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# Strong splitting in stable homogeneous models

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**Abstract**

In this paper we study elementary submodels of a stable homogeneous structure. We improve the independence relation defined in Hyttinen (Fund. Math. 156 (1998) 167–182). We apply this to prove a structure theorem. We also show that dop and sdop are essentially equivalent, where the negation of dop is the property we use in our structure theorem and sdop implies nonstructure, see Hyttinen (1998). © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Basic definitions and spectrum of stability

The purpose of this paper is to develop theory of independence for elementary submodels of a homogeneous structure. We get a model class of this kind if in addition to its first-order theory we require that the models omit some (reasonable) set of types, see [2]. If the set is empty, then we are in the 'classical situation' from [3]. In other words, we study stability theory without the compactness theorem. So e.g. the theory of $\Delta$-ranks is lost and so we do not get an independence notion from ranks. Our independence notion is based on strong splitting. It satisfies the basic properties of forking in a rather weak form. The main problem is finding free extensions. So the arguments are often based on the definition of the independence notion instead of the 'independence-calculus'.

Throughout this paper we assume that $\mathbf{M}$ is a homogeneous model of similarity type (=language) $L$ and that $\mathbf{M}$ is $\xi$-stable for some $\xi < |\mathbf{M}|$ (see [3, Definition 2.2]). Let

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$\lambda(\mathbf{M})$ be the least such $\xi$. By [2], $\lambda(\mathbf{M}) < \beth((2^{|L|+\omega})^+)$. We use $\mathbf{M}$ as a monster model and so we assume that the cardinality of $\mathbf{M}$ is large enough for all constructions we do in this paper. In fact, we assume that $|\mathbf{M}|$ is strongly inaccessible. Alternatively, we could assume less about $|\mathbf{M}|$ and instead of studying all elementary submodels of $\mathbf{M}$, we could study suitably small ones.

Note $Th(\mathbf{M})$ may well be unstable. Note also that if $\Delta$ is a stable finite diagram, then $\Delta$ has a monster model like $\mathbf{M}$, see [2].

By a model we mean an elementary submodel of $\mathbf{M}$ of cardinality $<|\mathbf{M}|$, we write $\mathscr{A}$, $\mathscr{B}$ and so on for these. So if $\mathscr{A} \subseteq \mathscr{B}$ are models, then $\mathscr{A}$ is an elementary submodel of $\mathscr{B}$. Similarly by a set we mean a subset of $\mathbf{M}$ of cardinality $<|\mathbf{M}|$, unless we explicitly say otherwise. We write $A$, $B$ and so on for these. By $a$, $b$ and so on we mean a finite sequence of elements of $\mathbf{M}$. By $a \in A$ we mean $a \in A^{length(a)}$.

By an automorphism we mean an automorphism of $\mathbf{M}$. We write $Aut(A)$ for the set of all automorphisms of $\mathbf{M}$ such that $f \restriction A = id_A$. By $S^*(A)$ we mean the set of all consistent complete types over $A$ and by $\mathrm{t}(a,A)$ we mean the type of $a$ over $A$ in $\mathbf{M}$. $S^m(A)$ means the set $\{\mathrm{t}(a,A) \mid a \in \mathbf{M},\ length(a) = m\}$ and $S(A) = \bigcup_{m<\omega} S^m(A)$.

We define $\kappa(\mathbf{M})$ as $\kappa(T)$ is defined in the case of stable theories but for strong splitting, i.e. we let $\kappa(\mathbf{M})$ be the least cardinal such that there are no $a$, $b_i$ and $c_i$, $i < \kappa(\mathbf{M})$, such that

(i) for all $i < \kappa(\mathbf{M})$, there is an infinite indiscernible set $I_i$ over $\bigcup_{j<i}(b_j \cup c_j)$ such that $b_i, c_i \in I_i$,

(ii) for all $i < \kappa(\mathbf{M})$, there is $\phi_i(x,y)$ such that $\models \phi_i(a,b_i) \wedge \neg\phi_i(a,c_i)$.

We say that a type $p$ over $A$ is $\mathbf{M}$-consistent if there is $a \in \mathbf{M}$ such that $p \subseteq \mathrm{t}(a,A)$ (i.e. there is $q \in S(A)$ such that $p \subseteq q$).

**Lemma 1.1** (Hyttinen [1]). *If $p \in S^*(A)$ is not $\mathbf{M}$-consistent, then there is finite $B \subseteq A$ such that $p \restriction B$ is not $\mathbf{M}$-consistent.*

**Lemma 1.2.** (i) *If $(a_i)_{i<\omega}$ is order-indiscernible over $A$ then it is indiscernible over $A$.*

(ii) *Assume $\mathbf{M}$ is $\xi$-stable and $|I| > \xi \geqslant |A|$. Then there is $J \subseteq I$ of power $> \xi$ such that it is indiscernible over $A$.*

(iii) *If $I$ is infinite indiscernible over $A$ then for all $\xi \leqslant |\mathbf{M}|$ there is $J \supseteq I$ of power $\geqslant \xi$ such that $J$ is indiscernible over $A$.*

(iv) *For all indiscernible $I$ and $\phi(x,a)$, either $X = \{b \in I \mid \models \phi(b,a)\}$ or $Y = \{b \in I \mid \models \neg\phi(b,a)\}$ is of power $< \lambda(\mathbf{M})$.*

(v) *There are no increasing sequence of sets $A_i$, $i < \lambda(\mathbf{M})$, and $a$ such that for all $i < \lambda(\mathbf{M})$, $\mathrm{t}(a, A_{i+1})$ splits over $A_i$. So for all $A$ and $p \in S(A)$, there is $B \subseteq A$ of power $< \lambda(\mathbf{M})$, such that $p$ does not split over $B$.*

(vi) *For all $A$ and $p \in S(A)$, there is $B \subseteq A$ of power $< \kappa(\mathbf{M})$, such that $p$ does not split strongly over $B$.*

**Proof.** Conditions (i), (ii) and (v) as in [1]. Condition (iii) follows immediately from the homogeneity of $\mathbf{M}$. Condition (vi) is trivial.

We prove (iv): Assume not. Let $I$ be a counterexample. Clearly, we may assume that $|I|=\lambda(\mathbf{M})$. Then By Lemma 1.1, for every $J\subseteq I$, the type

$$p_J=\{\phi(b,y)\mid b\in J\}\cup\{\neg\phi(b,y)\mid b\in I-J\}$$

is $\mathbf{M}$-consistent. Clearly, this contradicts $\lambda(\mathbf{M})$-stability of $\mathbf{M}$. □

**Corollary 1.3.** $\kappa(\mathbf{M})\leqslant\lambda(\mathbf{M})$.

**Proof.** Follows immediately from Lemma 1.2(v). □

We will use Lascar strong types instead of strong types:

**Definition 1.4.** Let $SE^n(A)$ be the set of all equivalence relation $E$ in $\mathbf{M}^n$, such that the number of equivalence classes is $<|\mathbf{M}|$ and for all $f\in Aut(A)$, $a\,E\,b$ iff $f(a)\,E\,f(b)$. Let $SE(A)=\bigcup_{n<\omega} SE^n(A)$.

Note that $E\in SE(A)$ need not be definable but an indiscernible set over $A$ is also an indiscernible set for all $E\in SE(A)$.

Usually, we either do not mention the arities of the equivalence relations we work with, or we mention that the arity is, e.g. $m$, but we do not specify what $m$ is. This is harmless since usually there is no danger of confusion.

**Lemma 1.5.** *If $I$ is an infinite indiscernible set over $A$, then for all $E\in SE(A)$ and $a,b\in I$, $a\,E\,b$.*

**Proof.** Assume not. Let $E\in SE(A)$ be a counterexample. Then for all $a,b\in I$, $a\neq b$, $\neg(a\,E\,b)$. Then Lemma 1.2(iii) implies a contradiction with the number of equivalence classes of $E$. □

**Lemma 1.6.** *If $E\in SE(A)$, $|A|\leqslant\xi$ and $\mathbf{M}$ is $\xi$-stable, then the number of equivalence classes of $E$ is $\leqslant\xi$.*

**Proof.** Assume not. Then by Lemma 1.2(ii), we can find $I$ such that it is infinite indiscernible over $A$ and for all $a,b\in I$, if $a\neq b$ then $\neg(a\,E\,b)$. This contradicts Lemma 1.5. □

**Corollary 1.7.** *For all $A$ and $n<\omega$, there is $E^n_{min,A}\in SE^n(A)$ such that for all $a,b$ and $E\in SE^n(A)$, $a\,E^n_{min,A}\,b$ implies $a\,E\,b$.*

**Proof.** Clearly $|SE^n(A)|$ is restricted ($\leqslant 2^{|S(A)|}$) and so $\cap SE^n(A)\in SE(A)$. Trivially $\cap SE^n(A)$ has the wanted property. □

**Definition 1.8.** (i) We say that $\mathscr{A}$ is $F^{\mathbf{M}}_{\kappa}$-saturated if for all $A\subseteq\mathscr{A}$ of power $<\kappa$ and $a$, there is $b\in\mathscr{A}$ such that $\mathrm{t}(b,A)=\mathrm{t}(a,A)$.

(ii) We say that $\mathscr{A}$ is strongly $F^{\mathbf{M}}_{\kappa}$-saturated if for all $A \subseteq \mathscr{A}$ of power $<\kappa$ and $a$ of length $m$, there is $b \in \mathscr{A}$ such that $b\,E\,a$ for all $E \in SE^m(A)$. We write $a$-saturated for strongly $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated.

**Lemma 1.9.** (i) *If $\mathscr{A}$ is strongly $F^{\mathbf{M}}_{\kappa}$-saturated then it is $F^{\mathbf{M}}_{\kappa}$-saturated.*

(ii) *Assume $|A| \leqslant \xi$, $\mathbf{M}$ is $\xi$-stable, $\xi^{<\kappa} = \xi$ and there is a regular cardinal $\delta$ such that $\kappa \leqslant \delta \leqslant \xi$. Then there is strongly $F^{\mathbf{M}}_{\kappa}$-saturated $\mathscr{A} \supseteq A$ such that $|\mathscr{A}| \leqslant \xi$. Furthermore if $\mathscr{B} \supseteq A$ is strongly $F^{\mathbf{M}}_{\kappa}$-saturated, then we can choose $\mathscr{A}$ so that $\mathscr{A} \subseteq \mathscr{B}$.*

(iii) *Assume $\mathbf{M}$ is $\xi$-stable, $\mathscr{A}$ is $F^{\mathbf{M}}_{\xi}$-saturated, $A \subseteq \mathscr{A}$ is of power $<\xi$ and $m<\omega$. Then there are $a_i \in \mathscr{A}$, $i<\xi$, such that for all $b$ of length $m$, there is $i<\xi$ such that $a_i\;E\,b$, for all $E \in SE^m(A)$, i.e. $\mathscr{A}$ is strongly $F^{\mathbf{M}}_{\xi}$-saturated.*

(iv) *If $\mathscr{A}$ is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated, then it is $a$-saturated.*

(v) *Assume $\mathscr{A}$ is strongly $F^{\mathbf{M}}_{\xi}$-saturated and $A \subseteq \mathscr{A}$ is of power $<\xi$. Then for all $B$ of power $<\xi$, there is $f \in Aut(A)$ such that $f(B) \subseteq \mathscr{A}$ and for all (finite sequences) $b \in B$, $f(b)\,E^m_{min,A}\,b$.*

**Proof.** Condition (i) is trivial.

(ii) For all $i \leqslant \delta$, choose sets $A_i$ of power $\leqslant \xi$ as follows: Let $A_0 = A$ and if $i$ is limit then $A_i = \bigcup_{j<i} A_j$. If $A_i$ is defined, then we let $A_{i+1} \supseteq A_i$ be such that for all $B \subseteq A_i$ of power $<\kappa$ and $a$ there is $b \in A_{i+1}$ such that $b\,E^m_{min,B}\,a$. By Lemma 1.6, we can find $A_{i+1}$ so that $|A_{i+1}| \leqslant \xi$. By Lemma 1.7, $A_\delta$ is as wanted.

(iii) By Lemma 1.6, choose $b_i$, $i<\xi$, so that for all $b$ there is $i<\xi$ such that $b\,E^m_{min,A}\,b_i$. Since $\mathscr{A}$ is $F^{\mathbf{M}}_{\xi}$-saturated, we can choose $a_i \in \mathscr{A}$ so that there is $f \in Aut(A)$ such that for all $i<\xi$, $f(b_i) = a_i$. Clearly this implies the claim.

(iv) Immediate by (iii).

(v) For all $c \in B$, choose $a_c \in \mathscr{A}$ so that $a_c\,E^m_{min,A}\,c$. Since $\mathscr{A}$ is $F^{\mathbf{M}}_{\xi}$-saturated, there is $f \in Aut(A \cup \{a_c \mid c \in B\})$ such that $f(B) \subseteq \mathscr{A}$. Clearly $f$ is as wanted. □

**Definition 1.10.** We write $f \in Saut(A)$ if $f \in Aut(A)$ and for all $a$, $f(a)\,E^m_{min,A}\,a$.

**Lemma 1.11.** *Assume $\mathbf{M}$ is $\xi$-stable and $|A|<\xi$. If $a\,E^m_{min,A}\,b$, then there is $f \in Saut(A)$ such that $f(a) = b$.*

**Proof.** We define $a\,E\,b$ if there is $f \in Saut(A)$ such that $f(a) = b$. Clearly it is enough to show that $E \in SE(A)$. For a contradiction, assume that this is not the case. Since $E$ is an equivalence relation and $f(E) = E$ for all $f \in Aut(A)$, there are $a_i$, $i<\xi^+$, such that for all $i \neq j$, $\neg(a_i\;E\,a_j)$. Choose $B \supseteq A$ of power $\xi$ such that every $E^m_{min,A}$-equivalence class is represented in $B$. Since $\mathbf{M}$ is $\xi$-stable, there are $i<j<\xi^+$, such that $\mathrm{t}(a_i,B) = \mathrm{t}(a_j,B)$. Then there is $f \in Aut(B)$ such that $f(a_i) = f(a_j)$. By the choice of $B$, $f \in Saut(A)$, a contradiction. □

**Lemma 1.12.** *Assume $\xi$ is such that for some $\xi' \geqslant \xi$, $\mathbf{M}$ is $\xi'$-stable. If $\mathscr{A}$ is $F^{\mathbf{M}}_{\xi}$-saturated and $A \subseteq \mathscr{A}$ has power $<\xi$, then $\mathrm{t}(a,\mathscr{A})$ does not split strongly over $A$ iff for all $b,c \in \mathscr{A}$ and $\phi$, $b\,E^m_{min,A}\,c$ implies $\models \phi(a,b) \leftrightarrow \phi(a,c)$.*

**Proof.** If $t(a,\mathscr{A})$ splits strongly over $A$, then by Lemma 1.5, there are $b,c\in\mathscr{A}$ and $\phi$, such that $b\,E^m_{min,A}\,c$ and $\models\neg(\phi(a,b)\leftrightarrow\phi(a,c))$. So we have proved the claim from right to left. We prove the other direction: For a contradiction assume that there are $b,c\in\mathscr{A}$ and $\phi$, such that $b\,E^m_{min,A}\,c$ and $\models\phi(a,b)\wedge\neg\phi(a,c)$.

We define an equivalence relation $E$ on $\mathbf{M}^m$ as follows: $a\,E\,b$ if $a=b$ or there are $I_i$, $i<n<\omega$, such that they are infinite indiscernible over $A$, $a\in I_0$, $b\in I_{n-1}$ and for all $i<n-1$, $I_i\cap I_{i+1}\neq\emptyset$. Clearly $E$ is an equivalence relation and for all $f\in Aut(A)$, $f(E)=E$. By Lemma 1.2(ii), the number of equivalence classes of $E$ is $<|\mathbf{M}|$. So $E\in SE^m(A)$.

Then $b\,E\,c$ and $b\neq c$. Let $I_i$, $i<n$, be as in the definition of $E$. Since $\mathscr{A}$ is $F^{\mathbf{M}}_{|A|^++\omega}$-saturated, we may assume that for all $i<n$, $I_i\subseteq\mathscr{A}$. Since $t(a,\mathscr{A})$ does not split strongly over $A$, for all $d\in I_0$, $\models\phi(a,d)$. So there is $d\in I_1$ such that $\models\phi(a,d)$. Again since $t(a,\mathscr{A})$ does not split strongly over $A$, for all $d\in I_1$, $\models\phi(a,d)$. We can carry this on and finally we get that $\models\phi(a,c)$, a contradiction. □

**Lemma 1.13.** *Assume $A\subseteq\mathscr{A}$, $|A|<\kappa(\mathbf{M})$, $\mathscr{A}$ is a-saturated and $p\in S(\mathscr{A})$ does not split strongly over A. Then for all $B\supseteq\mathscr{A}$, there is $q\in S(B)$ such that $p\subseteq q$ and for all $C\supseteq B$ there is $r\in S(C)$, which satisfies $q\subseteq r$ and $r$ does not split strongly over A.*

**Proof.** We define $q\in S^*(B)$ as follows: $\phi(x,b)\in q$, $b\in B$, if there is $a\in\mathscr{A}$ such that $a\,E^m_{min,A}\,b$ and $\phi(x,a)\in p$, where $m=length(b)$. By Lemma 1.12, it is enough to show that $q$ is $\mathbf{M}$-consistent. By Lemma 1.1, it is enough to show that for all $a,a'\in\mathscr{A}$, if $a\,E^m_{min,A}\,a'$, then $\phi(x,a)\in p$ implies $\phi(x,a')\in p$. This follows from Lemma 1.12, since by Lemma 1.9(i), $\mathscr{A}$ is $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated. □

**Lemma 1.14.** *Assume $A\subseteq\mathscr{A}\subseteq\mathscr{B}$, $|A|<\kappa(\mathbf{M})$, $\mathscr{B}$ is $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated and for every $c\in\mathscr{B}$ there is $d\in\mathscr{A}$ such that $d\,E^m_{min,A}\,c$. If $t(a,\mathscr{A})=t(b,\mathscr{A})$ and both $t(a,\mathscr{B})$ and $t(b,\mathscr{B})$ do not split strongly over A, then $t(a,\mathscr{B})=t(b,\mathscr{B})$.*

**Proof.** For a contradiction, assume $c\in\mathscr{B}$ and $\models\phi(a,c)\wedge\neg\phi(b,c)$. Choose $d\in\mathscr{A}$ such that $d\,E^m_{min,A}\,c$. By Lemma 1.12, $\models\phi(a,d)\wedge\neg\phi(b,d)$, a contradiction. □

**Lemma 1.15.** *If $\xi=\lambda(\mathbf{M})+\xi^{<\kappa(\mathbf{M})}$, then $\mathbf{M}$ is $\xi$-stable.*

**Proof.** Clearly, we may assume that $\xi>\lambda(\mathbf{M})$ and so by Corollary 1.3, $\xi\geqslant\kappa(\mathbf{M})^+$. Let $A$ be a set of power $\xi$. We show that $|S(A)|\leqslant\xi$.

**Claim.** *There is $\mathscr{A}\supseteq A$ such that*

(i) *$\mathscr{A}$ is $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated,*

(ii) $|\mathscr{A}|\leqslant\xi$,

(iii) *for all $B\subseteq\mathscr{A}$ of power $<\kappa(\mathbf{M})$ there is $\mathscr{A}_B\subseteq\mathscr{A}$ of power $\lambda(\mathbf{M})$ satisfying: $B\subseteq\mathscr{A}_B$ and for all $c\in\mathbf{M}$ there is $d\in\mathscr{A}_B$ such that $d\,E^m_{min,A}\,c$.*

**Proof.** By induction on $i<\kappa(\mathbf{M})^+$, we define $\mathscr{A}_i$ so that $|\mathscr{A}_i|\leqslant\xi$, $A\subseteq\mathscr{A}_0$, for $i<j$, $\mathscr{A}_j\subseteq\mathscr{A}_i$ and

(1) if $i$ is odd then for all $B\subseteq\bigcup_{j<i}\mathscr{A}_j$ of power $<\kappa(\mathbf{M})$, there is $\mathscr{A}_B\subseteq\mathscr{A}_i$ of power $\leqslant\lambda(\mathbf{M})$ satisfying: $B\subseteq\mathscr{A}_B$ and for all $c\in\mathbf{M}$ there is $d\in\mathscr{A}_B$ such that $dE^m_{min,A}\,c$,

(2) if $i$ is even then for all $B\subseteq\bigcup_{j<i}\mathscr{A}_j$ of power $<\kappa(\mathbf{M})$, every $p\in S(B)$ is realized in $\mathscr{A}_i$.

By Corollary 1.3, Lemma 1.6 and the fact that $|S(B)|\leqslant\lambda(\mathbf{M})$ for all $B$ of power $<\kappa(\mathbf{M})^+$, it is easy to see that such $\mathscr{A}_i$, $i<\kappa(\mathbf{M})$, exist. Clearly $\mathscr{A}=\bigcup_{i<\kappa(\mathbf{M})^+}\mathscr{A}_i$ is as wanted. □

So it is enough to show that $|S(\mathscr{A})|\leqslant\xi$. By Lemma 1.2(vi), for each $p\in S(\mathscr{A})$, choose $B_p$ so that $p$ does not split strongly over $B_p$ and $|B_p|<\kappa(\mathbf{M})$. Then by Lemma 1.14, every type $p\in S(\mathscr{A})$ is determined by $p\restriction\mathscr{A}_{B_p}$ and the fact that it does not split strongly over $B$. Since the number of possible $B$ is $\xi^{<\kappa(\mathbf{M})}=\xi$ and for each such $B$, $|S(\mathscr{A}_B)|\leqslant\lambda(\mathbf{M})$, $|S(\mathscr{A})|\leqslant\xi\times\lambda(\mathbf{M})=\xi$. □

**Lemma 1.16.** *If $\xi^{<\kappa(\mathbf{M})}>\xi$, then $\mathbf{M}$ is not $\xi$-stable.*

**Proof.** By the definition of $\lambda(\mathbf{M})$, we may assume that $\xi\geqslant\lambda(\mathbf{M})$. Let $\kappa<\kappa(\mathbf{M})$ be the least cardinal such that $\xi^\kappa>\xi$. By the definition of $\kappa(\mathbf{M})$, there are $a, b_i$ and $c_i$, $i<\kappa$, such that

(i) for all $i<\kappa$, there is an infinite indiscernible set $I'_i$ over $\bigcup_{j<i}(b_j\cup c_j)$ such that $b_i, c_i\in I'_i$,

(ii) for all $i<\kappa$, there is $\phi_i(x,y)$ such that $\models\phi_i(a,b_i)\wedge\neg\phi_i(a,c_i)$.

**Claim.** *There are $I_i$, $i<\kappa$, such that for all $i<\kappa$, $I_i=\{d^i_k\,|\,k<\xi\}$ is indiscernible over $\bigcup_{j<i}I_j$, $b_i,c_i\in I_i$ and for $k<k'<\xi$, $d^i_k\neq d^i_{k'}$.*

**Proof.** By induction on $0<\alpha\leqslant\kappa$, we define $I^\alpha_i=\{d^{\alpha,i}_k\,|\,k<\xi\}$, $i<\alpha$, such that

(1) for all $i<\alpha$, $I^\alpha_i$ is indiscernible over $\bigcup_{j<i}I^\alpha_j$ and $b_i,c_i\in I^\alpha_i$,

(2) for all $\beta<\alpha$, there is an automorphism $f$ such that $f\restriction\bigcup_{j<\beta}(b_j\cup c_j)= id_{\bigcup_{j<\beta}(b_j\cup c_j)}$ and for all $j<\beta$, $f(d^{\beta,j}_k)=d^{\alpha,j}_k$, $k<\xi$,

(3) for all $i<\alpha$ and $k<k'<\xi$, $d^{\alpha,i}_k\neq d^{\alpha,i}_{k'}$.

Clearly this is enough, since then $I^\kappa_i$, $i<\kappa$, are as wanted.

By (2) and homogeneity of $\mathbf{M}$, limits are trivial, so we assume that $\alpha=\beta+1$ and that $I^\beta_j$, $j<\beta$, are defined. By Lemma 1.15, there is $\delta>\xi$ such that $\mathbf{M}$ is $\delta$-stable. By the assumptions and Lemma 1.2(iii), there is $J=\{d_k\,|\,k<\delta^+\}$ such that it is indiscernible over $\bigcup_{j<\beta}(b_j\cup c_j)$ and $b_\beta,c_\beta\in J$. By Lemma 1.2(ii), there is $I\subseteq J$ of power $\xi$, such that it is indiscernible over $\bigcup_{j<\beta}I^\beta_j$. Since $J$ is indiscernible over $\bigcup_{j<\beta}(b_j\cup c_j)$, there is an automorphism $f$ such that $f\restriction\bigcup_{j<\beta}(b_j\cup c_j)=id_{\bigcup_{j<\beta}(b_j\cup c_j)}$ and $b_\beta,c_\beta\in\{f(d)\,|\,d\in I\}$. We let $I^\alpha_\beta=f(I)$ and if $i<\beta$, then $I^\alpha_i=f(I^\beta_i)$. Clearly these are as required. □

By Lemma 1.2(iv) we may assume that for all $i<\kappa$, $\models \phi_i(a, d^i_k)$ iff $k=0$. Then for all $\eta \in \xi^\kappa$ and $0<\alpha\leqslant\kappa$, we define function $f^\eta_\alpha$ so that the following holds ($f^\eta_0 = id_{\mathbf{M}}$):
(a) for all $i<\beta<\alpha$ and $\eta\in\xi^\kappa$, $f^\eta_\alpha \restriction I_i = f^\eta_\beta \restriction I_i$,
(b) if $\alpha=\beta+1$ and $\eta\in\xi^\kappa$, then

$$f^\eta_\alpha(f^\eta_\beta(d^\beta_0)) = f^\eta_\beta(d^\beta_{\eta(\beta)}),$$

$$f^\eta_\alpha(f^\eta_\beta(d^\beta_{\eta(\beta)})) = f^\eta_\beta(d^\beta_0)$$

and for all $i<\xi$, $i\neq 0, \eta(\beta)$,

$$f^\eta_\alpha(f^\eta_\beta(d^\beta_i)) = f^\eta_\beta(d^\beta_i),$$

(c) if $\eta\restriction\alpha = \eta'\restriction\alpha$ then $f^\eta_\alpha = f^{\eta'}_\alpha$.
It is easy to see that such $f^\eta_\alpha$ exist. For limit $\alpha$ this follows from the homogeneity of $\mathbf{M}$ and for successors this follows from the fact that $f^\eta_\beta(I_\beta)$ is indiscernible over $\bigcup_{i<\beta} f^\eta_\beta(I_i)$.

For all $\eta\in\xi^\kappa$, let $a_\eta = f^\eta_\kappa(a)$. Then clearly for $\eta\neq\eta'$, the types of $a_\eta$ and $a_{\eta'}$ over $A=\bigcup\{f^\nu_{\alpha+1}(I_\alpha) \mid \nu\in\xi^\kappa,\ \alpha<\kappa\}$ are different. By the choice of $\kappa$, $\xi^{<\kappa}=\xi$ and so by (c), $|A|=\xi$. Since $\xi^\kappa > \xi$, $\mathbf{M}$ is not $\xi$-stable. □

So we have proved the following theorem. With slightly different definitions this theorem is already proved in [2].

**Theorem 1.17.** $\mathbf{M}$ *is* $\xi$*-stable iff* $\xi = \lambda(\mathbf{M}) + \xi^{<\kappa(\mathbf{M})}$.

**Proof.** Follows from Lemmas 1.15 and 1.16. □

Let $\kappa_r(\mathbf{M})$ be the least regular $\kappa\geqslant\kappa(\mathbf{M})$. By Lemma 1.16, $\lambda(\mathbf{M})^{<\kappa(\mathbf{M})} = \lambda(\mathbf{M})$ and so $cf(\lambda(\mathbf{M}))\geqslant\kappa(\mathbf{M})$. Because $cf(\lambda(\mathbf{M}))$ is regular, $\kappa_r(\mathbf{M})\leqslant\lambda(\mathbf{M})$.

## 2. Indiscernible sets

In this section we prove basic properties of indiscernible sets. We start by improving Lemma 1.2(iv).

**Lemma 2.1.** *For all infinite indiscernible* $I$ *and* $a$ *there is* $p\in S(a)$ *such that*

$$|\{b\in I \mid \mathrm{t}(b,a)\neq p\}| < \kappa(\mathbf{M}).$$

**Proof.** Assume not. By Lemma 1.2(iii), we may assume that $I$ and $a$ are such that $I=\{b_i \mid i<\kappa(\mathbf{M})+\omega\cdot\kappa(\mathbf{M})\}$, $b_i\neq b_j$ for $i\neq j$ and for some $p\in S(a)$, $\mathrm{t}(b_i,a)=p$ iff $i\geqslant\kappa(\mathbf{M})$. For all $i<\kappa(\mathbf{M})$, we define $A_i$ as follows:
(i) $A_0=\emptyset$,

(ii) $A_{i+1}=A_i\cup\{b_{i-1}\}\cup\{b_j \mid \omega\cdot i\leqslant j<\omega\cdot(i+1)\}$,
(iii) for limit $i$, $A_i=\bigcup_{j<i}A_j$.
Then it is easy to see that for all $i<\kappa(\mathbf{M})$ $\mathrm{t}(a,A_{i+1})$ splits strongly over $A_i$, a contradiction. □

**Corollary 2.2.** *For all indiscernible $I$ and $\phi(x,a)$, either $X=\{b\in I \mid \models\phi(b,a)\}$ or $Y=\{b\in I \mid \models\neg\phi(b,a)\}$ is of power $<\kappa(\mathbf{M})$.*

**Proof.** Follows immediately from Lemma 2.1. □

**Definition 2.3.** If $I$ is indiscernible and of power $\geqslant\kappa(\mathbf{M})$, we write $Av(I,A)$ for $\{\phi(x,a) \mid a\in A,\ \phi\in L,\ |\{b\in I \mid \models\neg\phi(b,a)\}|<\kappa(\mathbf{M})\}$.

**Lemma 2.4.** (i) *If $I$ is indiscernible over $A$ and of power $\geqslant\kappa(\mathbf{M})$, then $I\cup\{b\}$ is indiscernible over $A$ iff* $\mathrm{t}(b,I\cup A)=Av(I,I\cup A)$.

(ii) *If $I$ and $J$ are of power $\geqslant\kappa(\mathbf{M})$ and $I\cup J$ is indiscernible, then for all $A$, $Av(I,A)=Av(J,A)$.*

(iii) *If $I$ is indiscernible and of power $\geqslant\kappa(\mathbf{M})$, then for all $A$, $Av(I,A)$ is* **M**-*consistent.*

**Proof.** Conditions (i) and (ii) are trivial. We prove (iii): By (ii) and Lemma 1.2(iii), we may assume that $|I|>|L\cup A|+\kappa_r(\mathbf{M})$. Then the claim follows by the pigeon hole principle from (i). □

**Definition 2.5.** Assume $I$ and $J$ are indiscernible sets of power $\geqslant\kappa(\mathbf{M})$.

(i) We say that $I$ is based on $A$ if for all $B\supseteq A\cup I$, $Av(I,B)$ does not split strongly over $A$.

(ii) We say that $I$ and $J$ are equivalent if for all $B$, $Av(I,B)=Av(J,B)$.

(iii) We say that $I$ is stationary over $A$ if $I$ is based on $A$ and for all $f\in Aut(A)$, $f(I)$ and $I$ are equivalent.

**Lemma 2.6.** *Assume $I$ is an indiscernible set of power $\geqslant\kappa(\mathbf{M})$, $|A|<\xi$ and* **M** *is $\xi$-stable. Then the following are equivalent*:

(i) *$I$ is based on $A$,*
(ii) *the number of non-equivalent indiscernible sets in $\{f(I) \mid f\in Aut(A)\}$ is $\leqslant\xi$,*
(iii) *the number of non-equivalent indiscernible sets in $\{f(I) \mid f\in Aut(A)\}$ is $<|\mathbf{M}|$.*

**Proof.** (i)$\Rightarrow$(ii) Assume not. Let $f_i(I)$, $i<\xi^+$, be a counterexample. For all $i<\lambda(\mathbf{M})$, choose $\mathscr{A}_i$ so that

(a) $A\subseteq\mathscr{A}_0$ and every type $p\in S(A)$ is realized in $\mathscr{A}_0$,
(b) if $i<j$, then $\mathscr{A}_i\subseteq\mathscr{A}_j$ and for limit $i$, $\mathscr{A}_i=\bigcup_{j<i}\mathscr{A}_j$,
(c) every type $p\in S(\mathscr{A}_i)$ is realized in $\mathscr{A}_{i+1}$,
(d) $|\mathscr{A}_i|\leqslant\xi$.

Let $\mathscr{A} = \bigcup_{i<\lambda(\mathbf{M})} \mathscr{A}_i$. Since $\mathbf{M}$ is $\xi$-stable there are $i \neq j$ such that $Av(f_i(I), \mathscr{A}) = Av(f_j(I), \mathscr{A})$. Let $a$ be such that $Av(f_i(I), \mathscr{A} \cup \{a\}) \neq Av(f_j(I), \mathscr{A} \cup \{a\})$. By Lemma 1.2(v), choose $i < \lambda(\mathbf{M})$ so that $t(a, \mathscr{A}_{i+\omega})$ does not split over $\mathscr{A}_i$. Without loss of generality, we may assume that $i = 0$. For all $i < \omega$, choose $a_i \in \mathscr{A}_{i+1}$ so that $t(a_i, \bigcup_{j \leqslant i} \mathscr{A}_j) = t(a, \bigcup_{j \leqslant i} \mathscr{A}_j)$. By an easy induction, we see that $\{a\} \cup \{a_i \mid i < \omega\}$ is order-indiscernible over $\mathscr{A}$ and so also over $A$. By Lemma 1.2(i), $\{a\} \cup \{a_i \mid i < \omega\}$ is indiscernible over $A$. But then clearly either $Av(f_i(I), \mathscr{A} \cup \{a\})$ or $Av(f_j(I), \mathscr{A} \cup \{a\})$ splits strongly over $A$, a contradiction.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Assume not. Then by Lemma 1.2(iii), we can find $J = \{a_i \mid i < |\mathbf{M}|\}$ and $\phi(x, y)$ such that $J$ is indiscernible over $A$, for $i \neq j$, $a_i \neq a_j$, and $\phi(x, a_i) \in Av(I, J)$ iff $i = 0$. But then for all $i < |\mathbf{M}|$, we can find $f_i \in Aut(A)$ such that for all $j < i$, $\phi(x, a_j) \notin Av(f_i(I), J)$ but $\phi(x, a_i) \in Av(f_i(I), J)$. Clearly these $f_i(I)$ are not equivalent, a contradiction. □

## 3. Independence

In this chapter we define an independence relation and prove the basic properties of it. This independence notion is an improved version of the one defined in [1]. It satisfies weak versions of the basic properties of forking, e.g. $a \downarrow_A A$ holds assuming $A$ is $a$-saturated.

**Definition 3.1.** (i) We write $a \downarrow_A B$ if there is $C \subseteq A$ of power $< \kappa(\mathbf{M})$ such that for all $D \supseteq A \cup B$ there is $b$ which satisfies: $t(b, A \cup B) = t(a, A \cup B)$ and $t(b, D)$ does not split strongly over $C$. We write $C \downarrow_A B$ if for all $a \in C$, $a \downarrow_A B$.

(ii) We say that $t(a, A)$ is bounded if $|\{b \mid t(b, A) = t(a, A)\}| < |\mathbf{M}|$. If $t(a, A)$ is not bounded, we say that it is unbounded.

**Lemma 3.2.** (i) *If $A \subseteq A' \subseteq B' \subseteq B$ and $a \downarrow_A B$ then $a \downarrow_{A'} B'$.*

(ii) *If $A \subseteq B$ and $a \downarrow_A B$ then for all $C \supseteq B$ there is $b$ such that $t(b, B) = t(a, B)$ and $b \downarrow_A C$.*

(iii) *Assume that $\mathscr{A}$ is a-saturated. If $A \subseteq \mathscr{A}$ is such that $t(a, \mathscr{A})$ does not split strongly over $A$ then for all $B$ such that $A \subseteq B \subseteq \mathscr{A}$, $a \downarrow_B \mathscr{A}$. Especially $a \downarrow_{\mathscr{A}} \mathscr{A}$.*

(iv) *Assume $a$ and $A$ are such that $t(a, A)$ is bounded. Then for all $B \supseteq A$, $t(a, B)$ does not split strongly over $A$.*

(v) *Assume $A \subseteq B$ and $t(a, A)$ is unbounded. If $t(a, B)$ is bounded, then $a \not\downarrow_A B$.*

(vi) *Assume $\mathscr{A}$ is a-saturated and $a \notin \mathscr{A}$. Then $t(a, \mathscr{A})$ is unbounded.*

(vii) *Let $\xi$ be a cardinal. Assume $a$ and $A$ are such that $t(a, A)$ is unbounded and $a \downarrow_A A$. If $a_i$, $i < \xi$, are such that for all $i < \xi$, $t(a_i, A) = t(a, A)$ and $a_i \downarrow_A \bigcup_{j<i} a_j$, then $|\{a_i \mid i < \xi\}| = \xi$.*

(viii) *Assume $A \subseteq B$, $a \downarrow_A A$ and $t(a, A)$ is unbounded. Then there is $b$ such that $b \downarrow_A B$ and $b E^m_{min,A} a$.*

(ix) *If $a \downarrow_A b \cup c$ and $b E^m_{min,A} c$, then $t(b, A \cup a) = t(c, A \cup a)$.*

**Proof.** (i) is immediate by the definition of $\downarrow$.

(ii) Choose $a$-saturated $\mathscr{D} \supseteq C$. Since $a \downarrow_A B$, there are $b$ and $A' \subseteq A$ such that $t(b,B) = t(a,B)$, $|A'| < \kappa(\mathbf{M})$ and $t(b,\mathscr{D})$ does not split strongly over $A'$. By Lemma 1.13, $b$ is as wanted.

(iii) By Lemmas 1.2(vi) and 1.13, $a \downarrow_A \mathscr{A}$ and so by (i), $a \downarrow_B \mathscr{A}$.

(iv) Assume not. Then there are distinct $a_i$, $i < |\mathbf{M}|$, and $\phi$, such that $\{a_i \mid i < |\mathbf{M}|\}$ is indiscernible over $A$ and $\models \phi(a, a_i)$ iff $i = 0$. For all $\kappa(\mathbf{M}) \leqslant i < |\mathbf{M}|$, find an automorphism $f_i \in Aut(A)$ such that $f_i(a_0) = a_i$, $f(a_i) = a_0$ and for all $0 < j < i$, $f_i(a_j) = a_j$. By Corollary 2.2, it is easy to see that $\{f_i(a) \mid \kappa(\mathbf{M}) \leqslant i < |\mathbf{M}|\}$ contains $|\mathbf{M}|$ distinct elements, a contradiction.

(v) Assume not. Then by (ii) we can find $C \supseteq B$ and $b$ such that $t(b,B) = t(a,B)$, $b \downarrow_A C$ and $b \in C$. By Lemma 1.2(ii), there is an infinite indiscernible set $I$ over $A$ such that $b \in I$. Clearly, we cannot find $c$ such that $t(c,C) = t(b,C)$ and $t(c, C \cup I)$ does not split strongly over some $A' \subseteq A$, a contradiction.

(vi) Follows immediately from (iii) and (v).

(vii) Immediate by (v).

(viii) Let $\xi > |A|$ be such that $\mathbf{M}$ is $\xi$-stable. Choose $a_i$, $i < \xi^+$ so that $t(a_i, A) = t(a, A)$ and $a_i \downarrow_A \bigcup_{j<i} a_j$. By (vii) and Lemma 1.2(ii), we may assume that $\{a_i \mid i < \omega\}$ is infinite indiscernible over $A$. Clearly we may also assume that $a = a_0$. Let $d = a_1$. Then $t(d,A) = t(a,A)$, $d \downarrow_A a$ and by Lemma 1.5, $d\, E^m_{min,A}\, a$. Then we can choose $b$ so that $t(b, A \cup a) = t(d, A \cup a)$ and $b \downarrow_A a \cup B$. Clearly then $b$ is as wanted.

(ix) Follows immediately from Lemma 1.12. Note that if $b\, E^m_{min,A}\, c$, then for all $d \in A$, $b \cup d\, E^{m+k}_{min,A}\, c \cup d$. $\square$

**Definition 3.3.** (i) We say that $\mathbf{M}$-consistent $p \in S(A)$ is stationary if for all $a$, $b$ and $B \supseteq A$ the following holds: if $t(a,A) = t(b,A) = p$, $a \downarrow_A B$ and $b \downarrow_A B$ then $t(a,B) = t(b,B)$.

(ii) We say that $I$ is $A$-independent if for all $a \in I$, $a \downarrow_A I - \{a\}$.

**Lemma 3.4.** *If $\mathscr{A}$ is a-saturated, then every $\mathbf{M}$-consistent $p \in S(\mathscr{A})$ is stationary.*

**Proof.** Assume not. Choose $\mathscr{B} \supseteq \mathscr{A}$, $a$ and $b$ so that $t(a,\mathscr{A}) = t(b,\mathscr{A})$, $a \downarrow_{\mathscr{A}} \mathscr{B}$, $b \downarrow_{\mathscr{A}} \mathscr{B}$ and $t(a,\mathscr{B}) \neq t(b,\mathscr{B})$. By Lemma 3.2(ii) we may assume that $\mathscr{B}$ is $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated. Choose $c \in \mathscr{B}$ and $\phi$ so that $\models \phi(a,c) \wedge \neg\phi(b,c)$. Let $A \subseteq \mathscr{A}$ be such that $|A| < \kappa(\mathbf{M})$ and both $t(a,\mathscr{B})$ and $t(b,\mathscr{B})$ do not split strongly over $A$. Choose $d \in \mathscr{A}$ so that $dE^m_{min,A}\, c$. By Lemma 1.12, a contradiction follows. $\square$

**Corollary 3.5.** (i) *Assume $\mathscr{A}$ is a-saturated. If $a \not\downarrow_{\mathscr{A}} B$, then there is $b \in B$ such that $a \not\downarrow_{\mathscr{A}} b$.*

(ii) *If $\mathscr{A}$ is a-saturated and $a_i$, $i < \alpha$, are such that $a_0 \notin \mathscr{A}$, for all $i, j$, $t(a_i, \mathscr{A}) = t(a_j, \mathscr{A})$ and $a_i \downarrow_{\mathscr{A}} \bigcup_{j<i} a_j$, then $\{a_i \mid i < \alpha\}$ is indiscernible over $\mathscr{A}$ and $\mathscr{A}$-independent and if $i \neq j$, then $a_i \neq a_j$.*

(iii) *Assume $\mathscr{A}$ is a-saturated. Then for all $B \supseteq \mathscr{A}$ and $C$ there is $D$ such that* $\mathrm{t}(D,\mathscr{A})=\mathrm{t}(C,\mathscr{A})$ *and* $D \downarrow_{\mathscr{A}} B$.

(iv) *If $A \subseteq \mathscr{B} \subseteq C$, $\mathscr{B}$ is a-saturated, $a \downarrow_A \mathscr{B}$ and $a \downarrow_{\mathscr{B}} C$, then $a \downarrow_A C$.*

(v) *Assume $\mathscr{A}$ is a-saturated,* $\mathrm{t}(a,\mathscr{A})$ *does not split strongly over $A \subseteq \mathscr{A}$ and $|A| < \kappa(\mathbf{M})$. Then $a \not\downarrow_{\mathscr{A}} B$ iff there is finite $b \in \mathscr{A} \cup B$ such that $a \not\downarrow_A b$.*

**Proof.** (i) follows immediately from Lemma 3.4 (if $a \not\downarrow_{\mathscr{A}} B$, then $\mathrm{t}(a,\mathscr{A}\cup B)$ is not the unique free extension of $\mathrm{t}(a,\mathscr{A})$, which can be detected from a finite sequence).

(ii) By Lemma 3.4, it is easy to see that $\{a_i \mid i<\alpha\}$ is order-indiscernible over $\mathscr{A}$. By Lemma 1.2(i), $\{a_i \mid i<\alpha\}$ is indiscernible over $\mathscr{A}$. Clearly, this implies that $\{a_i \mid i<\alpha\}$ is $\mathscr{A}$-independent. The last claim follows from Lemma 3.2(v).

(iii) Clearly, it is enough to prove the following: If $D \downarrow_{\mathscr{A}} B$, then for all $c$ there is $d$ such that $\mathrm{t}(d,\mathscr{A}\cup D)=\mathrm{t}(c,\mathscr{A}\cup D)$ and $d \cup D \downarrow_{\mathscr{A}} B$. This follows from Lemmas 1.1, 3.2(ii) and 3.4.

(iv) Choose $b$ so that $\mathrm{t}(b,\mathscr{B})=\mathrm{t}(a,\mathscr{B})$ and $b \downarrow_A C$. Then $b \downarrow_{\mathscr{B}} C$ and so by Lemma 3.4, we get $\mathrm{t}(b,C)=\mathrm{t}(a,C)$. Clearly this implies the claim.

(v) If $a \downarrow_{\mathscr{A}} B$ then by (iv), $a \downarrow_A \mathscr{A} \cup B$ from which it follows that there are no finite $b \in \mathscr{A}\cup B$ such that $a \not\downarrow_A b$. On the other hand if $a \not\downarrow_{\mathscr{A}} B$, then $\mathrm{t}(a,\mathscr{A}\cup B)$ is not the unique 'free' extension of $\mathrm{t}(a,\mathscr{A})$ defined in the proof of Lemma 1.13. This means that there are $c \in \mathscr{A}$ and $d \in \mathscr{A}\cup B$ such that $c\, E^m_{min,A}\, d$ and $\mathrm{t}(c,A\cup a)\neq \mathrm{t}(d,A\cup a)$. Clearly $a \not\downarrow_A c \cup d$. □

**Lemma 3.6.** *If $\mathscr{A}$ is a-saturated and $a \downarrow_{\mathscr{A}} b$, then $b \downarrow_{\mathscr{A}} a$.*

**Proof.** Assume not. Let $\xi > |\mathscr{A}|$ be such that $\mathbf{M}$ is $\xi$-stable. For all $i<\xi^+$, choose $a_i$ and $b_i$ so that $\mathrm{t}(a_i,\mathscr{A})=\mathrm{t}(a,\mathscr{A})$, $a_i \downarrow_{\mathscr{A}} \bigcup_{j<i}(a_j \cup b_j)$, $\mathrm{t}(b_i,\mathscr{A})=\mathrm{t}(b,\mathscr{A})$ and $b_i \downarrow_{\mathscr{A}} a_i \cup \bigcup_{j<i} (a_j \cup b_j)$. Then by Lemma 3.4, $b_i \not\downarrow_{\mathscr{A}} a_j$ iff $j>i$. Clearly this contradicts Lemma 1.2(ii). □

**Corollary 3.7.** *For all $a,b$ and $A$, $b \downarrow_A A$ and $a \downarrow_A b$ implies $b \downarrow_A a$.*

**Proof.** Assume not. Choose $a$-saturated $\mathscr{A} \supseteq A$ and $b'$ so that $\mathrm{t}(b',A)=\mathrm{t}(b,A)$ and $b' \downarrow_A \mathscr{A}$. We may assume that $b'=b$. Then choose $a'$ so that $\mathrm{t}(a'.A\cup b)=\mathrm{t}(a,A\cup b)$ and $a' \downarrow_A \mathscr{A}\cup b$. By Lemma 3.6, $b \downarrow_{\mathscr{A}} a'$. By Corollary 3.5(iii), $b \downarrow_A a'$ and so $b \downarrow_A a$. □

**Lemma 3.8.** (i) *If $b \downarrow_A D$ and $c \downarrow_{A\cup b} D$, then $b \cup c \downarrow_A D$.*

(ii) *If $\mathscr{A}$ is a-saturated, $B \downarrow_{\mathscr{A}} D$ and $C \downarrow_{\mathscr{A}\cup B} D$, then $B \cup C \downarrow_{\mathscr{A}} D$.*

(iii) *Assume $\mathscr{A}$ is a-saturated and $B \supseteq \mathscr{A}$. If $a \downarrow_{\mathscr{A}} B$, $a \downarrow_B C$ and there is $D \subseteq B$ (e.g. $D=B$) such that $C \downarrow_D B$, then $a \downarrow_{\mathscr{A}} B \cup C$.*

(iv) *Assume $\mathscr{A}$ is a-saturated. If $a \downarrow_{\mathscr{A}} b$ and $a \cup b \downarrow_{\mathscr{A}} B$, then $a \downarrow_{\mathscr{A}} B \cup b$.*

(v) *Assume $a \downarrow_A A$, for all $i<\omega$, $\mathrm{t}(a_i,A)=\mathrm{t}(a,A)$ and $a_i \downarrow_A \bigcup_{j<i} a_j$. Then for all $n<\omega$, $\{a_i \mid i<n\}$ is $A$-independent.*

**Proof.** (i) Choose $B \subseteq A$ of power $<\kappa(\mathbf{M})$ such that

(a) for all $C \supseteq A \cup D$ there is $b'$ which satisfies: $t(b', A \cup D) = t(b, A \cup D)$ and $t(b', C)$ does not split strongly over $B$ and

(b) for all $C \supseteq A \cup D \cup b$ there is $c'$ which satisfies: $t(c', A \cup D \cup b) = t(c, A \cup D \cup b)$ and $t(c', C)$ does not split strongly over $B \cup b$.

Let $C \supseteq A \cup D$ be arbitrary. Choose $b'$ as in (a) above. By (b) above we can find $c'$ such that $t(c' \cup b', A \cup D) = t(c \cup b, A \cup D)$ and $t(c', C \cup b')$ does not split strongly over $B \cup b'$.

For a contradiction, assume $t(b' \cup c', C)$ splits strongly over $B$. Let $I = \{a_i \mid i < \omega\} \subseteq C$ and $\phi$ be such that $I$ is indiscernible over $B$ and

(c) $\models \phi(c', b', a_0) \wedge \neg\phi(c', b', a_1)$.

**Claim.** *$I$ is indiscernible over $B \cup b'$.*

**Proof.** If not, then (change the enumeration if necessary) there is $\psi$ over $B$ such that $\models \psi(b', a_0, \ldots, a_{n-1}) \wedge \neg\psi(b', a_n, \ldots, a_{2n-1})$. Since

$$\{(a_{m \cdot n}, \ldots, a_{(m+1) \cdot n - 1}) \mid m < \omega\}$$

is indiscernible over $B$, we have a contradiction with the choice of $b'$. □

By Claim and (c), $t(c', C \cup b')$ splits strongly over $B \cup b'$. This contradicts the choice of $c'$.

(ii) Clearly we may assume that $C$ is finite. Let $b \in B$ be arbitrary. We show that $C \cup b \downarrow_{\mathscr{A}} D$. Choose $A \subseteq \mathscr{A}$ and $A' \subseteq B$ such that

(a) $b \in A'$, $|A \cup A'| < \kappa(\mathbf{M})$,

(b) for all $D' \supseteq \mathscr{A} \cup B \cup D$ there is $C'$ which satisfies: $t(C', \mathscr{A} \cup B \cup D) = t(C, A \cup B \cup D)$ and $t(C', D')$ does not split strongly over $A \cup A'$

(c) for all $D' \supseteq \mathscr{A} \cup D$ and $a \in A'$, there is $a'$ which satisfies: $t(a', \mathscr{A} \cup D) = t(a, A \cup D)$ and $t(a', D')$ does not split strongly over $A$.

Then we can proceed as in (i). (We assume that $\mathscr{A}$ is $a$-saturated in order to be able to use Corollary 3.5(iii).)

(iii) By Lemma 3.6, $B \downarrow_{\mathscr{A}} a$. By Corollary 3.7, $C \downarrow_B a$. By (ii), these imply $B \cup C \downarrow_{\mathscr{A}} a$, from which we get the claim by Lemma 3.6.

(iv) Choose $a'$ so that $t(a', \mathscr{A} \cup b) = t(a, \mathscr{A} \cup b)$ and $a' \downarrow_{\mathscr{A}} B \cup b$. By (i) and Lemma 3.4, $t(a' \cup b, \mathscr{A} \cup B) = t(a \cup b, \mathscr{A} \cup B)$.

(v) By (i) it is easy to see that

(*) for all $n < \omega$, $\bigcup_{i<n} a_i \downarrow_A A$.

We prove the claim by induction on $n$. For $n = 1$ the claim follows immediately from the assumptions. Let $i < n$. We show that $a_i \downarrow_A \cup \{a_j \mid j < n,\ j \neq i\}$. If $i = n-1$, then this is assumption. So assume that $i < n-1$. By the choice of $a_{n-1}$,

$$a_{n-1} \downarrow_{A \cup \bigcup \{a_j \mid j < n-1,\ j \neq i\}} a_i.$$

By the induction assumption

$$a_i \downarrow_A \cup \{a_j \mid j<n-1,\ j\neq i\}$$

and by (*) and Corollary 3.7

$$\cup \{a_j \mid j<n-1,\ j\neq i\} \downarrow_A a_i.$$

By (i),

$$a_{n-1} \cup \bigcup \{a_j \mid j<n-1,\ j\neq i\} \downarrow_A a_i.$$

By Corollary 3.7, the claim follows. □

**Lemma 3.9.** *Assume $B \supseteq A$ and $\mathrm{t}(a,A)$ is unbounded. Then $a \downarrow_A B$ iff there is an indiscernible set $I$ over $A$ such that $|I| \geqslant \kappa(\mathbf{M})$, $I$ is based on some $A' \subseteq A$ of power $<\kappa(\mathbf{M})$ and $Av(I,B) = \mathrm{t}(a,B)$.*

**Proof.** From right to left the claim is trivial. So we prove the other direction. Without loss of generality, we may assume that $B$ is $a$-saturated. Let $A' \subseteq A$ be such that $|A'| < \kappa(\mathbf{M})$ and for all $C \supseteq B$ there is $b$ such that $\mathrm{t}(b,B) = \mathrm{t}(a,B)$ and $\mathrm{t}(b,C)$ does not split strongly over $A'$. Let $\xi > |B|$ be a regular cardinal such that $\mathbf{M}$ is $\xi$-stable. For all $i < \xi^+$ we define $\mathscr{B}_i$ and $a_i$ so that

(i) $\mathscr{B}_i$, $i<\xi^+$, is an increasing sequence of $\xi$-saturated models of power $\xi$ and $B \subseteq \mathscr{B}_0$,

(ii) for all $i<\xi^+$, $\mathrm{t}(a_i,B) = \mathrm{t}(a,B)$, $a_i \in \mathscr{B}_{i+1} - \mathscr{B}_i$ and $\mathrm{t}(a_i,\mathscr{B}_i)$ does not split strongly over $A'$ (so $a_i \downarrow_{A'} \mathscr{B}_i$).

By Lemma 3.2(v) and Corollary 3.5(ii), $\{a_i \mid i<\xi^+\}$ is indiscernible over $B$ and $a_j \neq a_j$ for all $i<j<\xi^+$. We prove that $I = \{a_i \mid i < \kappa(\mathbf{M})\}$ is as wanted.

Clearly it is enough to show that $I$ is based on $A'$. For a contradiction, assume that $C \supseteq B$ is such that $Av(I,C)$ splits strongly over $A'$. Clearly, we may assume that $C \subseteq \mathscr{B}_{\kappa(\mathbf{M})+1}$. By Lemma 1.2(ii) there is $J \subseteq \xi^+ - (\kappa(\mathbf{M})+1)$, such that $|J| = \xi^+$ and $\{a_i \mid i \in J\}$ is indiscernible over $C$. Then $\mathrm{t}(a_i,C) = Av(I,C)$ for all $i \in J$. By (ii) above, for all $i \in J$, $\mathrm{t}(a_i,C)$ does not split strongly over $A'$, a contradiction. □

**Lemma 3.10.** *Assume $a\, E^m_{min,A}\, b$, $a \downarrow_A c$ and $b \downarrow_A c$. If $c \downarrow_A A$ or $\mathrm{t}(a,A)$ is bounded or $\mathrm{t}(c,A)$ is bounded, then $\mathrm{t}(a, A \cup c) = \mathrm{t}(b, A \cup c)$.*

**Proof.** We divide the proof to three cases:

*Case* 1: $\mathrm{t}(c,A)$ is bounded: Let $B$ be the set of all $e$ such that $\mathrm{t}(e,A)$ is bounded. Then $|B| < |\mathbf{M}|$ and so $|S(A \cup B)| < |\mathbf{M}|$. We define $E$ so that $x\ E\ y$ if $\mathrm{t}(x, A\cup B) = \mathrm{t}(y, A \cup B)$. Since for all $f \in Aut(A)$, $f(A\cup B) = A \cup B$, $E \in SE(A)$. Clearly this implies the claim.

*Case* 2: $\mathrm{t}(a,A)$ is bounded: Define $E$ so that $x\ E\ y$ if $x = y$ or $\mathrm{t}(x,A) \neq \mathrm{t}(a,A)$ and $\mathrm{t}(y,A) \neq \mathrm{t}(a,A)$. Clearly $E \in SE^m(A)$, and so $a = b$ from which the claim follows.

*Case* 3: $t(a,A)$ is unbounded and $c \downarrow_A A$: Assume the claim is not true. Let $\xi > |A|$ be such that $\mathbf{M}$ is $\xi$-stable. Choose $a_i$, $i < \xi^+$ so that $t(a_i, A \cup c) = t(a, A \cup c)$ and $a_i \downarrow_A c \cup \bigcup_{j<i} a_j$. By Lemmas 3.2(vii) and 1.2(ii), we may assume that $\{a_i \mid i < \omega\}$ is infinite indiscernible over $A$. Clearly we may also assume that $a = a_0$. Let $d = a_1$. Then $t(d, A \cup c) = t(a, A \cup c)$, $d \downarrow_A a \cup c$ and by Lemma 1.5, $d\, E^m_{min,A}\, a$. Then we can choose this $d$ so that in addition, $d \downarrow_A a \cup c \cup b$. By Lemma 3.8(i), $b \cup d \downarrow_A c$. By Corollary 3.7, $c \downarrow_A b \cup d$. Since $dE^m_{min,A}\, b$, this contradicts Lemma 3.2(ix). □

Note that in the case(s) 1 (and 2) above the assumptions $a \downarrow_A c$ and $b \downarrow_A c$ are not used.

**Corollary 3.11.** *Assume $a_i$, $i < \omega$, are such that for all $i, j < \omega$, $a_i\, E^m_{min,A}\, a_j$ and for all $i < \omega$, $a_i \downarrow_A \bigcup_{j<i} a_j$. Then for all $i \neq j$, $a_i \neq a_j$ and $\{a_i \mid i < \omega\}$ is indiscernible over $A$.*

**Proof.** By Lemma 3.2(vii), for all $i \neq j$, $a_i \neq a_j$. We show that for all $i_0 < i_1 < \cdots < i_n < \omega$, $t(a_0 \cup \cdots \cup a_n, A) = t(a_{i_0} \cup \cdots \cup a_{i_n}, A)$. By Lemma 1.2(i), this is enough.

By Lemma 3.8(v), $\{a_i \mid i \leqslant i_n\}$ is $A$-independent and by Lemma 3.8(i), it is easy to see that $\cup\{a_i \mid i \leqslant i_n\} \downarrow_A A$. So by Lemma 3.10, $t(a_0, A \cup \bigcup_{0<k\leqslant n} a_{i_k}) = t(a_{i_0}, A \cup \bigcup_{0<k\leqslant n} a_{i_k})$. So it is enough to show that $t(a_0 \cup \cdots \cup a_n, A) = t(a_0 \cup a_{i_1} \cup \cdots \cup a_{i_n}, A)$. As above we can see that $t(a_1, A \cup a_0 \cup \bigcup_{1<k\leqslant n} a_{i_k}) = t(a_{i_1}, A \cup a_0 \cup \bigcup_{1<k\leqslant n} a_{i_k})$. So it is enough to show that $t(a_0 \cup \cdots \cup a_n, A) = t(a_0 \cup a_1 \cup a_{i_2} \cup \cdots \cup a_{i_n}, A)$. We can carry this on and get the claim. □

**Theorem 3.12.** *Assume $a \downarrow_A c$, $b \downarrow_A c$ and $a\, E^m_{min,A}\, b$. Then $t(a, A \cup c) = t(b, A \cup c)$.*

**Proof.** Assume not. As in the proof of Lemma 3.10 (Case 3.), we can find $a'$ and $b'$ such that $t(a', A \cup c) = t(a, A \cup c)$, $t(b', A \cup c) = t(b, A \cup c)$, $a' \downarrow_A c \cup a$, $b' \downarrow_A c \cup b$, $a'\, E^m_{min,A}\, a$ and $b'\, E^m_{min,A}\, b$. For all $i < \kappa(\mathbf{M})$, choose $a_i$ so that $a_i \downarrow_A c \cup a \cup b \cup \bigcup_{j<i} a_j$, if $i$ is odd, then $t(a_i, A \cup c \cup a) = t(a', A \cup c \cup a)$ and if $i$ is even, then $t(a_i, A \cup c \cup b) = t(b', A \cup c \cup b)$. By Corollary 3.11, for all $i \neq j$, $a_i \neq a_j$ and $\{a_i \mid i < \kappa(\mathbf{M})\}$ is indiscernible over $A$. Clearly this contradicts Lemma 2.1. □

**Lemma 3.13.** *Assume $\mathbf{M}$ is $\xi$-stable and $|A| \leqslant \xi$. Then there is a-saturated $\mathscr{A} \supseteq A$ of power $\leqslant \xi$.*

**Proof.** Immediate by Lemma 1.9(ii) and the fact that $\kappa_r(\mathbf{M}) \leqslant \lambda(\mathbf{M})$ is regular. □

**Theorem 3.14.** *Assume $\mathbf{M}$ is $\xi$-stable and $|A| \leqslant \xi$. Then there is $F^{\mathbf{M}}_{\xi}$-saturated $\mathscr{A} \supseteq A$ of power $\leqslant \xi$.*

**Proof.** By Lemma 3.13, there is an increasing continuous sequence $A_i$, $i \leqslant \xi \cdot \xi$, of models of power $\leqslant \xi$ such that

(i) $A \subseteq A_0$ and for all $i \leqslant \xi \cdot \xi$, $A_{i+1}$ is $a$-saturated,

(ii) for all $i < \xi \cdot \xi$ and $a$, there is $b \in A_{i+1}$ such that $t(b, A_i) = t(a, A_i)$.

We show that $\mathscr{A} = A_{\xi\cdot\xi}$ is as wanted. For this let $B \subseteq \mathscr{A}$ of power $<\xi$ and $b$ be arbitrary. We show that $t(b,B)$ is realized in $\mathscr{A}$.

By Theorem 1.17, $cf(\xi) \geqslant \kappa_r(\mathbf{M})$ and so $\mathscr{A}$ is $a$-saturated and there is $\alpha' < \xi$ such that $b \downarrow_{A_{\xi\cdot\alpha'}} \mathscr{A}$. By the pigeon hole principle there is $\alpha < \xi$ such that $\alpha \geqslant \alpha'$ and $(A_{\xi\cdot(\alpha+1)} - A_{\xi\cdot\alpha}) \cap B = \emptyset$.

**Claim.** *There is $\beta < \xi$ such that $B \downarrow_{A_{\xi\cdot\alpha+\beta}} A_{\xi\cdot\alpha+\beta+1}$.*

**Proof.** Assume not. Then by the pigeon hole principle, we can find $c \in B$ such that

$$|\{\gamma < \xi \mid c \not\downarrow_{A_{\xi\cdot\alpha+\gamma}} A_{\xi\cdot\alpha+\gamma+1}\}| \geqslant cf(\xi).$$

But this is impossible by Lemma 3.2(iii), because $cf(\xi) \geqslant \kappa_r(\mathbf{M})$ and $\mathscr{A}_{\xi\cdot\gamma}$ is $a$-saturated for all $\gamma \leqslant \xi$ such that $cf(\gamma) \geqslant \kappa_r(\mathbf{M})$. □

Choose $c \in A_{\xi\cdot\alpha+\beta+1}$ so that $t(c, A_{\xi\cdot\alpha+\beta}) = t(b, A_{\xi\cdot\alpha+\beta})$. By Claim, $B \downarrow_{A_{\xi\cdot\alpha+\beta}} c$ and so $c \downarrow_{A_{\xi\cdot\alpha+\beta}} B$. Since $b \downarrow_{A_{\xi\cdot\alpha+\beta}} B$, Lemma 3.4 implies, $t(c, A_{\xi\cdot\alpha+\beta} \cup B) = t(b, A_{\xi\cdot\alpha+\beta} \cup B)$. □

We finish this chapter by proving that over $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models our independence notion is equivalent with the notion used in [1].

**Lemma 3.15.** *Assume $\mathscr{A}$ is $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated model and $B \supseteq \mathscr{A}$. Then the following are equivalent*:

(i) $a \downarrow_{\mathscr{A}} B$.

(ii) *For all $b \in B$ there is $A \subseteq \mathscr{A}$ of power $<\lambda(\mathbf{M})$ such that $t(a, \mathscr{A} \cup b)$ does not split over $A$.*

**Proof.** Let $p \in S(\mathscr{A})$ be arbitrary $\mathbf{M}$-consistent type. Let $a$ be such that $t(a,\mathscr{A}) = p$ and $a \downarrow_{\mathscr{A}} B$. Let $a'$ be such that $t(a',\mathscr{A}) = p$ and for all $b \in B$ there is $A \subseteq \mathscr{A}$ of power $<\lambda(\mathbf{M})$ such that $t(a', \mathscr{A} \cup b)$ does not split over $A$. We show that then $t(a,B) = t(a',B)$. This implies the claim, since for all $\mathbf{M}$-consistent $p \in S(\mathscr{A})$ such $a$ and $a'$ exist: The existence of $a$ follows from Lemma 3.2(ii) and (iii) and the existence of $a'$ can be seen as in [1].

For a contradiction, assume that there is $b \in B$ such that $t(a, \mathscr{A} \cup b) \neq t(a', \mathscr{A} \cup b)$. By the choice of $a$ and $a'$ and Lemma 1.2(vi), there is $A \subseteq \mathscr{A}$ of power $<\lambda(\mathbf{M})$ such that $t(a, \mathscr{A} \cup b)$ does not split strongly over $A$, $t(a', \mathscr{A} \cup b)$ and $t(b, \mathscr{A})$ do not split over $A$ and $t(a, A \cup b) \neq t(a', A \cup b)$. For all $i < \omega$, choose $b_i \in \mathscr{A}$ so that $t(b_i, A \cup \bigcup_{j<i} b_j) = t(b, A \cup \bigcup_{j<i} b_j)$. Since $t(b,\mathscr{A})$ does not split over $A$, by Lemma 1.2(i), it is easy to see that $\{b_i \mid i < \omega\} \cup \{b\}$ is infinite indiscernible over $A$. Since $t(a,\mathscr{A}) = t(a',\mathscr{A})$, either $t(a, \mathscr{A} \cup b)$ or $t(a', \mathscr{A} \cup b)$ splits strongly over $A$, a contradiction. □

## 4. Orthogonality

In this section we study orthogonality. Since we do not have full transitivity of $\downarrow$, we need stationary pairs:

**Definition 4.1.** Assume $A \subseteq B$ and $p \in S(B)$. We say that $(p, A)$ is stationary pair if for all $a$, $t(a,B) = p$ implies $a \downarrow_A B$ and for all $C \supseteq B$, $a$ and $b$, the following holds: if $a \downarrow_A C$, $b \downarrow_A C$ and $t(a,B) = t(b,B) = p$, then $t(a,C) = t(b,C)$.

**Lemma 4.2.** (i) *Assume $A \subseteq B \subseteq C$, $a \downarrow_A C$ and $(t(a,B), A)$ is a stationary pair. Then $(t(a,C), A)$ is a stationary pair.*

(ii) *Assume $A \subseteq B \subseteq C \subseteq D$, $a \downarrow_A C$, $a \downarrow_B D$ and $(t(a,C), B)$ is a stationary pair. Then $a \downarrow_A D$.*

**Proof.** (i) is trivial, so we prove (ii): Choose $a'$ so that $t(a',C) = t(a,C)$ and $a' \downarrow_A D$. Then $a' \downarrow_B D$ and so $t(a',D) = t(a,D)$ from which the claim follows. □

**Lemma 4.3.** *Assume $\mathscr{A}$ is a-saturated, $t(a,\mathscr{A})$ does not split strongly over $A \subseteq \mathscr{A}$ and $|A| < \kappa(\mathbf{M})$. Then there is $B \subseteq \mathscr{A}$ such that $A \subseteq B$, $|B - A| < \omega$, $B \downarrow_A A$ and $(t(a,B), A)$ is a stationary pair.*

**Proof.** By Lemma 1.13, $a \downarrow_A \mathscr{A}$. Choose $b_i$, $i \leqslant \omega$, so that for all $i \leqslant \omega$, $t(b_i, \mathscr{A}) = t(a, \mathscr{A})$ and $b_i \downarrow_A \mathscr{A} \cup \bigcup_{j<i} b_j$. Then $\{b_i \mid i \leqslant \omega\}$ is indiscernible over $A$ and by Lemma 3.8(ii),

$$(*) \quad \{b_i \mid i < \omega\} \downarrow_A \mathscr{A}.$$

Especially,

$$(**) \quad \{b_i \mid i < \omega\} \downarrow_A A.$$

Without loss of generality, we may assume that $b_\omega = a$. Choose $a^* \in \mathscr{A}$ so that $a^* E^m_{min,A}\, a$. Let $B = A \cup a^*$ and $I = \{b_i \mid i < \omega\}$. Then $B \downarrow_A A$.

**Claim.** Assume $J \supseteq I$ is indiscernible over $A$, $t(b,B) = t(a,B)$ and $b \downarrow_A B \cup J \cup a$. Then $J \cup \{b\}$ is indiscernible over $A$.

**Proof.** By Lemmas 1.12 and 1.5 it is enough to show that $t(b, A \cup I) = t(a, A \cup I)$. By $(*)$, $I \downarrow_A a^*$. By the choice of $a^*$, $a^* \downarrow_A A$ and so by Corollary 3.7, $a^* \downarrow_A I$. By the choice of $b$ and Lemma 3.2(i), $b \downarrow_{A \cup a^*} I$. By Lemma 3.8(i), $b \cup a^* \downarrow_A I$. By $(**)$ and Corollary 3.7, $I \downarrow_A a^* \cup b$. So by Lemma 3.2(ix), $t(a^*, A \cup I) = t(b, A \cup I)$. Similarly we can see that $I \downarrow_A a^* \cup a$ and so by Lemma 3.2(ix), $t(a^*, A \cup I) = t(a, A \cup I)$. □

We show that $(t(a,B), A)$ is a stationary pair. Assume not. Since $\mathscr{A}$ is $F^{\mathbf{M}}_{\kappa(\mathbf{M})}$-saturated, we can find $b$ such that $b \downarrow_A \mathscr{A}$, $t(b,B) = t(a,B)$ and $t(b,\mathscr{A}) \neq t(a,\mathscr{A})$. Choose $c_i$, $i < \kappa(\mathbf{M})$, so that for all $i < \kappa(\mathbf{M})$, $t(c_i, \mathscr{A}) = t(b, \mathscr{A})$ if $i$ is odd, $t(c_i, \mathscr{A}) = t(a, \mathscr{A})$ if $i$ is even and for all $i < \kappa(\mathbf{M})$, $c_i \downarrow_A \mathscr{A} \cup I \cup \bigcup_{j<i} c_j$. By Claim $\{c_i \mid i < \kappa(\mathbf{M})\}$ is indiscernible. This contradicts Corollary 2.2.

**Definition 4.4.** (i) We say that $p \in S(A)$ is orthogonal to $q \in S(C)$ if for all $a$-saturated $\mathscr{A} \supseteq A \cup C$ the following holds: if $t(b,C) = q$, $b \downarrow_C \mathscr{A}$, $t(a,A) = p$ and $a \downarrow_A \mathscr{A}$, then

$a \downarrow_{\mathscr{A}} b$. We say that $p \in S(A)$ is orthogonal to $C$ if it is orthogonal to every $q \in S(C)$.

(ii) We say that a stationary pair $(p, A)$ is orthogonal to $q \in S(C)$ if for all $a$-saturated $\mathscr{A} \supseteq C \cup dom(p)$ the following holds: if $t(b, C) = q$, $b \downarrow_C \mathscr{A}$, $t(a, dom(p)) = p$ and $a \downarrow_A \mathscr{A}$, then $a \downarrow_{\mathscr{A}} b$. We say that a stationary pair $(p, A)$ is orthogonal to $C$ if it is orthogonal to every $q \in S(C)$.

**Lemma 4.5.** *Assume $\mathscr{A}$ is a-saturated, $A \subseteq B \subseteq \mathscr{A}$, $a \downarrow_A \mathscr{A}$ and $(t(a, B), A)$ is a stationary pair. Then $t(a, \mathscr{A})$ is orthogonal to $C$ iff $(t(a, B), A)$ is orthogonal to $C$.*

**Proof.** Immediate. □

**Lemma 4.6.** *Assume $A \subseteq \mathscr{A}$, $\mathscr{A}$ is a-saturated and $p \in S(\mathscr{A})$. Then the following are equivalent.*
(i) *$p$ is orthogonal to $A$.*
(ii) *For all $a$ and $b$, if $t(a, \mathscr{A}) = p$ and $b \downarrow_A \mathscr{A}$, then $a \downarrow_{\mathscr{A}} b$.*

**Proof.** Clearly (i) implies (ii) and so we prove the other direction. Assume (ii) and for a contradiction assume that there is $a$-saturated $\mathscr{C} \supseteq \mathscr{A}$ and $a$ and $b$ such that $t(a, \mathscr{A}) = p$, $a \downarrow_{\mathscr{A}} \mathscr{C}$, $b \downarrow_A \mathscr{C}$ and $a \not\downarrow_{\mathscr{C}} b$.

Choose $B_0 \subseteq B_1 \subseteq \mathscr{A}$ so that
(1) $|B_1| < \kappa(\mathbf{M})$,
(2) $a \downarrow_{B_0} \mathscr{A}$ and $b \downarrow_{B_0 \cap A} \mathscr{A}$,
(3) $(t(a, B_1), B_0)$ is a stationary pair.
By Corollary 3.5(v), choose finite $d \in \mathscr{C}$ such that $a \not\downarrow_{B_1} d \cup b$. Choose $B_2 \supseteq B_1 \cup d$ of power $< \kappa(\mathbf{M})$ such that $B_2 \subseteq \mathscr{C}$ and $t(a \cup b, \mathscr{C})$ does not split strongly over $B_2$. Since $t(a, \mathscr{C})$ and $t(b, \mathscr{C})$ do not split strongly over $B_2$ we can find by Lemmas 4.3 and 4.2(i) $B_3 \supseteq B_2$ of power $< \kappa(\mathbf{M})$ such that $B_3 \subseteq \mathscr{C}$ and both $(t(a, B_3), B_2)$ and $(t(b, B_3), B_2)$ are stationary pairs. Then

(*) $a \downarrow_{B_0} B_3$ and $b \downarrow_{B_0 \cap A} B_3$.

Choose $f \in Aut(B_1)$ so that $f(B_3) \subseteq \mathscr{A}$ and for all $c \in B_3$, $f(c) E^m_{min, B_1} c$. Then $t(f(a), f(B_3)) = t(a, f(B_3))$ and so we may assume that $f(a) = a$. Now $a \cup f(b) \downarrow_{f(B_2)} f(B_3)$, and so we can find $a'$ and $b'$ so that $t(a' \cup b', f(B_3)) = t(a \cup f(b), f(B_3))$ and $a' \cup b' \downarrow_{f(B_2)} \mathscr{A}$. Then by (*) and Lemma 4.2(ii), $a' \downarrow_{B_0} \mathscr{A}$ and so $t(a', \mathscr{A}) = t(a, \mathscr{A})$ and we may assume that $a' = a$. Also by Lemma 4.2(ii) and (*), $b' \downarrow_{B_0 \cap A} \mathscr{A}$ and so $b' \downarrow_A \mathscr{A}$. Because $a \not\downarrow_{B_1} f(c) \cup b'$, by Corollary 3.5(v), $a \not\downarrow_{\mathscr{A}} b'$. Clearly this contradicts (ii). □

**Lemma 4.7.** *Let $\xi \geqslant \kappa_r(\mathbf{M})$ be a cardinal. Assume $D \subseteq C$, $p \in S(C)$, $(p, D)$ is a stationary pair and orthogonal to $\mathscr{A}$, $|C| < \xi$, $\mathscr{A} \subseteq \mathscr{B}$ are strongly $F^{\mathbf{M}}_{\xi}$-saturated and $C \downarrow_{\mathscr{A}} \mathscr{B}$. Then $(p, D)$ is orthogonal to $\mathscr{B}$.*

**Proof.** For a contradiction, assume that $q \in S(\mathscr{B})$ is not orthogonal to $(p, D)$. Choose $B \subseteq \mathscr{B}$ of power $< \kappa(\mathbf{M})$ so that $q$ does not split strongly over $B$. Choose $A \subseteq \mathscr{A}$ so

that
(i) $|A| < \xi$,
(ii) for all $c \in C$, $t(c, \mathscr{A} \cup \mathscr{B})$ does not split strongly over $A$.
By Lemma 1.9(v), we can find $B' \subseteq \mathscr{A}$ and $f \in Aut(A)$ so that $f(B) = B'$ and for all $b \in B$, $b\, E^m_{min,A}\, f(b)$. By Lemma 1.12, $t(B', C) = t(B, C)$. Let $q' = f(q) \restriction B'$. Then it is easy to see that $q'$ and $(p, C)$ are not orthogonal, a contradiction. □

**Corollary 4.8.** *Assume $\mathscr{A} \subseteq \mathscr{B} \cap \mathscr{C}$ are strongly $F^{\mathbf{M}}_{\kappa_r(\mathbf{M})}$-saturated, $\mathscr{B} \downarrow_{\mathscr{A}} \mathscr{C}$ and $p \in S(\mathscr{C})$ is orthogonal to $\mathscr{A}$. Then $p$ is orthogonal to $\mathscr{B}$.*

**Proof.** Follows immediately from Lemmas 4.3, 4.5 and 4.7. □

## 5. Structure of *s*-saturated models

We say that $\mathbf{M}$ is superstable if $\kappa(\mathbf{M}) = \omega$.

**Lemma 5.1.** *The following are equivalent*:
(i) $\kappa(\mathbf{M}) = \omega$.
(ii) *There are no increasing sequence $\mathscr{A}_i$, $i < \omega$, of a-saturated models and $a$ such that for all $i < \omega$, $a \not\downarrow_{\mathscr{A}_i} \mathscr{A}_{i+1}$.*
(iii) *There are no increasing sequence $\mathscr{A}_i$, $i < \omega$, of $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models and $a$ such that for all $i < \omega$, $a \not\downarrow_{\mathscr{A}_i} \mathscr{A}_{i+1}$.*

**Proof.** Clearly (i) implies (ii) and (ii) implies (iii). So we assume that (i) does not hold and prove that (iii) does not hold either. For this, choose an increasing sequence of regular cardinals $\xi_i$, $i < \omega$, such that for all $i < \omega$, $\mathbf{M}$ is $\xi_i$-stable. Let $\xi = sup_{i<\omega}\, \xi_i$. By Theorem 1.17, $\mathbf{M}$ is not $\xi$-stable. Let $A$ be such that $|A| \leqslant \xi$ and $|S(A)| > \xi$. Then choose an increasing sequence $\mathscr{A}_i$, $i < \omega$, of $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated models of power $\xi_i$ such that $A \subseteq \bigcup_{i<\omega} \mathscr{A}_i$. Then $|S(\bigcup_{i<\omega} \mathscr{A}_i)| > \xi$. By Corollary 3.5(i), it is enough to show that there is $a$ such that for all $i < \omega$, $a \not\downarrow_{\mathscr{A}_i} \bigcup_{i<\omega} \mathscr{A}_i$. For a contradiction, assume not. Then for all $a$ there is $i_a < \omega$, such that $a \downarrow_{\mathscr{A}_{i_a}} \bigcup_{i<\omega} \mathscr{A}_i$. Then by Lemma 3.4, for all $a$, $t(a, \bigcup_{i<\omega} \mathscr{A}_i)$ is determined by $t(a, \mathscr{A}_{i_a})$. Since for all $i < \omega$, $|S(\mathscr{A}_i)| \leqslant \xi$, this implies that $|S(\bigcup_{i<\omega} \mathscr{A}_i)| \leqslant \xi$, a contradiction. □

**Definition 5.2.** We say that $t(a, A)$ is $F^{\mathbf{M}}_{\xi}$-isolated if there is $B \subseteq A$ of power $< \xi$, such that for all $b$, $t(b, B) = t(a, B)$ implies $t(b, A) = t(a, A)$. We define $F^{\mathbf{M}}_{\xi}$-construction, $F^{\mathbf{M}}_{\xi}$-primary, etc., as in [3]. Instead of $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-saturated, $F^{\mathbf{M}}_{\lambda(\mathbf{M})}$-isolated, etc., we write *s*-saturated, *s*-isolated, etc.

In slightly different context, the following theorem is proved in [2].

**Theorem 5.3** (Shelah [2]). *Assume $\xi \geqslant \lambda(\mathbf{M})$*:
(i) *For all $A$ there is an $F^{\mathbf{M}}_{\xi}$-primary model over $A$.*

(ii) *If* $\mathscr{A}$ *is* $F^{\mathbf{M}}_{\xi}$*-primary over* $A$ *then it is* $F^{\mathbf{M}}_{\xi}$*-prime over* $A$.

(iii) *If* $\mathscr{A}$ *is* $F^{\mathbf{M}}_{\xi}$*-primary over* $A$ *and* $\xi \geqslant \lambda(\mathbf{M})$ *is regular, then* $\mathscr{A}$ *is* $F^{\mathbf{M}}_{\xi}$*-atomic over* $A$.

(iv) *If* $\xi \geqslant \lambda(\mathbf{M})$ *is regular, then* $F^{\mathbf{M}}_{\xi}$*-primary models over any set* $A$ *are unique up to isomorphism over* $A$.

As usual we write $A \rhd_C B$ if for all $a$, $a \downarrow_C A$ implies $a \downarrow_C B$.

**Lemma 5.4.** (i) *Assume* $\mathscr{A}$ *is s-saturated and* $\mathscr{B}$ *is s-primary over* $\mathscr{A} \cup B$. *Then* $B \rhd_{\mathscr{A}} \mathscr{B}$.

(ii) *Assume* $\mathscr{A} \subseteq B \cap C$, $\mathscr{A}$ *is s-saturated and* $B \downarrow_{\mathscr{A}} C$. *If* $(B, \{b_i \mid i<\gamma\}, (B_i \mid i<\gamma))$ *is an s-construction over* $B$, *then* $(B \cup C, \{b_i \mid i<\gamma\}, (B_i \mid i<\gamma))$ *is an s-construction over* $B \cup C$.

(iii) *Assume* $\xi \geqslant \lambda(\mathbf{M})$, $\mathscr{A}$ *is* $F^{\mathbf{M}}_{\xi}$*-saturated and* $\mathscr{B}$ *is* $F^{\mathbf{M}}_{\xi}$*-primary over* $\mathscr{A} \cup B$. *Then* $B \rhd_{\mathscr{A}} \mathscr{B}$.

**Proof.** (i) Assume not. Then we can find $s$-saturated $\mathscr{A}$, $B$, $b$ and $a$ so that $t(b, \mathscr{A} \cup B)$ is $s$-isolated, $a \downarrow_{\mathscr{A}} B$ and $a \not\downarrow_{\mathscr{A}} b$ (if $(\mathscr{A} \cup B, \{b_i \mid i<\gamma\}, (B_i \mid i<\gamma))$ is an $s$-construction of $\mathscr{B}$, then let $b = b_i$, where $i$ is the least ordinal such that $a \not\downarrow_{\mathscr{A}} B \cup \bigcup_{j\leqslant i} b_j$ and rename $B \cup \bigcup_{j<i} b_j$ as $B$; $i$ exists by Corollary 3.5(v)). Without loss of generality we may assume that $|B| < \lambda(\mathbf{M})$. Choose $A \subseteq \mathscr{A}$ so that

(i) $t(b, A \cup B)$ $s$-isolates $t(b, \mathscr{A} \cup B)$,

(ii) for all $c \in B$, $t(c, \mathscr{A} \cup a)$ does not split strongly over some $A' \subseteq A$ of power $< \kappa(\mathbf{M})$,

(iii) $t(b, \mathscr{A})$ does not split strongly over some $A' \subseteq A$ of power $< \kappa(\mathbf{M})$,

(iv) $|A| < \lambda(\mathbf{M})$.

This is possible since $\kappa_r(\mathbf{M}) \leqslant \lambda(\mathbf{M})$: Let $\delta = |B| + 1 < \lambda(\mathbf{M})$. Clearly, we can choose $A$ so that it of the form $A' \cup A''$ where $A'$ is of power $< \lambda(\mathbf{M})$ and $A''$ is a union of $\delta$ many sets of power $< \kappa_r(\mathbf{M}) \leqslant \lambda(\mathbf{M})$. If $\lambda(\mathbf{M})$ is regular, then clearly $|A| < \lambda(\mathbf{M})$. Otherwise $\kappa_r(\mathbf{M}) < \lambda(\mathbf{M})$ in which case $|A| \leqslant |A'| + max(\delta, \kappa_r(\mathbf{M})) < \lambda(\mathbf{M})$.

By Lemma 1.9(iii), the proof of Lemma 1.13 and (iii) above, there are $c, c', a' \in \mathscr{A}$ such that $c \cup a \, E^m_{min,A} \, c' \cup a'$ and $t(b \cup c \cup a, A) \neq t(b \cup c' \cup a', A)$. By (ii), $t(B \cup c \cup a, A) = t(B \cup c' \cup a', A)$. So there is $f \in Aut(A \cup B)$ such that $f(c) = c'$ and $f(a) = a'$. Then $f(b)$ contradicts (i) above.

(ii) As (i) above.

(iii) By (i) we may assume that $\xi > \lambda(\mathbf{M})$. For a contradiction, assume that the claim does not hold. As in (i), we can find $s$-saturated $\mathscr{A}$, $B$, $b$ and $a$ so that $t(b, \mathscr{A} \cup B)$ is $F^{\mathbf{M}}_{\xi}$-isolated, $a \downarrow_{\mathscr{A}} B$, $a \not\downarrow_{\mathscr{A}} b$ and $|B| < \xi$. Let $A \subseteq \mathscr{A}$ be such that $t(a, A \cup B)$ $F^{\mathbf{M}}_{\xi}$-isolates $t(b, \mathscr{A} \cup B)$. Choose $s$-saturated $\mathscr{C} \subseteq \mathscr{A}$ so that $|\mathscr{C}| = \lambda(\mathbf{M})$ and $a \downarrow_{\mathscr{C}} \mathscr{A} \cup \mathscr{B}$. For $i < \xi$, choose $a_i \in \mathscr{A}$ such that $(a_i)_{i<\xi}$ is $\mathscr{C}$-independent and for all $i < \xi$, $t(a_i, \mathscr{C}) = t(a, \mathscr{C})$. As in (i), it is enough to show that there is $i < \xi$ such that $a_i \downarrow_{\mathscr{C}} A \cup B$. For this we choose maximal sequence of models $\mathscr{A}_j$ and sets $I_j \subseteq \xi$, $j \leqslant j^*$, such that

(a) $\mathscr{A}_0 = \mathscr{C}$ and $I_0 = \emptyset$,

(b) $I_{j+1} - I_j$ is finite, $\mathscr{A}_{j+1}$ is $s$-primary over $\mathscr{A}_j \cup (I_{j+1} - I_j)$ and for some $c \in A \cup B$, $c \not\downarrow_{\mathscr{A}_j} I_{j+1} - I_j$,

(c) if $j$ is limit, then $I_j = \bigcup_{k<j} I_k$ and $\mathscr{A}_j$ is $s$-primary over $\bigcup_{k<j} \mathscr{A}_k$.

Since $\kappa_r(\mathbf{M}) \leqslant |A \cup B| < \xi$, $I_{j^*} \neq \xi$. Let $i \in \xi - I_{j^*}$. By (i) and (ii), it is easy to see that for all $j \leqslant j^*$, $\mathscr{A}_j$ is $s$-primary over $\mathscr{A} \cup I_j$. Then by (i), $a_i \downarrow_{\mathscr{C}} \mathscr{A}_{j^*}$ and because the sequence was maximal, $A \cup B \downarrow_{\mathscr{A}_{j^*}} a_i$. So $a_i \downarrow_{\mathscr{C}} A \cup B$ as wanted. □

**Corollary 5.5.** (i) *Assume $A \subseteq \mathscr{A}$ and $\mathscr{A}$ is s-saturated. If $p \in S(\mathscr{A})$ is orthogonal to $A$, then for all $C \supseteq \mathscr{A}$, $a$ and $b$ the following holds: if $a \downarrow_{\mathscr{A}} C$, $\mathrm{t}(a, \mathscr{A}) = p$ and $b \downarrow_A C$, then $a \downarrow_{\mathscr{A}} C \cup b$.*

(ii) *Assume $\mathbf{M}$ is superstable and $\gamma$ is a limit ordinal. Let $\mathscr{A}_i$, $i < \gamma$, be an increasing sequence of s-saturated models and $\mathscr{A}$ be s-primary over $\bigcup_{i<\gamma} \mathscr{A}_i$. If $a \notin \mathscr{A}$ then there is $i < \gamma$ such that $\mathrm{t}(a, \mathscr{A})$ is not orthogonal to $\mathscr{A}_i$.*

(iii) *Assume $\mathscr{A}$ is s-saturated and $p \in S(\mathscr{A})$ is orthogonal to $A \subseteq \mathscr{A}$. If $a_i$, $i < \omega$, are such that for all $i < \omega$, $\mathrm{t}(a_i, \mathscr{A}) = p$ and $a_i \downarrow_{\mathscr{A}} \bigcup_{j<i} a_j$, then for all $n < \omega$, $\mathrm{t}(\bigcup_{i<n} a_i, \mathscr{A})$ is orthogonal to $A$.*

**Proof.** (i) Assume not. Let $\mathscr{C}$ be $s$-primary over $\mathscr{A} \cup C$. Then by Lemma 5.4(i) and Corollary 3.5(iv), $a \downarrow_{\mathscr{A}} \mathscr{C}$, $b \downarrow_A \mathscr{C}$ and $a \not\downarrow_{\mathscr{C}} b$, a contradiction.

(ii) Clearly, we may assume that if $i < j$ then $\mathscr{A}_i \neq \mathscr{A}_j$. Since $\kappa(\mathbf{M}) = \omega$, there is $i < \gamma$ such that $a \downarrow_{\mathscr{A}_i} \bigcup_{j<\gamma} \mathscr{A}_j$. By (i), $a \downarrow_{\mathscr{A}_i} \mathscr{A}$. By Lemma 3.2(v), this is more that required.

(iii) Assume not. Then by Lemma 4.6, there is $b$ such that $b \downarrow_A \mathscr{B}$ and $\bigcup_{i<n} a_i \not\downarrow_{\mathscr{A}} b$. Let $m \leqslant n$ be the least such that $\bigcup_{i<m} a_i \not\downarrow_{\mathscr{A}} b$. By Lemma 3.8(i), $a_{m-1} \not\downarrow_{\mathscr{A} \cup \bigcup_{i<m-1} a_i} b$. Clearly this contradicts (i). □

Let $P$ be a tree without branches of length $> \omega$. Then by $t^-$ we mean the immediate predecessor of $t$ if $t \in P$ is not the root. For all $t \in P$, by $t^1_>$ we mean the set of immediate successors of $t$.

**Definition 5.6** (*Shelah [3]*). We say that $(P, f, g) = ((P, \prec), f, g)$ is an $s$-free tree of $s$-saturated $\mathscr{A}$ if the following holds:

(i) $(P, \prec)$ is a tree without branches of length $> \omega$, $f : (P - \{r\}) \to \mathscr{A}$ and $g : P \to P(\mathscr{A})$, where $r \in P$ is the root of $P$ and $P(\mathscr{A})$ is the power set of $\mathscr{A}$ – in order to simplify the notation we write $a_t$ for $f(t)$ and $\mathscr{A}_t$ for $g(t)$,

(ii) $\mathscr{A}_r$ is $s$-primary model (over $\emptyset$),

(iii) if $t$ is not the root and $u^- = t$ then $\mathrm{t}(a_u, \mathscr{A}_t)$ is orthogonal to $\mathscr{A}_{t^-}$,

(iv) if $t = u^-$ then $\mathscr{A}_u$ is $s$-primary over $\mathscr{A}_t \cup a_u$,

(v) Assume $T, V \subseteq P$ and $u \in P$ are such that

(a) for all $t \in T$, $t$ is comparable with $u$,

(b) $T$ is downwards closed.

(c) if $v \in V$ then for all $t$ such that $v \succcurlyeq t \succ u$, $t \notin T$.

Then

$$\bigcup_{t\in T} \mathscr{A}_t \downarrow_{\mathscr{A}_u} \bigcup_{v\in V} \mathscr{A}_v.$$

**Definition 5.7.** We say that $(P,f,g)$ is an $s$-decomposition of $\mathscr{A}$ if it is a maximal $s$-free tree of $\mathscr{A}$.

Note that 'the finite character of dependence' implies, that unions of increasing sequences of $s$-free trees of $\mathscr{A}$ are $s$-free trees of $\mathscr{A}$. So for all $s$-saturated $\mathscr{A}$ there is an $s$-decomposition of $\mathscr{A}$.

We say that $\mathscr{A}$ is $s$-primary over an $s$-free tree $(P,f,g)$ if $\mathscr{A}$ is $s$-primary over $\bigcup\{\mathscr{A}_t \mid t\in P\}$.

**Definition 5.8.** Assume that $(P,f,g)$ is an $s$-decomposition of $\mathscr{A}$, $\mathscr{A}$ is $s$-saturated. Let $P=\{t_i \mid i<\alpha\}$ be an enumeration of $P$ such that if $t_i \prec t_j$ then $i<j$. Then we say that $(\mathscr{A}_i)_{i\leqslant\alpha}$ is a generating sequence if the following holds:

(i) for all $i\leqslant\alpha$, $\mathscr{A}_i\subseteq\mathscr{A}$,

(ii) $\mathscr{A}_0=\emptyset$,

(iii) $\mathscr{A}_{i+1}$ is $s$-primary over $\mathscr{A}_i\cup\mathscr{A}_{t_i}$,

(iv) if $0<i\leqslant\alpha$ is limit then $\mathscr{A}_i$ is $s$-primary over $\bigcup_{j<i}\mathscr{A}_j$.

**Lemma 5.9.** *Assume that $(P,f,g)$ is an $s$-free tree of $\mathscr{A}$, $\mathscr{A}$ is $s$-saturated and $(\mathscr{A}_i)_{i\leqslant\alpha}$ is a generating sequence. Then for all $0<i<\alpha$, $\mathscr{A}_{t_i}\downarrow_{\mathscr{A}_{t_i^-}}\mathscr{A}_i$.*

**Proof.** By Lemma 5.4(i), it is enough to prove that for all $i<\alpha$, $\mathscr{A}_i$ is $s$-primary over $\bigcup_{j<i}\mathscr{A}_{t_j}$. We prove this by induction on $i$. In fact, we need to prove slightly more to keep the induction going: We show that $\mathscr{A}_i$ is not only $s$-constructible over $\bigcup_{j<i}\mathscr{A}_{t_j}$ but that the natural construction works. Then the limit cases are trivial and the successor cases follow from Lemma 5.4(ii). □

**Definition 5.10.** Assume $\mathscr{A}$ is $s$-saturated. We say that $\mathrm{t}(a,\mathscr{A})$ is a $c$-type if for all $s$-saturated $\mathscr{C}$ and $\mathscr{B}$ the following holds: If $\mathscr{C}\subseteq\mathscr{A}$ is such that $\mathrm{t}(a,\mathscr{A})$ is not orthogonal to $\mathscr{C}$ and $\mathscr{A}\cup a\subseteq\mathscr{B}$, then there is $b\in\mathscr{B}-\mathscr{A}$ such that $b\downarrow_{\mathscr{C}}\mathscr{A}$.

Note that the notion of $c$-type is a generalization of regular type.

**Lemma 5.11.** *Assume $\mathbf{M}$ is superstable. Let $\mathscr{A}\subseteq\mathscr{B}$ be $s$-saturated and $\mathscr{A}\neq\mathscr{B}$. Then there is a singleton $a\in\mathscr{B}-\mathscr{A}$ such that $\mathrm{t}(a,\mathscr{A})$ is a $c$-type.*

**Proof.** Since $\kappa(\mathbf{M})=\omega$, by Lemma 1.1 it is easy to see that there is a singleton $a\in\mathscr{B}-\mathscr{A}$ and finite $A\subseteq\mathscr{A}$ such that the following holds: for all $b\in\mathscr{B}-\mathscr{A}$ and $B\subseteq\mathscr{A}$, if there is an automorphism $f$ of $\mathbf{M}$ such that $f(a)=b$ and $f(A)=B$, then $\mathrm{t}(b,\mathscr{A})$ does not split strongly over $B$ (and so $b\downarrow_B\mathscr{A}$). We show that $a$ is as wanted. Let $s$-saturated $\mathscr{C}\subseteq\mathscr{A}$ be such that $\mathrm{t}(a,\mathscr{A})$ is not orthogonal to $\mathscr{C}$. Since $\mathscr{B}$ can now be any $s$-saturated model such that $\mathscr{A}\cup a\subseteq\mathscr{B}$, it is enough to show that there is $b\in\mathscr{B}-\mathscr{A}$ such that $b\downarrow_{\mathscr{C}}\mathscr{A}$.

By Lemma 4.6, find $d$ such that $d\downarrow_{\mathscr{C}}\mathscr{A}$ and $a\not\downarrow_{\mathscr{A}} d$. Let $\mathscr{D}$ be $s$-primary over $\mathscr{C}\cup d$. Then $\mathscr{D}\downarrow_{\mathscr{C}}\mathscr{A}$ and $a\not\downarrow_{\mathscr{A}}\mathscr{D}$. For all $i<\omega$, choose $\mathscr{A}_i$ and $a_i$ so that $t(a_i\cup\mathscr{A}_i,\mathscr{D})=t(a\cup\mathscr{A},\mathscr{D})$ and $a_i\cup\mathscr{A}_i\downarrow_{\mathscr{D}} a\cup\mathscr{A}\cup\bigcup_{j<i}(a_j\cup\mathscr{A}_j)$.

**Claim.** $\{a\cup\mathscr{A}\}\cup\{a_i\cup\mathscr{A}_i\mid i<\omega\}$ *is indiscernible over* $\mathscr{C}$ *and* $a\cup\mathscr{A}\not\downarrow_{\mathscr{C}}\bigcup_{i<\omega}(a_i\cup\mathscr{A}_i)$.

**Proof.** The first of the claims follow immediately from Corollary 3.5(ii). For a contradiction, assume that the second claim is not true. For all $i<\omega$, we define $\mathscr{B}_i$ as follows: We let $\mathscr{B}_0$ be $s$-primary over $\mathscr{A}\cup a$ and $\mathscr{B}_{i+1}$ be $s$-primary over $\mathscr{B}_i\cup\mathscr{A}_i\cup a_i$. By Lemma 5.1, there is $i<\omega$ such that $d\downarrow_{\mathscr{B}_i}\mathscr{A}_i\cup a_i$. Since $\{a\cup\mathscr{A}\}\cup\{a_i\cup\mathscr{A}_i\mid i<\omega\}$ is indiscernible over $\mathscr{C}$ and $a\cup\mathscr{A}\downarrow_{\mathscr{C}}\bigcup_{i<\omega}(a_i\cup\mathscr{A}_i)$, $\mathscr{A}_i\cup a_i\downarrow_{\mathscr{C}} a\cup\mathscr{A}\cup\bigcup_{j<i}(a_j\cup\mathscr{A}_j)$. By Lemma 5.4(ii), $\mathscr{A}_i\cup a_i\downarrow_{\mathscr{C}}\mathscr{B}_i$. But then $\mathscr{A}_i\cup a_i\downarrow_{\mathscr{C}} d$, a contradiction. □

By Claim and Corollary 3.5(v), let $n<\omega$ be the least such that $a\cup\mathscr{A}\not\downarrow_{\mathscr{C}}\bigcup_{i<n}(a_i\cup\mathscr{A}_i)$. Let $\mathscr{A}^*$ be $s$-primary over $\mathscr{A}\cup\mathscr{A}_0\cup\bigcup_{0<i<n}(\mathscr{A}_i\cup a_i)$. It is easy to see that $\mathscr{A}_n\downarrow_{\mathscr{C}}\mathscr{A}\cup\bigcup_{0<i<n}(\mathscr{A}_i\cup a_i)$. By Claim, $\mathscr{A}_0\downarrow_{\mathscr{C}}\mathscr{A}\cup\bigcup_{0<i<n}(\mathscr{A}_i\cup a_i)$ and so by Lemma 3.8(iv) and the choice of $n$, $\mathscr{A}\cup a\downarrow_{\mathscr{C}}\mathscr{A}_0\cup\bigcup_{0<i<n}(\mathscr{A}_i\cup a_i)$ and so by Lemmas 3.6 and 3.2(i), $a\downarrow_{\mathscr{A}}\mathscr{A}_0\cup\bigcup_{0<i<n}(\mathscr{A}_i\cup a_i)$. By Lemma 5.4(i), $a\downarrow_{\mathscr{A}}\mathscr{A}^*$. Similarly we see that $a_0\downarrow_{\mathscr{A}_0}\mathscr{A}^*$. Then also $a\not\downarrow_{\mathscr{A}^*} a_1$.

By the choice of $\mathscr{A}_0$ and $a_0$ there is $f\in Aut(\mathscr{C})$ such that $f(\mathscr{A})=\mathscr{A}_0$ and $f(a)=a_0$. Let $A_0=f(A)$. By Corollary 3.5(v) there is finite $C\subseteq\mathscr{A}^*$ such that $a\not\downarrow_A A_0\cup C\cup a_0$. Choose $B\subseteq\mathscr{C}$ such that $t(A\cup a,\mathscr{C})$ does not split strongly over $B$. Then there is $g\in Saut(B)$ such that $g(A_0)\subseteq\mathscr{C}$. Since $a\cup\mathscr{A}\downarrow_{\mathscr{C}}\mathscr{A}^*$ and every $h\in Aut(\mathscr{A}^*)$ belongs to $Saut(B)$, we may assume that

$$(*)\quad a\cup\mathscr{A}\downarrow_{\mathscr{C}} g(C)\cup A_0\cup C.$$

Then $t(g(A_0\cup C),A\cup a)=t(A_0\cup C,A\cup a)$. Choose $h\in Saut(A\cup g(A_0))$ such that $h(g(C))\subseteq\mathscr{A}$. By (*), $t(a,\mathscr{A}\cup g(C))$ does not split strongly over $A$ and so it does not split strongly over $A\cup g(A_0)$. Then $t(g(A_0)\cup h(g(C)),A\cup a)=t(A_0\cup C,A\cup a)$. Choose $b\in\mathscr{B}$ such that $t(g(A_0)\cup h(g(C))\cup b,A\cup a)=t(A_0\cup C\cup a_0,A\cup a)$. Then by Corollary 3.5(v) and the choice of $C$, $a\not\downarrow_{\mathscr{A}} b$ and so by Lemma 3.2(iii), $b\in\mathscr{B}-\mathscr{A}$ ($b$ is a singleton). By the choice of $A$, $t(b,\mathscr{A})$ does not split strongly over $g(A_0)$. By Lemma 3.2(iii), $b\downarrow_{\mathscr{C}}\mathscr{A}$. □

**Definition 5.12.** (i) We say that $\mathbf{M}$ has $s$-SP (structure property) if every $s$-saturated $\mathscr{A}$ is $s$-primary over any $s$-decomposition of $\mathscr{A}$.

(ii) Let $\kappa\geqslant\lambda(\mathbf{M})$. We say that $\mathbf{M}$ has $\kappa$-dop if there are $F^{\mathbf{M}}_{\kappa}$-saturated $\mathscr{A}_i$, $i<4$, and $a\notin\mathscr{A}_3$ such that

(a) $\mathscr{A}_0\subseteq\mathscr{A}_1\cap\mathscr{A}_2$,

(b) $\mathscr{A}_1\downarrow_{\mathscr{A}_0}\mathscr{A}_2$,

(c) $\mathscr{A}_3$ is $F^{\mathbf{M}}_{\kappa}$-primary over $\mathscr{A}_1 \cup \mathscr{A}_2$,
(d) $\mathrm{t}(a, \mathscr{A}_3)$ is orthogonal to $\mathscr{A}_1$ and to $\mathscr{A}_2$.
We say that $\mathbf{M}$ has $\kappa$-ndop if it does not have $\kappa$-dop.

**Theorem 5.13.** *Assume* $\mathbf{M}$ *is superstable and has* $\lambda(\mathbf{M})$*-ndop. Then* $\mathbf{M}$ *has s-SP.*

**Proof.** Let $\mathscr{A}$ be $s$-saturated and $(P, f, g)$ an $s$-decomposition of $\mathscr{A}$. Let $(\mathscr{A}_i)_{i \leqslant \alpha}$ be a generating sequence and $P = \{t_i \mid i < \alpha\}$ be the enumeration of $P$ from the definition of a generating sequence.

**Claim.** $\mathscr{A}_\alpha = \mathscr{A}$.

**Proof.** Assume not. For all $a \in \mathscr{A} - \mathscr{A}_\alpha$ let $i_a$ be the least ordinal such that $\mathrm{t}(a, \mathscr{A}_\alpha)$ is not orthogonal to $\mathscr{A}_{i_a}$. Let $a \in \mathscr{A} - \mathscr{A}_\alpha$ be any sequence such that

(i) for some $l \leqslant \alpha$ either $\mathrm{t}(a, \mathscr{A}_l)$ is a $c$-type and $a \downarrow_{\mathscr{A}_l} \mathscr{A}_\alpha$ or $\mathrm{t}(a, \mathscr{A}_{t_l})$ is a $c$-type and $a \downarrow_{\mathscr{A}_{t_l}} \mathscr{A}_\alpha$ and

(ii) among these $a$, $i = i_a$ is the least.

By Lemma 5.11 there is at least one such $a$.

There are two cases:

*Case* 1: For some $l < \alpha$ $\mathrm{t}(a, \mathscr{A}_{t_l})$ is a $c$-type and $a \downarrow_{\mathscr{A}_{t_l}} \mathscr{A}_\alpha$. Let $t^* \leqslant t_l$ be the least $t$ such that $\mathrm{t}(a, \mathscr{A}_{t_l})$ is not orthogonal to $\mathscr{A}_t$. Since $\mathrm{t}(a, \mathscr{A}_{t_l})$ is a $c$-type choose $b$ so that

(1) $b \downarrow_{\mathscr{A}_{t^*}} \mathscr{A}_{t_l}$

and

(2) $b \in \mathscr{A}_{t_l}[a] - \mathscr{A}_{t_l}$, where $\mathscr{A}_{t_l}[a] \subseteq \mathscr{A}$ is $s$-primary over $\mathscr{A}_{t_l} \cup a$.

Then if $(t^*)^-$ exists, by (2) and Lemmas 4.6 and 5.4(i), $\mathrm{t}(b, \mathscr{A}_{t_l})$ is orthogonal to $\mathscr{A}_{(t^*)^-}$ and so by (1) and Lemma 4.6 it is easy to see that $\mathrm{t}(b, \mathscr{A}_{t^*})$ is orthogonal to $\mathscr{A}_{(t^*)^-}$.

By (1), (2) and Lemma 5.4(i), $b \downarrow_{\mathscr{A}_{t^*}} \mathscr{A}_\alpha$.

We define $((P', \prec'), f', g')$ as follows:

(i) $P' = P \cup \{t\}$, $t$ a new node,

(ii) for all $u \in P$, $u \prec' t$ iff $u \preccurlyeq t^*$

(iii) $f' \restriction P = f$ and $f'(t) = b$,

(iv) $g' \restriction P = g$ and $g'(t) \subseteq \mathscr{A}$ is $s$-primary over $A_{t^*} \cup b$.

**Subclaim.** *$((P', \prec'), f', g')$ is an s-free tree of $\mathscr{A}$.*

**Proof.** (i), (ii), (iii) and (iv) in the Definition 5.6 are clear. So we prove (v):

Let $T \subseteq P'$, $u \in P'$ and $V \subseteq P'$ be as in Definition 5.6(v). There are four cases:

*Case* a: $t \in T - V$. Let $T' = T - \{t\}$ and $\mathscr{A}_{T'} \subseteq \mathscr{A}_\alpha$ be $s$-primary over $\bigcup\{\mathscr{A}_d \mid d \in T'\}$. By the choice of $b$ and Lemma 5.4(i),

$$A_t \downarrow_{\mathscr{A}_{t^*}} \mathscr{A}_{T'} \cup \bigcup_{v \in V} \mathscr{A}_v.$$

By Lemmas 3.2(i) and 3.6,

$$\bigcup_{v\in V} \mathscr{A}_v \downarrow_{\mathscr{A}_{T'}} \mathscr{A}_t.$$

By Corollary 3.5(iv), the assumption that $(P, f, g)$ is $s$-free tree of $\mathscr{A}$ and Lemma 5.4(i),

$$\bigcup_{v\in V} \mathscr{A}_v \downarrow_{\mathscr{A}_u} \mathscr{A}_{T'} \cup \mathscr{A}_t.$$

By Lemma 3.6,

$$\bigcup_{d\in T} \mathscr{A}_d \downarrow_{\mathscr{A}_u} \bigcup_{v\in V} \mathscr{A}_v.$$

*Case* b: $t\in V - T$: Exactly as the Case a.

*Case* c: $t\in V\cap T$: Because $t\in T-P$, $u\leqslant t$. Since $t\in V$, $u=t$. Then because $u\notin P$, $\bigcup_{d\in T}\mathscr{A}_d=\mathscr{A}_u$, and the claim follows from Lemma 3.2(iv).

*Case* d: $t\notin T\cup V$: Immediate by the assumption that $(P, f, g)$ is an $s$-free tree of $\mathscr{A}$. □

Subclaim contradicts the maximality of $P$. So Case 1 is impossible and we are in the Case 2:

*Case* 2: $l\leqslant\alpha$ is such that $\mathrm{t}(a,\mathscr{A}_l)$ is a $c$-type and $a\downarrow_{\mathscr{A}_l}\mathscr{A}_\alpha$. Let $\mathscr{B}\subseteq\mathscr{A}$ be $s$-primary over $\mathscr{A}_\alpha\cup a$. Clearly $i(=i_a)\leqslant l$ and so let $b'$ be the element given by $\mathrm{t}(a,\mathscr{A}_l)$ being a $c$-type: $b'\downarrow_{\mathscr{A}_i}\mathscr{A}_l$ and $b'\in\mathscr{A}_l[a]-\mathscr{A}_l$, where $\mathscr{A}_l[a]\subseteq\mathscr{B}$ is $s$-primary over $\mathscr{A}_l\cup a$. By Lemma 5.11 we may choose $b$ so that $\mathrm{t}(b,\mathscr{A}_i)$ is a $c$-type and $b\in\mathscr{A}_i[b']-\mathscr{A}_i$, where $\mathscr{A}_i[b']\subseteq\mathscr{B}$ is $s$-primary over $\mathscr{A}_i\cup b'$. Then $b\downarrow_{\mathscr{A}_i}\mathscr{A}_\alpha$, $b\notin\mathscr{A}_i$ and $i_b\leqslant i(=i_a)$.

(1) $i$ is not a limit $>0$. This is because otherwise by Lemma 5.5(ii), $\mathrm{t}(b,\mathscr{A}_i)$ is not orthogonal to $\mathscr{A}_j$ for some $j<i$. Then $\mathrm{t}(b,\mathscr{A}_\alpha)$ is not orthogonal to $\mathscr{A}_j$, i.e., $i_b<i_a$. This contradicts the choice of $a$.

(2) $i$ is not a successor $>1$. Assume it is, $i=j+1$. Then $\mathscr{A}_i$ is $s$-primary over $\mathscr{A}_j\cup\mathscr{A}_{t_j}$ and by Lemma 5.9, $\mathscr{A}_j\downarrow_{\mathscr{A}_{t_j^-}}\mathscr{A}_{t_j}$. (Note that since Case 1 is not possible, $A_{j+1}\neq\mathscr{A}_{t_j}$.) By the choice of $a$ $\mathrm{t}(b,\mathscr{A}_i)$ is orthogonal to $\mathscr{A}_j$. So by $\lambda(\mathbf{M})$-ndop $\mathrm{t}(b,\mathscr{A}_i)$ is not orthogonal to $\mathscr{A}_{t_j}$. Then as in Case 1 we get a contradiction with the maximality of $(P, f, g)$. Alternatively, we can find $c$ such that it satisfies the assumptions of Case 1, which is a contradiction.

(3) $i$ is not 0 or 1. Immediate, since Case 1 is not possible.

Clearly (1) and (2) above contradict (3). So also Case 2 imply a contradiction. □

Let $\mathscr{C}\subseteq\mathscr{A}$ be $F^{\mathbf{M}}_{\kappa}$-primary over $\bigcup\{\mathscr{A}_t\mid t\in P\}$. We want to show that $\mathscr{C}=\mathscr{A}$. For this we choose a generating sequence $(\mathscr{A}_i)_{i\leqslant\alpha}$, so that $\mathscr{A}_i\subseteq\mathscr{B}$ for all $i\leqslant\alpha$. By the claim above $\mathscr{A}_\alpha=\mathscr{A}$ and so $\mathscr{C}=\mathscr{A}$. □

## 6. On non-structure

**Definition 6.1.** We say that $\mathbf{M}$ has $\kappa$-sdop if the following holds: there are $F^{\mathbf{M}}_{\kappa}$-saturated $\mathscr{A}_i$, $i<4$, and $I=\{a_i \mid i<\lambda(\mathbf{M})\}$, $a_i \in \mathscr{A}_3$, such that

(a) $\mathscr{A}_0 \subseteq \mathscr{A}_1 \cap \mathscr{A}_2$, $\mathscr{A}_3$ is $F^{\mathbf{M}}_{\kappa}$-primary over $\mathscr{A}_1 \cup \mathscr{A}_2$,

(b) $\mathscr{A}_1 \downarrow_{\mathscr{A}_0} \mathscr{A}_2$,

(c) $I$ is an indiscernible sequence over $\mathscr{A}_1 \cup \mathscr{A}_2$ and if $i<j<\lambda(\mathbf{M})$ then $a_i \neq a_j$.

As in [1], we can prove non-structure theorems from $\kappa$-sdop. (In [3], this was the formulation of dop, which was used to get non-structure.)

In this section we show that dop and sdop are essentially equivalent, i.e. $\lambda(\mathbf{M})^+$-sdop implies $\lambda(\mathbf{M})^+$-dop and $\lambda(\mathbf{M})$-dop implies $\lambda_r(\mathbf{M})^+$-sdop, where $\lambda_r(\mathbf{M})$ is the least regular cardinal $\geqslant \lambda(\mathbf{M})$.

**Lemma 6.2.** *Assume* $\mathbf{M}$ *is* $\xi$*-stable and* $\kappa=\xi^+$. *If* $\mathbf{M}$ *has* $\kappa$*-sdop then it has* $\kappa$*-dop.*

**Proof.** Let $I$ and $\mathscr{A}_i$, $i<4$, be as in the definition of $\kappa$-sdop. We need to show that there is $\mathbf{M}$-consistent type $p$ over $\mathscr{A}_3$ such that (d) in Definition 5.12(ii) is satisfied. We show that $Av(I,\mathscr{A}_3)$ is the required type.

By Lemma 2.4(iii), let $a$ be such that $t(a,\mathscr{A}_3)=Av(I,\mathscr{A}_3)$. For a contradiction, by Lemma 4.6, let $b$ be such that

(i) $a \not\downarrow_{\mathscr{A}_3} b$,

(ii) $b \downarrow_{\mathscr{A}_1} \mathscr{A}_3$.

Let $\mathscr{C}_i \subseteq \mathscr{A}_i$, $i<4$ be $F^{\mathbf{M}}_{\xi}$-saturated models of cardinality $\xi$ such that

(1) $\mathscr{C}_i \subseteq \mathscr{A}_i$, $\mathscr{C}_1 \cap \mathscr{C}_2 = \mathscr{C}_0$, $\mathscr{C}_3 \cap \mathscr{A}_1 = \mathscr{C}_1$, $\mathscr{C}_3 \cap \mathscr{A}_2 = \mathscr{C}_2$ and $I \subseteq \mathscr{C}_3$,

(2) $a \cup b \downarrow_{\mathscr{C}_3} \mathscr{A}_3$ and $a \not\downarrow_{\mathscr{C}_3} b$,

(3) $a \cup b \cup \mathscr{C}_3 \downarrow_{\mathscr{C}_1} \mathscr{A}_1$ and $a \cup b \cup \mathscr{C}_3 \downarrow_{\mathscr{C}_2} \mathscr{A}_2$,

(4) for all $c \in \mathscr{C}_3$ there is $D \subseteq \mathscr{C}_1 \cup \mathscr{C}_2$ of power $\xi$, such that $t(c,D)$ $F^{\mathbf{M}}_{\kappa}$-isolated $t(c,\mathscr{A}_1 \cup \mathscr{A}_2)$.

We can see the existence of the sets as in the proof of Theorem 3.14 (the only non-trivial part being to guarantee that the models are $F^{\mathbf{M}}_{\xi}$-saturated).

Let $a^* \in \mathscr{A}_3$ be such that it realizes $Av(I,\mathscr{C}_3)$.

**Claim.** $t(a^*,\mathscr{C}_3)$ $F^{\mathbf{M}}_{\kappa}$*-isolates* $t(a^*,\mathscr{C}_3 \cup \mathscr{A}_1 \cup \mathscr{A}_2)$.

**Proof.** Assume not. Then there is $d \in \mathscr{C}_3$ such that $t(a^* \cup d, \mathscr{C}_1 \cup \mathscr{C}_2)$ does not $F^{\mathbf{M}}_{\kappa}$-isolate $t(a^* \cup d, \mathscr{A}_1 \cup \mathscr{A}_2)$.

**Subclaim.** *There is* $a' \in I$ *such that* $t(a' \cup d, \mathscr{C}_1 \cup \mathscr{C}_2) = t(a^* \cup d, \mathscr{C}_1 \cup \mathscr{C}_2)$.

**Proof.** By Lemma 1.2(v), there is $i<\lambda(\mathbf{M})$ such that $t(d, \mathscr{C}_1 \cup \mathscr{C}_2 \cup I)$ does not split over $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \{a_j \mid j<i\}$. Since $I$ is indiscernible over $C_1 \cup \mathscr{C}_2$, $a' = a_i$ is as wanted. □

Clearly Subclaim contradicts (4) above.

Choose $b^* \in \mathscr{A}_1$ so that $t(b^*, \mathscr{C}_1) = t(b, \mathscr{C}_1)$. By (3), $t(b^*, \mathscr{C}_3) = t(b, \mathscr{C}_3)$. By Claim, $a^* \downarrow_{\mathscr{C}_3} b^*$. Let $f$ be an automorphism such that $f(b) = b^*$ and $f \restriction \mathscr{C}_3 = id_{\mathscr{C}_3}$. Then $f(a)$ contradicts Claim. □

**Theorem 6.3.** *Let $\lambda \geqslant \lambda_r(\mathbf{M})$ be such that $\mathbf{M}$ is $\lambda$-stable. Then $\lambda_r(\mathbf{M})$-dop implies $\lambda^+$-sdop.*

**Proof.** Let $\mathscr{A}_i$, $i < 4$, and $p \in S(\mathscr{A}_3)$ be as in the the definition of $\lambda_r(\mathbf{M})$-dop. By Lemma 4.5, as in the proof of Lemma 3.14, we find these so that $|\mathscr{A}_3| \leqslant \lambda_r(\mathbf{M})$. Let $\mathscr{B}_0 \supseteq \mathscr{A}_0$ be $F^{\mathbf{M}}_{\lambda^+}$-saturated such that $\mathscr{B}_0 \downarrow_{\mathscr{A}_0} \mathscr{A}_3$. Let $\mathscr{B}_i$, $i \in \{1,2\}$ be $s$-primary over $\mathscr{A}_i \cup \mathscr{B}_0$. Let $\mathscr{B}_3$ be $s$-primary over $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{A}_3$. Let $\mathscr{C}_i$, $i \in \{1,2\}$, be $F^{\mathbf{M}}_{\lambda^+}$-primary over $\mathscr{B}_i$ such that $\mathscr{C}_1 \downarrow_{\mathscr{B}_1} \mathscr{B}_3$ and $\mathscr{C}_2 \downarrow_{\mathscr{B}_2} \mathscr{B}_3 \cup \mathscr{C}_1$.

Let $q \in S(\mathscr{A}_3)$ be any type such that it is orthogonal to $\mathscr{A}_1$ and $\mathscr{A}_2$. Our first goal is to show that there is only one $q^* \in S(\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{B}_3)$ which extends $q$.

**Claim 1.** *$\mathscr{B}_1 \cup \mathscr{A}_2$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{A}_1 \cup \mathscr{B}_0 \cup \mathscr{A}_2$ and for all $b \in \mathscr{B}_1$, there is $B \subseteq \mathscr{A}_1 \cup \mathscr{B}_0$ of power $< \lambda_r(\mathbf{M})$ such that $t(b, B)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $t(b, \mathscr{B}_0 \cup \mathscr{A}_1 \cup \mathscr{A}_2)$.*

**Proof.** Follows immediately from the proof of Lemma 5.4(ii) and Theorem 5.3(iii). □

**Claim 2.** *$\mathscr{B}_1 \cup \mathscr{B}_2$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{B}_1 \cup \mathscr{B}_0 \cup \mathscr{A}_2$ and for all $b \in \mathscr{B}_2$, there is $B \subseteq \mathscr{A}_2 \cup \mathscr{B}_0$ of power $< \lambda_r(\mathbf{M})$ such that $t(b, B)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $t(b, \mathscr{B}_1 \cup \mathscr{A}_2)$.*

**Proof.** As Claim 1. □

**Claim 3.** *$\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{A}_3$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{A}_3 \cup \mathscr{B}_0$.*

**Proof.** By Claims 1 and 2, $\mathscr{B}_1 \cup \mathscr{B}_2$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{B}_0$. So it is enough to show that for all $a \in \mathscr{A}_3$, $t(a, \mathscr{A}_1 \cup \mathscr{A}_2)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})^+}$-isolates $t(a, \mathscr{B}_1 \cup \mathscr{B}_2)$.

Assume not. Choose $b_1 \in \mathscr{B}_1$ and $b_2 \in \mathscr{B}_2$ so that $t(a, \mathscr{A}_1 \cup \mathscr{A}_2)$ does not $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})^+}$-isolate $t(a, \mathscr{A}_1 \cup \mathscr{A}_2 \cup b_1 \cup b_2)$. Choose $A_1 \subseteq \mathscr{A}_1$, $A_2 \subseteq \mathscr{A}_2$ and $B_0 \subseteq \mathscr{B}_0$ of power $< \lambda_r(\mathbf{M})$ such that

(i) $t(a, A_1 \cup A_2)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $t(a, \mathscr{A}_1 \cup \mathscr{A}_2)$,

(ii) $t(b_1, A_1 \cup B_0)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $t(a, \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{B}_0)$ and $t(b_2, A_2 \cup B_0)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $t(b_2, \mathscr{B}_1 \cup \mathscr{A}_2 \cup \mathscr{B}_0)$,

(iii) $A_1 \cap \mathscr{A}_0 = A_2 \cap \mathscr{A}_0 = A_0$ and for all $c \in A_1 \cup A_2$, $t(c, \mathscr{B}_0)$ does not split strongly over $A_0$.

By Lemma 1.9(v), choose $f \in Aut(A_0)$ so that $f(B_0) \subseteq \mathscr{A}_0$ and for all $c \in B_0$, $f(c)$ $E^m_{min, A_0} c$. Let $B'_0 = f(B_0)$. Then by (iii), $t(B'_0, A_1 \cup A_2) = t(B_0, A_1 \cup A_2)$. Choose $b'_i \in \mathscr{A}_i$ so that $t(b'_i \cup B'_0, A_i) = t(b_i \cup B_0, A_i)$, $i \in \{1,2\}$. By (ii) $t(b'_1 \cup b'_2 \cup B'_0, A_i) = t(b_1 \cup b_2 \cup B_0, A_i)$. Clearly this contradicts (i). □

**Claim 4.** *$\mathscr{B}_3$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-primary over $\mathscr{A}_3 \cup \mathscr{B}_0$.*

**Proof.** Immediate by Claim 3 and the choice of $\mathscr{B}_3$. □

By Claim 4 and Lemma 5.4, there is exactly one $q' \in S(\mathscr{B}_3)$ such that $q \subseteq q'$. By Corollary 4.8, $q'$ is orthogonal to $\mathscr{B}_1$ and $\mathscr{B}_2$. So if $a$ realizes $q'$, then $a \downarrow_{\mathscr{B}_3} \mathscr{C}_1$. Then by Corollary 5.5(i), there is exactly one $q^* \in S(\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{B}_3)$, which extends $q$.

Now choose $a_i$, $i < \lambda_r(\mathbf{M})$, so that for all $i$, $\mathrm{t}(a_i, \mathscr{A}_3) = p$ and $a_i \downarrow_{\mathscr{A}_3} \bigcup_{j<i} a_j$. Then $I = \{a_i \mid i < \lambda_r(\mathbf{M})\}$ is indiscernible over $\mathscr{A}_3$ and by Corollary 5.5(iii), for all $n < \omega$, $\mathrm{t}(a_0 \cup \cdots \cup a_n, \mathscr{A}_3)$ is orthogonal to $\mathscr{A}_1$ and $\mathscr{A}_2$. So, by what we showed above, $I$ is indiscernible over $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{B}_3$ and for all $i < \lambda_r(\mathbf{M})$, $\mathrm{t}(a_i, \mathscr{A}_3 \cup \bigcup_{j<i} a_j)$ $F^{\mathbf{M}}_{\lambda^+}$-isolates $\mathrm{t}(a_i, \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{B}_3 \cup \bigcup_{j<i} a_j)$. So there is an $F^{\mathbf{M}}_{\lambda^+}$-primary model $\mathscr{C}_3$ over $\mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{B}_3$ such that $I \subseteq \mathscr{C}_3$.

So to get $\lambda^+$-sdop, it is enough to show that $\mathscr{C}_3$ is $F^{\mathbf{M}}_{\lambda^+}$-primary over $\mathscr{C}_1 \cup \mathscr{C}_2$. By Claim 3 and the choice of $\mathscr{B}_3$, $\mathscr{B}_3$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{B}_1 \cup \mathscr{B}_2$.

**Claim 5.** *For all $c \in \mathscr{C}_1$ there is $B \subseteq \mathscr{B}_1$ of power $\leqslant \lambda$ such that $\mathrm{t}(c, B)$ $F^{\mathbf{M}}_{\lambda^+}$-isolates $\mathrm{t}(c, \mathscr{B}_3)$.*

**Proof.** Assume not. Choose $B_1 \subseteq \mathscr{B}_1$ of power $\leqslant \lambda$ and $c \in \mathscr{C}_1$ so that

(i) $\mathrm{t}(c, B_1)$ $F^{\mathbf{M}}_{\lambda^+}$-isolates $\mathrm{t}(c, \mathscr{B}_1)$,

(ii) $\mathrm{t}(c, B_1)$ does not $F^{\mathbf{M}}_{\lambda^+}$-isolate $\mathrm{t}(c, \mathscr{B}_3)$.

By (ii) above, choose $b \in \mathscr{B}_3$, $B_0 \subseteq \mathscr{B}_0$ $C_1 \subseteq \mathscr{B}_1$ and $C_2 \subseteq \mathscr{B}_2$

(iii) $|C_1 \cup C_2| < \lambda_r(\mathbf{M})$,

(iv) $\mathrm{t}(b, C_1 \cup C_2)$ $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-isolates $\mathrm{t}(b, \mathscr{B}_1 \cup \mathscr{B}_2)$,

(v) $\mathrm{t}(c, B_1)$ does not $F^{\mathbf{M}}_{\lambda^+}$-isolate $\mathrm{t}(c, B_1 \cup C_1 \cup C_2 \cup b)$,

(vi) for all $a \in B_1 \cup C_1$, $\mathrm{t}(a, \mathscr{B}_2)$ does not split strongly over $B_0$ and $|B_0| \leqslant \lambda$.

Since $\mathscr{B}_0$ is $F^{\mathbf{M}}_{\lambda^+}$-saturated and $\mathbf{M}$ is $\lambda$-stable, we can find $f \in Aut(B_0)$ such that $f(C_2) \subseteq \mathscr{B}_0$ and for all $a \in C_2$, $f(a) E^m_{min, B_0^*}\, a$. Then by (vi), $\mathrm{t}(f(C_2), B_0 \cup B_1 \cup C_1) = \mathrm{t}(C_2, B_0 \cup B_1 \cup C_1)$. Choose $b' \in \mathscr{B}_1$ so that $\mathrm{t}(b' \cup f(C_2), C_1) = \mathrm{t}(b \cup C_2, C_1)$. Then by (iv), $\mathrm{t}(b' \cup f(C_2), B_0 \cup B_1 \cup C_1) = \mathrm{t}(b \cup C_2, B_0 \cup B_1 \cup C_1)$. By (vi) $\mathrm{t}(c, B_1)$ does not $F^{\mathbf{M}}_{\lambda^+}$-isolate $\mathrm{t}(c, B_1 \cup \mathscr{C}_1 \cup f(C_2) \cup b')$. Clearly this contradicts (i). □

So $\mathscr{B}_3 \cup \mathscr{C}_1$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible over $\mathscr{C}_1 \cup \mathscr{B}_2$.

**Claim 6.** *For all $c \in \mathscr{C}_2$ there is $B \subseteq \mathscr{B}_2$ of power $\leqslant \lambda$ such that $\mathrm{t}(c, B)$ $F^{\mathbf{M}}_{\lambda^+}$-isolates $\mathrm{t}(c, \mathscr{C}_1 \cup \mathscr{B}_3)$.*

**Proof.** As Claim 5 above. □

So $\mathscr{B}_3 \cup \mathscr{C}_1 \cup C_2$ is $F^{\mathbf{M}}_{\lambda_r(\mathbf{M})}$-constructible (and so $F^{\mathbf{M}}_{\lambda^+}$-constructible) over $\mathscr{C}_1 \cup \mathscr{C}_2$. By the choice of $\mathscr{C}_3$, this implies that $\mathscr{C}_3$ is $F^{\mathbf{M}}_{\lambda^+}$-primary over $\mathscr{C}_1 \cup \mathscr{C}_2$. □

Note that in Theorem 6.3 the assumption, $\mathbf{M}$ is $\lambda$-stable, is not necessary. We can avoid the use of it by Lemma 3.15.

**Lemma 6.4.** *Assume $\kappa > \lambda \geqslant \lambda(\mathbf{M})$. Then $\lambda$-dop implies $\kappa$-dop.*

**Proof.** Let $\mathscr{A}_i$, $i<4$, and $a$ as in the definition of $\lambda$-dop. Choose $F_\kappa^{\mathbf{M}}$-saturated $\mathscr{B}_0 \supseteq \mathscr{A}_0$ such that $\mathscr{B}_0 \downarrow_{\mathscr{A}_0} \mathscr{A}_1 \cup \mathscr{A}_2$. Let $\mathscr{B}_1$ be $F_\kappa^{\mathbf{M}}$-primary over $\mathscr{B}_0 \cup \mathscr{A}_1$, $\mathscr{B}_2$ be $F_\kappa^{\mathbf{M}}$-primary over $\mathscr{B}_0 \cup \mathscr{A}_2$ and $\mathscr{B}_3$ be $F_\kappa^{\mathbf{M}}$-primary over $\mathscr{B}_1 \cup \mathscr{B}_2$. Clearly we can choose the sets so that $\mathscr{A}_3 \subseteq \mathscr{B}_3$ and $a \downarrow_{\mathscr{A}_3} \mathscr{B}_3$. By Lemmas 5.4(iii) and 3.8(iv), $\mathscr{B}_1 \downarrow_{\mathscr{B}_0} \mathscr{A}_2$. Then $\mathscr{A}_2 \downarrow_{\mathscr{A}_0} \mathscr{B}_1$ and so $\mathscr{A}_2 \downarrow_{\mathscr{A}_1} \mathscr{B}_1$. By Lemma 5.4(iii),

(1) $\mathscr{A}_3 \downarrow_{\mathscr{A}_1} \mathscr{B}_1$.

Similarly,

(2) $\mathscr{A}_3 \downarrow_{\mathscr{A}_2} \mathscr{B}_2$.

Also by Lemmas 5.4(iii) and 3.8(iv), $\mathscr{B}_1 \downarrow_{\mathscr{B}_0} \mathscr{B}_2$.

By (1), (2), Lemma 4.5 and Corollary 4.8, $\mathrm{t}(a, \mathscr{B}_3)$ is orthogonal to $\mathscr{B}_1$ and to $\mathscr{B}_2$. □

**Corollary 6.5.** *$\lambda(\mathbf{M})$-dop implies $\lambda_r(\mathbf{M})^+$-sdop.*

**Proof.** Immediate by Lemma 6.4 and Theorem 6.3. □

We finish this paper by giving open problems:

**Question 6.6.** *What are the relationships among the following properties*:
(1) $a \downarrow_A A$,
(2) $a \not\downarrow_A a$,
(3) $t(a,A)$ *is unbounded*?

Note that (1) does not imply (2) nor (3) (fails already in the ‘classical’ case), (3) implies (2) (Lemma 3.2(v)) and $(1) \wedge (2)$ implies (3) (just choose $a_i$, $i<|\mathbf{M}|$, so that $t(a_i,A) = t(a,A)$ and $a_i \downarrow_A \bigcup_{j<i} a_j$).

**Question 6.7.** *Does Corollary* 4.8 *hold without the assumption that the sets are strongly $F^{\mathbf{M}}_{\kappa_r(\mathbf{M})}$-saturated*?

**Question 6.8.** *Does the following hold*: *If* $\mathbf{M}$ *is superstable*, *then for all $A$ there exists an ‘a-primary’ set over $A$*?

## References

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