proof of the Lemma. (i) Let p be any member of P_λ^κ and $\alpha<\lambda_1^+$ if $h(T_p)<\alpha$. We can add to each node of the maximal level of T_p a linearly ordered extension of length $\alpha+1$ above it and define a condition of extending p s.t. T_q will be T_p with these extensions.

(ii) Let $(p_i : i<\mu<\kappa)$ be an increasing set of elements of P_λ^κ. Define a condition p by letting $T'_p=\bigcup_{i<\mu} T_{p_i}$ and adding to T'_p a maximal level by adjoining one node x_i for every $\langle x, i\rangle$ s.t. $x\in T'_p$ and $i\in\bigcap_{\alpha<\beta<\mu}\{j : f_x^\alpha(j) < f_x^\beta(j)\}=Ax$, x_i is above $\{f_\alpha^x(i) : \alpha<\mu\}$ (this is a branch in T'_p), $T_p=T'_p\cup\{x_i : i\in A_x,\ x\in T'_p\}$ and $F_p=(\bigcup_{i<\mu} F_{p_i})\cup\{f_\gamma^x\, x\in T'_p\}$ where γ is $f(T'_p)+1$ and $f_\gamma^x(i)=x_i$ (actually $f_\gamma^x(i)$ is defined only for $i\in A$ and A is a club in κ). We can change the indices of κ-many elements of A s.t. $\{x_i : i\in A\}$ will be $\{f_\gamma^x(i) : i<\kappa\}$ and $\{i : x_i=f_\gamma^x(i)\}$ is still a club in κ, so no harm is done. It is easy to check that $P=\langle T_p, F_p\rangle$ is a member of P_λ above each p_i $(i<\mu)$.

(iii) *Definition.* A forcing notion P is strategically-complete if player I has a wining strategy in the following game: Each player in his turn picks a condition p_i from P above all the conditions that where already picked, player I gets all the even moves (including all the limit stages) and player II the odd moves. Player I wins if $\bigcup_{i<\mu} p_i$ has an upper bound in P. A standard argument shows that if P is μ-strategically-complete, then forcing with P adds no μ-sequences to the universe. Let us prove that P_λ^κ is μ-strategically-complete for every $\mu<\lambda^+$. The strategy for player I will be as follows: At a successor stage $i+2$ let $T_{p_{i+1}}$ be the tree of the last condition chosen by player II. Player I will extend $T_{p_{i-1}}$ by one level, for every $x\in T_{p_{i+1}}$ this level will contain κ-many extensions $\langle x^j : j<\kappa\rangle$ s.t. for $x\in T_{p_i}$ for all $j<\kappa$, $f_x^{\delta_i}(j)<x^j$ and whenever $f_x^{\delta_i}(j)<f_x^{\delta_{i+1}}(j)$, then $f_x^{\delta_{i+1}}(j)<x^j$ (where δ_i, δ_{i+1} are the last levels of T_{p_i}, $T_{p_{i+1}}$ respectively).

To complete the definition of p_{i+2} we have to define $F_{p_{i+2}} : F_{p_{i+2}}=F_{p_{i+1}}\cup\{f_x^{\delta_{i+2}} : x\in T_{p_{i+1}}\}$ and the $f_x^{\delta_{i+2}}$ are naturally defined by $f_x^{\delta_{i+2}}(j)=x^j$ for all $x\in T_{p_{i+1}}$, $j<k$. It is easy to check that p_{i+2} is a member of P_λ and that it extends p_{i+1}. We are left with the case i limit, in such a case T_{p_i} will be a one-level-extension of $\bigcup_{\alpha<i} T_{p_\alpha}$. Let δ be $\bigcup_{j<i} h(T_{p_j})$, let A be $\{h(T_{p_j}) : p_j$ was picked by player I$\}$. For each $x\in\bigcup_{j<i} T_{p_j}$, for all $j<\kappa$, $\bigcup_{\alpha\in A} f_x^\alpha(j)$ is a branch. Let $f_x^\delta(j)$ be a one-node extension of that branch and $F_{p_i}=\bigcup_{\alpha<i} F_{p_\alpha}\cup\{f_x^\delta : x\in\bigcup_{\alpha<i} T_{p_\alpha}\}$.

(iv) Let G be generic for P_λ^κ and $T_G=\bigcup_{p\in G} T_p$. By (i), T_G is a tree of height λ^+, each level of T_G is a level of some T_p, $p\in P_\lambda$ so its cardinality is $\leqslant\lambda$ so, once we show that each set of pairwise incomparable members of T_G has cardinality less than λ^+, we will know that T_G is a λ^+-Souslin tree.

Let $A \subseteq T_G$ be a maximal p.i.s. (pairwise incomparable set) in $V[G]$, let $\tilde{A}$ be a name for A and $p^* \in G$ s.t. $p^* \Vdash$ "$\tilde{A}$ is a maximal p.i.s. in T_G", denote $A \cap \alpha = \{x \in A : l(x) < \alpha\}$. We work in V. $\{p \in P^\kappa_\lambda : p \Vdash$ "$\tilde{A} \cap h(T_p) = a$" for some set $a \in V\}$ is dense in P^κ_λ (as P^κ_λ adds no sets of power less than λ^+ to V). We shall restrict ourselves to members of this set.

Let T^- be $\{x \in T : l(x) < h(T)\}$.

Claim (a). $D^-_A = \{p : \Vdash$ "$\tilde{A} \cap h(T_p)$ is a maximal p.i.s. in T^-_p"$\}$ is dense above p^* in P^κ_λ.

Proof. For each $p^* \leqslant q$ construct an increasing sequence $\langle q_i : i < \omega\rangle$ s.t. $q = q_0$ and $q_{i+1} \Vdash$ "each member of T_{q_i} is comparable with a member of $A \cap h(T_{q_{i+1}})$" and let $\bar{q}$ be $\sup\{q_i : i < \kappa\}$ (recall P^κ_λ is $<\kappa$-complete), $\bar{q} \in D^-_A$.

Claim (b). $D_A = \{p : p \Vdash$ "$\tilde{A} \cap (h(T_p) + 1)$ is a maximal p.i.s. in T_p"$\}$ is dense above p^* in P^κ_λ.

Proof. For $p^* \leqslant q$ construct an increasing sequence $\langle q_i : i < \kappa\rangle$ s.t. $q_0 = q$ and for $\delta_i = h(T_{q_i})$, $q_{i+1} \Vdash$ "$\forall x \in T_{q_i}\, \forall j < i$ ($f^{\delta_{i+1}}_x(j)$ is above a member of $\tilde{A} \cap \delta_{i+1}$)". We choose the q_i's by induction on i, at stage $i+1$ pick a member q^*_{i+1} of D^-_A above q_i and change the last level of $T_{q_{i+1}}$ and the $f^{*\delta_{i+1}}_x$ s.t. for all $x \in T_{q_i}$ and $j < i$, $f^{\delta_i}_x(j) < f^{\delta_{i+1}}_x(j)$ and $f^{\delta_{i+1}}_x(i)$ is above a member of $A \cap h(T_{q_{i+1}})$ (as forced by q^*_{i+1}), as for all $x \in T_{q_i}$, $\{j : f^{*\delta_{i+1}}_x(j) \neq f^{\delta_{i+1}}_x(j)\} \equiv \emptyset \pmod{D_\kappa}$. The q_{i+1} obtained this way is a condition above q_i. At limit stages we take sup. Let $\bar{q}$ be the natural one-level extension of $\bigcup_{i<\kappa} q_i$, that is, for each $x \in \bigcup_{i<\kappa} T_{q_i}$, $x^j = \{f^{\delta_i}_x(j) : j < i\}$ is a branch above x cofinal in $\bigcup_{i<\kappa} T_{q_i}$ and we let $f^{\delta_{\bar{q}}}_x(j)$ be a one-node extension of it. It is easy to check that $\bar{q} \in D_A$.

Claim (c). $P^* \Vdash$ "$\tilde{A}$ is a set of power $\leqslant\lambda$".

Proof. D_a is dense above P^* and each member $p \in D_A$ forces "$\tilde{A} \subseteq \tilde{A} \cap (h(T_p) + 1)$", because if $p \in G$, then T_G is an end extension of T_p, so for $x \in T_G$, $l(x) > h(T_p)$ implies x is above a member of T_p's last level which is forced by p to be above a member of $\tilde{A} \cap h(T_p)$ and as p^* forces A to be a p.i.s., no such node can belong to A. □

Theorem 2. *If* "ZFC + *there is a super-compact cardinal*" *is consistent, then so is* "ZFC + *there is no* $\aleph_{\omega+1}$*-special Aronszajn tree* + *there exists* $\aleph_{\omega+1}$*-Souslin trees*".

Proof. Let κ be a super-compact cardinal and let λ be the first singular cardinal above κ. Let us first extend the universe by forcing with P^κ_λ. In $V[P^\kappa_\lambda]$, κ is still a super-compact cardinal and there is a λ^+-Souslin tree. By [10] there is some regular $\mu < \kappa$ s.t. if we collapse by finite conditions μ to $\aleph_0$ and then Levy-collapse κ to $\aleph_1$, we end up with a model in which λ is $\aleph_\omega$ and there are no $\aleph_{\omega+1}$-special Aronszajn trees. As the iteration of the two collapsing partial orders is of size less than λ (in $V[P^\kappa_\lambda]$), any set of size λ^+ in our last model has a subset of size λ^+ in $V[P^\kappa_\lambda]$. It follows that any λ^+-Souslin tree in $V[P^\kappa_\lambda]$ remains a λ^+-Souslin tree. □

The 'classical' methods for constructiong λ^+-Souslin trees use Jensen's principles $\square_\lambda$ and $\diamondsuit_{\lambda^+}$ (see Jensen [7] and Gregory [5]).

We want to construct a Souslin tree on λ^+ for a singular λ where there exists a super-compact cardinal κ below it. In such cases if $\mathrm{cof}(\lambda) < \kappa$ then $\square_\lambda$ fails, so a different construction has to be developed.

We present here two weaker versions of $\square_\lambda$, $\square_{\lambda,\mu}$ and $\square_\lambda^\mu$. For $\mu < \lambda$ these principles are consistent with the existence of super-compact cardinals below λ (even when λ has small cofinality).

Theorem 3 states that (for any $\mu < \lambda$) "$\square_{\lambda,\mu} + \diamondsuit_{\lambda^+}$" implies the existence of a λ^+-Souslin tree. (Recall that by Shelah [11] and Gregory [5], $\diamondsuit_{\lambda^+}$ is not a strong demand and it holds whenever λ is a strong limit cardinal and $2^\lambda = \lambda^+$.)

$\square_{\lambda,\mu}$ fails for all $\mu < \lambda$ if λ is a limit of super-compact cardinals, in such a case $\square_\lambda^\mu$ comes to our rescue, this principle is consistent for such λ's as well, and assuming further that μ is a λ-strongly compact cardinal we get in Theorem 5 a Souslin tree on λ^+.

Definition. $\square_{\lambda,\mu}$ for cardinals $\mu < \lambda$, asserts the existence of a sequence $\langle C_\alpha : \alpha < \lambda^+, \text{ cof } \alpha > \mu \rangle$ satisfying: (i) C_α is a closed and unbounded subset of α; (ii) for $\alpha > \lambda$, $\mathrm{otp}(C_\alpha) < \alpha$; (iii) if γ is a limit point of both C_α and C_β, then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

Remark. Note that for $\mu < \mu' < \lambda$, $\square_{\lambda,\mu} \Rightarrow \square_{\lambda,\mu'}$.

Theorem 3. *If λ is a singular strong limit cardinal, $2^\lambda = \lambda^+$ and $\square_{\lambda,\mu}$ holds for some $\mu < \lambda$, then there exists a Souslin tree on λ^+.*

Proof. By Shelah [11] our assumptions imply $\diamondsuit(S)$ for any stationary $S \subseteq \lambda^+$ s.t. $S \subseteq \{\alpha < \lambda^+ : \mathrm{cof}\, \alpha \neq \mathrm{cof}\, \lambda\}$. As $\square_{\mu,\lambda} \Rightarrow \square_{\mu',\lambda}$ for any $\mu < \mu' < \lambda$ we may assume $\mathrm{cof}\, \lambda \neq \mu^+$.

Lemma 3.1 (Shelah [11]). *There is a stationary $S \subseteq \{\alpha < \lambda^+ : \mathrm{cof}\, \alpha = \mu^+\}$ and a $\square_{\lambda,\mu}$-sequence $\langle C_\alpha : \alpha < \lambda^+, \mathrm{cof}\, \alpha > \mu \rangle$ s.t. if β is a limit point of C_α, then $\beta \in S$ only if $\beta = \alpha$ (and, of course, $\alpha \in S$).*

Proof of the Lemma. Let $\langle C'_\alpha : \alpha < \lambda^+, \text{ cof } \alpha \geq \mu \rangle$ exemplify $\square_{\lambda,\mu}$ (from our assumptions). Let δ be the first ordinal s.t. $S_\delta = \{\alpha : \mathrm{cof}\, \alpha = \mu^+ \text{ and } \mathrm{otp}(C'_\alpha) = \delta\}$ is stationary (such δ exists by Fodor's Lemma as $\mathrm{otp}(C'_\alpha) < \alpha$ and $\{\alpha : \mathrm{cof}\, \alpha = \mu^+\}$ is stationary in λ^+). Let S be S_δ and $C'_\alpha = C_\alpha$ if $\mathrm{otp}(C'_\alpha) < \delta$ or $\alpha \in S$, and $C_\alpha = C'_\alpha \setminus (C'_\alpha$ first $\delta + 1$ elements) if $\mathrm{otp}(C'_\alpha) > \delta$ and $\alpha \notin S$. It is easy to check that if β is a limit point of any C_α, then $\beta \in S$ or $\beta = \alpha$.

Let us fix a set S, a $\diamondsuit(S)$-sequence $\langle A_\alpha : \alpha \in S \rangle$ and a $\square_{\lambda,\mu}$-sequence $\langle C_\alpha : \alpha < \lambda^+, \text{ cof } \lambda \geq \mu, \ \alpha \notin S \rangle$ satisfying the demands of the Lemma.

The construction of the tree. We define by induction on $\alpha<\lambda^+$, $T\restriction(\alpha+1)$ carrying the following induction hypothesis:

(i) $T\restriction(\alpha+1)$ is a tree of height $\alpha+1$ and each of its levels has no more than λ nodes.

(ii) For each node $x\in T\restriction\alpha$, and for each $\beta\leq\alpha+1$ there is a one-one function $f_x^\beta:\mu^+\to(T\restriction(\alpha+1))_\beta$ (the β's level of the tree) s.t. $\forall i<\mu^+\ x<_T f_x^\beta(i)$.

(iii) $x\in T\restriction\alpha$ and $h(x)<\beta<\gamma\leq\alpha$ imply $\{i<\mu^+:f_x^\beta(i)<_T f_x^\gamma(i)\}$ belongs to D_{μ^+}.

(iv) $x\in T\restriction\alpha$ and $h(x)<\beta<\gamma\leq\alpha$ imply that whenever C_α is defined and $\beta<\gamma$ are limit points of C_α, then $\{i<\mu^+:f_x^\beta(i)<_T f_x^\gamma(i)\}=\mu^+$.

(v) $\alpha\in S$ implies that if A_α is a maximal antichain in $T\restriction\alpha$ then each member of $(T\restriction(\alpha+1))_\alpha$ is above a member of A_α.

Claim. *If each* $(T\restriction\alpha+1)$ *satisfies the demands* (i)–(v) *and* $T=\bigcup_{\alpha<\lambda^+}(T\restriction\alpha+1)$, *then* T *is a* λ^+*-Souslin tree.*

Proof. By (i), T is a λ^+-tree and the cardinality of each of its levels is less than λ^+, by (v) we can show it has no λ^+-antichain. Let $A\subseteq T$ be an antichain. As $\langle A_\alpha:\alpha\in S\rangle$ is a $\diamondsuit$-sequence, there exists some $\beta\in S$ s.t. $A\restriction\beta=A_\beta$. By claim (v) any $x\in T$ s.t. $h(x)>\beta$ cannot belong to A as any such x is above a member of $T_{\beta+1}$ so it is above a member of $A_\beta=A\restriction\beta$ (and A is assumed to be an antichain) so $|A|\leq\lambda$.

Let us show that the construction can be carried on for each $\alpha<\lambda^+$.

We assume $T\restriction\alpha$ is defined and satisfies (i)–(v) and define $T\restriction\alpha+1$.

Case (a) If α is a successor, then there are no difficulties as no new cases are added to the demands (iv), (v). We just split each node of T_β (where $a=\beta+1$) to λ many successors and for each $x\in T\restriction\beta$ and $i<\mu^+$ we pick an arbitrary successor of $f_x^\beta(i)$ to be $f_x^\alpha(i)$, for $x\in T_\beta$ we pick as $f_x^\alpha(i)$ an arbitrary successor of x.

Case (b) α is a limit ordinal, $\operatorname{cof}(\alpha)\leq\mu$.

(i) For each $x\in T\restriction\alpha$ let $C=\langle\alpha_\rho:\rho<\operatorname{cof}\alpha\rangle$ be an increasing and continuous sequence of ordinals increasing to α s.t. $\alpha_0=l(x)$. Now for each $i\in\bigcap_{\gamma<\delta\in C}\{j:f_x^\gamma(j)<f_x^\delta(j)<f_x^\delta(j)\}$ we add a node $y_x^\alpha(i)$ above $\bigcup_{\rho<\operatorname{cof}\alpha}f_x^\alpha(i)$.

Now if α is not a limit point of any C_β, then we let $(T\restriction\alpha+1)_\alpha$ be $\bigcup_{x\in T\restriction\alpha,i<\mu^+}\{y_x^\alpha(i)\}$ and define $f_x^\alpha(i)$ to be $y_x^\alpha(i)$ for all $x\in T\restriction\alpha$ and for almost all i (as $y_x^\alpha(i)$ is defined for a closed unbounded subset of μ^+).

(ii) When α is a limit point of some C_β we should take care of demand (iv). Let β_0 be the minimal β s.t. α is a limit point of C_β. If α is the first limit point of C_β, we may just repeat the construction for $(T\restriction\alpha+1)_\alpha$ of case (b)(i). Otherwise, either α is a limit of limit points of C_{β_0}, or there is some β_1 the maximal limi point of C_{β_0} below α. In the first case by the induction hypothesis for each $i<\mu^+$, $\bigcup\{f_x^\gamma(i):\gamma$ is a limit point of C_{β_0}, $h(x)<j<\alpha\}$ is a branch and we just add to it a maximal node and define $f_x^\alpha(i)$ to be this node. In the second case we first repeat our basic construction of $(T\restriction(\alpha+1))_{\alpha+1}$ (case (b)(i)). Let $f_x'^{\alpha}(i)$ be the set of functions we get this way, so for each x, $\{i : f_x^{\beta_1}(i) <_T f_x'^{\alpha}(x)\} \in D_{\mu}$, and let C_x be its complement. For each $i \in C_x$ there is an unbounded branch in $T\restriction\alpha$ above $f_x^{\beta_1}(i)$ (as $\operatorname{cof}(\alpha) \leq \mu$); let $t_x^{\alpha}(i)$ be a new node at the top of such a branch and define

$$f_x^{\alpha}(i) = \begin{cases} f_x'^{\alpha}(i) & \text{if } i \notin C_x, \\ t_x^{\alpha}(i) & \text{if } i \in C_x. \end{cases}$$

Case (c): $\operatorname{cof}(\alpha) > \mu$.

(i) $\alpha \notin S$, by the induction hypothesis for each $i < \mu^+$, $A_x^{\alpha} = \{f_x^{\gamma}(i) : \gamma$ is a limit point of $C_\alpha\}$ is a branch for all $x \in T\restriction\alpha$. For each such x and i we add a node at level α above A_x^{α} and define $f_x^{\alpha}(i)$ to be this node. (The α's level of $T\restriction(\alpha+1)$ will be the set of all these nodes.)

(ii) $\alpha \in S$, now we consider the $\diamondsuit$-sequence. Let A_α be its α's element, if A_α is not a maximal antichain of $T\restriction\alpha$. We repeat the construction of case (c)(i). We are left with the case $A_\alpha \subseteq T\restriction\alpha$ and is a maximal antichain there and $\operatorname{cof}\alpha = \mu^+$. First, note that as $\operatorname{cof}(\alpha) = \mu^+$, C_α exists so by the induction hypothesis for each $x \in T\restriction\alpha$ there is a cofinal branch above it (in fact μ^+ many), as for all $i < \mu^+$, $\{f_x^{\beta}(i) : h(x) < \beta$, β a limit point of $C_\alpha\}$ is such a branch. Note also that as A_α is a maximal antichain in $T\restriction\alpha$ each node of $T\restriction\alpha$ is either above a node in A_α or it has an extension in A_α.

We will define for each $x \in T\restriction\alpha$, f_x^{α} in such a way that for all $i < \mu^+$, $f_x^{\alpha}(i)$ is above a node in A_α, and let $(T\restriction(\alpha+1))_\alpha$ be $\bigcup_{x\in T\restriction\alpha}\{\operatorname{Range}(f_x^{\alpha})\}$. Let $B_\alpha = \{\alpha_i : i < \mu^+\}$ be a cofinal subset of C_α or order type $\operatorname{cof}\alpha = \mu^+$. For each x and i let y_x^i be a node above $f_x^{\alpha_i}(i)$ that either belongs to A_α or extends a node in A_α. As noted above such y_x^i always exists and there is always a cofinal (in $T\restriction\alpha$) branch above it, so let $f_x(i)$ be a new node at the top of such a branch. As $\alpha \in S$ it is not a limit point of any C_β for $\beta > \alpha$, so we do not have to worry about demand (iv) at higher stages, and demand (iii) is satisfied as for all x and i, $\{j : f_x^{\alpha_i}(j) \leq_T f_x^{\alpha}(j)\}$ is at least $\{j < \mu^+ : i \leq j\}$ and as $\langle \alpha_i : i < \mu^+\rangle$ is cofinal in α for any $\beta < \alpha$ there is some α_i above it and

$$\{j : f_x^{\beta}(j) <_T f_x^{\alpha}(j)\} \supseteq \{j : f_x^{\beta}(j) <_T f_x^{\alpha_i}(j)\} \cap \{j : f_x^{\alpha_i}(j) <_T f_x^{\alpha}(j)\}$$

so it is a member of D_{μ^+}.

This completes the definition of T. It is easy to check that the induction hypotheses hold all through the way. □

Definition. $\square_\lambda^\mu$ asserts the existence of a sequence $\langle C_\alpha : \lambda < \alpha < \lambda^+,\ \operatorname{cof}(\alpha) < \mu\rangle$ s.t. C_α is a club in α of order type less than α and if δ is a limit point of C_α, then $C_\delta = C_\alpha \cap \delta$.

Theorem 4. *Let λ be a singular cardinal. Let κ be a λ-strongly compact cardinal s.t.* $\operatorname{cof}\lambda < \kappa$ *and assume $\square_\lambda^\kappa$ and $\diamondsuit_\lambda$, then there is an Aronszajn tree on λ^+.*

Remark. The significance of the theorem is that the assumptions are consistent with the existence of arbitrary many super-compact cardinals between κ and λ. We may start with any model in which λ is a strongly compact cardinal and $\lambda^+ = 2^\lambda$ (so $\diamondsuit_\lambda$ holds) and κ a λ-strongly compact cardinal, and then force the $\square^\kappa_\lambda$-sequence. Such forcing can be done without disturbing the super-compactness of any cardinal between κ and λ, and without adding subsets to λ so the λ-compactness of κ still holds in the generic extension. (For details about forcing $\square$-sequences above super-compact cardinals consider [2].)

Proof of the Theorem. Let us fix a sequence of regular cardinals $\langle \kappa_i : i < \operatorname{cof} \lambda \rangle$ s.t. $\kappa_0 = \kappa$ and the sequence is increasing and λ is its supremum, and a $\square^\kappa_\lambda$-sequence $\langle C_\alpha : \operatorname{cof} \alpha < \kappa, \lambda < \alpha < \lambda^+ \rangle$. Finally let $\langle A_\alpha : \alpha \in S \rangle$ be a $\diamondsuit_\lambda$-sequence where $S \subseteq \lambda^+$ is stationary in λ^+ and $\alpha \in S \Rightarrow \operatorname{cof}(\alpha) > \kappa$.

We define $T \restriction (\alpha + 1)$ by induction on $\alpha < \lambda^+$ such that $T \restriction (\alpha + 1)$ should satisfy:

(i) $T \restriction (\alpha + 1)$ is a tree if its height is α and each of its levels has less than λ^+ nodes.

(ii) For each $x \in T \restriction \alpha$ and β s.t. $h(x) < \beta \leqslant \alpha$ there are one-one functions $\langle f^\beta_{x,i} : i < \operatorname{cof} \lambda \rangle$ s.t. $f^\beta_{x,i} : \kappa_i \to (\beta\text{'s level of } T \restriction (\alpha + 1))$, and for all $j < \kappa_i$, $x <_T f^\beta_{x,i}(j)$.

(iii) For $\beta_1 < \beta_2 \leqslant \alpha$ and $x \in T \restriction \alpha$ s.t. $h(x)\kappa < \beta_1$, there is $i_x(\beta_1, \beta_2) < \operatorname{cof}(\lambda)$ s.t. $i_x(\beta_1, \beta_2) < i$ implies $\{j : f^{\beta_1}_{x_i}(j) <_T f^{\beta_2}_{x_i}(j)\} \in D_{\kappa_i}$.

(iv) If $\operatorname{cof}(\alpha) < \kappa$ and $\beta < \gamma < \alpha$ are limit points of C_α, then for all $i < \operatorname{cof} \lambda$ and for all $x \in T \restriction \alpha$ s.t. $l(x) < \beta$ and for all $j < \kappa$, $f^\beta_{x,i}(j) <_T f^\gamma_{x,i}(j)$.

(v) If A_α is a cofinal branch of $T \restriction \alpha$, then in $T \restriction (\alpha + 1)$, A_α has no node above it (recall that if $\operatorname{cof}(\alpha) \leqslant \kappa$, then A_α is not defined).

It is easy to see that if the construction can be carried away for all $\alpha < \lambda^+$, then $\bigcup_{\alpha < \lambda^+} T \restriction (\alpha + 1)$ is a λ^+-Aronszajn tree. Assume $T \restriction (\beta + 1)$ is defined and satisfies (i)–(v) for all $\beta < \alpha$ and let us define $T \restriction (\alpha + 1)$. Whenever $\operatorname{cof}(\alpha) < \kappa$, the situation in analogous to cases we have dealt with in the proof of Theorem 3. We are left with the case $\operatorname{cof}(\alpha) > \kappa$.

Claim. *For each $x \in T \restriction \alpha$ there is an unbounded $B \subseteq \alpha$ on which $i_x(\beta_1, \beta_2)$ is bounded.*

Proof. As κ is a λ-strongly compact cardinal and $\kappa \leqslant \operatorname{cof}(\alpha) < \lambda$, there is a κ-complete ultrafilter U on $\operatorname{cof}(\alpha)$. Let $\langle \beta_i : i < \operatorname{cof}(\alpha) \rangle$ be an increasing sequence cofinal in α, $x \in T \restriction \alpha$. The function $i_x(\beta_1, \beta_2)$ is, by (iii) of the induction hypothesis, defined for all $\beta_1, \beta_2 < \alpha$. Define $g : \operatorname{cof}(\alpha)^2 \to \operatorname{cof}(\lambda)$ by $g(\rho, \gamma) \overset{\text{def}}{=} i_x(\beta_\rho, \beta_\gamma)$. As U is κ-complete and $\operatorname{cof} \lambda < \kappa$, for each γ there is some $\delta_\gamma < \operatorname{cof}(\lambda)$ s.t. $A_\gamma = \{\rho : g(\rho, \gamma) = \delta_\gamma\} \in U$. And there is a set $B \leqslant \operatorname{cof}(\alpha)$, $B \in U$ s.t. $\gamma_1, \gamma_2 \in B$ implies $\delta_{\gamma_1} = \delta_{\gamma_2}$. We claim that g is bounded on B^2. Let γ_1, γ_2 be any two members of B, as $A_{\gamma_1}, A_{\gamma_2} \in U$ there is some $\gamma_3 \in A_{\gamma_1} \cap A_{\gamma_2}$ s.t. γ_1,

$\gamma_2 < \gamma_3$. Now $g(\gamma_1, \gamma_3) = \delta_{\gamma_1}$, $g(\gamma_2, \gamma_3) = \delta_{\gamma_2} = \delta_{\gamma_1}$ (as $\gamma_1, \gamma_2 \in B$) and we claim that $g(\gamma_1, \gamma_2) \leqslant \delta_{\gamma_1}$. To see why this is true recall that $g(\gamma_1, \gamma_3) = \delta_{\gamma_1} \Rightarrow$ for all $i > \delta_{\gamma_1}$, $\{j : f^{\beta_{\gamma_1}}_{xi}(j) <_T f^{\beta_{\gamma_3}}_{xi}(j)\} \in D_{\kappa_i}$. $g(\gamma_2, \gamma_3) = \delta_{\gamma_1}$ implies that for all $i > \delta_{\gamma_1}$, $\{j : f^{\beta_{\gamma_2}}_{xi}(j) <_T f^{\beta_{\gamma_3}}_{xi}(j)\} \in D_{\kappa_i}$. As $T \restriction \alpha$ is a tree, $\gamma_1 < \gamma_2 < \gamma_3$ and $f^{\beta_{\gamma_1}}_{xi}(j) <_T f^{\beta_{\gamma_3}}_{xi}(j)$ and $f^{\beta_{\gamma_2}}_{xi} <_T f^{\beta_{\gamma_3}}_{xi}(j)$ imply $f^{\beta_{\gamma_1}}_{xi}(j) < f^{\beta_{\gamma_2}}_{xi}(j)$. So, for $i > \delta_{\gamma_1}$, $\{j : f^{\beta_{\gamma_1}}_{xi}(j) <_T f^{\beta_{\gamma_2}}_{xi}(j)\} \in D_{\kappa_i}$, so $i_x(\beta_{\gamma_1}, \beta_{\gamma_2}) \leqslant \delta_{\gamma_1}$, so $g(\gamma_1 \gamma_2) \leqslant \delta_{\gamma_1}$. As U is uniform, B is unbounded in $\mathrm{cof}(\alpha)$, so $\langle \beta_\rho : \rho \in B \rangle$ is cofinal in α. This completes the proof of the claim.

Now we can define $T \restriction (\alpha + 1)$: Let $i_0 < \mathrm{cof}\, \lambda$ be an upper bound of $\mathrm{Rang}_e(g \restriction B^2)$ s.t. $\mathrm{cof}(\alpha) < \kappa_{i_0}$. As D_{κ_i} is a $\mathrm{cof}(\alpha)$-complete filter for all $i_0 < i < \mathrm{cof}(\lambda)$, and for all $\rho_1, \rho_2 \in B$, $i_0 < i < \mathrm{cof}(\lambda)$, $C_{\rho_1 \rho_2} = \{j : f^{\rho_i}_{x_i}(j) <_T f^{\rho_2}_{xi}(j)\} \in D_{\kappa_i}$ and $\bigcap_{\rho_1 < \rho_2 \in B} C_{\rho_1 \rho_2} \in D_{\kappa_i}$, we can now define the σ's level of $T \restriction (\alpha + 1)$ by adding a node on top of each branch that has the form $\bigcup_{\rho \in B} f^{\rho}_{xi}(j)$ $(j \in \bigcap_{\rho_1 < \rho_2 \in B} C_{\rho_1 \rho_2})$. For all $i_0 < i < \mathrm{cof}\, \lambda$, define $f^{\alpha}_{xi}(j)$ to be this node and define $f^{\alpha}_{xi}(j)$ for $i \leqslant i_0$ or j's outside this set in any arbitrary way that satisfies demand (ii). We repeat the construction for each $x \in T \restriction \alpha$ and let $(T \restriction (\alpha + 1))_\alpha$ be the union of the ranges of all the f^{α}_{xi} $(x \in T \restriction \alpha,\ i < \mathrm{cof}(\lambda))$. We do need one more modification. In order to satisfy demand (v) we may have to eliminate one node at level α, so we eliminate it. No harm can be done as $\mathrm{cof}(\alpha) > \kappa$ so C_α is not defined, demand (iv) is empty of content, and all other demands are formulated in terms of belonging to some D_{κ_i} which is not affected by such a small change. □

Theorem 5. *For a strong limit singular λ s.t. $2^\lambda = \lambda^+$: if there is a λ-strongly compact cardinal κ,* $\mathrm{cof}(\lambda) > \kappa > \lambda$, *and $\square^\kappa_\lambda$ holds, then there is a λ^+-Souslin tree.*

Proof. By Shelah [11] for a strong limit singular λ, $2^\lambda = \lambda^+$ and $\square^\kappa_\lambda$ for some κ, $\mathrm{cof}(\lambda) < \kappa < \lambda$, imply the existence of a set $S \subseteq \lambda^+$ s.t. S is stationary, $\alpha \in S$ implies $\mathrm{cof}(\alpha) = \mathrm{cof}(\lambda)$, $\diamondsuit_S$ holds and there is a $\square^\kappa_\lambda$-sequence $\langle C_\alpha : \alpha < \lambda^+, \mathrm{cof}\, \alpha < \kappa \rangle$ such that for no α there is a limit point of C_α in S except maybe α itself.

We assume the existence of a such an S and such a $\square^\lambda_\kappa$-sequence and repeat the proof of Theorem 4 (using this $\diamondsuit^\lambda_\kappa$-sequence) with the following additional demand: If $\alpha \in S$ and A_α is a maximal antichain in T_α, then every member of T_α is above a member of A_α.

Let us show that it is possible to cary out the inductive construction of our tree in such a way that this demand is met.

We repeat the construction from the proof of Theorem 4 for all the stages α s.t. our new demand does not apply to α.

Let $\alpha \in S$, A_α a maximal antichain in T_α.

For every $x \in T_\alpha$ let $\langle \alpha_i : i < \mathrm{cof}(\lambda) \rangle$ be an increasing and cofinal subset of C_α s.t. $l(x) < \alpha_0$. For each $i < \mathrm{cof}(\lambda)$ and each $j \in \kappa_i$, let $f^{\alpha}_{x},(j)$ be any extension of a cofinal branch of T_α above $f^{\alpha_i}_{x,i}(j)$ through a point in A_α. Such an extension always exists as A_α is a maximal antichain.

Let T_α be the set of all the $f^{\alpha}_{x,i}(j)$ for all $i<\mathrm{cof}(\lambda)$, $j<\kappa_i$, $x\in T_\alpha$. It is straightforward to check that $\langle T(\alpha+1), F_{\alpha+1}\rangle$ defined here satisfy all of our inductive demands (note that $\{\alpha_i : i<\mathrm{cof}(X)\}\subseteq C_\alpha$ and for $f_{x,y}\in C_\alpha$, $f^{\beta}_{x,i}(j)<f^{\gamma}_{x,i}(j)$ for all x, i, j).

Why is the tree T constructed in such a way a Souslin tree? Let X be any maximal antichain in T, there is a closed unbounded subset of λ^+, C, s.t. $\alpha\in C\Rightarrow X\cap\alpha$ is a maximal antichain in T_α. There is also a stationary $D\subseteq S$ s.t. $\alpha\in D\Rightarrow X\cap\alpha=A_\alpha$. Let α belong to $D\cap C$, A_α is a maximal antichain in T_α, so by our last demand all members of T_α are above members of $A_\alpha=X\cap\alpha$, so as X is an antichain $X\subseteq T_{(\alpha+1)}$, so $|X|\leqslant\lambda$. □

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