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ON THEORIES T CATEGORICAL IN $|T|$

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ABSTRACT. Morley conjectured that if an infinite first-order theory T is categorical in the power $|T| > \aleph_0$, then it has a model of power $< |T|$. Here we affirm this conjecture for the case $|T|^{\aleph_0} = |T|$.

§0. Introduction. Morley conjectured in [3] that if a theory T is categorical in the power $|T|$, $|T| > \aleph_0$, then it has a model of power $< |T|$. (The power $|T|$ of a theory is the number of its sentences plus $\aleph_0$.) Keisler (as mentioned in Ressayre [5]) proved this conjecture for the case $|T| < 2^{\aleph_0}$, $|T|$ regular.

The aim of this article is to prove the following theorem:

THEOREM 0.1. *If a theory T is categorical in the power $|T|$ and $|T| = |T|^{\aleph_0}$, then T has a model of power $< |T|$.*

DEFINITION 0.1. Theory T is categorical in power λ if all its models of this power are isomorphic.

In this article we will not attempt to present all the possible results of the methods employed here. Fuller developments will appear in [6], [7], [8]. It can be proved, by Theorem 6.3, that if T is categorical in $|T|^{\aleph_0} = |T|$, then T is a definable extension of a theory of smaller power. Also, by slight changes in §§2, 3, it can be proved that if T has only homogeneous models in the power $|T| > \aleph_0$, then T has a model of power $< |T|$.

In §1 we will define our notation and mention several known theorems which we shall use. In §2 we define the rank of a type, and we define a theory to be superstable if every type has rank $< \infty$. (This is a generalization of Morley's definition in [3].)

In §3 we will show that a nonsuperstable theory has a model of power $|T|$ which is not $\aleph_1$-saturated. From this we conclude that if T is categorical in $|T| = |T|^{\aleph_0}$, then T is superstable.

In §4 we define prime model over a set A, $A \subseteq |M|$ (M a model of T) and we prove the existence of prime models for superstable theories.

In §5 we prove some properties of indiscernible sequences.

In §6 we use the results of §5 to prove the main theorem.

§1. Notations. Every ordinal is the set of all smaller ordinals, and every cardinal (power) is the first ordinal of its power. We shall use $\alpha, \beta, \gamma, i, j, k, l$, for ordinals, κ, λ for cardinals, m, n for natural numbers. δ will be a limit ordinal. If A is a set, its power is denoted by $|A|$. The domain of a function F is denoted by Dom(F)

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[1] I would like to thank my friend Leo Marcus for translating this paper and finding many errors. I would like to thank Mr. Victor Harnik for suggesting the simplified proof of Theorem 6.2, which appears here.

and its range by Rang(F). If F, G are functions, then F is said to extend G, or to be a continuation of G if Dom(G) $\subseteq$ Dom(F) and for all a in Dom(G), $F(a) = G(a)$. If F is one-one, F^{-1} will denote the inverse function. $F = G/A$ if Dom(F) $= A$ and G extends F. A sequence $\bar{t}$ is a function whose domain is an ordinal, which is called its length and will be denoted $l(\bar{t})$. If $\bar{t}$ is a sequence, then $\bar{t}_i = \bar{t}(i)$ (=the value of the function at i). The sequence $\bar{t}$ will sometimes be denoted and defined as $\langle \bar{t}_i : i < l(\bar{t}) \rangle$. We shall frequently not distinguish between t_0 and $\langle t_i : i < 1 \rangle$. If A, B are sets $A - B = \{a \in A : a \notin B\}$.

T will designate a fixed first order theory in the language L with the equality sign. We shall use x, y, z for variables, $\bar{x}, \bar{y}, \bar{z}$ for finite sequences of variables, ϕ, ψ for formulas of L. We shall write $\phi(x_0, \cdots, x_{n-1})$ for ϕ if all the free variables occurring in ϕ are in $\{x_0, \cdots, x_{n-1}\}$. M, N will denote models of T. If M is a model, $|M|$ will be the set of its elements, and thus $\|M\|$ will be its power. We write $M \vDash \phi[a_0, \cdots, a_{n-1}]$ if $a_0, \cdots, a_{n-1} \in |M|$ and $\phi[a_0, \cdots, a_{n-1}]$ is satisfied by M. The model M is said to be λ-saturated if for every sequence of formulas $\langle \phi_i(x, \bar{y}) : i < i_0 < \lambda \rangle$ and sequence $\langle \bar{b}_i : i < i_0 < \lambda \rangle$, of sequences of elements of M, if for every finite set $I \subseteq i_0$ there is a $c \in |M|$ s.t. $i \in I$ implies $M \vDash \phi_i[c, \bar{b}_i]$, then there is a $c \in |M|$ s.t. for all $i < i_0$, $M \vDash \phi_i[c, \bar{b}_i]$. Let λ_0 be $2^{2^{|T|}}$ and let M_0 be a $(2^{\lambda_0})^+$-saturated model of T of power $> |T|$. (The proof of the existence of such a model, and information about other properties of saturation can be found in [4], [2]. The definition of saturation in [4] is slightly different from ours.)

M_1 is an elementary submodel of M_2 if $|M_1| \subseteq |M_2|$ and for every formula $\phi(\bar{x})$ and every sequence $\bar{b}$ of elements of M_1, $M_1 \vDash \phi[\bar{b}]$ iff $M_2 \vDash \phi[\bar{b}]$. If we do not specify otherwise, every model will be an elementary submodel of M_0. It is easy to see that M_1 is an elementary submodel of M_2 iff $|M_1| \subseteq |M_2|$.

A, B, C will designate sets included in $|M_0|$; a, b, c will denote elements of $|M_0|$, and $\bar{a}, \bar{b}, \bar{c}$ will denote finite sequences of elements of $|M_0|$. $\bar{b}$ is "from A" if all the members of the sequence belong to A. Instead of $M \vDash \phi[\bar{b}]$ we can write $\vDash \phi[\bar{b}]$ since the particular model M (which is an elementary submodel of M_0) does not matter. For a set A there is a model M such that $|M| = A$ (i.e., such that M is an elementary submodel of M_0) iff for every sequence $\bar{b}$ and formula $\phi(x, \bar{y})$ if $\vDash (\exists x)\phi(x, \bar{b})$ then there is $a \in A$ such that $\vDash \phi[a, \bar{b}]$. (This is the Tarski-Vaught test.) If not mentioned otherwise, when writing M or A we demand implicitly $\|M\|$, $|A| \leq 2^{\lambda_0}$.

A function F will be called an elementary mapping if Dom(F), Rang(F) $\subseteq |M_0|$ and for every formula $\phi(x_0, \cdots, x_{n-1})$ and $b_0, \cdots, b_{n-1} \in$ Dom(F), $\vDash \phi[b_0, \cdots, b_{n-1}]$ iff $\vDash \phi[F(b_0), \cdots, F(b_{n-1})]$. (Clearly an elementary mapping must be a one-one function.) F, G will denote elementary mappings. From the properties of saturated models it is known that if M_0 is λ_1-saturated, $|A| \leq \lambda_1$, $|\text{Dom}(G)| < \lambda_1$ then there is an extension F of G s.t. Dom(F) $= A \cup$ Dom(G). F is an automorphism of M if Dom(F) $=$ Rang(F) $= |M|$. Without loss of generality we assume that every elementary mapping F with finite domain can be extended to an automorphism of M_0(see [4]).

Actually, we shall frequently use a language wider than L which contains, in addition, for every $a \in |M_0|$ a name which will be a itself. p is an n-type over A if p is a set of formulas of the form $\phi(x_0, \cdots, x_{n-1}, \bar{b})$ where $\bar{b}$ is a sequence from A.

q, p, r will designate types over $|M_0|$. p extends q if $q \subseteq p$. $\bar{c}$ realizes p if for every $\phi(x_0, \cdots, x_{n-1}, \bar{b}) \in p$, $\vDash\phi[\bar{c}, \bar{b}]$. When we say type we shall mean a noncontradictory type, i.e., for every finite subset of it, there is an element which realizes the finite subset. (From the definition of M_0 it follows that if p is a type over A, $|A| < (2^{\lambda_0})^+$, then there is a sequence $\bar{b}$ realizing p.) p is a complete n-type if for every sequence $\bar{b}$ from A and every formula ϕ, $\phi(x_0, \cdots, x_{n-1}, \bar{b}) \in p$ or $\neg\phi(x_0, \cdots, x_{n-1}, \bar{b}) \in p$. Unless mentioned otherwise, every type is a 1-type. $S(A)$ will be the set of complete (1-) types over A. Every sequence $\langle b_0, \cdots, b_{n-1}\rangle$ realizes a complete n-type over A, which will be called "the type that $\langle b_0, \cdots, b_{n-1}\rangle$ realizes over A." If $n = 1$, this type belongs to $S(A)$, i.e., every element b realizes a type in $S(A)$. It is known that every n-type over a set A has an extension which is a complete n-type over A. Define p/A as $\{\psi \in p : \{\psi\}$ is a type over $A\}$. Define $F(p)$ as

$$\{\psi(x_0, \cdots, x_{n-1}, F(a_0), \cdots, F(a_m)) : \psi(x_0, \cdots, x_{n-1}, a_0, \cdots, a_m) \in p; \\ a_0, \cdots, a_m \in \mathrm{Dom}(F)\}.$$

The definition and properties of ultrapowers can be found in [1], [2]. Assume D is a nonprincipal ultrafilter on ω (the set of the natural numbers). It is known that there is an elementary mapping from M into M^ω/D; thus without loss of generality we can assume that this mapping is the identity, and that M^ω/D is an elementary submodel of M_0. It is known that M^ω/D is an $\aleph_1$-saturated model. It is also known that if p_n is a type over $|M|$ which is realized in M and $p_{n+1} \supseteq p_n$ (this for all $n < \omega$), then $\bigcup_{n<\omega} p_n$ is realized in M^ω/D.

It is known that if M is a model of T, $A \subseteq |M|$, then M has an elementary submodel M_1, $A \subseteq |M_1|$, of power $\leq |T| + |A|$. Clearly if M omits (i.e., does not realize) a type p over A, then M_1 also omits it.

§2. Ranks of types.

DEFINITION 2.1. (1) We define TR_α by induction on α: TR_α is the set of types p over $|M_0|$ such that there is $q \subseteq p$, $|q| < \aleph_0$, and q has $\leq \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\beta<\alpha} TR_\beta$. (Clearly $\alpha > \beta$ implies $TR_\alpha \supseteq TR_\beta$.)

(2) If $p \in TR_\beta - \bigcup_{\gamma<\beta} TR_\gamma$ then $\mathrm{Rank}(p) = \beta$ and if for all β, $p \notin TR_\beta$, then $\mathrm{Rank}(p) = \infty$.

(3) The ranks are ordered by the natural ordering of the ordinals with the additional stipulation that $\infty > \alpha$ for all ordinals α.

(4) If for all p $\mathrm{Rank}(p) < \infty$, we say that T is superstable.

THEOREM 2.1. (1) *If $p \subseteq q$ then* $\mathrm{Rank}(p) \geq \mathrm{Rank}(q)$.

(2) *For every type p there is $q \subseteq p$, $|q| < \aleph_0$, such that* $\mathrm{Rank}(p) = \mathrm{Rank}(q)$.

(3) *There is α_0 such that for every p,* $\mathrm{Rank}(p) < \alpha_0$ *or* $\mathrm{Rank}(p) = \infty$.

(4) *If* $\mathrm{Rank}(p) = \infty$, $|p| < \aleph_0$, *then p has $>\lambda_0$ continuations in $S(|M_0|)$ which are of rank ∞.*

(5) *If $\{p_i : i < i_0\}$ is a set of $>\lambda_0$ continuations of p in $S(M_0)$,* $\mathrm{Rank}(p) < \infty$, *then there is an $i < i_0$ such that* $\mathrm{Rank}(p) > \mathrm{Rank}(p_i)$.

PROOF. (1) If $\mathrm{Rank}(q) = \alpha$, then for every $\beta < \alpha$, $q \notin TR_\beta$. Thus for all $q_1 \subseteq q$ such that $|q_1| < \aleph_0$, q_1 has $>\lambda_0$ continuations in $S(|M_0|) - \bigcup_{\gamma<\beta} TR_\gamma$. Thus for every $q_1 \subseteq p$, $|q_1| < \aleph_0$, q_1 has $>\lambda_0$ continuations in $S(|M_0|) - \bigcup_{\gamma<\beta} TR_\gamma$. Thus $p \notin TR_\beta$, and $\mathrm{Rank}(p) > \beta$. Since this is true for all $\beta < \alpha$, $\mathrm{Rank}(p) \geq \alpha = \mathrm{Rank}(q)$.

(2) If $\operatorname{Rank}(p) = \alpha$, then by definition there is $q \subseteq p$, $|q| < \aleph_0$, such that q has $\leq \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\gamma<\alpha} TR_\gamma$. Thus $\operatorname{Rank}(q) \leq \alpha$. On the other hand, since $q \subseteq p$, $\operatorname{Rank}(q) \geq \alpha$. This proves $\operatorname{Rank}(q) = \operatorname{Rank}(p)$.

(3) Since the types over $|M_0|$ form a set, the α's for which $TR_\alpha - \bigcup_{\beta<\alpha} TR_\beta \neq 0$ form a set, and thus there is an ordinal α_0 larger than all of them. This α_0 satisfies the required condition. (It is not hard to find a bound for α_0 and to show that if it is the last one to satisfy the condition then for every $\alpha < \alpha_0$ there is a type p with $\operatorname{Rank}(p) = \alpha$.)

(4) Since $\operatorname{Rank}(p) = \infty$, $p \notin TR_{\alpha_0}$; and thus every $q \subseteq p$ with $|q| < \aleph_0$, in particular p, has more than λ_0 continuations in $S(|M_0|) - TR_{\alpha_0}$. Each one of them has rank ∞, since its rank is $> \alpha_0$.

(5) Assume $\operatorname{Rank}(p) = \alpha$ and for all $i < i_0$, $\operatorname{Rank}(p_i) \geq \alpha$; thus by (1) $\operatorname{Rank}(p_i) = \alpha$. By definition there is a $q \subseteq p$, $|q| < \aleph_0$, which has $\leq \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\gamma<\alpha} TR_\gamma$. Since all the p_i's are such continuations, we have a contradiction, thus proving (5).

THEOREM 2.2. *If p is a type over A, then there is a type q, $|q| < \aleph_0$, such that $p \cup q$ is a type over A which has no continuation of smaller rank in $S(A)$. (Clearly if p is finite then $p \cup q$ is finite, and if p is a type over a finite set then $p \cup q$ is a type over a finite set.)*

PROOF. Since the ranks are well ordered, there is a $q_1 \supseteq p$ in $S(A)$ of minimal rank. By Theorem 2.1(2) there is $q \subseteq q_1$, $|q| < \aleph_0$, such that $\operatorname{Rank}(q) = \operatorname{Rank}(q_1)$. Clearly $p \cup q$ is a type over A. If $q_2 \supseteq p \cup q$, $q_2 \in S(A)$, then

$$\begin{aligned}\operatorname{Rank}(q_2) \leq \operatorname{Rank}(p \cup q) &= \operatorname{Rank}(q_1)\\ &= \inf\{\operatorname{Rank}(q_3): q_3 \supseteq p \cup q, q_3 \in S(A)\} \leq \operatorname{Rank}(q_2).\end{aligned}$$

Thus $\operatorname{Rank}(q_2) = \operatorname{Rank}(p \cup q)$. Thus all the continuations of $p \cup q$ in $S(A)$ are of the rank $\operatorname{Rank}(p \cup q)$. This clearly implies also that all the continuations of $p \cup q$ to a type over A are of the same rank.

THEOREM 2.3. (1) *If F is an automorphism of M_0 and p is a type, then* $\operatorname{Rank}(p) = \operatorname{Rank}(F(p))$.

(2) *If F is an elementary mapping and p is a type, then* $\operatorname{Rank}(p) = \operatorname{Rank}(F(p))$, *where p is a type over* Dom F.

PROOF. (1) We will prove by induction on α that $p \in TR_\alpha$ iff $F(p) \in TR_\alpha$. Assume that this is true for all $\beta < \alpha$ (for $\alpha = 0$ this is a void assumption). If $p \in TR_\alpha$ then there is $p_1 \subseteq p$, $|p_1| < \aleph_0$, and p_1 has $\leq \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\beta<\alpha} TR_\beta$. If $F(p_1)$ has $> \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\beta<\alpha} TR_\beta$, and $\{p^i: i < \lambda_0^+\}$ is a set of λ_0^+ such continuations, then, since F^{-1} is also an automorphism, $F^{-1}(p^i) \supseteq p_1$ and $i \neq j$ implies $p^i \neq p^j$ for all $i, j < \lambda_0^+$.

$p^i \in S(|M_0|)$ and $p^i \notin \bigcup_{\beta<\alpha} TR_\beta$, and hence by induction hypothesis $F^{-1}(p^i) \in S(|M_0|)$, $F^{-1}(p^i) \notin \bigcup_{\beta<\alpha} TR_\beta$. Thus p_1 has $> \lambda_0$ continuations in $S(|M_0|) - \bigcup_{\beta<\alpha} TR_\beta$, contradiction. Thus $p \in TR_\alpha$ implies $F(p) \in TR_\alpha$. In the same way $F(p) \in TR_\alpha$ implies $p = F^{-1}(F(p)) \in TR_\alpha$. Thus Theorem 2.3(1) is proved.

(2) By Theorem 2.1(2) there is $p_1 \subseteq p$, $|p_1| < \aleph_0$, such that $\operatorname{Rank}(p_1) = \operatorname{Rank}(p)$. Clearly there exists a finite set A s.t. p_1 is a type over A, and thus F/A can be extended to an automorphism G of M_0. Thus, using (1),

$$\operatorname{Rank}(p) = \operatorname{Rank}(p_1) = \operatorname{Rank}(G(p_1)) = \operatorname{Rank}(F(p_1)) \geq \operatorname{Rank}(F(p)).$$

Thus Rank(p) $\geq$ Rank($F(p)$). In the same way, since F^{-1} is also an elementary mapping, Rank($F(p)$) $\geq$ Rank($F^{-1}(F(p))$) $=$ Rank(p). Thus we have proved Theorem 2.3(2).

THEOREM 2.4. (1) *If p is a type, there exists an $r \in S(|M_0|)$ such that* Rank(r) $=$ Rank(p), $p \subseteq r$.

(2) *If p is a type over A, and p has $>\lambda_0$ continuations in $S(A)$,* Rank(p) $< \infty$, *then at least one of them is of rank* $<$Rank(p).

PROOF. (1) Suppose there is no such r. Let $\alpha =$ Rank(p), and let $\{p_i : i < k \leq \|M_0\|\}$ be the set of types such that $|p_i| < \aleph_0$, $\alpha_i =$ Rank$(p_i) < \alpha$; and let $\psi_i = \bigwedge_{\phi \in p_i} \phi$. Suppose $q \supseteq p$, $q \supseteq \{\neg\psi_i : i < k\}$ and $q \in S(|M_0|)$. Since $q \supseteq p$, Rank$(q) \leq \alpha$. If Rank$(q) < \alpha$ then there is a $q_1 \subseteq q$, $|q_1| < \aleph_0$, Rank$(q_1) < \alpha$. Thus $q_1 = p_{i_0}$ for some $i_0 < k$, and $(\neg\bigwedge_{\phi \in p_{i_0}} \phi) \in q$, $\{\phi : \phi \in p_{i_0} = q_1\} \subseteq q$. Hence q is not consistent, so Rank$(q) = \alpha$, a contradiction.

Therefore there is no such q, and by the compactness theorem there exists a finite contradictory subset of $p \cup \{\neg\psi_i : i < k\}$. Hence there exist $\phi_1, \cdots, \phi_n \in p$, $\phi^1 = \psi_{i_1}, \cdots, \phi^m = \psi_{i_m}$, such that $\bigwedge_{i=1}^{n} \phi_i \to \bigvee_{j=1}^{m} \phi^j$. Let $\beta = \max_{j=1,m} \alpha_{i_j} = \max_{j=1,m}(\text{Rank}(p_{i_j})) < \alpha$. Every $\{\phi^j\}$ has $\leq \lambda_0$ continuation in $S(|M_0|) - \bigcup_{\gamma<\beta} TR_\gamma$. Hence also $\{\bigvee_{j=1}^{m} \phi^j\}$ and $\{\phi_i : i = 1, \cdots, n\}$ have $\leq \lambda_0$ continuation in $S(|M_0|) - \bigcup_{\gamma<\beta} TR_\gamma$. We can conclude Rank$(q) \leq$ Rank$(\{\phi_i : i = 1, \cdots, n\}) \leq \beta < \alpha$, a contradiction.

(2) If $\{p_i : i < \lambda_0^+\}$ are continuations of p in $S(A)$, such that Rank$(p_i) =$ Rank(p), then by (1) each of them has a continuation of the same rank in $S(|M_0|)$, and we get a contradiction by Theorem 2.1(5).

§3. A theory which is not superstable has a model which is not $\aleph_1$-saturated.

THEOREM 3.1. *If* Rank$(\bar{p}) = \infty$, $|\bar{p}| < \aleph_0$, *then there is a sequence of formulas* $\{\psi_i(x_0, \bar{a}_i) : i < (2^{|T|})^+\}$ *such that for all* $j < (2^{|T|})^+$

$$\text{Rank}(\bar{p} \cup \{\psi_i(x_0, \bar{a}_i) : i < j\} \cup \{\neg\psi_j(x_0, \bar{a}_j)\}) = \infty.$$

PROOF. By Theorem 2.1(4) $\bar{p}$ has $>\lambda_0$ continuations of rank ∞ in $S(|M_0|)$. Let $\{p_i : i < \lambda_0^+\}$ be λ_0^+ of these continuations (all different). For a formula ψ define $\psi^0 = \psi$, $\psi^1 = \neg\psi$ and let η denote a sequence of ones and zeroes.

Define $\psi_\eta = \phi_\eta(x_0, \bar{a}_\eta)$ by induction on $l(\eta)$ so that if the formula $\psi_{\eta/i}$ is defined for all $i \leq l(\eta)$ then the type $q_\eta = \{(\psi_{\eta/i})^{\eta(i)} : i < l(\eta)\}$ is contained in one of the p_i's.

If $l(\eta) = 0$, since $p_1 \neq p_2$, there is ψ, $\psi \in p_1$, $\neg\psi \in p_2$. Let $\psi_\eta = \psi$. If $\psi_{\eta/j}$ is defined for all $j < l(\eta)$, and there are at least two types $p^1, p^2 \in \{p_i : i < \lambda_0^+\}$ which continue q_η, then there is a ψ, $\psi \in p^1$, $\neg\psi \in p^2$. Then we define $\psi_\eta = \psi$. If there are no two such types, ψ_η will be undefined.

Now for all $p \in \{p_i : i < \lambda_0^+\}$ we can easily find $\eta = \eta_p$ such that $q_\eta \subseteq p$ and is the only continuation of q_η among the p_i's. If for all p, $l(\eta_p) < (2^{|T|})^+$, then

$$\lambda_0^+ = |\{p_i : i < \lambda_0^+\}| \leq |\{\eta : l(\eta) < (2^{|T|})^+\}| = \sum \{2^{|i|} : i < (2^{|T|})^+\} = 2^{2^{|T|}} \leq \lambda_0.$$

(Here we used the definition of λ_0 as $2^{2^{|T|}}$.) Contradiction. Thus there is a p such that $l(\eta_p) \geq (2^{|T|})^+$. Define, for $i < (2^{|T|})^+$, $\psi_i(x_0, \bar{a}_i) = [\phi_{\eta_p/i}(x_0, \bar{a}_{\eta_p/i})]^{\eta_p(i)}$. Since for all $j < (2^{|T|})^+$, $\bar{p} \cup \{\psi_i(x_0, \bar{a}_i) : i < j\} \cup \{\neg\psi_j(x_0, \bar{a}_j)\}$ is included in one of the p_i's, its rank is $\geq$Rank$(p_i) = \infty$. This proves the theorem.

THEOREM 3.2. *If T is not superstable then it has a non-$\aleph_1$-saturated model of power $|T|$.*

PROOF. Define by induction N_n and p_n for $n < \omega$ such that N_n is an elementary submodel of N_{n+1}, $\|N_n\| = |T|$. p_n is a type over $|N_n|$, $|p_n| < \aleph_0$, there is no element in N_{n-1} which realizes p_n, $p_n \subseteq p_{n+1}$, and Rank$(p_n) = \infty$. It is easy to see that in this case $\bigcup_{n<\omega} N_n$ is a model of T of power $|T|$, and $\bigcup_{n<\omega} p_n$ is a type of power $\aleph_0$ over $\bigcup_{n<\omega} N_n$ which is not realized there. This will prove the theorem.

For $n = 0$ let N_0 be an elementary submodel of M_0 of power $|T|$, $p_0 = \{ \}$. (Since T is not superstable, there is a p with Rank$(p) = \infty$. But $p_0 = \{ \} \subseteq p$ and thus Rank$(p_0) \geq$ Rank$(p) = \infty$, Rank$(p_0) = \infty$.)

Assume that N_n, p_n are defined. Since Rank$(p_n) \doteq \infty$, by Theorem 3.1 there is $\{\psi_i(x_0, \bar{a}_i): i < (2^{|T|})^+\}$ such that for all $j < (2^{|T|})^+$, Rank$(q_j) = \infty$ where q_j is defined as $q_j = p_n \cup \{\psi_i(x_0, \bar{a}_i) : i < j\} \cup \{\neg\psi_j(x_0, \bar{a}_j)\}$. (And, of course, q_j is a type, i.e. consistent.) For $a \in N_n$ define $I_a = \{i < (2^{|T|})^+ : \ \vDash \psi_i(a, \bar{a}_i)\}$. Since there are $\leq |T|$ sets I_a and $(2^{|T|})^+$ i's, there are $k, l < (2^{|T|})^+$, $k < l$, and $a \in |N_n|$ implies $k \in I_a$ iff $l \in I_a$. Define $p_{n+1} = p_n \cup \{\psi_k(x_0, \bar{a}_k), \neg\psi_l(x_0, \bar{a}_l)\}$, and let N_{n+1} be an elementary submodel of M_0 of power $|T|$ which includes $|N_n| \cup$ Rang$(\bar{a}_k) \cup$ Rang$(\bar{a}_l)$.

Since for all $a \in |N_n|$ we have $\vDash \psi_k(a, \bar{a}_k) \leftrightarrow \psi_l(a, \bar{a}_l)$, it is clear that no a realizes p_{n+1}. Also $p_n \subseteq p_{n+1} \subseteq q_l$ and thus p_{n+1} is a consistent type of rank ∞. Of course $|p_{n+1}| < \aleph_0$. Thus all the conditions of the definition are satisfied and the theorem is proved.

THEOREM 3.3. *If $|T|^{\aleph_0} = |T|$ then T has a $\aleph_1$-saturated model of power $|T|$.*

PROOF. Let M be a model of T of power $|T|$, and let D be a nonprincipal ultrafilter on ω (the set of natural numbers). Then, as stated in the introduction, M^ω/D is a $\aleph_1$-saturated model, and $\|M^\omega/D\| \leq \|M\|^{\aleph_0} = |T|^{\aleph_0} = |T|$.

THEOREM 3.4. *If T is categorical in $|T|^{\aleph_0} = |T|$ then T is superstable.*

PROOF. Immediate from Theorems 3.2 and 3.3.

§4. On prime models.

DEFINITION 4.1. (1) p will be called an isolated type over A if p is a type over A and there is a finite set $B \subseteq A$ such that Rank$(p) =$ Rank(p/B) and p/B has no continuation in $S(A)$ of smaller rank, and Rank$(p) < \infty$.

(2) $B \supseteq A$ will be called prime over A if $B = A \cup \{a_i : i < i_0\}$ and for every $i < i_0$, a_i realizes an isolated type over $A \cup \{a_j : j < i\}$.

(3) M will be called a prime model over A if $|M|$ is prime over A and every type over a finite set contained in $|M|$ is realized in M.

THEOREM 4.1. (1) *If T is superstable, then over every A there is a prime model M (such that $|M| = A \cup \{a_i : i < i^0\}$, where a_j realizes an isolated type over $A_j = A \cup \{a_i : i < j\}$).*

(2) *In the above model for every $l < i^0$ there is n and $j_0 < j_1 < \cdots < j_n = l$ such that for all $m \leq n$ a_{j_m} realizes an isolated type over $A \cup \{a_{j_0}, \cdots, a_{j_{m-1}}\}$.*

PROOF. (1) Define i_k for $k \leq \omega$ and a_i for $i < i_k$ by induction on k such that $i_\omega = \bigcup_{k<\omega} i_k$ and $k < l$ implies $i_k < i_l$. Assume that it is defined for all $k_1 < k$. If $k = 0$, $i_0 = 0$. If $k = l + 1$, let i_k be any ordinal such that $\{p_j : i_l \leq j < i_k\}$ is the set of types over finite subsets of $A \cup \{a_j : j < i_l\}$. Define by induction a_i for

$i_l \leq i < i_k$. Assume that it is defined for all $i < j$ and let q be an isolated type in $S(A \cup \{a_i: i < j\})$ which continues p_j (the existence of such a type was proved in Theorem 2.2). Let a_j be an element which realizes q. By definition, $A_1 = A \cup \{a_i: i < i_\omega\}$ is prime over A.

Assume q is a type over a finite set $B \subseteq A_1$. We must show that q is realized by a member of A_1. Since B is finite there is $n < \omega$ such that $B \subseteq A \cup \{a_i: i < i_n\}$, and thus there is j, $i_n \leq j < i_{n+1}$, with $q = p_j$. Clearly a_j will realize q.

It remains to be proved that there is an M with $|M| = A_1$. By the Tarski-Vaught test (see §1) it is sufficient to prove that if $\vDash(\exists x)\phi(x, \bar{a})$, and $\bar{a}$ is a sequence from A_1, then there is $b \in A_1$ such that $\vDash\phi[b, \bar{a}]$. But $\{\phi(x, \bar{a})\}$ is a type over a finite set $\subseteq A_1$. Thus, by the above, there is a b as required and that proves (1).

(2) By the definition of $\{a_i: i < i^0\}$, for every i there is a finite set $I_i \subseteq \{k: k < i\}$ such that if p^i is the type that a_i realizes over $A \cup \{a_j: j < i\}$ then $p^i/(A \cup \{a_j: j \in I_i\})$ has no continuation in $S(A \cup \{a_j: j < i\})$ of smaller rank. If we find $j_0, \cdots, j_n = l$ such that for all $m \leq n$, $I_{j_m} \subseteq \{j_0, \cdots, j_{m-1}\}$ this will clearly prove the theorem.

Define E_n by induction on $n < \omega$: $E_0 = \{l\}$ and $E_{n+1} = \bigcup\{I_i: i \in E_n\}$. It is easy to see that E_n is a finite set since the finite union of finite sets is finite. Let $\alpha_n = \max\{i: i \in E_n\}$. Since E_n is finite, the maximum exists. Also for every n, $\alpha_{n+1} < \alpha_n$ since $\alpha_{n+1} \in E_{n+1} = \bigcup\{I_i: i \in E_n\}$ and thus there is an $i \in E_n$, $\alpha_{n+1} \in I_i$ and hence $\alpha_{n+1} < i \leq \max E_n = \alpha_n$. (We have used the fact that $I_i \subseteq \{k: k < i\}$.) Since the ordinals are well ordered, there is no descending sequence of ordinals, and therefore there is n_0 for which E_{n_0} is the empty set. If $\bigcup_{n<n_0} E_n = \{j_0, \cdots, j_m\}$ and $j_0 < j_1 < \cdots < j_m$ then $j_m = l$ and this proves Theorem 4.1(2).

§5. Indiscernible sequences.

DEFINITION 5.1. (1) The sequence $\bar{a} = \langle a_i: i < k\rangle$ is indiscernible over A if for all $n < \omega$ and every $i_0 < \cdots < i_n < k$, $j_0 < \cdots < j_n < k$, the sequences $\langle a_{i_0}, \cdots, a_{i_n}\rangle$, $\langle a_{j_0}, \cdots, a_{j_n}\rangle$ realize the same type over A. We shall always assume that $k \geq \omega$ and for $l \leq k$ let $\bar{a}^l = \langle a_i: i < l\rangle$ and $A_l = A \cup \{a_i: i < l\}$. We use this notation when it is clear what A and $\bar{a}$ are. Also, we shall always assume that $a_0 \neq a_1$.

(2) In the above notation $p_l(\bar{a})$ will be the type $\{\psi(x_0, a_{i_0}, \cdots, a_{i_n}, \bar{b}) : \bar{b}$ a sequence from A; for $m \leq n$, $i_m \leq l$ and $j_m \leq n$, for $m_1, m_2 \leq n$, $i_{m_1} > i_{m_2}$ iff $j_{m_1} > j_{m_2}$ and $\vDash\psi[a_{n+1}, a_{j_0}, \cdots, a_{j_n}, \bar{b}]\}$. (For $l < k$ $p_l(\bar{a})$ is the type that a_l realizes over A_l.)

THEOREM 5.1. *If $\bar{a} = \langle a_i: i < k\rangle$ is an indiscernible sequence over A then $p_l(\bar{a})$ is a complete (consistent) type over A_l for all $l \leq k$.*

PROOF. If $l < k$, a_l realizes $p_l(\bar{a})$ and the claim follows. Assume $l = k$. If there is a contradiction in $p_l(\bar{a})$ then there is a contradiction in a finite subtype q of $p_l(\bar{a})$. By changing the order of the variables of the formulas in q and adding dummy variables, there will be a contradiction in $\{\psi_m(x_0, a_{i_0}, \cdots, a_{i_n}, \bar{b}) : m < n_0\}$ where $\bar{b}$ is a sequence from A, $i_0 < \cdots < i_n$. Thus $\vDash\neg(\exists x)(\bigwedge_{m<n_0}\psi_m(x_0, a_{i_0}, \cdots, a_{i_n}, \bar{b}))$ and $\vDash\neg(\exists x)(\bigwedge_{m<n_0}\psi_m(x_0, a_0, \cdots, a_n, \bar{b}))$. But $\vDash\bigwedge_{m<n_0}\psi_m(a_{n+1}, a_0, \cdots, a_n, \bar{b})$, contradiction. Thus Theorem 5.1 is proved.

THEOREM 5.2. *Assume T is superstable.*

(1) *If $\langle a_i: i < k\rangle$ is an indiscernible sequence over A and $k_0 + \omega \leq k$, then $\langle a_i: k_0 \leq i < k\rangle$ is an indiscernible sequence over A_{k_0}.*

(2) *If p is a type over A, and $\langle a_i: i < k\rangle$ is an indiscernible sequence over A, then p has no continuation in $S(A_\omega)$ of smaller rank iff p has no continuation in $S(A_k)$ of smaller rank.*

(3) *If $p \in S(A)$, $\langle a_i: i < k\rangle$ an indiscernible sequence over A, p has no continuation in $S(A_k)$ of smaller rank, $p \subseteq q \in S(A_k)$, $i_0 < \cdots < i_n < k$, $j_0 < \cdots < j_n < k$, and $\bar{b}$ a sequence from A then*

$$\psi(x_0, a_{i_0}, \cdots, a_{i_n}, \bar{b}) \in q \quad \textit{iff} \quad \psi(x_0, a_{j_0}, \cdots, a_{j_n}, \bar{b}) \in q.$$

(4) *If a realizes type $p \in S(A_k)$ and p/A has no continuation in $S(A_k)$ of smaller rank where $\langle a_i: i < k\rangle$ is an indiscernible sequence over A, then $\langle a_i: i < k\rangle$ is an indiscernible sequence over $A \cup \{a\}$.*

PROOF. (1) Immediate.

(2) Assume that p has a continuation q in $S(A_\omega)$ of smaller rank. Let q_1 be any continuation of q in $S(A_k)$. Then $\text{Rank}(q_1) \leq \text{Rank}(q) < \text{Rank}(p)$ and thus p has a continuation in $S(A_k)$ of smaller rank.

Assume p has a continuation q in $S(A_k)$ of smaller rank. In this case there is $q_1 \subseteq q$, $|q_1| < \aleph_0$, such that $\text{Rank}(q_1) = \text{Rank}(q)$ (by Theorem 2.1(2)). Thus there exist $i_0 < \cdots < i_n$ such that q_1 is a type over $B = A \cup \{a_{i_m}: m \leq n\}$. Define an elementary mapping F such that $\text{Dom}(F) = B$, for a in A $F(a) = a$ and $F(a_{i_m}) = a_m$ for $m \leq n$. It is easy to see that $F(q_1) \supseteq p$ and $F(q_1)$ is a type over A_ω; thus it has a continuation $q_2 \in S(A_\omega)$. We have $\text{Rank}(q_2) \leq \text{Rank}(F(q_1)) = \text{Rank}(q_1) = \text{Rank}(q) < \text{Rank}(p)$, i.e. p has a continuation in $S(A_\omega)$ of smaller rank. Thus we have finished the proof of (2).

(3) Without loss of generality, assume $k < (2^{\lambda_0})^+$. By Theorem 5.1(2), and since M_0 is $(2^{\lambda_0})^+$-saturated, it is easy to see that it is possible to define a_i for $k \leq i < (2^{\lambda_0})^+ = k_1$ such that $\langle a_i: i < k_1\rangle$ is an indiscernible sequence over A (a_i will be an element realizing $p_i(\langle a_j: j < i\rangle)$).

Assume $\psi(x_0, a_{i_0}, \cdots, a_{i_n}, \bar{b}) \in q$, $\neg\psi(x_0, a_{j_0}, \cdots, a_{j_n}, \bar{b}) \in q$. Let q_1 be a continuation of q in $S(A_{k_1})$, l an ordinal $< k_1$, $i_n, j_n < l$. Without loss of generality assume $\psi(x_0, a_l, \cdots, a_{l+n}, \bar{b}) \in q_1$ (for otherwise take $\neg\psi$ instead of ψ and interchange the i's and j's). For every $\delta < k_1$ let $\phi_\delta = \psi(x_0, a_\delta, \cdots, a_{\delta+n}, \bar{b})$. Since for every $\delta_1 < \delta_2 < k_1$ the sequence

$$\langle a_{\delta_1}, \cdots, a_{\delta_1+n}, a_{\delta_2}, \cdots, a_{\delta_2+n}\rangle, \langle a_{j_0}, \cdots, a_{j_n}, a_l, \cdots, a_{l+n}\rangle$$

realize the same type over A, we have that for all $\delta_1 < \delta_2 < k_1$, $q_1 \cup \{\neg\phi_{\delta_1}, \phi_{\delta_2}\}$ is a consistent type over A_{k_1}, thus having a continuation $q(\delta_1, \delta_2)$ in $S(A_{k_1})$.

On the other hand p has no continuation in $S(A_{k_1})$ of smaller rank (by part 2 of this theorem) and thus by Theorem 2.4 p has $\leq \lambda_0$ continuations in $S(A_{k_1})$. For every such continuation $\bar{p}$ corresponds a set $I(\bar{p}) = \{\delta: \delta < k_1, \phi_\delta \in \bar{p}\}$. Since there are $\leq \lambda_0$ such sets and $|\{\delta: \delta < k_1\}| = (2^{\lambda_0})^+$, there are $\delta_1 < \delta_2 < k_1$ such that for every continuation $\bar{p}$ of p in $S(A_{k_1})$, $\phi_{\delta_1} \in \bar{p}$ iff $\phi_{\delta_2} \in \bar{p}$. But $q(\delta_1, \delta_2)$ constitutes a counterexample. Thus, the claim is proved.

(4) Follows directly from (3).

§6. On theories T categorical in $|T|$.

THEOREM 6.1. *There is an indiscernible sequence $\bar{a} = \langle a_i: i < \delta\rangle$ over $A = 0$.*

PROOF. See [9].

THEOREM 6.2. *If T is superstable and $\bar{a}^\delta = \langle a_i : i < \delta \rangle$ is the sequence mentioned in Theorem 6.1 then there is a prime model over $A_\omega = \{a_i : i < \omega\}$ omitting $p_\omega(\langle a_i : i < \omega \rangle)$.*

PROOF. By Theorem 4.1(1) there is a prime model M over A_ω such that $|M| = \{b_i : i < i_0\} \cup A_\omega$ and for all $j < i_0$, b_j realizes an isolated type over $\{b_i : i < j\} \cup A_\omega$. If $p_\omega(\bar{a}^\omega)$ is realized in M then there is $j < i_0$ such that b_j realizes it, and thus by Theorem 4.1(2) there is a sequence $j_0 < \cdots < j_n = j$ such that for all $m \leq n$, $c_m = b_{j_m}$ realizes an isolated type q_m over $A_\omega \cup \{c_0, \cdots, c_{m-1}\}$. Thus, there are finite sets $B_m \subseteq A_\omega$ such that $q_m/(B_m \cup \{c_0, \cdots, c_{m-1}\})$ has no continuation in $S(A_\omega \cup \{c_0, \cdots, c_{m-1}\})$ of smaller rank. Let $B = \bigcup_{m \leq n} B_m \subseteq \{a_0, \cdots, a_l\}$, $l < \omega$.

Let $a_\omega = c_n$; since c_n realizes $p_\omega(\bar{a}^\omega)$, $\langle a_i : i < \omega + 1 \rangle$ is an indiscernible sequence. We shall show by induction on $m \leq n + 1$ that $\langle a_i : l < i < \omega + 1 \rangle$ is indiscernible over $\{c_i : i < m\} \cup \{a_i : i \leq l\}$. For $m = 0$, by Theorem 5.2(1), $\langle a_i : l < i < \omega + 1 \rangle$ is an indiscernible sequence over $\{a_i : i \leq l\} = \{c_i : i < m = 0\} \cup \{a_i : i \leq l\}$. Now assume validity for m and we shall prove for $m + 1$. By definition, the type which c_m realizes over $\{c_i : i < m\} \cup \{a_i : i \leq l\}$ has no extension in $S(A_\omega \cup \{c_i : i < m\})$ of smaller rank. Thus, by Theorem 5.2(2), this type has no extension in $S(A_{\omega+1} \cup \{c_i : i < m\})$ of smaller rank. By Theorem 5.2(4) it follows that $\langle a_i : l < i < \omega + 1 \rangle$ is an indiscernible sequence over $\{a_i : i \leq l\} \cup \{c_i : i < m\} \cup \{c_m\} = \{a_i : i \leq l\} \cup \{c_i : i < m + 1\}$.

We have proved that for every $m \leq n + 1$, $\langle a_i : l < i < \omega + 1 \rangle$ is an indiscernible sequence over $\{a_i : i \leq l\} \cup \{c_i : i < m\}$. In particular, this is true for $m = n + 1$, and since $c_n \in \{a_i : i \leq l\} \cup \{c_i : i < n + 1\}$ it follows that for all $l < i$, $j < \omega + 1$, $\models a_i = c_n$ iff $\models a_j = c_n$. But $c_n = a_\omega$; therefore $a_{l+1} = c_n = a_\omega$. Since $\langle a_i : i < \omega + 1 \rangle$ is an indiscernible sequence, $a_0 = a_1$, in contradiction to the definition. Thus $p_\omega(\bar{a}^\omega)$ is omitted in M.

THEOREM 6.3. *If T is categorical in $|T| = |T|^{\aleph_0}$ then T has a model of power $< |T|$. Furthermore, this model can be chosen so that every type over a finite set is realized.*

PROOF. By Theorem 3.4, T is superstable. By Theorem 6.2, T has a prime model M in which there is a sequence $\bar{a}^\omega = \langle a_i : i < \omega \rangle$ indiscernible over the empty set A, and in which $p_\omega(\bar{a}^\omega)$ is not realized. Since M is a prime model every type over a finite set is realized in it and thus if $\|M\| < |T|$ the theorem is proved. Assume $\|M\| \geq |T|$. Then by the downward Lowenheim-Skolem theorem there is an elementary submodel M_1 of M such that $\{a_i : i < \omega\} \subseteq |M_1|$, $\|M_1\| = |T|$, and $p_\omega(\bar{a}^\omega)$ is omitted. We will show that T has a model of power $|T|$ which cannot be isomorphic to M_1.

Let N_0 be any model of T of power $|T|$. By induction define N_i for $i \leq N_1$: N_0 is defined, $N_{i+1} = N_i^\omega / D$ (D is a nonprincipal ultrafilter over ω), and if $i = \delta$ limit number then $|N_\delta| = \bigcup_{i<\delta} |N_i|$. Clearly $\|N_{\aleph_1}\| = |T|$. Assume $\bar{a}^\omega = \langle a_i : i < \omega \rangle$ is an indiscernible sequence in $N_{\aleph_1}$. Then there is $i < \aleph_1$ such that $|N_i| \supseteq \{a_i : i < \omega\}$. Since $p_\omega(\bar{a}^\omega) = \bigcup_{n<\omega} p_n(\bar{a}^\omega)$ and $\bigcup_{n<m} p_n(\bar{a}^\omega)$ is realized in N_i for all $m < \omega$, clearly $p_\omega(\bar{a}^\omega)$ is realized in N_{i+1}. Therefore $N_{\aleph_1}$ is not isomorphic to M_1 in contradiction to the categoricity of T in $|T|$. Thus the theorem is proved.

REMARK. It can be proved similarly that if T is categorical in λ, $\lambda^{\aleph_0} = \lambda$, then T has a model of power $< \lambda$.

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