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ON MEASURE AND CATEGORY†

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ABSTRACT

We show that under ZF + DC, even if every set of reals is measurable, not necessarily every set of reals has the Baire property. This was somewhat surprising, as for the Σ^1_2 set the implication holds.

Recently, following a proof in Raisonnier [1] which follows Shelah [3] §5, Raisonnier and Stern have proved: if the union of any κ zero measure sets (of reals) has measure zero *then* the union of κ meager sets (in ${}^\omega 2$) is meager; and if every $\mathbf{\Sigma}^1_2$ set of reals is (Lebesgue) measurable then any $\mathbf{\Sigma}^1_2$ set of reals has the Baire property, and M.U.P.-perfect set theorem. Those results were independently proved by Bartosynski. The following answers the question they have asked. I thank Magidor for a very helpful discussion.

THEOREM. *If in L there is an inaccessable cardinal, then in some forcing extension $L[G]$ of L the following holds: ZF + DC + "Every set of reals is measurable" + "there is a set of reals without the Baire property" + "there is an uncountable set of reals with no perfect subset."*

PROOF.

(1) *Scheme.* We start with $V = L$, κ an inaccessible (or just $V \models \mathrm{ZFC}$ + "κ strongly inaccessible"). We want to build a forcing notion B, which will be just the Levi collapse of κ to $\aleph_1$ which Solovay used, and a special set P of B-names of reals. Later we force by B, let G be the generic set, $P[G] = \{\underset{\sim}{r}[G] : \underset{\sim}{r} \in P\}$

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and the desired universe is the family of sets which hereditarily are definable in $V[G]=L[G]$, from a real, an ordinal and $P[G]$.

(2) *Notation.* Here *a* real is a function from ω to ω. We say r_1 dominates r_2 if for every large enough n, $r_2(n)\leqq r_1(n)$. Call $r\in{}^{\omega}\omega$ quasi-generic over V, if no $\tau'\in({}^{\omega}\omega)^V$ dominates r. In forcing notions, bigger means giving more information; using a Boolean algebra we omit the zero and invert the order so 1 becomes the minimal element.

(3) *Definition.* We define what is an approximation: it is a pair (B,P) such that: B is a complete Boolean algebra of power $<\kappa$ (and $B\in H(\kappa)$ for simplicity), P a set of B-names of reals (here functions from ω to ω), more formally such a B-name $\underset{\sim}{r}$ consists of ω maximal antichains of B; $\langle b^{\underset{\sim}{r}}_{n,i}: i<\alpha_n\rangle$, and function $f^{\underset{\sim}{r}}$ such that $b^{\underset{\sim}{r}}_{n,i}\Vdash$ "$\underset{\sim}{r}(n)=f^{\underset{\sim}{r}}(n,i)$". Let AP be the set of approximations.

(4) *Definition.* We define a partial order on (AP): $(B_1,P_1)\leqq(B_2,P_2)$ if: $B_1\lessdot B_2$, i.e., B_1 is a complete (Boolean) subalgebra of B_2, $P_1\subseteq P_2$, and if $\underset{\sim}{r}\in P_2-P_1$ then $\Vdash_{B_2}$ "$\underset{\sim}{r}$ is quasi generic over V^{B_1}".

Clearly:

(4A) $\leqq$ is a partial order,

(4B) if $\langle(B_i,P_i): i<\alpha\rangle$ is increasing *then* it has a natural upper bound $\bigcup_{i<\alpha}(B_i,P_i)\stackrel{\text{def}}{=}((\bigcup_{i<\alpha}B_i)^c,\bigcup_{i<\alpha}P_i)$ (where the c denotes completion).

(5) Let us force with AP, and get a generic set H; clearly no cardinal is collapsed or changes its cofinality, and no bounded subset of κ is added. Let

$$B^H=\bigcup\{B:(\exists P)[(B,P)\in H]\},\quad P^H=\bigcup\{P:(\exists B)[(B,P)\in H]\}.$$

Easily B^H is a complete Boolean algebra of power κ, collapsing any $\lambda<\kappa$ to $\aleph_0$, satisfying the κ-chain condition, and P is a set of B-names, and $[(B,P)\in H\Rightarrow B$ is a complete subalgebra of B^H and for $\underset{\sim}{r}\in P$, $\Vdash_{B^H}$ "$\underset{\sim}{r}$ is a real"].

(6) Next, over $L[H]$ force by B^H, get a generic set G, and let $V^*=\{a\in L[H,G]: a$ is hereditarily definable from a real, H, an ordinal and $\underset{\sim}{P}[H,G]\}$ where $\underset{\sim}{P}[H,G]=\{\underset{\sim}{r}[G]:\underset{\sim}{r}\in P^H\}$. By Solovay [4], $V^*\vDash$ "ZF + DC + κ is $\aleph_1$".

(7) $V^*\vDash$ "$\underset{\sim}{P}[H,G]$ is an uncountable set of reals which contains no perfect set".

The first part is by the genericity of H. For the second part, suppose not, then for some $p \in B^H$, and B^H-name $\underset{\sim}{T}$ of a downward closed perfect subset of ${}^{\omega>}\omega$, $L[H] \models$ "$p \Vdash_{B^H}$ every branch of $\underset{\sim}{T}$ is in $\underset{\sim}{P}[H, \underset{\sim}{G}]$".

As B^H satisfies the κ-chain condition, for some $(B_0, P_0) \in H$, $\underset{\sim}{T}$ is a B_0-name, $p \in B_0$ (remember H is directed) so w.l.o.g. $(B_0, P_0) \Vdash_{AP}$ "in $L[\underset{\sim}{H}]$, $p \Vdash_{B^H}$ (every branch of $\underset{\sim}{T}$ is in $\underset{\sim}{P}[\underset{\sim}{H}, \underset{\sim}{G}]$)".

We find $B_1, B_0 \lessdot B_1 \in H(\kappa)$, and a B_1-name $\underset{\sim}{r}$ of a branch of $\underset{\sim}{T}$, which is not in $L[H]^{B_0}$. Then $(B_0, P_0) \leqq (B_1, P_0) \in AP$ and $(B_1, P_0) \Vdash_{AP}$ "$p \Vdash_{B^H}$ ($\underset{\sim}{r}$ is a branch of $\underset{\sim}{T}$ and $\underset{\sim}{r} \notin \underset{\sim}{P}[\underset{\sim}{H}, \underset{\sim}{G}]$)" (the $\underset{\sim}{r} \notin \underset{\sim}{P}[\underset{\sim}{H}, \underset{\sim}{G}]$ holds because, for any $\underset{\sim}{s} \in P^H$, either $\underset{\sim}{s}$ is a B_0-name and then cannot be forced to be equal to $\underset{\sim}{r}$ by its choice, or $\underset{\sim}{s} \notin P_0$, hence, if $(B_1, P_0) \in \underset{\sim}{H}$, $\underset{\sim}{s}$ is forced to be quasi-generic over $L[H]^{B_1}$ (equivalently over L^{B_1}), hence cannot be equal to any member of $V[H]^{B_1}$, in particular to $\underset{\sim}{r}$).

(8) $V^* \models$ "${}^\omega\omega - \underset{\sim}{P}[H, G]$" is of the second category in every $N_s = \{r \in {}^\omega\omega : r \restriction l(s) = s\}$ $(s \in {}^{\omega>}\omega)$.

The proof is similar to (7) for we could have chosen $\underset{\sim}{r}$ a B_1-name of a real in N_s, generic over L^{B_0} equivalently over $L[H]^{B_0}$.

(9) Remember $G \subseteq B^H$ is generic over $L[H]$. Now $V^* \models$ "$\underset{\sim}{P}[H, G]$ is of the second category in every N_s $(s \in {}^{\omega>}\omega)$". We proceed as in (8), the only difference is that we use $(B_1, P_0, \bigcup\{\underset{\sim}{r}\})$ (instead of (B_1, P_0)) where $\underset{\sim}{r}$ is a B_1-name of a real generic over V^{B_0}. The point is that as $\underset{\sim}{r}$ is generic (hence quasi-generic) over V^{B_0}, clearly $(B_0, P_0) \leqq (B_1, P_0 \cup \{\underset{\sim}{r}\})$.

(10) The main point: $V^* \Vdash$ "every set of reals is measurable".

Let $A \in V^*$, $A \subseteq \mathbf{R}^{V^*} = \mathbb{R}^{L[H,G]}$, so there is a formula $\varphi(x, \ , \ , \ )$ and $\mathrm{AP} * B^H$-name $\underset{\sim}{r}$ of a real and ordinal α such that

$$A = \{x \in \mathbf{R} : L[H, G] \models \psi[x, \underset{\sim}{r}[H, G], \alpha, P]\}.$$

As AP is κ-complete, $B^{\underset{\sim}{H}}$ satisfies the κ-chain condition, clearly there is $(B_0, P_0) \in H$ such that $(B_0, P_0) \Vdash_{AP}$ "$\underset{\sim}{r} = \underset{\sim}{s}$, $\underset{\sim}{r}$ a B_0-name of a real". We know that almost all reals of V^* (in the measure sense) are random over $L[H]^{B_0}$ (as for any $(B, P) \in \mathrm{AP}$, $(B * \underset{\sim}{\text{Amoeba}}, P)$ is $\geqq (B, P)$ (and is in AP)). So as in Solovay [4], it is enough to prove:

(*) if $B_0 \lessdot B_1^l \lessdot B_2^l$, $(B_0, P_0) \leqq (B_2^l, P_2^l)$, B_1^l / B_0 is random real forcing, for $l = 1, 2$ and f is an isomorphism from B_1^1 onto B_1^2, $f \restriction B_0 =$ the identity, *then* we can amalgamate in AP (B_2^1, P_2^1), (B_2^2, P_2^2) over f

[i.e., there is $(B, P) \in \mathrm{AP}$ and isomorphisms g_l from B_2^l onto B_2^{l+2} mapping P_2^l onto P_2^{l+2}, such that $(B_2^{l+2}, P_2^{l+2}) \leqq (B, P)$, and $g_2 f = g_1 \restriction B_1^1$]. [Note that where

Solovay uses actual automorphism of B^H, we use automorphism of names, i.e., its genericity; it doesn't matter.] For this we need

(11) *Key Fact.* If $(B_1, P_1) \leqq (B_3, P_3)$, $B_1 \lessdot B_2 \lessdot B_3$, B_2/B_1 is random real forcing, then $(B_1, P_1) \leqq (B_2, P_1) \leqq (B_3, P_3)$.

Proof of Key Fact. The first inequality is trivial; for the second we have to prove: if $\underset{\sim}{r} \in P_3 - P_1$ then $\Vdash_{B_3}$ "r_3 is not dominated by any real in L^{B_2}". However it is well known that every $x \in ({}^\omega\omega)^{LB_2}$ is dominated by some $x^1 \in ({}^\omega\omega)^{LB_1}$ [as B_2/B_1 is random real forcing] and $\underset{\sim}{r}$ is not dominated by x^1 as $(B_1, P_1) \leqq (B_3, P_3)$.

(12) *Proof of* (*) *of* (10) *from the Key Fact.* We can find B_2^3 ($\in H(\kappa)$) and g such that $B_1^2 \lessdot B_2^3$, g an isomorphism from B_1^2 onto B_2^3 extending f, and $B_2^3 \cap B_2^2 = B_1^2$.

Let

$$Q = \{(p_2, p_3): p_2 \in B_2^2,\ p_3 \in B_2^3,\ \text{and for some } r \in B_1^2,\ (\forall q \in B_1^2)[r \leqq q \rightarrow (r, p_2 \text{ are compatible in } B_2^2 \text{ and } r, p_3 \text{ are compatible in } B_2^3)]\}$$

with the order:

$$(p_2, p_3) \leqq (p_2', p_3') \qquad \text{iff } p_2 \leqq p_2',\ p_3 \leqq {}'_3.$$

We identify $(p_2, 1)$ with p_2, $(1, p_3)$ with p_3. Now (as forcing notions) $B_2 \lessdot Q$, $B_2^3 \lessdot Q$, and let B be the completion of Q (to a Boolean algebra); now (see e.g. [3] §6) $B_2^2 \lessdot B$, $B_2^3 \lessdot P$ (and elements of $B_2^3 - B_1^2$, $B_2^2 - B_1^2$ are not identified with elements of B_2^2, B_2^3 resp.). Let P_2^3 be the image under g of P_2^1, and $P = P_2^2 \cup P_2^3$. We choose g_1, g_2, B_2^3, P_2^3, B_2^4, P_2^1 in (*) as id, g, B_2^3, P_2^3, B_2^2, P_2^2 here resp. What we want is $(B_2^2, P_2^2) \leqq (B, P)$, $(B_2^3, P_2^3) \leqq (B, P)$. By the symmetry in the situation it is enough to prove:

(**) if $\underset{\sim}{r} \in P - P_2^2$, then in $L[H]^B$, $\underset{\sim}{r}$ is quasi-generic over $L[H]^{B_2^3}$.

By the Key Fact (11), $\underset{\sim}{r}$ is quasi-generic over $L[H]^{B_1^2}$. Let $G_1^2 \subseteq B_1^2$ be generic over $L[H]$. Now in $L[H, G_1^2]$, B/G_1^2 is equivalent to $(B_2^2/G_1^2) \times (B_2^3/G_1^2)$, and $\underset{\sim}{r}$ is (essentially) a B_2^2/G_1^2-name of a real. Let $\underset{\sim}{s}$ be a (B_2^3/C_1^2)-name of a real, and it suffices to prove

(***) in $L[H, G_1^2]$, $\Vdash_{B/G_1^2}$ "$\underset{\sim}{r}$ is not dominated by $\underset{\sim}{s}$".

If not, then for some $(p_2, p_3) \in (B_2^2/G_1^2) \times (B_2^3/G_1^2)$, and $k < \omega$,

$$(p_2, p_3) \Vdash_{B/G_1^2} \text{“}(\forall n)(k \leqq n < \omega \rightarrow \underset{\sim}{r}(m) \leqq \underset{\sim}{s}(n))\text{”}.$$

For every $l < \omega$ there are $m_l < \omega$ and p_3^l, $p_3 \leqq p_3^l \in B_2^3/G_1^2$, $p_3^l \Vdash_{B_2^3/G_1^2}$ “$\underset{\sim}{s}(l) = m$”. Clearly $\langle m_l : l < \omega \rangle$ is in $L[H, G_1^2]$ hence $p_2 \nVdash_{B_2^3/G_1^2}$ “$(\forall l)(k \leqq l < \omega \rightarrow \underset{\sim}{r}(l) \leqq m_l)$”. Hence for some p_2^1, $p_2 \leqq p_2^1 \in B_2^2/G_1^2$ and $l, k < l < \omega$, $p_2 \Vdash$ “$\underset{\sim}{r}(l) > m_l$”. Now $(p_2^1, p_3^l) \in (B_2^2/G_1^2) \times (B_2^3/G_1^2)$ contradicts the choice of (p_2, p_3) and k. So we have proved (**) hence (*) of (10).

REMARK. What happens if, in the theorem, we change in the conclusion $V^* \vDash$ “every set of reals has the Baire property”?

It seems that a different method is necessary (non-κ-chain condition).

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