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Mathematical Logic

Saharon Shelah

PCF and infinite free subsets in an algebra

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Annotated content

1 Other variants of "G.C.H. holds almost always"

[We give another way to prove that for every $\lambda \geq \beth_\omega$ for every large enough regular $\kappa < \beth_\omega$ we have $\lambda^{[\kappa]} = \lambda$, dealing with sufficient conditions for replacing $\beth_\omega$ by $\aleph_\omega$. This continues [Sh 460].]

2 Large pcf($\mathfrak{a}$) implies the existence of free sets

[A nice example of the implication stated in the title is that if $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$ then for every algebra M of cardinality $\aleph_\omega$ with countably many functions (or just $< \aleph_\omega$ many), for some $a_n \in M$ (for $n < \omega$) we have $a_n \notin c\ell_M(\{a_\ell : \ell \neq n, \ell < \omega\})$. Generally if pcf($\mathfrak{a}$) is not just of cardinality $> |\mathfrak{a}|$, but $\langle J_{<\theta}[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ has large rank (as defined below) then a relevant instance of IND connected to sup ($\mathfrak{a}$) holds.]

3 Existence of free subsets implies restrictions on pcf

[We have results of forms complementary to those of §2 (though not close enough). So if IND(κ, σ) (in every algebra with universe λ and $\leq \sigma$ functions there is an infinite independent subset) then for no distinct regular $\lambda_i \in \mathrm{Reg}\backslash\kappa^+$ (for $i < \kappa$) does $\prod\limits_{i<\kappa} \lambda_i/[\kappa]^{\leq\sigma}$ have true cofinality. We also look at IND$(\langle J^{\mathrm{bd}}_{\kappa_n} : n < \omega\rangle)$, J_n an ideal on κ_n (we ask for $\bar{\alpha} \in \prod\limits_{n<\omega} \kappa_n$ such that $n < \omega \Rightarrow \alpha_n > \sup(c\ell_M\{\alpha_\ell : \ell \in (n, \omega)\})$ and more general version, and from assumptions as in §2 get results even for the non stationary ideal.]

4 Sticks and Boolean Algebras

[We deal with some other measurements of $[\lambda]^\theta$. We also give an application by a construction of a Boolean Algebra: one into which the free algebra with λ generators is embedded but no homomorphic image is the free Boolean Algebra with κ generators; with even weakened demand we can replace free by finite/cofinite Boolean Algebras.]

S. Shelah: Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel (e-mail: shelah@math.huji.ac.il); and Rutgers University, Mathematics Department, New Brunswick, NJ 08854, USA

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5 More on free subsets and pcf
6 Odds and ends
[In 6.1 we deal with a replacement for Δ-system lemma; with $> 2^\kappa$ sequences of ordinals of length κ. In 6.3 we look at how we can divide $F \subseteq \Pi\mathfrak{a}$ to few bounded sets. In 6.4 we relook at the characterization of a property of [HJS], generalizing the questions somewhat. We then deal with freeness properties for $F \subseteq {}^\delta\mathrm{Ord}$ (modulo an ideal) and we give a correct version of [Sh:g, Ch. IX,3.5] on characterizing $\mathrm{cov}(\lambda, \lambda, \theta, \sigma)$ when $\sigma > \aleph_0$ concerning the obtainment of the pp version. We shall continue in [Sh 589].]

1. Other variants of "G.C.H. holds almost always"

We essentially redo the proof of [Sh 460], §2 in another more general way.

Notation 1.1. 1) $\mathfrak{F}_\kappa(A)$ is the family of κ-complete filters $\mathfrak{D}$ on $\mathcal{P}(A)$ so $\mathfrak{D} \subseteq \mathcal{P}(\mathcal{P}(A))$; so the points are subsets of A, and the members of $\mathfrak{D}$ (which are $\subseteq \mathcal{P}(A)$) which we shall be most interested in are ideals and their compliments.
2) We say $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$ has σ-complete character if for any $Y \subseteq \mathcal{P}(A)$ we have: $Y \in \mathfrak{D}$ iff $\mathrm{id}_\sigma(Y) \in \mathfrak{D}$ where $\mathrm{id}_\sigma(Y)$ is the σ-complete ideal on A generated by Y.
3) For an ideal I on X let $I^+ = \mathcal{P}(X) \backslash I$, similarly for a filter.

Definition 1.2. *1) For $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$, cardinals $\mu < \lambda$ and σ such that $|A| < \mu < \lambda$, we say that λ is $(\mathfrak{D}, \mu, \sigma)$-inaccessible when: if $\mathfrak{a}_t \subseteq (\mu, \lambda) \cap Reg$ for $t \in A$, $|\mathfrak{a}_t| < \sigma$ then $\{B \subseteq A : pcf_{\sigma-\mathrm{complete}}(\cup\{\mathfrak{a}_t : t \in B\}) \subseteq \lambda\} \in \mathfrak{D}$.*
2) If J is an ideal on A, we say λ is (J, μ, σ) inaccessible if $\mu < \lambda$ and for no $\theta_x \in (\mu, \lambda) \cap Reg$ for $x \in A$ and σ-complete ideal J_1 on A extending A is $\prod_x \theta_x / J$, λ-directed.
3) If we omit μ we mean: for $\mu = (|A| + \sigma)^+$.

Theorem 1.3. *Suppose $\langle \kappa_n : n < \omega \rangle$ is a strictly increasing sequence of regular cardinals $> \aleph_2$. Stipulate $\kappa_{-1} = \aleph_1$ and assume $\mathfrak{D}_n \in \mathfrak{F}_{\kappa_{n-1}}(\kappa_n)$ for $n < \omega$ and $\kappa = \sum_{n<\omega} \kappa_n$ satisfies:*

> $\otimes$ *if $n < \omega, \aleph_0 < \theta = cf(\theta) < \kappa_n, h : \kappa_{n+1} \to \theta$ and $Y \in \mathfrak{D}^+_{n+1}$ (so $Y \subseteq \mathcal{P}(\kappa_{n+1})$) then for some $\zeta < \theta$ we have*
> *$(*)_{Y,\zeta}$ $\{B \in Y : sup\ Rang(h \restriction B) < \zeta\} \in \mathfrak{D}^+_{n+1}$.*

If $\lambda > \kappa$ then for every $n < \omega$ large enough, λ is $(\mathfrak{D}_n, \kappa, \aleph_1)$-inaccessible.

Remark 1.4. 1) We can replace $\omega, \aleph_1$ by θ, θ^+ or $< \theta, \theta$ (when θ is regular uncountable) respectively (so $\kappa = \sum_{i<\theta} \kappa_i$, etc.) (why? repeat the proof or force by $\mathrm{Levy}(\aleph_0, < \sigma)$). Of course, we can replace $\mathfrak{F}(\kappa_n)$ by $\mathfrak{F}(A)$ if $|A| = \kappa_n$.
2) Note that the set defined in 1.1(2) is always an ideal on A.
3) We can assume that every $\mathfrak{D}_n$ has σ-complete character because: λ is $(\mathfrak{D}, \mu, \sigma)$-inaccessible iff λ is $(\mathfrak{D}', \mu, \sigma)$ whenever $\mathfrak{D} \in \mathfrak{F}_\kappa(A)$ and $\mathfrak{D}' = \{Y \subseteq \mathcal{P}(A) : \mathrm{id}_\sigma(Y) \in \mathfrak{D}\}$.

Proof. We prove this by induction on λ. If $\lambda = \kappa^+$ this is an empty statement (as $(\kappa, \lambda) = \emptyset$). Also if $\lambda < \kappa^{+\omega_1}$ this is trivial, as $\bigcup_{t \in A} \mathfrak{a}_t$ is countable ($\subseteq \{\kappa^{+(\alpha+1)} : \alpha < \omega_1\} \cap \lambda$) hence $\mathrm{pcf}_{\aleph_1\text{-complete}}(\bigcup_{t \in A} \mathfrak{a}_t) = \bigcup_{t \in A} \mathfrak{a}_t \subseteq \{\kappa^{+(\alpha+1)} : \alpha < \omega_1\} \cap \lambda$. Also if this holds for λ it holds for λ^+ because $\mathrm{pcf}(\mathfrak{a} \cup \{\lambda\}) \subseteq \mathrm{pcf}(\mathfrak{a}) \cup \{\lambda\}$. So we can assume that λ is a limit cardinal. If the conclusion fails then for some infinite $W \subseteq \omega$, for each $n \in W$ there is a sequence $\langle \mathfrak{a}^n_\alpha : \alpha < \kappa_n \rangle$ (where $\mathfrak{a}^n_\alpha \subseteq (\kappa, \lambda) \cap \mathrm{Reg}$) which is a counterexample, i.e. $Y_n =: \{B \subseteq \kappa_n : \lambda \not\supseteq \mathrm{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}^n_\alpha)\} \in \mathfrak{D}^+_n$. If $\mathrm{cf}(\lambda) > \kappa$ then $\cup\{\mathfrak{a}^n_\alpha : n < \omega, \alpha < \kappa_n\}$ is a subset of λ of cardinality $\leq \kappa$, hence is bounded by some $\lambda' < \lambda$, so apply the induction hypothesis on λ'. If $\aleph_0 < \mathrm{cf}(\lambda) < \kappa$ let $\lambda = \sum\{\lambda_\zeta : \zeta < \mathrm{cf}(\lambda)\}$, $\bigwedge_{\zeta < \xi} \lambda_\zeta < \lambda_\xi < \lambda$. Now as $\mathrm{cf}(\kappa) = \aleph_0$ and $\kappa = \sum_{n < \omega} \kappa_n$, for some $n(*) < \omega$ we have $\mathrm{cf}(\lambda) < \kappa_{n(*)}$. For every $n \in W \backslash (n(*) + 2)$, we define a function $h_n : \kappa_n \to \mathrm{cf}(\lambda)$ by[†] $h_n(\alpha) = \mathrm{Min}\{\zeta < \kappa_{n(*)} : \mathfrak{a}^n_\alpha \subseteq \lambda_\zeta\}$. Hence by the assumption $\otimes$, as $\mathrm{cf}(\lambda) < \kappa_{n(*)} < \kappa_{n-1}$, for some $\zeta_n < \mathrm{cf}(\lambda)$ we have $\{B \subseteq \kappa_n : \bigwedge_{\alpha \in B} \mathfrak{a}^n_\alpha \subseteq \lambda_{\zeta_n}\} \in \mathfrak{D}^+_n$. Now we can contradict the induction hypothesis for $\lambda' = \sup\{\lambda_{\zeta_n} : n \in W \backslash (n(*) + 2)\}$. We are left with the case $\mathrm{cf}(\lambda) = \aleph_0$ so let $\lambda = \sum_{n < \omega} \lambda_n$, $\lambda_0 = \kappa^+$, $\lambda_n < \lambda_{n+1}$. For each $n, k < \omega$ define $Y^k_n = \{B \subseteq \kappa_n : \lambda \not\supseteq \mathrm{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}^n_i \cap [\lambda_k, \lambda_{k+1})\}$. So $Y_n = \bigcup_{k < \omega} Y^k_n$, but $Y_n \in \mathfrak{D}^+_n$, and $\mathfrak{D}^+_n$ is $\aleph_1$- complete hence for some $k_n < \omega$, $Y^{k_n}_n \in \mathfrak{D}^+_n$, so possibly shrinking W we get $\langle k_n : n \in W \rangle$ is constant or strictly increasing, the former contradicts the induction hypothesis on $\lambda_{k_{\mathrm{Min}(W)+1}}$. Now renaming the λ_k's we get $k_n = n$ and we can replace $\mathfrak{a}^n_\alpha$ by $\mathfrak{a}^n_\alpha \cap [\lambda_n, \lambda_{n+1})$. So without loss of generality $\mathrm{Min}(W) > 4$ and for $n \in W$ we have $\bigcup_\alpha \mathfrak{a}^n_\alpha \subseteq [\lambda_n, \lambda_{n+1})$ and $\lambda_n < \lambda$, of course.

Let $n(*) = \min(W)$. We try to define by induction on $k < \omega$, $\langle \theta_t : t \in w_k \rangle$, $w_k = \bigcup_{i < \kappa_{n(*)}} w_{k,i}$, J_k and h_{k-1} if $k > 0$ such that:

(a) $\theta_t \in \mathrm{Reg} \cap \lambda \backslash \kappa$ for $t \in w_k$
(b) $w_k = \bigcup_{i < \kappa_{n(*)}} w_{k,i}$ is disjoint to $\bigcup_{\ell < k} w_\ell$
(c) $\langle w_{k,i} : i < \kappa_{n(*)} \rangle$ is a sequence of pairwise disjoint, countable sets
(d) $w_{0,i} = \{i\} \times \mathfrak{a}^{n(*)}_i$ and $\theta_{(i,\tau)} = \tau$ for $\tau \in \mathfrak{a}^{n(*)}_i$
(e) h_k is a function from w_{k+1} to w_k mapping $w_{k+1,i}$ into $w_{k,i}$
(f) $J_k = \{w \subseteq w_k : \lambda \supseteq \mathrm{pcf}_{\aleph_2\text{-complete}}(\{\theta_t : t \in w\})\}$ is a proper ideal
(g) if $w \in J^+_k$ then $\{t \in w_{k+1} : h_k(t) \in w\} \in J^+_{k+1}$
(h) $t \in w_{k+1} \Rightarrow \theta_t < \theta_{h_k(t)}$.

During the induction, h_k is defined in the k-th step.

If we succeed, we shortly get a contradiction (by observation [Sh 460, 2.2]). For $k = 0$ define $w_{0,i}$, θ_τ for $\tau \in w_0 = \bigcup_{i < \kappa_{n(*)}} w_{0,i}$ by clause (d) and the clause (f) holds

[†] Well defined as $\mathrm{cf}(\lambda) \geq \aleph_1 = \sigma$

as otherwise $\{\theta_t : t \in w_0\}$ can be represented as $\bigcup_{\varepsilon<\omega_1} \mathfrak{b}_\varepsilon$ with $\max \mathrm{pcf}(\mathfrak{b}_\varepsilon) < \lambda$, let $h : \kappa_{n(*)} \to \omega_1$ be $h(i) = \sup\{\min\{\varepsilon : \tau \in \mathfrak{b}_\varepsilon\} : \tau \in \mathfrak{a}_i^{n(*)}\}$ and apply $\otimes$ from the hypothesis to get $Y \subseteq Y_{n(*)}$ and $\zeta < \omega_1$ such that $Y \in \mathfrak{D}^+_{n(*)}$ and $\{B \in Y : \sup \mathrm{Rang}(h \restriction B) < \zeta\} \in \mathfrak{D}^+_{n(*)}$, but $B \in Y$ implies $\lambda \not\supseteq \mathrm{pcf}_{\aleph_1\text{-complete}}(\bigcup_{\alpha \in B} \mathfrak{a}_\alpha^{n(*)})$ because $B \in Y_{n(*)}$ and

$$\begin{aligned}\mathrm{pcf}_{\aleph_1\text{-complete}}\left(\bigcup_{\alpha\in B} \mathfrak{a}_\alpha^{n(*)}\right) &\subseteq \mathrm{pcf}_{\aleph_1\text{-complete}}\left(\bigcup_{\varepsilon<\zeta} \mathfrak{b}_\varepsilon\right)\\ &\subseteq \bigcup_{\varepsilon<\zeta} \mathrm{pcf}_{\aleph_1\text{-complete}}(\mathfrak{b}_\varepsilon) \subseteq \lambda;\end{aligned}$$

contradiction.

So assume $w_k, \langle w_{k,i} : i < \kappa_{n(*)}\rangle, J_k$ are as required and we shall define $w_{k+1}, \langle w_{k+1,i} : i < \kappa_{n(*)}\rangle, J_{k+1}, h_k$.

Now for any $t \in w_k$ by the induction hypothesis for some $g(t) < \omega$ we have

$(*)_1$ if $m \in [g(t), \omega)$ and $\mathfrak{b}_i \subseteq \mathrm{Reg} \cap \theta_t \backslash \kappa$ is countable for $i < \kappa_m$ then

$$\{B \subseteq \kappa_m : \mathrm{pcf}_{\aleph_1\text{-complete}}(\cup\{\mathfrak{b}_i : i \in B\}) \subseteq \theta_t\} \in \mathfrak{D}_m$$

$(*)_2$ $g(t) > n(*) + 1$.

Let $u_{k,m} = \{t \in w_k : g(t) = m \text{ and } \theta_t > \kappa^+\}$. We shall prove

$\boxtimes_{k,m}$ if $u_{k,m} \notin J_k$, then we can find $\langle \mathfrak{c}_t : t \in u_{k,m}\rangle$, $\mathfrak{c}_t \subseteq \mathrm{Reg} \cap \theta_t \backslash \kappa^+$ countable such that:
$u \subseteq u_{k,m}, u \in J_k^+$ implies $\mathrm{pcf}_{\aleph_2\text{-complete}}(\bigcup_{t\in u} \mathfrak{c}_t) \not\subseteq \lambda$.

As J_k is $\aleph_1$-complete (by its definition; even more) this suffices for carrying the induction.
[Why? Let $\langle \mathfrak{c}_t^m : t \in u_{k,m}\rangle$ for $m < \omega$ such that $u_{k,m} \notin J_k$ be as above, let

$$w_{k+1,i} = \bigcup\{\{t\} \times \mathfrak{c}_t^m : \text{for some } m, u_{k,m} \notin J_k \text{ and } t \in u_{k,m} \cap w_{k,i}\},$$

$\theta_{(t,\tau)} = \tau$, $w_{k+1} = \bigcup_{i<\kappa_{n(*)}} w_{k+1,i}$ and we define the function $h_{k+1} : w_{k+1} \to w_k$ by $h_{k+1}((t,\sigma)) = t$ (note: every t belongs to at most one $u_{k,m}$ $(m < \omega)$ and $w_k \backslash \cup\{u_{k,m} : u_{k,m} \notin J_k\} = \emptyset \bmod J_k$.]

Proof of $\boxtimes_{k,m}$. For each $\tau \in \mathfrak{a}_m$, we apply [Sh:g, Ch. I,1.6] or [Sh:g, Ch. IX,4.1] on $\langle \theta_t : t \in u_{k,m}\rangle$, $J = J_k \restriction u_{k,m}$ and τ (possible as $|u_{k,m}| < \kappa < \min\{\theta_t : t \in u_{k,m}\}$), each θ_t (for $t \in u_{k,m}$) is regular and $\prod_{t\in u_{k,m}} \theta_t/J$ is λ^+-directed, $\tau < \lambda^+$ (the cases $\tau < \theta_t$ can be ignored for several reasons, e.g. $\mathfrak{a}_m = \bigcup_\alpha \mathfrak{a}_\alpha^m \subseteq [\lambda_m, \lambda_{m+1})$). So we can find $\langle \theta_{t,\tau} : t \in u_{k,m}, \tau \in \mathfrak{a}_m\rangle$ such that:

(α) $\theta_{t,\tau}$ is regular and $\kappa^+ \leq \theta_{t,\tau} < \theta_t$

(β) $\prod_{t \in u_{k,m}} \theta_{t,\tau}/J_k$ has true cofinality τ

(note that $t \in u_{k,m} \Rightarrow \kappa^+ < \theta_t$, so we can assume $\theta_{t,\tau} \geq \kappa^+$).

Now for each $t \in u_{k,m}, \theta_{t,\tau} \in \mathrm{Reg} \cap \theta_t \backslash \kappa$ for $\tau \in \mathfrak{a}_m$, but $g(t) = m$ (as $t \in u_{k,m}$), hence by the definition of g, letting $\mathfrak{a}_i^{m,t} = \{\theta_{t,\tau} : \tau \in \mathfrak{a}_i^m\}$, for $i < \kappa_m$, we have

$$\Gamma_t^m =: \left\{ B \subseteq \kappa_m : \theta_t \supseteq \mathrm{pcf}_{\aleph_1\text{-complete}} \left(\bigcup_{i \in B} \mathfrak{a}_i^{m,t} \right) \right\} \in \mathfrak{D}_m.$$

But $\mathfrak{D}_m$ is $\kappa_{n(*)+1}$-complete (as $m = g(t) > n(*)+1$) and $|u_{k,m}| \leq \kappa_{n(*)} < \kappa_{n(*)+1}$ hence $\Gamma^* = \bigcap_{t \in u_{k,m}} \Gamma_t^m \in \mathfrak{D}_m$. On the other hand (by the choice of $\mathfrak{a}_m$)

$$\Gamma_m =: \left\{ B \subseteq \kappa_m : \lambda \supseteq \mathrm{pcf}_{\aleph_1\text{-complete}} \left(\bigcup_{i \in B} \mathfrak{a}_t^m \right) \right\} \notin \mathfrak{D}_m.$$

So there is $B \in \Gamma_m \cap \Gamma^* = \Gamma_m \cap \bigcap_{t \in u_{k,m}} \Gamma_t^m$.

Let

$$\mathfrak{a}^* = \{\theta_t : t \in u_{k,m}\} \cup \{\theta_{t,\tau} : t \in u_{k,m}, \tau \in \mathfrak{a}_m\} \cup \mathfrak{a}_m,$$

and we can for simplicity assume $|\lambda \cap \mathrm{pcf}(\mathfrak{a}^*)| < \min(\mathfrak{a}^*)$.
[Why? By [Sh 460, Th.2.5].] Hence there is a smooth close generating sequence $\langle \mathfrak{b}_\sigma[\mathfrak{a}^*] : \sigma \in \lambda \cap \mathrm{pcf}(\mathfrak{a}^*) \rangle$ (see e.g. [Sh 430, 6.7], if not use [Sh 430, 6.7,6.7F]). Clearly $B \neq \emptyset$. For each $t \in u_{k,m}$ we know $B \in \Gamma_t^m$ hence $\theta_t \supseteq \mathrm{pcf}_{\aleph_1\text{-complete}}$ $(\bigcup_{i \in B} \mathfrak{a}_i^{m,t})$. So we can find countable $\mathfrak{c}_t \subseteq \theta_t \cap \mathrm{pcf}(\bigcup_{i \in B} \mathfrak{a}_i^{m,t})$ such that

$$\bigcup_{i \in B} \mathfrak{a}_i^{m,t} \subseteq \bigcup_{\sigma \in \mathfrak{c}_t} \mathfrak{b}_\sigma[\mathfrak{a}^*].$$

The pcf calculus verifies clause (g) (as in [Sh 460, §2]). $\square_{1.3}$

It is now natural to look for suitable filters $\mathfrak{D}$, the simplest ones are:

Definition 1.5. *For $\sigma < \theta < \kappa$ and $\mu \leq \kappa$ (always θ regular) let $\mathfrak{D} = \mathfrak{D}_{\sigma,\theta,\kappa,\mu}$ be the following filter on $\mathcal{P}(\kappa) : Y \in \mathfrak{D}$ iff there are functions $f_\alpha : \kappa \to \theta_\alpha$ for $\alpha < \alpha(*) < \theta$ where $\theta_\alpha \in [\sigma, \theta) \cap Reg$ such that $Y \supseteq \{a \subseteq \kappa : |a| \geq \mu$ and for every $\alpha < \alpha(*)$ for some $\zeta < \theta_\alpha$ we have $Rang(f_\alpha \restriction a) \subseteq \zeta\}$. If $\mu = \theta$ we may omit it.*

Observation 1.6. 1) If $\sigma < \theta < \kappa_1 \leq \kappa_2$ and $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa_1,\mu}$ then $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa_2,\mu}$.
2) $\mathfrak{D}_{\sigma,\theta,\kappa}$ is a θ-complete filter.
3) If $2^{<\theta} < \kappa$ then $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa}$ and this is preserved by σ-c.c. forcing.

Proof. Straight.

Conclusion 1.7. Let μ be a limit singular cardinal of cofinality $< \sigma = \mathrm{cf}(\sigma) < \mu$ and:

$\otimes$ for every $\theta \in (\sigma, \mu) \cap \mathrm{Reg}$ for some $\kappa \in (\theta, \mu)$ we have: $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa}$.

Then for every $\lambda > \mu$, for some $\theta = \theta_\mu \in (\sigma, \mu) \cap \mathrm{Reg}$ for every $\kappa \in (\theta, \mu)$ we have

$(*)$ if $\lambda_i \in (\mu, \lambda) \cap \mathrm{Reg}$ for $i < \kappa$ then

$$\{a \subseteq \kappa : \mathrm{pcf}_{\sigma\text{-complete}}\{\lambda_i : i \in a\} \subseteq \lambda\} \in \mathfrak{D}_{\sigma,\theta,\kappa}.$$

Proof. Assume λ is a counterexample. Without loss of generality σ is regular, choose by induction on $\zeta < \sigma$, $\kappa_\zeta \in (\sigma, \mu) \cap \mathrm{Reg}$ as follows

$\kappa_0 \in (\sigma, \mu) \cap$ Reg arbitrary;

$\kappa_\zeta \in (\bigcup_{\epsilon<\zeta} \kappa_\epsilon^+, \mu) \cap$ Reg is minimal κ which is a witness to $\otimes$ for

$\theta_\zeta = \left(\bigcup_{\epsilon<\zeta} \kappa_\epsilon\right)^+$ (in particular $\emptyset \notin \mathfrak{D}_{\sigma,\theta_\zeta,\kappa_\zeta}$, so $\kappa_\zeta < \mu$).

Let $\kappa = \bigcup_{\zeta<\sigma} \kappa_\zeta$ and apply 1.3 for $\mathfrak{D}_{\sigma,\theta_\zeta,\kappa_\zeta}$ from Definition 1.5, more exactly the variant with σ instead $\aleph_1$ (see 1.4); alternatively use only $\langle \kappa_n : n < \omega \rangle$. $\square_{1.7}$

Remark 1.8. 1) In the proof of 1.3 we can change the universe during the proof, so weaken the demand $\otimes$.

2) The problematic example is: $T \subseteq {}^{\omega_1}\omega$ a Kurepa tree, say $T \cap {}^\alpha 2 = \{\gamma_\alpha(n) : m < \omega\}$, $\eta_j \in \lim_{\omega_1}(T)$ for $j < j^*$ and $\{a \subseteq \omega_1 \cup j^* : \delta =: a \cap \omega_1$ a limit ordinal and for every $n < \omega$ for some $j \in j^* \cap a$ we have $\gamma_\alpha(n) = \eta_j(\alpha)\} \in \mathcal{D}_{<\aleph_1}(\omega_1 \cup j^*)$.

3) We can replace in 1.7 above $\mathrm{cf}(\mu) < \sigma$ by $\mathrm{cf}(\mu) \neq \sigma$. We can replace $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa}$ by $\emptyset \notin \mathfrak{D}_{\sigma,\theta,\kappa;\Upsilon}$ for any fix $\Upsilon \in [\sigma, \mu)$. Note that the case $\Upsilon < \sigma$ is not interesting.

4) Note that the meaning of $\emptyset \in \mathfrak{D}_{\sigma,\theta,\kappa;\Upsilon}$ is that there are $\alpha(*) < \theta$ and functions $f_\alpha : \kappa \to \theta_\alpha$ where $\theta_\alpha \in \mathrm{Reg} \cap [\sigma, \theta)$ such that for no $u \in [\kappa]^\Upsilon$ do we have $\alpha < \alpha(*) \Rightarrow \sup \mathrm{Rang}(f_\alpha \restriction u) < \theta_\alpha$. Recall if Υ is omitted it means $\Upsilon = \theta$.

Definition 1.9. *Assume* $\mathbf{J} \subseteq Id(\kappa)$ *(= the family of ideals on* κ*).*

1) We say (λ, μ) is $\mathbf{J}$-inaccessible if $\kappa \leq \mu < \lambda$ and there are no $\lambda_i \in \mathrm{Reg} \cap (\mu, \lambda)$ for $i < \kappa$ and $J \in \mathbf{J}$ such that $\prod_{i<\kappa} \lambda_i/\mathbf{J}$ is λ-directed (equivalently, for some such λ_i's, $\prod_{i<\kappa} \lambda_i/J$ has true cofinality and it is $\geq \lambda$).
2) $(\lambda, *)$ means (λ, μ) for some $\mu \in [\kappa, \lambda)$, λ means (λ, κ).
3) $\mathbf{J}$ is σ-indecomposable when: if $J \in \mathbf{J}$ and $h : \mathrm{Dom}(J) \to \sigma$ then for some $\zeta < \sigma$ and $I \in \mathbf{J}$ we have $J \restriction h^{-1}\{\varepsilon : \varepsilon < \zeta\} \leq^* I$ (see below).
4) For ideals I_ℓ, on A_ℓ $(\ell = 0, 1)$ $I_0 \leq^* I_1$ if there is $B_0 \in I_0^+$ and $B_1 \in I_1^+$ and one-to-one function g from B_0 into B_1 such that
$$Y \cap B_0 \in I_0 \Rightarrow g''(Y \cap B_0) \in J_1$$
5) $\mathbf{J}$ is $[\sigma, \kappa)$-indecomposable if it is θ-indecomposable for every $\theta \in [\sigma, \kappa) \cap \mathrm{Reg}$.

Claim 1.10. Assume J is a σ-complete ideal on κ and $X \in I^+ \Rightarrow I \restriction X \cong I$ (e.g. $I = J_\kappa^{\mathrm{bd}}$, $\mathrm{cf}\,\kappa \geq \sigma$). Then (λ, μ) is $\{J\}$-inaccessible if λ is (J, μ, σ)-inaccessible.

Proof. Compare Definition 1.9(1) and 1.2(2).

2. Large pcf($\mathfrak{a}$) implies the existence of free sets

Definition 2.1. *1) Let $\bar{A} = \langle A_\alpha : \alpha < \alpha^* \rangle$ be a sequence of subsets of κ, no A_α in the ideal generated by $\{A_\beta : \beta < \alpha\}$. We define functions $rk = rk_{\bar{A}}$, $rk' = rk'_{\bar{A}}$ from $\mathcal{P}(\kappa)$ to the ordinals by:*

> *$rk(A) \geq \zeta$ <u>iff</u> for every $\xi < \zeta$ for some α, $A \neq A \cap A_\alpha$ and $rk(A \cap A_\alpha) \geq \zeta$*
> *$rk'(A) \geq \zeta$ <u>iff</u> for every $\xi < \zeta$ for some α, we have $rk'(A \cap A_\alpha) \geq \xi$ and $A \backslash A_\alpha$, $A \cap A_\alpha$ are not in $id_{\bar{A} \restriction \alpha}$ = the ideal generated by $\{A_\beta : \beta < \alpha\}$.*

2) Let $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ be a sequence of pairs of ideals on κ such that $[\alpha < \beta \Rightarrow J_\alpha \subseteq J'_\alpha \subseteq J_\beta \subseteq J'_\beta]$ and for some $A_\alpha \in J^+_\alpha$, $J'_\alpha = J_\alpha + A_\alpha$ we define $rk'_{\bar{J}}(A)$ for $A \subseteq \kappa$ by:

> *$rk'_{\bar{J}}(A) \geq \zeta$ <u>iff</u> for every $\xi < \zeta$ for some $\alpha < \alpha^*$ we have: $rk'_{\bar{J}}(A \cap A_\alpha) \geq \xi$ and $A \backslash A_\alpha$, $A \cap A_\alpha$ are not in J_α.*

3) We identify $\bar{A}$ with $\langle (id_{\bar{A} \restriction \alpha}, id_{\bar{A} \restriction (\alpha+1)}) : \alpha < \alpha^* \rangle$ (see 2.2(3) below). If $\bar{J} = \langle (J_\alpha, J_{\alpha+1}) : \alpha < \alpha^* \rangle$ is as required in (2) we may write $\langle J_\alpha : \alpha < \alpha^* \rangle$ instead $\bar{J}$. We can replace κ by any other set. We may write $rk^{(\prime)}(A, \bar{A})$ or $rk'(A, \bar{J})$ instead $rk^{(\prime)}_{\bar{A}}(A)$ or $rk'_{\bar{J}}(A)$.

Claim 2.2. 1) $\mathrm{rk}_{\bar{A}}$, $\mathrm{rk}'_{\bar{A}}$ are well defined (values: ordinals or ∞) and nondecreasing in A (under $\subseteq$).
2) $\mathrm{rk}_{\bar{A}}(A) \geq \mathrm{rk}'_{\bar{A}}(A)$.
3) $\mathrm{rk}'_{\bar{A}}$ depend just on $\langle \mathrm{id}_{\bar{A} \restriction \alpha} : \alpha \leq \alpha^* \rangle$ and for $A \subseteq \kappa$, we have $\mathrm{rk}'_{\bar{A}}(A) = \mathrm{rk}'_{\bar{J}}(A)$ where $\bar{J} = \langle (\mathrm{id}_{\bar{A} \restriction \alpha}, \mathrm{id}_{\bar{A} \restriction (\alpha+1)}) : \alpha \leq \alpha^* \rangle$ (so we may write $\mathrm{rk}'_{\langle J_\alpha : \alpha \leq \alpha^* \rangle}(A)$ with $J_\alpha = \mathrm{id}_{\bar{A} \restriction \alpha}$).
4) If $\mathrm{rk}'_{\bar{A}}(\kappa) = \zeta$, <u>then</u> we can find $\bar{Y} = \langle Y_\varepsilon : \varepsilon < \zeta \rangle$, an increasing sequence of subsets of α^*, $\bigcup_{\varepsilon < \zeta} Y_\varepsilon = \alpha^*$ and for each $\varepsilon < \zeta$, $\{A_\alpha : \alpha \in Y_\varepsilon \backslash \bigcup_{\xi < \varepsilon} Y_\xi\}$ are almost disjoint and positive modulo the ideal generated by $\{A_\alpha : \alpha \in \bigcup_{\xi < \varepsilon} Y_\xi\}$.
5) If $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ is as in 2.1(2), where J_α, J'_α are ideals on κ, $J'_\alpha = J_\alpha + A_\alpha$ and $\bar{A} = \langle A_\alpha : \alpha < \alpha^* \rangle$ <u>then</u> for every $B \subseteq \kappa$ we have $\mathrm{rk}_{\bar{A}}(B) \geq \mathrm{rk}'_{\bar{A}}(B) \geq \mathrm{rk}'_{\bar{J}}(B)$.

Proof. Straight: e.g. for the fourth part use $Y_\varepsilon =: \{\alpha : \mathrm{rk}'_{\bar{A}}(A_\alpha) < \varepsilon\}$. $\square_{2.2}$

Claim 2.3. 1) If $\mathrm{rk}_{\bar{A}}(\kappa) \geq \kappa^+$ <u>then</u> for some $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ we have $\alpha_n < \alpha_{n+1} < \kappa$ and for every $\ell < k < \omega$ for some $\alpha < \alpha^*$ we have $A_\alpha \cap \{\alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_k\} = \{\alpha_{\ell+1}, \ldots, \alpha_k\}$.
2) If $\mathrm{rk}'_{\bar{J}}(\kappa) \geq \beta$ and $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ as in 2.1(2) <u>then</u> for some $\Gamma \subseteq \alpha^*$, $|\Gamma| \leq |\beta|$ we have $\mathrm{rk}'_{\bar{J} \restriction \Gamma}(\kappa) \geq \beta$.

3) If $\mathrm{rk}_{\bar{J}}(B) \geq \beta$, $\bar{J} = \langle (J_\alpha, J'_\alpha) : \alpha < \alpha^* \rangle$ as in 2.1(2) and $J'_\alpha = J_\alpha + A_\alpha$ then we can find $\Gamma \subseteq \alpha^*$ such that

(∗) $|\Gamma| \leq |\beta| + \aleph_0$ (even $|\Gamma| < |\beta|^+ + \aleph_0$) and if $A_\alpha \subseteq A'_\alpha \in J'_\alpha$ then $\mathrm{rk}_{\langle A'_\alpha : \alpha \in \Gamma \rangle}(B) \geq \beta$.

Proof. 1) Let rk $= \mathrm{rk}_{\bar{A}}$; choose by induction on n an ordinal $\alpha_n < \kappa$ and for every $\zeta < \kappa^+$ a decreasing sequence $\langle B^n_{\zeta,0}, \ldots, B^n_{\zeta,n} \rangle$ of sets such that

(α) $(\forall \ell \leq n)(\forall m \leq n)[\alpha_\ell \in B^n_{\zeta,m} \Leftrightarrow \ell > m]$,
(β) $\mathrm{rk}(B^n_{\zeta,n}) \geq \zeta$
(γ) each $B^m_{\zeta,n}$ is the intersection of finitely many A_α's

For $n = 0$, for every $\zeta < \kappa^+$ there is $\alpha_\zeta < \alpha^*$ such that $\kappa \cap A_{\alpha_\zeta} \neq \kappa$, $\mathrm{rk}(\kappa \cap A_{\alpha_\zeta}) \geq \zeta$, and choose $\alpha^0_\zeta \in \kappa \backslash A_{\alpha_\zeta}$. So for some $\alpha_0 < \kappa$ we have $\kappa^+ = \sup\{\zeta < \kappa^+ : \alpha^0_\zeta = \alpha_0\}$ and let $B^0_{\zeta,0} = A_{\alpha_{\xi(\zeta)}}$ where $\xi(\zeta) < \kappa^+$ is the minimal $\xi \geq \zeta$ such that $\alpha^0_\xi = \alpha_0$, as in demand (β) we ask "$\geq \zeta$" not "$= \zeta$", we succeed. If we have defined for n, for each $\zeta < \kappa^+$, as $\mathrm{rk}(B^n_{\zeta+1,n}) \geq \zeta + 1$, there is $\beta(\zeta, n) < \ell g(\bar{A})$ such that $\neg[B^n_{\zeta+1,n} \subseteq A_{\beta(\zeta,n)}]$ but $\mathrm{rk}_{\bar{A}}(B^n_{\zeta+1,n} \cap A_{\beta(\zeta,n)}) \geq \zeta$, and choose $\gamma(\zeta, n) \in B^n_{\zeta+1,n} \backslash A_{\beta(\zeta,n)}$ so for some $\alpha_{n+1} < \kappa$, the set $S_n = \{\zeta < \kappa^+ : \gamma(\zeta, n) = \alpha_{n+1}\}$ is unbounded in κ^+. For every $\zeta < \kappa^+$ let $\xi(\zeta, n) = \min\{\xi : \xi \in S_n, \xi > \zeta\}$, let $B^{n+1}_{\zeta,\ell}$ be $B^n_{\xi(\zeta,n),\ell}$ if $\ell \leq n$ and $B^n_{\xi(\zeta,n)+1,n} \cap A_{\beta(\zeta,n)}$ if $\ell = n+1$. In the end we know that for $\ell < k < \omega$, for every $\zeta < \kappa^+$ we have $B^k_{\zeta,\ell} \cap \{\alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_k\} = \{\alpha_{\ell+1}, \ldots, \alpha_k\}$; also $B^k_{\zeta,\ell}$ has the form $\bigcap_{m<m(*)} A_{\alpha(m)}$ for some $m(*)$ and $\alpha(m) < \ell g(\bar{A})$, so for some m we have $\alpha_\ell \notin A_{\alpha(m)}$, but $\{\alpha_{\ell+1}, \ldots, \alpha_k\} \subseteq B^k_{\zeta,\ell} \subseteq A_{\alpha(m)}$ so $\alpha(m)$ is as required in 2.3(1) for our $\ell < k < \omega$. Lastly $\bigwedge_{n<m} \alpha_n \neq \alpha_m$ hence by Ramsey theorem without loss of generality $\alpha_n < \alpha_{n+1}$ and we are done.
2) By induction on β or by part (3).
3) We can find $\langle (B_\eta, j_\eta) : \eta \in ds(\beta) \rangle$ where $ds(\beta) = \{\eta : \eta$ is a (strictly) decreasing sequence of cardinals $< \beta\}$, $B_{\langle\,\rangle} = B$,$\mathrm{rk}'_{\bar{J}}(B_\eta) \geq \min(\{\beta\} \cup \{\eta(\ell) : \ell < \ell g(\eta)\}$ and $j_\eta < \alpha^*$ and if $\nu = \eta\hat{\ }\langle \gamma \rangle \in ds(\beta)$ then $B_\nu \neq B_\eta$, $B_\nu = B_\eta \cap A_{j_\eta} \notin J_{j_\eta}$ and $B_\eta \backslash B_\nu \notin J_{j_\eta}$. Let $\Gamma = \{j_\eta : \eta \in ds(\eta)\}$, now if $A_\alpha \subseteq A'_\alpha \in J_\alpha$ for $\alpha \in \Gamma$ then we can prove by induction on $\gamma < \beta$ that: $\eta \in ds(\beta) \Rightarrow \mathrm{rk}'_{\langle A'_\alpha : \alpha \in \Gamma \rangle}(B_\eta) \geq \max(\{\beta\} \cup \{\eta(\ell) : \ell < \ell g(\eta)\}$. $\square_{2.3}$

Definition 2.4. *1) For $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ strictly increasing, let IND($\bar{\lambda}$) mean:*
$(*)_{\bar{\lambda}}$ *for every algebra M with universe $\bigcup_{n<\omega} \lambda_n$ and $\aleph_0$ functions (all finitary) there is $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ such that:*

(a) $\alpha_n < \lambda_n$
(b) α_n is not in the M-closure of

$$\{\alpha_\ell : \ell \in (n, \omega)\} \cup \{i : \bigvee_{m<n} i < \lambda_m\}.$$

2) IND(λ) means that $\lambda > cf(\lambda) = \aleph_0$ and for every (equivalently some, see below) $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ strictly increasing with limit λ we have $(*)'_{\bar{\lambda}}$ for every algebra M with universe λ and countably many functions there is a sequence $\langle \alpha_n : n \in w \rangle$ such that:

(a) $w \subseteq \omega$ is infinite
(b) $\alpha_n < \lambda_n$ for $n \in w$
(c) for $n \in w$, α_n is not in the M-closure of

$$\{\alpha_\ell : \ell \in w, \ell > n\} \cup \{i : \bigvee_{\ell \in n \cap w} i < \lambda_\ell\}$$

3) IND(λ, κ) = IND$^0(\lambda, \kappa)$ means: if M is a model with universe λ and κ functions we can find $\bar{\alpha} = \langle \alpha_n : n < \omega \rangle$ such that

$$\alpha_n < \lambda, \alpha_n \notin c\ell_M\{\alpha_\ell : \ell < \omega, \ell > n\}.$$

4) IND$^1(\lambda, \kappa)$ is defined similarly but demanding

$$\alpha_n \notin c\ell_M\{\alpha_\ell : \ell < \omega, \ell \neq n\}.$$

Observation 2.5. 1) In 2.4(2), if $(*)'_{\bar{\lambda}}$ holds for one strictly increasing $\bar{\lambda}$ with limit λ, then it holds for every strictly increasing $\bar{\lambda}' = \langle \lambda'_n : n < \omega \rangle$ with limit λ.
2) If λ is uncountable with cofinality $\aleph_0$, $\mathbb{P}$ a forcing notion of cardinality $\leq \mu < \lambda$ or satisfying the μ^+-c.c. for some $\mu < \lambda$, or λ-complete then: IND(λ) $\Leftrightarrow$ $\Vdash_{\mathbb{P}}$"IND(λ)" and if $\kappa \in [\mu, \lambda)$, μ as above then IND(λ, κ) $\Leftrightarrow$ $\Vdash_{\mathbb{P}}$"IND(λ, κ)" and if in addition $\mu < \lambda_n < \lambda_{n+1}$, then IND($\langle \lambda_n : n < \omega \rangle$) $\Leftrightarrow$ $\Vdash_{\mathbb{P}}$"IND($\langle \lambda_n : n < \omega \rangle$)".
3) IND($\bar{\lambda}$) $\Rightarrow$ IND(λ) $\Rightarrow$ IND$^1(\lambda, \kappa)$ $\Rightarrow$ IND$^0(\lambda, \kappa)$ if $\lambda = \bigcup_{n<\omega} \lambda_n$ and $\lambda_n < \lambda_{n+1}$ and $\lambda_0 > \kappa$. If $\kappa < \lambda \leq \lambda'$ and $i \in \{0, 1\}$ then

$$\text{IND}^i(\lambda, \kappa) \Rightarrow \text{IND}^i(\lambda', \kappa).$$

4) If ($i \in \{0, 1\}$ and) IND$^i(\lambda, \kappa)$, λ minimal for this κ then

(a) $\kappa \leq \kappa_1 < \lambda \Rightarrow \text{IND}^i(\lambda, \kappa_1)$
(b) cf(λ) $= \aleph_0$ and IND$^1(\lambda, \kappa)$ or λ is inaccessible
(c) if $\lambda = \sum_{n<\omega} \lambda_n$ and $\lambda_n < \lambda_{n+1}$ then not only $(*)'_{\bar{\lambda}}$ (from 2.4(2)), where $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$) but if $\mathbb{P}$ is a c.c.c. forcing adding a dominating real then in $V^{\mathbb{P}}$ for some infinite $w \subseteq \omega$ we have $(*)_{\bar{\lambda} \restriction w}$ from 2.4(1) holds.

5) IND$^1(\lambda, \kappa)$ is equivalent to IND$^0(\lambda, \kappa)$.

Proof. 1, 2), 3) Check.
4) Clause (a) of (4): Assume not and first let $i = 0$. Let $\chi = \beth_3(\lambda)^+$ and let M be the model with universe λ and the functions (n-place from λ to λ for some n) definable in $(\mathcal{H}(\chi), \in, <^*_\chi)$ with the additional individual constants $\lambda, \kappa, \kappa_1$. Clearly

$(M,\alpha)_{\alpha<\kappa_1}$ exemplifies $\neg\mathrm{IND}^i(\lambda,\kappa_1)$. Let F_n^-, F_n^+ be such that for $\bar\beta = \langle \beta_\ell : \ell < n\rangle$, $\beta_\ell < \lambda$ we have: $F_n^+(-,\bar\beta)$ is a one-to-one function from $c\ell_M(\{\beta_0,\dots,\beta_{n-1}\}\cup\kappa_1)$ onto κ_1 and $F_n^-(-,\beta)$ is its inverse. We can apply the assumption $\mathrm{IND}^i(\lambda,\kappa)$ to the model $(M, F_n^+, F_n^-, \beta)_{n<\omega,\beta<\kappa}$, so there are $\alpha_n (n<\omega)$ as in 2.4(3). By the assumption for no infinite $w\subseteq\omega$ is $\{\alpha_n : n\in w\}$ as required in 2.4(3) for $(M,\beta)_{\beta<\kappa_1}$. We claim

(∗) for some infinite $w\subseteq\omega$, $\bigwedge_{n\in w} \alpha_n \in c\ell_M(\{\alpha_\ell : n<\ell\in w\}\cup\kappa_1)$.

[Why? Try to choose by induction on $\ell<\omega, u_\ell, n_\ell$ such that: $\bigwedge_{m<\ell} n_m < n_\ell < \omega, u_\ell \subseteq (n_\ell,\omega)$ is infinite, $u_{\ell+1}\subseteq u_\ell\subseteq w$ and $\alpha_{n_\ell}\notin c\ell_M(\{\alpha_n : n\in u_\ell\})$, we cannot succeed so w or some u_ℓ is as required.]

By renaming $w=\omega$, so for every n for some $k_n\in(n+1,\omega)$ we have $\alpha_n\in c\ell_M(\{\alpha_{n+1},\dots,\alpha_{k_n}\}\cup\kappa_1)$; as we can increase k_n, without loss of generality $k_n<k_{n+1}$ hence $m\le n\Rightarrow \alpha_m\in c\ell_M(\{\alpha_{n+1},\dots,\alpha_{k_n}\}\cup\kappa_1)$ (just prove this by induction on n), so $\gamma_{n,m} =: F^+_{k_n-n}(\alpha_m,\alpha_{n+1},\dots,\alpha_{k_n}) < \kappa_1$ and for each n the sequence $\langle\gamma_{n,m} : m\le n\rangle$ is with no repetitions. Choose by induction on ℓ, $m_\ell\in[\ell^2,(\ell+1)^2)$ such that $\gamma_{(\ell+1)^2,m_\ell}\notin\{\gamma_{(q+1)^2,m_q} : q<\ell\}$. But as $\neg\mathrm{IND}^i(\kappa_1,\kappa)$ (because $\lambda>\kappa_1$ was minimal such that ...) for some $\ell<p<\omega$ we have $\gamma_{(\ell+1)^2,m_\ell}\in c\ell_M(\{\gamma_{(q+1)^2,m_q} : \ell<q<p\}\cup\kappa)$ and using some $F^-_{p^2-(\ell+1)^2}$ we have $\alpha_{m_\ell}\in c\ell_M(\{\alpha_q : q \text{ is } \ge(\ell+1)^2 \text{ but } \le k_{(p^2)}\}\cup\kappa)$.
[Why? First note that $\gamma_{(q+1)^2,m_q}$ belong to this model for $q=\ell+1,\dots,p-1$, (using $F_{k_{(q+1)^2}}$) hence also $\gamma_{(\ell+1)^2,m_\ell}$ belongs to this model by the choice of ℓ and p; a contradiction.] So we have proved clause (a) for $i=0$.

If $i=1$ the proof is similar: choose, by induction on $\ell, k_\ell, m_\ell, m_\ell$ such that $k_\ell<m_\ell<k_{\ell+1}$ and $\alpha_{m_\ell}\in c\ell_M(\{\alpha_n : n\in[k_\ell,m_\ell) \text{ or } n\in(m_\ell,k_{\ell+1})\}\cup\kappa_1)$, this is possible as otherwise $\{\alpha_n : n\in[k_\ell,\omega)\}$ contradict "M exemplifies $\neg\mathrm{IND}^1(\lambda,\kappa_1)$". Let

$$\gamma_\ell = F^+_{k_{\ell+1}-k_\ell-1}(\alpha_{m_\ell};\alpha_{k_\ell},\alpha_{k_\ell+1},\dots,\alpha_{m_\ell-1},\alpha_{m_\ell+1},\dots,\alpha_{k_{\ell+1}-1}) < \kappa_1.$$

For some $\ell<\ell(*)<\omega$ we have $\gamma_\ell\in c\ell_M(\{\gamma_0,\dots,\gamma_{\ell-1},\gamma_{\ell+1},\dots,\gamma_{\ell(*)-1}\}\cup\kappa)$ (because $\neg\mathrm{IND}^1(\kappa_1,\kappa)$ as λ is first and the choice of M), hence $\alpha_\ell\in c\ell_M(\{\alpha_n : n<\omega, n\le\ell\}\cup\kappa)$; a contradiction.

Clause (b) of (4): By the definition easily $\aleph_0<\mathrm{cf}(\lambda)\le\kappa$ is impossible.
[Why? Let $\lambda=\sum_{i<\kappa}\lambda_i$, with $\lambda_i<\lambda$, and by the minimality of λ let M_i be a model with universe λ_i and $\le\kappa$ functions exemplifying $\neg\mathrm{IND}(\lambda_i,\kappa)$ and lastly let M be the model with universe λ and the functions of all the M_i; check that M exemplifies $\neg\mathrm{IND}(\lambda,\kappa)$.]
By 2.5(4) clause (a) it follows that

$$[\mathrm{cf}(\lambda)>\aleph_0 \,\&\, \kappa_1<\lambda\Rightarrow\kappa_1<\mathrm{cf}(\lambda)].$$

So if $\mathrm{cf}(\lambda)>\aleph_0$ then λ is regular, it is inaccessible as it is not a successor as trivially $\neg\mathrm{IND}^i(\mu^+,\mu)$ so by clause (a) we have

$$\neg\mathrm{IND}^i(\mu,\kappa)\Rightarrow\neg\mathrm{IND}^i(\mu^+,\kappa).$$

We still have to prove $\mathrm{IND}^1(\lambda,\kappa)$ when $\mathrm{cf}(\lambda)=\aleph_0$; if $i=1$ this is trivial so assume $i=0$. So assume $\mathrm{cf}(\lambda)=\aleph_0$, $\bar\lambda=\langle\lambda_n : n<\omega\rangle$, $\lambda_n<\lambda_{n+1}$ and $\lambda=\sum\limits_{n<\omega}\lambda_n$. We should prove $\mathrm{IND}^1(\lambda,\kappa)$; it follows from part (5).

<u>Clause (c) of (4)</u>: Left to the reader.
5) By 2.5(3),

$$\mathrm{IND}^1(\lambda,\kappa)\Rightarrow \mathrm{IND}^0(\lambda,\kappa)\text{and } \lambda\le\lambda' \,\&\, \mathrm{IND}^i(\lambda,\kappa)\Rightarrow \mathrm{IND}^i(\lambda',\kappa).$$

Hence it suffices to prove: if λ is minimal such that $\mathrm{IND}^0(\lambda,\kappa)$ then $\mathrm{IND}^1(\lambda,\kappa)$. Let M be a model with universe λ and vocabulary of cardinality $\le\kappa$ and we shall prove the conclusion of 2.4(4) (= the Definition of $\mathrm{IND}^1(\lambda,\kappa)$). Let for $n<\omega$, F_n^+, F_n^- be $(n+2)$-place functions from λ to λ such that

- $(*)$ if $\gamma<\lambda$ and $\bar\beta\in {}^n\lambda$ then $F_n^+(-,\bar\beta,\gamma)$ is a one-to-one function from $c\ell_M(\{\beta_\ell : \ell<\ell g(\bar\beta)\}\cup\{i : i\le\gamma \text{ or } i<\kappa\})$ onto $|\kappa+\gamma|$ and $F_n^-(-,\bar\beta,\gamma)$ be the inverse function.

Let $M^*=(M,F_n^+,F_n^-)_{n<\omega}$. For each $\beta<\lambda$ there is a model M_β with inverse $|\kappa+\beta|$ and functions $H_{i,n}$ for $i<\kappa, n<\omega$, $H_{i,n}$ is n-place, which exemplify that $\mathrm{IND}^0(|\kappa+\beta|,\kappa)$, and $\{H_{i,n} : i<\kappa, m<\omega\}$ is closed under composition.

Let $H_{i,n}^+$ be an $(n+1)-$ place function $H_{i,n}^+(\gamma_0,\dots,\gamma_{n-1},\beta)=H_{i,n}^{M\beta}(\gamma_0,\dots,\gamma_{n-1})$ when $\gamma_0,\dots,\gamma_{n-1}<\beta<\lambda$, and $M^+=(M^*,i,H_{i,n}^+)_{i\le\kappa,n<\omega}$. Now as $\mathrm{IND}^0(\lambda,\kappa)$ we can apply Definition 2.4(3) and find a sequence $\langle\alpha_n : n<\omega\rangle$ satisfying $\alpha_n<\lambda, \alpha_n\notin c\ell_{M^+}(\{\alpha_\ell : \ell\in(n,\omega)\})$. Without loss of generality $\langle\alpha_n : n<\omega\rangle$ is strictly increasing; $\alpha_n>\kappa$ (as each $i\le\kappa$ is an individual constant of M^+), clearly it suffices to prove

- $(**)$ for any $n<\omega$ for some $m\in(n,\omega)$ we have $\alpha_m\notin c\ell_M(\{\alpha_\ell : \ell<n$ or $\ell>m\})$
  hence it suffices to prove
- $(**)\prime$ for any $n<\omega$ for some $m\in(n,\omega)$ we have

$$\alpha_m\notin c\ell_M(\{\alpha_\ell : \ell>m\}\cup\{i : i\le\alpha_n\}).$$

Toward contradiction assume that $(**)'$ fail for $n=n(*)$. So for every $m\in(n(*),\omega)$ there is $k_m\in(m,\omega)$ st $\alpha_m\in c\ell_M\big(\{\alpha_\ell : \ell\in(m,k_m)\}\cup\{i : i\le\alpha_{n(*)}\}\big)$. Wlog $k_{m+1}>k_m$ hence we can show that for any m,p is satisfying $n(*)<m<p<\omega$ we have $\alpha_m\in c\ell_M\big(\{\alpha_\ell : \ell\in(p,k_p)\}\cup\{i : i\le\alpha_{n(*)}\}\big)$.

Let $\gamma_m=F^+_{k_m-m}(\alpha_m,\langle\alpha_{m+1}\dots,\alpha_{k_m}\rangle,\alpha_{n(*)})$ so $\gamma_m<|\kappa+\gamma_{n(*)}|$. By the choise of $M_{\alpha_{n(*)}}$ for some m,r,i we have $\gamma_m=H_{i,zr}^{M_{\alpha_{n(*)}}}(\gamma_{m+1,\dots,}\gamma_n)$ hence $\gamma_m=H^+_{i,r}(\gamma_{m+1}\dots\gamma_{k_m},\alpha^n_{m(*)})$. Using the F^+-s, $\{\gamma_{m+1}\dots\gamma_{k(m)}\}\subseteq c\ell_{M^+}\{\alpha_{m+1},\alpha_{m+2},\dots|\cup\{\alpha_{n(*)}\}|$ hence $\gamma_m\in c\ell_{M^+}(\{\alpha_{m+1},\alpha_{m+2},\dots\}\cup\alpha_{n(*)}\})$, hence using $F^-_{k_m-m}$ we have $\alpha_m\in c\ell_{M^+}\{\alpha_{m+1},\dots 1\}$.

Contradiction to the choice of $\langle\alpha_m : m<\omega\rangle$. $\square_{2.5}$

Claim 2.6. 1) Assume $\lambda>\mathrm{cf}(\lambda), |\mathfrak{a}|^+<\min(\mathfrak{a})$ and $\sup(\mathfrak{a})=\lambda$ and $\mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{a}]:\theta\in\mathrm{pcf}(\mathfrak{a})\rangle}(\mathfrak{a})\ge|\mathfrak{a}|^+$. <u>Then</u> $\mathrm{IND}(\lambda,|\mathfrak{a}|)$.

2) Moreover, for any model M with universe λ and $|\mathfrak{a}|$ functions and $\mathfrak{c}$ such that $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, $|\mathfrak{c}| < \min(\mathfrak{a})$ and $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ a generating sequence we can find $\bar{\alpha} = \langle \alpha_\theta : \theta \in \mathfrak{a} \rangle \in \Pi\mathfrak{a}$ such that, defining for $\mathfrak{b} \subseteq \mathfrak{a}$:

$$c\ell_{M,\bar{\alpha}}(\mathfrak{b}) = \mathfrak{b} \cup \{\theta \in \mathfrak{a} : \alpha_\theta \in c\ell_M(\{\alpha_\mu : \mu \in \mathfrak{b}\})\};$$

we have

$\otimes_1$ $[\theta \in \mathfrak{c} \Rightarrow c\ell_{M,\bar{\alpha}}(\mathfrak{b}_\theta[\mathfrak{a}]) \in J_{\leq\theta}[\mathfrak{a}]]$;
$\otimes_2$ $c\ell_{M,\bar{\alpha}}(-)$ is a closure operation on $\mathfrak{a}$, i.e.

$$\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow c\ell_{M,\bar{\alpha}}(\mathfrak{b}_1) \subseteq c\ell_{M,\bar{\alpha}}(\mathfrak{b}_2),$$

$$\mathfrak{b} \subseteq c\ell_{M,\alpha}(\mathfrak{b}),$$

$$c\ell_{M,\bar{\alpha}}(c\ell_{M,\bar{\alpha}}(\mathfrak{b})) = c\ell_{M,\bar{\alpha}}(\mathfrak{b}).$$

Remark. See 3.17 – 3.20 for more.

Proof. 1) Let us define $\bar{J} = \langle (J_{<\theta}[\mathfrak{a}], J_{\leq\theta}[\mathfrak{a}]) : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$. We prove part (1) assuming part (2). Choose $\mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, $|\mathfrak{c}| = |\mathfrak{a}|^+$ such that $\mathrm{rk}'_{\bar{J}\restriction\mathfrak{c}}(\mathfrak{a}) \geq |\mathfrak{a}|^+$ (this is possible by 2.3(2)) and without loss of generality $\mathfrak{a} \subseteq \mathfrak{c}$ and let $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a generating sequence for $\mathfrak{a}$ (exists by [Sh 371, 2.6]). For proving $\mathrm{IND}(\lambda, |\mathfrak{a}|)$ let M be a model with universe λ and $\leq |\mathfrak{a}|$ functions, by part (2) there is a sequence $\bar{\lambda} = \langle \alpha_\tau : \tau \in \mathfrak{a} \rangle$ as there. Let for $\theta \in \mathfrak{c}$, $\mathfrak{d}_\theta =: c\ell_{M,\bar{\alpha}}(\mathfrak{b}_\theta[\mathfrak{a}]) \subseteq \mathfrak{a}$ (as defined in part (2)) so $\mathfrak{b}_\theta[\mathfrak{a}] \subseteq \mathfrak{d}_\theta \in J_{\leq\theta}[\mathfrak{a}]$ hence $J_{<\theta}[\mathfrak{a}] + \mathfrak{d}_\theta = J_{\leq\theta}[\mathfrak{a}]$. So by 2.2(5) we know $\mathrm{rk}_{\langle \mathfrak{d}_\theta:\theta\in\mathfrak{c}\rangle}(\mathfrak{a}) \geq \mathrm{rk}'_{\langle \mathfrak{d}_\theta:\theta\in\mathfrak{c}\rangle}(\mathfrak{a}) \geq \mathrm{rk}'_{\bar{J}\restriction\mathfrak{c}}(\mathfrak{a}) \geq \kappa^+$ (by the choice of $\mathfrak{c}$ above). Now by 2.3(1) we can find $\tau_n \in \mathfrak{a}$ for $n < \omega$, pairwise distinct and strictly increasing with n such that for every $n < m < \omega$ for some $\theta_{n,m} \in \mathfrak{c}$ we have $\{\tau_n, \tau_{n+1}, \ldots, \tau_m\} \cap \mathfrak{d}_{\theta_{n,m}} = \{\tau_{n+1}, \ldots, \tau_m\}$, note: as $\tau_m \in \mathfrak{d}_{\theta_{n,m}}$ necessarily $\theta_{n,m} \geq \tau_m$. So by the choice of $\bar{\alpha}$ and the $\mathfrak{d}_\theta - s$, we have $\alpha_{\tau_n} \notin c\ell_M(\{\alpha_{\tau_{n+1}}, \alpha_{\tau_{n+2}}, \ldots, \alpha_{\tau_m}\})$. So $\langle \alpha_{\tau_n} : n < \omega\rangle$ are as required in the definition of $\mathrm{IND}(\lambda, |\mathfrak{a}|)$.
2) Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be the generating sequence for $\mathfrak{a}$; without loss of generality $\max \mathrm{pcf}(\mathfrak{a}) \in \mathfrak{c}$ and $\theta \in \mathfrak{c} \setminus \{\max \mathrm{pcf}(\mathfrak{a})\} \Rightarrow \min(\mathrm{pcf}(\mathfrak{a}) \setminus \theta^+) \in \mathfrak{c}$. We know that

$(*)_\mathfrak{a}$ there is $\bar{F} = \langle F_\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ such that:
(a) $F_\theta \subseteq \Pi\mathfrak{a}$ and $|F_\theta| \leq \theta$ and F_θ is $(< \theta)$-directed
(b) $\{f \restriction \mathfrak{b} : f \in F_\theta\}$ is cofinal in $\Pi\mathfrak{b}$ for every $\mathfrak{b} \in J_{\leq\theta}[\mathfrak{a}]$
(c) F_θ includes $\cup\{F_\tau : \tau \in \theta \cap \mathrm{pcf}(\mathfrak{a})\}$ and is closed under some natural operations
(d) if $\tau \in \theta \cap \mathfrak{c}$ then $f \in F_\theta \Rightarrow (\exists g \in F_\tau)(f \restriction \mathfrak{b}_\tau[\mathfrak{a}] = g \restriction \mathfrak{b}_\tau[\mathfrak{a}])$.

[Why? E.g. the proof of [Sh 355, 3.5].]

Let M be a model with universe λ and vocabulary of cardinality $|\mathfrak{a}|$. For every $f \in \Pi\mathfrak{a}$ (e.g., $f \in F_\theta, \theta \in \mathrm{pcf}(\mathfrak{a})$) we define $g_f \in \Pi\mathfrak{a}$ by $g_f(\tau) =: \sup[\tau \cap c\ell_M(\mathrm{Rang}\, f)]$. For every $\theta \in \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, $\{g_f : f \in \bigcup_{\tau\in\theta\cap\mathfrak{c}} F_\tau\}$ is a subset of $\Pi\mathfrak{a}$ of cardinality $< \theta$ (here instead $|\mathfrak{c}| < \min(\mathfrak{a})$, just $\theta \in \mathfrak{c} \Rightarrow |\mathfrak{c} \cap \theta| < \theta$ suffice), so there is $g^\theta \in \Pi\mathfrak{a}$ such that:

$(*)_1$ $f \in \bigcup_{\tau \in \theta \cap \mathfrak{c}} F_\tau \Rightarrow g_f < g^\theta \bmod J_{<\theta}[\mathfrak{a}]$.

Define $g^* \in \Pi\mathfrak{a}$ by $g^*(\tau) = \sup\{g^\theta(\tau) + 1 : \theta \in \mathfrak{c}\}$ (remember $|\mathfrak{c}| < \min(\mathfrak{a})$). So there is $h \in F_{\max \mathrm{pcf}(\mathfrak{a})}$ such that $g^* < h$ (see $(*)_\mathfrak{a}$ (b)); we shall show that:

$(*)_2$ for any such h the sequence $\langle h(\tau) : \tau \in \mathfrak{a}\rangle$ is as required.

Proof of $(*)_2$. So let $\alpha_\tau = h(\tau)$. Note that $\otimes_2$ from 2.6(2) is trivial so we shall prove $\otimes_1$. Assume $\theta \in \mathfrak{c}$ and let $\mathfrak{b} = \mathfrak{b}_\theta[\mathfrak{a}] \in J_{\leq\theta}[\mathfrak{a}]$ so by clause (d) of $(*)_\mathfrak{a}$ for some $f_1 \in F_\theta$ we have

$\oplus_1$ $h \restriction \mathfrak{b} = f_1 \restriction \mathfrak{b}$;

we can assume $\theta < \max \mathrm{pcf}(\mathfrak{a})$ (otherwise conclusion is trivial) and let $\sigma = \min(\mathrm{pcf}(\mathfrak{a}) \backslash \theta^+)$, by an assumption made in the beginning of the proof $\sigma \in \mathfrak{c}$ and so as $f_1 \in F_\theta$ by the choice of g^σ we have:

$$g_{f_1} < g^\sigma \bmod J_{<\sigma}[\mathfrak{a}]$$

but by the choice of g^*

$$g^\sigma < g^*$$

and by the demand of h

$$g^* < h$$

together

$$g_{f_1} < h \bmod J_{<\sigma}[\mathfrak{a}]$$

so for some $\mathfrak{d} \in J_{<\sigma}[\mathfrak{a}] = J_{\leq\theta}[\mathfrak{a}]$ we have:

$\oplus_2$ $g_{f_1} \restriction (\mathfrak{a} \backslash \mathfrak{d}) < h \restriction (\mathfrak{a} \backslash \mathfrak{d})$.

Now for any $\tau \in \mathfrak{a}$

$\oplus_3$ $\tau \in c\ell_{M,\bar{\alpha}}(\mathfrak{b}) \Rightarrow \alpha_\tau \in c\ell_M[\{\alpha_\kappa : \kappa \in \mathfrak{b}\}] \Rightarrow \alpha_\tau \in c\ell_M[\mathrm{Rang}(h \restriction \mathfrak{b})] \Rightarrow \alpha_\tau \in c\ell_M[(\mathrm{Rang}(f_1 \restriction \mathfrak{b})] \Rightarrow \alpha_\tau \in c\ell_M(\mathrm{Rang}(f_1)) \Rightarrow \alpha_\tau \leq g_{f_1}(\tau) \Rightarrow h(\tau) = \alpha_\tau \leq g_{f_1}(\tau)$.

By $\oplus_2 + \oplus_3$ we have $c\ell_{M,\bar{\alpha}}(\mathfrak{b}_\theta) \subseteq \mathfrak{d}$ so the required conclusion follows. $\square_{2.6}$

E.g.

Claim 2.7. If $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$ <u>then</u> $\mathrm{IND}(\aleph_\omega)$.

Proof. Let $\mathrm{pp}(\aleph_\omega) = \aleph_{\alpha^*}$ (so $\alpha^* < \omega_4, \alpha^* = \beta^* + 1$, see [Sh:g, Ch. IX,2.1]) let $\underline{\mathfrak{a}} = \{\aleph_{i+1} : 5 \leq i < \omega\}$ so we know $\mathrm{pcf}(\mathfrak{a}) = \{\aleph_{i+1} : 5 \leq i < \alpha^*\}$ and let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a normal generating sequence for $\mathrm{pcf}(\mathfrak{a})$ (not $\mathfrak{a}$!, exists as $|\mathrm{pcf}(\mathfrak{a})| < \min(\mathrm{pcf}(\mathfrak{a}))$; without loss of generality $\mathfrak{b}_{\aleph_{\alpha^*}} = \mathrm{pcf}(\mathfrak{a})$. Let $\bar{\mathfrak{c}} = \langle \mathfrak{b}_\lambda \cap \mathfrak{a} : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$. Now by localization ([Sh:g, Ch. VIII, 3.4, p. 337+ II, 2.1, p. 55]) we know that for some club E of $\omega_1 : \delta \in E \Rightarrow \mathrm{pp}(\aleph_\delta) > \aleph_{\omega_1}$ (so we can assume $\omega \in E$).

Let $E = \{\beta_\zeta : \zeta < \omega_1\}$ (increasing in ζ). Hence by [Sh:g, Ch. II,1.5A] we have

$(*)_0$ if $\delta < \omega_1$ is limit, $\delta < \beta < \omega_1$ and $\delta \in E$, then for some unbounded $\mathfrak{d} \in \aleph_\delta \cap \mathrm{Reg}$, we have $\aleph_{\beta+1} \in \mathrm{pcf}_{J^{\mathrm{bd}}_{\mathfrak{d}}}(\mathfrak{d})$.

Hence we can prove by induction on $\varepsilon < \omega_1$ that:

$(*)$ if $\beta_\varepsilon \leq \zeta < \omega_1$, $\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}_{\aleph_{\zeta+1}} \bmod J_{<\aleph_{\zeta+1}}[\mathfrak{a}]$ <u>then</u> $\mathrm{rk}'_{\bar{\zeta}}(\mathfrak{b}) \geq \varepsilon$.

[Why? For $\varepsilon = 0$ this is trivial and also for ε limit. If $\varepsilon = \xi + 1$ we can find $\mathfrak{d} \in \aleph_{\beta_\varepsilon} \cap \mathrm{Reg} \backslash \aleph_{\beta_\xi}$ such that $\aleph_{\zeta+1} \in \mathrm{pcf}_{J^{\mathrm{bd}}_{\mathfrak{d}}}(\mathfrak{d})$ so without loss of generality $\aleph_{\zeta+1} = \max \mathrm{pcf}(\mathfrak{d})$. By [Sh:g, Ch. I,1.12] we know that the set $\mathfrak{d}'$ of $\theta \in \mathfrak{d}$ such that $\mathfrak{b}_\theta \cap \mathfrak{a} \subseteq \mathfrak{b} \bmod J_{<\theta}[\mathfrak{a}]$ is $= \mathfrak{d} \bmod J_{<\aleph_{\zeta+1}}[\mathfrak{d}]$ hence is not bounded in $\mathfrak{d}$, hence is not bounded in $\aleph_{\beta_\varepsilon}$. But $\mathfrak{d}' \subseteq \mathfrak{d} \subseteq \aleph_{\beta_\varepsilon} \cap \mathrm{Reg} \backslash \aleph_{\beta_\varepsilon}$, hence by the induction hypothesis $\theta \in \mathfrak{d}' \Rightarrow \mathrm{rk}'_{\bar{\zeta}}(\mathfrak{b}_\theta \cap \mathfrak{b}) \geq \xi$, but of course $\mathfrak{b} \backslash \mathfrak{b}_\theta \neq \emptyset$.]

Now apply 2.6. $\square_{2.7}$

Claim 2.8. If $|\mathfrak{a}| < \min(\mathfrak{a})$, $\lambda = \sup(\mathfrak{a})$ is singular and $\mathrm{pcf}_{J^{\mathrm{bd}}_{\mathfrak{a}}}(\mathfrak{a})$ contains an interval of Reg of cardinality $|\mathfrak{a}|^+$ <u>then</u> $\mathrm{IND}(\lambda)$.

Proof. Similar to the proof of 2.7, (and even can weaken the assumption as in [Sh:410]).

Discussion 2.9. We can also prove e.g.: if $\lambda = \mathrm{tcf}(\prod_{\varepsilon<\kappa} \lambda_\varepsilon/[\kappa]^{<\aleph_0})$, satisfies $\lambda > \lambda_\varepsilon = \mathrm{cf}(\lambda_\varepsilon) > \kappa$, then for every algebra M on $\sum_{\varepsilon<\kappa} \lambda_\varepsilon$ with $< \min\{\lambda_\varepsilon : \varepsilon < \kappa\}$ functions there are $\alpha_\varepsilon < \lambda_\varepsilon (\varepsilon < \kappa)$ such that: for finite $u \subseteq \kappa$ we have $\{\zeta : \alpha_\zeta \in c\ell_M(\{\alpha_\varepsilon : \varepsilon \in u\})\}$ is finite (and more). Not clear how interesting is this statement and where it leads.

Claim 2.10. Assume $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$ and $|\mathfrak{c}| < \min(\mathfrak{a})$ (or $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, see [Sh:g, Ch. VIII,§3] and [Sh:E11]), so $\mathrm{pcf}(\mathfrak{a}) = \mathrm{pcf}(\mathfrak{c})$. <u>Then</u>

$$\mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{a}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}(\mathfrak{a}) = \mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{c}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}(\mathfrak{c}).$$

Proof. Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a generating sequence for $\mathfrak{a}$, hence we know that letting $\mathfrak{b}_\theta[\mathfrak{c}] =: \mathfrak{c} \cap \mathrm{pcf}(\mathfrak{b}_\theta[\mathfrak{a}])$, we have: $\bar{\mathfrak{b}}' = \langle \mathfrak{b}_\theta[\mathfrak{c}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ is a generating sequence for $\mathfrak{c}$. Without loss of generality $\mathfrak{b}_\theta[\mathfrak{c}] \cap \mathfrak{a} = \mathfrak{b}_\theta[\mathfrak{a}]$ and $\mathfrak{b}_{\max \mathrm{pcf}(\mathfrak{a})}[\mathfrak{a}] = \mathfrak{a}$ hence $\mathfrak{b}_{\max \mathrm{pcf}(\mathfrak{a})}[\mathfrak{c}] = \mathfrak{c}$. So we can prove easily:

$$\mathfrak{d} \subseteq \mathfrak{a} \Rightarrow \mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{a}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}(\mathfrak{d}) \leq \mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{c}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}(\mathfrak{d})$$

(as $J_{<\theta}[\mathfrak{a}] = J_{<\theta}[\mathfrak{c}] \restriction \mathfrak{a}$).

For the other direction we prove

$(*)$ if $n < \omega$, $\{\theta_1, \ldots, \theta_n\} \subseteq \mathrm{pcf}(\mathfrak{a})$ then

$$\mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{c}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}\left(\bigcap_{\ell=1}^{n} \mathfrak{b}_{\theta_\ell}[\mathfrak{c}]\right) \leq \mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{c}]:\theta \in \mathrm{pcf}(\mathfrak{a})\rangle}\left(\bigcap_{\ell=1}^{n} \mathfrak{b}_{\theta_\ell}[\mathfrak{a}]\right).$$

$\square_{2.10}$

3. Existence of free sets implies restrictions on pcf

Definition 3.1. *Suppose $\bar{\mathcal{I}} = \langle(\kappa_n, I_n) : n < \omega\rangle$ is such that: I_n is an ideal on κ_n.*

1) We define $J_n = J_n^{\bar{\mathcal{I}}}$ an ideal on $\prod_{\ell<n} \kappa_\ell$:

$(*)_0$ $J_0 =$ the empty ideal on $\{\langle\,\rangle\}$
$(*)_1$ $J_{n+1} = \{A \subseteq \prod_{\ell\le n} \kappa_\ell : \{\alpha < \kappa_n : \{\eta \in \prod_{\ell<n} \kappa_\ell : \eta\hat{}\langle\alpha\rangle \in A\} \notin J_n\} \in I_n\}$

we let $\bar{J}^{\bar{\mathcal{I}}} = \langle J_n^{\bar{\mathcal{I}}} : n < \omega\rangle$.
2) We say $\langle J_n : n < \omega\rangle$ is a candidate (for $\bar{\mathcal{I}}$) if in $(*)_1$ we weaken "$J_{n+1} = \ldots$" to "$J_{n+1} \subseteq \ldots$". [So there may be many candidates for a given $\bar{\mathcal{I}}$.]

Fact 3.2. 1) In definition 3.1(1) above, each J_n is an ideal on $\prod_{\ell<n} \kappa_\ell$.
2) If $I_0, \ldots, I_{n-1}$ are σ-complete <u>then</u> so is J_n.
3) If $\langle J_n' : n < \omega\rangle$ is a candidate for $\bar{\mathcal{I}}$, then $\bigwedge_{n<\omega} J_n' \subseteq J_n^{\bar{\mathcal{I}}}$.

Claim 3.3. For $\bar{\mathcal{I}}$, $\bar{J}^{\bar{\mathcal{I}}}$ as in 3.1(1) we have $A \in J_n$ <u>iff</u> for some functions $f_0, \ldots, f_{n-1}$ we have:

$$\mathrm{Dom}(f_\ell) = \prod_{m=\ell+1}^{n-1} \kappa_m \text{ and } \mathrm{Rang}(f_\ell) \subseteq I_\ell,$$

and $A \subseteq \bigcup_{\ell<n} A_\ell^n(f_\ell)$ where

$$A_\ell^n(f_\ell) = \{\eta \in \prod_{m<n} \kappa_m : \eta(m) \in f_m(\eta \restriction (m, n))\}.$$

Proof. By induction on n.

Theorem 3.4. *Let $\bar{\mathcal{I}} = \langle(\kappa_n, I_n) : n < \omega\rangle$ be as in 3.1, $\kappa = \sum_{n<\omega} \kappa_n \le \mu_0 = cf(\mu_0) < \mu_1 < \mu = cf(\mu)$. <u>Then</u> $\otimes_1 \Rightarrow \otimes_2$ where*

- $\otimes_1$ *for each n there is $\langle\lambda_i^n : i < \kappa_n\rangle$ such that: $\lambda_i^n \in [\mu_0, \mu_1) \cap \mathrm{Reg}$ and $\prod_{i<\kappa_n} \lambda_i^n / I_n$ is μ-directed*
- $\otimes_2$ *for some $\bar{\mathcal{I}}$-candidate, $\bar{J} = \langle J_n : n < \omega\rangle$ (except for clause (δ), $J_n = J_n^{\bar{\mathcal{I}}}$ is O.K.) we have:*
- $\otimes_{\bar{J}}^2$ *there are $\bar{\lambda}^n = \langle\lambda_\eta : \eta \in \prod_{\ell<n} \kappa_\ell\rangle$ for $n \in (0, \omega)$ such that for each n:*
    - (α) $(\Pi\{\lambda_\eta : \eta \in \prod_{\ell<n} \kappa_\ell\}/J_n)$ *has true cofinality μ*
    - (β) $\mu_0 \le \lambda_\eta = cf(\lambda_\eta) < \mu_1$ *(note that by clause (α) we have $\{\eta \in \prod_{\ell<n} \kappa_\ell : \lambda_\eta = \mu_0\} \in J_n$)*

(γ) *if* $0 < n < \omega, \alpha < \kappa_n$ *and* $\eta \in \prod_{\ell<n} \kappa_\ell$ *then* $\lambda_\eta > \mu_0 \Rightarrow \lambda_\eta > \lambda_{\eta\hat{}\langle\alpha\rangle}$ *so* $\{\eta \in \prod_{\ell<n} \kappa_\ell : \lambda_{\eta\hat{}\langle\alpha\rangle} \not< \lambda_\eta\} \in J_n$ *hence*

$(\gamma)'$ $\{\eta \in \prod_{\ell\le n} \kappa_\ell : \lambda_\eta \not< \lambda_{\eta\restriction n}\} \in J_{n+1}$

(δ) $J_n = \{A \subseteq \prod_{\ell<n} \kappa_\ell : \max \mathrm{pcf}\{\lambda_\eta : \eta \in A\} < \mu\}$.

Question. Can we prepare the ground to 3.8 with IND$^+$ instead IND?

Proof. We choose $\bar{\lambda}^n$ by induction on n. For $n = 1$ apply [Sh:g, Ch. II,1.5A] to the sequence $\langle\lambda_i^1 : i < \kappa_0\rangle$, the ideal $\{A \subseteq \kappa_1 : \max \mathrm{pcf}\{\lambda_i^1 : i < \kappa_0\} < \mu_0\}$ and the cardinal μ and get $\langle\lambda_{\langle i\rangle} : i < \kappa_0\rangle$. For $n+1$ for each $i < \lambda_n$ we apply [Sh:g, Ch. II,1.5A] to $\langle\lambda_n : \eta \in \prod_{\ell<n} \kappa_\ell\rangle$, the ideal $\{A \subseteq \prod_{\ell<n} \kappa_\ell : \max \mathrm{pcf}\{\lambda_\eta : \eta \in A\} < \mu\}$ and the cardinal λ_i^n and we get $\langle\lambda_{\eta\hat{}\langle i\rangle} : \eta \in \prod_{\ell<n} \kappa_\ell\rangle$. $\square_{3.4}$

Claim 3.5. In 3.4, from $\otimes_2$ we can deduce

$\otimes^3$ there are functions $f_{\ell,n} : \prod_{m=\ell+1}^{n} \kappa_m \to I_\ell$ (for $\ell < n < \omega$) such that for every $\eta \in \prod_{m<\omega} \kappa_m$ for some $\ell < n < \omega$ we have $\eta(\ell) \in f_{\ell,n}(\eta \restriction (\ell, n))$.

Proof. Otherwise $\langle\lambda_{\eta\restriction n} : n < \omega\rangle$ is a strictly decreasing sequence of cardinals. $\square_{3.5}$

Definition 3.6. *1) IND($\langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$) (note that $|Dom(J_\varepsilon)|$ is not necessarily increasing with ε) means that each J_ε is an ideal on $Dom(J_\varepsilon)$, say κ_ε and*

($*$) *for every sequence $\langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \backslash (\varepsilon+1)$ finite$\rangle$ such that*[†] *$f_{\varepsilon,u}$ a function from $\prod_{\zeta\in u} \kappa_\zeta$ to J_ε there is an increasing sequence $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n < \cdots < \varepsilon^*$ (for $n < \omega$) and $\alpha_\ell \in \kappa_{\varepsilon_\ell}$ (for $\ell < \omega$) such that:*

($**$) *for $\ell < n < \omega$ we have $\alpha_\ell \notin f_{\varepsilon_\ell,u}(\langle\alpha_{\varepsilon_{\ell+1}}, \ldots, \alpha_{\varepsilon_n}\rangle)$ for $u = \{\alpha_{\varepsilon_{\ell+1}}, \ldots, \alpha_{\varepsilon_n}\}$.*

2) IND$^+$($\langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$) means J_ε is an ideal (on $Dom(J_\varepsilon)$ which is say κ_ε) such that:

($*$) *for every sequence $\langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \backslash (\varepsilon+1)$ finite$\rangle$ such that $f_{\varepsilon,u} : \prod_{\zeta\in u} \kappa_\zeta \to J_\varepsilon$ there is $\langle\alpha_\varepsilon : \varepsilon < \varepsilon^*\rangle \in \prod_{\varepsilon<\varepsilon^*} \kappa_\varepsilon$ such that*

($**$) *for $\varepsilon < \varepsilon^*, u \subseteq \varepsilon^* \backslash (\varepsilon+1)$ finite we have: $\alpha_\varepsilon \notin f_{\varepsilon,u}(\ldots, \alpha_\zeta, \ldots)_{\zeta\in u}$.*

3) Let function$\langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$ be the set of $\bar{f}$ as in ($*$) of part (1), i.e. $\bar{f} = \langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*$ and $u \subseteq \varepsilon^* \backslash (\varepsilon+1)$ is finite$\rangle$ where $f_{\varepsilon,u}$ is a function with domain $\prod_{\zeta\in u} \kappa_\zeta$ and range $\subseteq J_\varepsilon$. We say for $\bar{f} \in$ function$\langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$, that $\bar{\varepsilon}$ is candidate if $\bar{\varepsilon}$ is an increasing sequence of length ω of ordinals $< \varepsilon^*$. In this case we say that $\bar{\alpha}$ is

[†] Of course Wlog $v \subseteq u \Rightarrow f_{\epsilon,v}(\langle\alpha_\zeta : \zeta \in v\rangle) \subseteq f_{\epsilon,u}(\langle\alpha_\zeta : \zeta \in u\rangle)$

$(\bar{f}, \bar{\varepsilon})$-free if $\bar{\alpha} \in \prod_{n<\omega} Dom(J_{\varepsilon_n})$ and the statement $(**)$ of part (1) holds.
4) Above if $J_\varepsilon = J^{bd}_{\lambda_\varepsilon}$ for $\varepsilon < \varepsilon^*$ then we may write $\langle \lambda_\varepsilon : \varepsilon < \varepsilon^* \rangle$ instead of $\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$.

Observation 3.7. 1) If IND$(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ is as in 3.6, each J_ε is $(|\varepsilon^*|^{\aleph_0})^+$-complete, then for some $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon^*$ we have IND$^+(\langle J_{\varepsilon_n} : n < \omega \rangle)$.
2) If IND$(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle$ as in 3.6, each J_ε is cov$(|\varepsilon^*|, \mu, \aleph_1, 2)^+$-complete then for some infinite $u \in \mathcal{S}_{<\mu}(\varepsilon^*)$ we have IND$(\bar{J} \restriction u)$.
3) Definition 3.6(4) and Definition 2.4(1) are compatible.
4) IND$(\langle \lambda_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ is equivalent to IND$(\langle [\lambda_\varepsilon]^{\leq \mu_\varepsilon} : \varepsilon < \varepsilon^* \rangle)$, similarly with IND$^+$.

Before we prove 3.7

Conjecture 3.8. if IND$(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$, $\varepsilon^* < \omega_1$ and each J_ε is $\aleph_1$-complete then for some c.c.c. forcing $\mathbb{P}$ we have:

$$\Vdash_{\mathbb{P}} \text{“for some } \varepsilon_0 < \cdots < \varepsilon_n < \varepsilon_{n+1} < \cdots < \varepsilon^*, \text{IND}(\langle J_{\varepsilon_n} : n < \omega \rangle)\text{”}.$$

Remark 3.9. In the proof of 3.7(2) it is enough to demand on $\mathcal{P}$:

(*) if $\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_n < \cdots < \varepsilon^*$ (for $n < \omega$) then for some $b \in \mathcal{P}$, $(\exists^\infty n)\varepsilon_n \in b$

this seems to weaken cov$(\ldots)$ but does not.

Proof of 3.7. 1) Similar to the proof of part (2), as cov$(\lambda, \aleph_1, \aleph_2, 2) \leq |\varepsilon^*|^{\aleph_0}$.
2) Let $\mu =:$ cov$(\varepsilon^*, \mu, \aleph_1, 2)$ and let $\mathcal{P} \subseteq [\varepsilon^*]^{<\mu}$ be of cardinality μ exemplifying its definition i.e. $(\forall a)[a \subseteq \varepsilon^* \;\&\; |a| = \aleph_0 \Rightarrow (\exists b \in \mathcal{P})[a \subseteq b]]$.

If for some $b \in \mathcal{P}$, IND$(\bar{J} \restriction u)$ hold then we are done. Otherwise for each $b \in \mathcal{P}$, we can find $\bar{f}^b = \langle f^b_{\varepsilon,u} : \varepsilon \in b$ and $b \subseteq u \backslash (\varepsilon + 1)$ is finite$\rangle \in$ function$(J \restriction u)$ such that for no $\bar{\varepsilon} = \langle \varepsilon_n : n < \omega \rangle$ strictly increasing sequence of ordinals from b and $\alpha_n \in$ Dom(J_{ε_n}) (for $n < \omega$) do we have $n < \omega \;\&\; u \subseteq \{\varepsilon_{n+1}, \varepsilon_{n+2}, \ldots\} \Rightarrow \alpha_n \notin f^b_{\varepsilon,u}(\{\alpha_m : m \in u\})$. Let us define for $\varepsilon < \varepsilon^*$ and $u \subseteq \varepsilon^* \backslash (\varepsilon + 1)$ finite a function $f_{\varepsilon,u}$ from $\prod_{\zeta \in u}$ Dom(J_ζ) to J_ε by:

(*) if $\alpha_\zeta \in$ Dom(J_ζ) for $\zeta \in u$ then $f_{\varepsilon,u}(\ldots, \alpha_\zeta, \ldots)_{\zeta \in u} = \bigcup \{ f^b_{\varepsilon,u}(\ldots, \alpha_\zeta, \ldots)_{\zeta \in u} : \varepsilon \notin b$ and $u \subseteq b$ and $b \in \mathcal{P}\}$.

(As each J_ζ is $|\mathcal{P}|$-complete (by assumption) Rang$(f_{\zeta,u}) \subseteq J_\zeta$. As IND$(\langle J_\varepsilon : \varepsilon < \varepsilon^* \rangle)$ necessarily there is a strictly increasing $\langle \varepsilon_n : n < \omega \rangle$, $\varepsilon_n < \varepsilon^*$, and $\alpha_n \in$ Dom(J_{ε_n}) (for $n < \omega$) such that:

(**) if $n < \omega$, $u \subset \{\varepsilon_{n+1}, \varepsilon_{n+2}, \ldots\}$ finite then $\alpha_n \notin f_{\varepsilon_n,u}(\ldots, \alpha_m, \ldots)_{\varepsilon_m \in u}$.

By the choice of $\mathcal{P}$ for some $b \in \mathcal{P}$ we have $\{\varepsilon_n : n < \omega\} \subseteq b$, but then $\langle (\varepsilon_n, \alpha_n) : n < \omega \rangle$ contradict the choice of $\bar{f}^b = \langle f^b_{\zeta,u} : \zeta \in b, u \subseteq b \backslash (\zeta + 1)$ finite$\rangle$.
3), 4) easy. $\square_{3.7}$

Conclusion 3.10. 1) If $\mathrm{IND}^+(\langle I_n : n < \omega\rangle)$ and $\mathrm{Dom}(I_n) = \kappa_n$ <u>then</u>

(a) the conclusion of 3.5 (i.e. $\otimes^3$ there) fails, hence $\otimes^2$ of 3.4 fails hence $\otimes^1$ of 3.4 fails

(b) if $\lambda > \sum_{n<\omega} \kappa_n$ and $\kappa_n < \mathrm{cf}(\kappa_{n+1})$, <u>then</u> for every n large enough for no $\lambda_i \in (\sum_{n<\omega} \kappa_n, \lambda) \cap \mathrm{Reg}$ (for $i < \kappa_n$) is $\prod_{i<\kappa_n} \lambda_i / I_n$ λ-directed.

2) If we weaken the assumption to $\mathrm{IND}(\langle I_n : n < \omega\rangle)$ then in (b) we have just for arbitrarily large $n < \omega$.
3) If in addition $X \in I_n^+ \Rightarrow I_n \restriction X \cong I_n$ (e.g. $I_n = J^{\mathrm{bd}}_{\kappa_n}$) <u>then</u> in clause (b) for n large enough λ is $(I_n, \aleph_0)$-inaccessible.

Proof. 1)

(a) straight
(b) our problem is to get μ_1, which is not serious.

2), 3) Similarly. $\square_{3.10}$

Conclusion 3.11. 1) Assume $\langle \kappa_\varepsilon : \varepsilon < \delta\rangle$ is strictly increasing, $|\delta| \le \sigma < \kappa_0, \kappa = \sum_{i<\delta} \kappa_i$ and $\mathrm{IND}(\kappa, \sigma)$. If $\lambda > \kappa$ <u>then</u> for every large enough $\varepsilon < \delta$, there are no $\lambda_\alpha \in (\kappa, \lambda) \cap \mathrm{Reg}$ for $\alpha < \kappa_\varepsilon$ such that $\prod_{\alpha<\kappa_\varepsilon} \lambda_\alpha / [\kappa_\varepsilon]^{\le\sigma}$ is λ-directed recalling $[\kappa_\varepsilon]^{\le\sigma} = \{a \subseteq \kappa_\varepsilon : |a| \le \sigma\}$.
2) If $\mathrm{IND}(\kappa)$, $\mathrm{cf}(\kappa) = \aleph_0 = \sigma, \kappa = \sum_{n<\omega} \kappa_n, \kappa_n < \kappa_{n+1}$ <u>then</u> the conclusion of part (1) holds.
3) If $\mathrm{IND}(\kappa, \sigma), \delta = \omega, \kappa_\varepsilon = \kappa$, <u>then</u> the conclusion of (1) holds.

Proof. Check.

Discussion 3.12. Let $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$ and assume $\mathrm{IND}(\bar{J})$.
1) Note that if P is a θ-c.c. forcing notion, each J_ε is θ-complete then for any $\bar{f} = \langle f_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \in [\varepsilon^* \backslash (\varepsilon + 1)]^{<\aleph_0}\rangle \in V^P$ as in Definition 3.6 we can find $\bar{f}' = \langle f'_{\varepsilon,u} : \varepsilon < \varepsilon^*, u \in [\varepsilon^* \backslash (\varepsilon + 1)]^{<\aleph_0}\rangle \in V$ such that for every $\bar{\alpha} \in \prod_{\zeta\in u} \mathrm{Dom}(J_\zeta)$ we have $f_{\varepsilon,u}(\bar{\alpha}) \subseteq f'_{\varepsilon,u}(\bar{\lambda})$, so we can consider only $\bar{f} \in V$. For each such $\bar{f}$ let $A_{\bar{f}} = \{v :$ for some strictly increasing sequence $\bar{\varepsilon} = \langle \varepsilon_n : n < \omega\rangle$ of ordinals $< \varepsilon^*$ and $\bar{\alpha} \in \prod_{n<\omega} \mathrm{Dom}(J_{\varepsilon_n})$ the conclusion $(**)$ of Definition 3.6 holds and $v = \{\varepsilon_\ell : \ell < m\}$ for some $m < \omega\}$.

For $\bar{g}, \bar{f} \in \mathrm{function}(\bar{J})$ where $\bar{J} = \langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle$, let $\bar{g} \le \bar{f}$ iff for every $\varepsilon < \varepsilon^*$ and $u \in [\varepsilon^* \backslash (\varepsilon + 1)]^{<\aleph_0}$ we have $\bar{\alpha} \in \prod_{\varepsilon\in u} \mathrm{Dom}(J_\varepsilon) \Rightarrow g_{\varepsilon,u}(\bar{\alpha}) \subseteq f_{\varepsilon,u}(\bar{\alpha})$.
Clearly $\bar{g} \le \bar{f} \Rightarrow A_{\bar{f}} \subseteq A_g$.
2) In $\mathbf{V}$ we can define a filter D on $[\bigcup_{\varepsilon<\varepsilon^*} \mathrm{Dom}(\bar{J}_\varepsilon)]^{<\aleph_0}$:

$$A \in D \text{ \underline{iff} for some } \bar{f} \in \mathrm{function}(\langle J_\varepsilon : \varepsilon < \varepsilon^*\rangle) \text{ we have } A_f \subseteq A.$$

Now $D \subseteq \mathcal{P}([\bigcup_{\varepsilon<\varepsilon^*} \mathrm{Dom}(J_\varepsilon)]^{<\aleph_0})$ and D is upward closed trivially. Also D is closed under intersection of countable many members if each J_n is $\aleph_1$-closed (similarly σ-closed) because if $A_n \in D$ let $\bar{f}^n \in \mathrm{function}(\bar{J})$ be such that $A_{\bar{f}^n} \subseteq A$. Now for some $g \in \mathrm{function}(J)$ $[n < \omega \Rightarrow \bar{f}^n \subseteq \bar{g}]$, hence $A_g \subseteq A_{\bar{f}^n}$, so $A_{\bar{g}} \subseteq A_n$ for $n < \omega$ and obviously $A_{\bar{g}} \in D$. Lastly $\emptyset \notin D$ as $\mathrm{IND}(\bar{J})$ holds.

Claim 3.13. Suppose for $\alpha < \alpha^*$, $\bar{I}^\alpha = \langle I^\alpha_n : n < \omega\rangle$, $\kappa = \sup\{|\mathrm{Dom}(I^\alpha_n)| : \alpha < \alpha^*\}$, $\mathrm{IND}^+(\langle I^\alpha_n : n < \omega\rangle)$, and:

(∗) if $\alpha < \alpha^*$, $f_n : \mathrm{Dom}(I^\alpha_n) \to \mathrm{Ord}$ then for some $n(*) < \omega$, $\beta < \alpha^*$ and ordinal γ

$$I^\beta_{1+n} \cong I^\alpha_{n(*)+1+n} \restriction \{x \in \mathrm{Dom}(I^\alpha_{n(*)+1+n}) : f_{n(*)+1+n}(x) > \gamma\}$$
$$I^\beta_0 = I^\alpha_{n(*)} \restriction \{x \in \mathrm{Dom}(I^\alpha_{n(*)}) : f_0(x) < \gamma\}.$$

Then for no $\lambda > \kappa$ and $\alpha < \alpha^*$ do we have for every $n < \omega$:

$$x \in \mathrm{Dom}(I^\alpha_n) \Rightarrow \lambda^n_x \in (\kappa, \lambda) \cap \mathrm{Reg} \text{ and } \prod_{x\in \mathrm{Dom}(I^\alpha_n)} \lambda^n_x / I^\alpha_n \text{ is } \lambda^+\text{-directed.}$$

Proof. No new point.

Remark 3.14. 1) This claim is used in the proof of 5.2.
2) If in 3.13, $\alpha^* = 1$ and I^α_{n+1} is λ_n-complete, $\lambda_n > |\mathrm{Dom}(I^\alpha_\ell))|$ for $\ell < n$ then (∗) there holds.

Question 3.15. If $\mathrm{IND}(\lambda, \sigma)$, $\mathrm{cf}(\lambda) = \aleph_0$ do we have $\mathrm{IND}(\langle J^{\mathrm{bd}}_{\lambda_n} : n < \omega\rangle)$ for some $\lambda_n = \mathrm{cf}(\lambda_n) < \lambda, \sigma < \lambda_n$?

Conclusion 3.16. 1) Assume $\mathrm{IND}(\langle J^{\mathrm{bd}}_{\kappa_n} : n < \omega\rangle)$ and $\kappa_n < \kappa_{n+1}, \kappa = \Sigma\{\kappa_n : n < \omega\}$. For any $\lambda > \kappa$ for infinitely many $n < \omega$, λ is $J^{\mathrm{bd}}_{\kappa_n}$-inaccessible.
2) If moreover $\mathrm{IND}^+(\langle J^{\mathrm{bd}}_{\kappa_n} : n < \omega\rangle)$, then the conclusion holds for every n large enough.

Theorem 3.17. *If* $|\mathfrak{a}| < Min(\mathfrak{a})$ *and* $\mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{a}]:\theta \in pcf(\mathfrak{a})\rangle}(\mathfrak{a}) \geq |\mathfrak{a}|^+$, *then* $IND(\langle J^{bd}_\theta : \theta \in \mathfrak{a}\rangle)$.

Proof. Reread the proof of 2.6: let $f_{u,\varepsilon}$ be as in Definition 3.6, so without loss of generality $\mathrm{Rang}(f_{\varepsilon,u}) \subseteq \kappa_\varepsilon$ (that is any value of $f_{\varepsilon,u}$ is an ordinal $< \kappa_\varepsilon$, remembering $\alpha = \{\beta : \beta < \alpha\}$) and $M = (\kappa, \kappa_\varepsilon, f_{u,\varepsilon})_{\varepsilon,u}$. Now repeat the proof of 2.6 or see below. $\square_{3.17}$

Next we improve the ideals from "bounded" to "nonstationary"

Theorem 3.18. *1) Assume* $\lambda > \mathrm{cf}(\lambda), |\mathfrak{a}| < \min(\mathfrak{a})$ *and* $\lambda = \sup(\mathfrak{a})$ *and* $\mathrm{rk}'_{\langle J_{<\theta}[\mathfrak{a}]:\theta\in pcf(\mathfrak{a})\rangle}(\mathfrak{a}) \geq |\mathfrak{a}|^+$ *and* $\sigma^* \in (|\mathfrak{a}|, \min(\mathfrak{a})) \cap \mathrm{Reg}$, *and*

$$I_\theta =: \{S : S \subseteq \theta \text{ and } \{\delta \in S : cf(\delta) = \sigma^*\} \text{ is not stationary}\}$$

then $IND(\langle I_\theta : \theta \in \mathfrak{a}\rangle)$.

2) Moreover for any sequence $\bar{H} = \langle H_\theta : \theta \in \mathfrak{a} \rangle$, *where* $\mathfrak{a} \subseteq \lambda$, H_θ *a function from* $[\lambda]^{<\aleph_0}$ *to* I_θ *and* $\mathfrak{c}$, $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, $|\mathfrak{c}| < \min(\mathfrak{a})$ *we can find* $\bar{\alpha} = \langle \alpha_\tau : \tau \in \mathfrak{a} \rangle \in \Pi\mathfrak{a}$ *such that defining for* $\mathfrak{b} \subseteq \mathfrak{a}$

$$c\ell_{\bar{H},\bar{\alpha}}(\mathfrak{b}) = \{\tau \in \mathfrak{a} : \alpha_\tau \in H_\tau(\bar{\alpha} \restriction \mathfrak{e}) \textit{ for some finite } \mathfrak{e} \subseteq \mathfrak{b}\}$$

we have

$(*)$ $\theta \in \mathfrak{c}$ & $\mathfrak{b} \in J_{<\theta}[\mathfrak{a}] \Rightarrow c\ell_{\bar{H},\bar{\alpha}}(\mathfrak{b}) \in J_{<\theta}[\mathfrak{a}]$.

Proof. 1) We can prove it from part 2) exactly as in the proof of 2.6(1).
2) Let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a generating sequence for $\mathfrak{a}$, without loss of generality $|\mathfrak{c}|^+ < \min(\mathfrak{a})$ and $\max \mathrm{pcf}(\mathfrak{a}) \in \mathfrak{c}$ and

$$[\theta \in \mathfrak{c} \ \& \ \theta \neq \max \mathrm{pcf}(\mathfrak{a}) \Rightarrow \min(\mathrm{pcf}(\mathfrak{a}) \backslash \theta^+) \in \mathfrak{c}].$$

Before we continue, recall that we know:

Fact 3.19. If $|\mathfrak{a}| \leq |\mathfrak{c}| < \min(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathrm{Reg}$, $\mathfrak{a} \subseteq \mathfrak{c} \subseteq \mathrm{pcf}(\mathfrak{a})$, and $\langle \mathfrak{b}_\sigma[\mathfrak{a}] : \sigma \in \mathrm{pcf}(\mathfrak{a})\rangle$ a generating sequence for $\mathfrak{a}$ then

$(*)_\mathfrak{a}$ there is $\langle \bar{f}^\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ such that

- (a) $\bar{f}^\theta = \langle f^\theta_\alpha : \alpha < \theta\rangle$ is $<_{J_{<\theta}[\mathfrak{a}]}$-increasing
- (b) $\bar{f}^\theta$ is cofinal in $(\Pi\mathfrak{a}, <_{J_\theta[\mathfrak{a}]})$ where $J_\theta[\mathfrak{a}] =: J_{<\theta}[\mathfrak{a}] + (\mathfrak{a} \backslash \mathfrak{b}_\theta[\mathfrak{a}])$
- (c) if $\sigma \in \theta \cap \mathfrak{c}$, $\sigma < \theta, \delta < \theta$ and $\mathfrak{b} = \mathfrak{b}_\sigma[\mathfrak{a}]$ <u>then</u> for some $n < \omega$ and $\alpha_\ell < \theta_\ell \leq \sigma$ (for $\ell \leq n$) we have $f^\theta_\delta \restriction \mathfrak{b} = (\max\{f^{\theta_\ell}_{\alpha_\ell} : \ell < n\}) \restriction \mathfrak{b}$ (the max is pointwise)
- (d) if $\delta < \theta \in \mathrm{pcf}(\mathfrak{a})$, $\mathrm{cf}(\delta) \in (|\mathfrak{a}|, \min(\mathfrak{a}))$ <u>then</u> for every $\tau \in \mathfrak{a}$

$$f^\theta_\delta(\tau) = \min\{\cup\{f^\theta_\alpha(\tau) : \alpha \in C\} : C \text{ a club of } \delta\}$$

<u>provided that</u> the function defined satisfies condition (c) above (exist by [Sh:g, Ch. VIII,§1] or we choose by induction on θ).

Let

$$S^{\mathrm{gd}}_\theta = S^{\mathrm{gd}}_{\bar{f}^\theta} =: \Big\{\delta < \theta : \mathrm{cf}(\delta) \in (|\mathfrak{a}|, \min(\mathfrak{a})),\ f^\theta_\delta \text{ is a } <_{J^{\mathrm{bd}}_{<\theta}[\mathfrak{a}]}\text{-eub of } \bar{f}^\theta \restriction \delta \text{ and }$$
$$\{\tau \in \mathfrak{a} : \mathrm{cf}(f^\theta_\alpha(\tau)) = \mathrm{cf}(\delta)\} = \mathfrak{b}_\theta[\mathfrak{a}] \bmod J_{<\theta}[\mathfrak{a}]\Big\}$$

(alternatively use simultaneous witnesses for $I[\theta]$ as in [Sh 420, §1].
Note:

$(*)_\mathfrak{a}$ (e) if E_τ is a club of τ for $\tau \in \mathfrak{a}$ and $\theta \in \mathrm{pcf}(\mathfrak{a})$ <u>then</u> for some club E of θ:

$$\delta \in E \cap S^{\mathrm{gd}}_\theta \Rightarrow \{\tau \in \mathfrak{a} : f^\theta_\delta(\tau) \in E_\tau\} = \mathfrak{a} \bmod J_\theta[\mathfrak{a}]$$

(f) if $\sigma^* \in (|\mathfrak{c}|, \mathrm{Min}(\mathfrak{a}))$ and $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ for $i \leq \delta^*$ is increasing continuous, $\langle N_j : j \leq i\rangle \in N_{i+1}$, $\|N_i\| = \sigma^*$, $\sigma^* + 1 \subseteq N_i$, $\langle \bar{f}^\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle \in N_i$, $\delta^* < (\sigma^*)^+$ has cofinality $> |\mathfrak{a}|$ and $\{\mathfrak{a}, \mathfrak{c}\} \in N_i$ <u>then</u> : $\theta \in \mathfrak{c}$ implies $\sup (N_\delta \cap \theta) \in S^{\mathrm{gd}}_\theta$ and $\{\sigma \in \mathfrak{a} : f_{\sup(N_\delta \cap \theta)}(\sigma) = \sup(N_\delta \cap \sigma)\} = \mathfrak{a} \bmod J_\theta[\mathfrak{a}]$.

Why? See [Sh:g, ChVIII,1.2,1.4].

Let $\bar{H} = \langle H_\tau : \tau \in \mathfrak{a} \rangle$ be as in the claim, so H_τ is a function from $[\lambda]^{<\aleph_0}$ to I_τ. Now for every $\theta \in \mathrm{pcf}(\mathfrak{a})$ and $\alpha < \theta$ and $\tau \in \mathfrak{a}$ we define $A^{\theta,\tau}_\alpha = A^\tau(f^\theta_\alpha) \in I_\tau$ as

$$\bigcup\{H_\tau(u) : u \subseteq \mathrm{Rang}(f^\theta_\alpha) \text{ is finite}\}$$

(as I_τ is τ-complete, $\tau > |\mathfrak{a}| = |\{u : u \subseteq \mathrm{Rang}\,(f^\theta_\alpha) \text{ finite}\}|$, really $A^\theta_\alpha \in I_\tau$).

Now for each $\alpha < \theta \in \mathrm{pcf}(\mathfrak{a})$ and $\sigma \in \mathrm{pcf}(\mathfrak{a})$ by $(*)_\mathfrak{a}$ (e) applied with σ, $\langle A^{\theta,\tau}_\alpha : \tau \in \mathfrak{a} \rangle$ here standing for θ, $\langle E_\tau : \tau \in \mathfrak{a} \rangle$ there, we get a club $C_{\theta,\sigma,\alpha}$ of σ.

For each $\sigma \in \mathrm{pcf}(\mathfrak{a})$ let $C_\sigma = \bigcap_{\theta \in \mathfrak{c} \cap \sigma, \alpha < \theta} C_{\theta,\sigma,\alpha}$ (note: $|\mathfrak{c} \cap \sigma| < \sigma$), so C_σ is a club of σ. Lastly let $\sigma^* = \mathrm{cf}(\sigma^*) \in (|\mathfrak{c}|, \min(\mathfrak{a}))$ and $\langle N_i : i \leq \sigma^* \rangle$ be as in clause (f) of $(*)_\mathfrak{a}$, so

$\oplus_1$ $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is increasing continuous, for $i \leq \sigma^*$ we have $\|N_i\| = \sigma^*$, $\langle N_j : j \leq i \rangle \in N_{i+1}$.

Let $\alpha_\theta =: \sup(N_{\sigma^*} \cap \theta)$ for $\theta \in \mathfrak{c}$; so $\alpha_\theta \in S^{\mathrm{gd}}_\theta$ for every $\theta \in \mathfrak{c}$, and we shall show that $\bar{\alpha} = \langle \alpha_\tau : \tau \in \mathfrak{a} \rangle$ is as required.

For each $\theta \in (N_{\sigma^*} \cap \mathrm{pcf}(\mathfrak{a}))$, $\sigma \in \mathfrak{c}$, $\sigma < \theta$ we have: $\alpha_\theta \in C_\theta$ hence $\alpha_\theta \in C_{\sigma,\theta,\alpha_\sigma}$ hence

$$\{\tau \in \mathfrak{a} : f^\sigma_{\alpha_\sigma}(\tau) \in \tau \backslash A^{\sigma,\tau}_{\alpha_\sigma}\} = \mathfrak{b}_\tau[\mathfrak{a}] \bmod J_\sigma[\mathfrak{a}].$$

The rest should be clear. $\square_{3.18}$

We next point out another connection; if the rank is small and $|\mathrm{pcf}(\mathfrak{a})|$ is large, then we have a case of "$\Pi\mathfrak{d} / \mathcal{S}_{\leq\lambda}(\mathfrak{d})$ has large true cofinality".

Claim 3.20. If $\zeta > \mathrm{rk}'(\mathfrak{a}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle)$ and $\bar{\lambda} = \langle \lambda_\varepsilon : \varepsilon \leq \zeta \rangle$ is strictly increasing and $|\mathrm{pcf}(\mathfrak{a})| \geq \lambda_\zeta$ <u>then</u> for some $\varepsilon < \zeta$ and $\mathfrak{c} \subseteq \mathfrak{a}$ we have $|\mathrm{pcf}(\mathfrak{c})| \geq \lambda_{\varepsilon+1}$ and $\mathfrak{d} \in J_*[\mathrm{pcf}(\mathfrak{c})] \Rightarrow \Pi\mathfrak{d} / [\mathfrak{d}]^{\leq \lambda_\varepsilon}$ has the true cofinality which is max pcf$(\mathfrak{c})$.

Proof. If not, prove by induction on $\varepsilon \leq \zeta$ that

$(*)$ if $\mathfrak{c} \subseteq \mathfrak{a}$, $\varepsilon = \mathrm{rk}'(\mathfrak{c}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle)$
<u>then</u> $|\mathrm{pcf}(\mathfrak{c})| < \lambda_{\varepsilon+1}$.

Let $J_0 = \{\mathfrak{d} : \mathfrak{d} \subseteq \mathfrak{c}, \varepsilon > \mathrm{rk}'(\mathfrak{d}, \langle J_{<\theta}[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle)\}$. By the induction hypothesis $[\mathfrak{d} \in J_0 \Rightarrow |\mathrm{pcf}(\mathfrak{d})| \leq \lambda_\varepsilon]$. Let J be the ideal on $\mathfrak{c}$ which J_0 generates. We have: $[\mathfrak{d} \in J \Rightarrow |\mathrm{pcf}(\mathfrak{d})| \leq \lambda_\varepsilon]$, so by the assumption toward contradiction $\mathfrak{c} \notin J$. Let $\theta = \max \mathrm{pcf}(\mathfrak{c})$. So by the definition of rk$'$ we have $J_{<\theta}[\mathfrak{c}] \subseteq J$. Hence (see [Sh:g, Ch. VIII,§3] or [Sh:E11]) for some $\mathfrak{d} \subseteq \mathrm{pcf}(\mathfrak{c})$, $\mathfrak{d} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ we have $\Pi\mathfrak{d} / J^*_{<\theta}[\mathrm{pcf}(\mathfrak{c})]$ has the true cofinality θ and $J_*[\mathrm{pcf}(\mathfrak{a})]$ is generated by $\{\mathrm{pcf}(\mathfrak{b}) : \mathfrak{b} \in J_{<\theta}[\mathfrak{c}]\}$. So the conclusion holds. $\square_{3.20}$

4. Sticks and BA's

Lemma 4.1. *Assume $\theta \leq \mu < \lambda \leq \lambda^*$, J an ideal on θ and assume*

$\otimes^J_{\theta,\mu,\lambda,\lambda^*}$ *if $n < \omega$, $\mathfrak{a}_i \in [\mathrm{Reg} \cap \lambda^+ \backslash \mu^+]^n$ for $i < \theta$ <u>then</u>*
$\{a \in J : \max \mathrm{pcf}\,(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*\}$ *is generated by $\leq \mu$ sets.*[1]

[1] See 4.2(3); we can use $\mathfrak{a}_i$ singletons.

Then there is a set H such that

(a) H a set of partial functions from θ to $[\lambda]^{\leq\mu}$
(b) $|H| \leq \lambda^*$
(c) for every function $g : \theta \to \lambda$ we can find h and $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i < \theta\rangle$ such that
(i) $\mathfrak{a}_i$ is a finite set of regular cardinals from $(\mu, \lambda]$
(ii) h is a function from θ to $[\lambda]^{\leq\mu}$ such that $i < \mu \Rightarrow g(i) \in h(i)$
(iii) for any $n < \omega$ and $a \in J$:
if $(\forall i \in a)[|\mathfrak{a}_i| \leq n]$ and $\max \mathrm{pcf}(\bigcup_{i\in a} \mathfrak{a}_i) \leq \lambda^*$ then for some b satisfying $a \subseteq b \subseteq \theta$ we have $h \restriction b \in H$.

Proof. Like [Sh 430, §2].

Remark 4.2. 1) But we can then change the bound (in clause (c) (ii)) to $h(i) \in [\lambda]^{<\mu}$. Then $\otimes_{\theta,\mu,\lambda,\lambda^*}$ is changed to

$$\otimes'_{J,\theta,\mu,\lambda,\lambda^*} \text{ if } n < \omega,\ \mathfrak{a}_i \in [\mathrm{Reg} \cap \lambda^+]^n \text{ for } i < \theta \text{ then } \{a \in J : \text{for some } \mu_0 < \mu,\ \max \mathrm{pcf}(\bigcup_{i\in a} \mathfrak{a}_i \setminus \mu_0^+) \leq \lambda^*\} \text{ is generated by } < \mu \text{ sets.}$$

2) We can weaken $\otimes$ to

$$\otimes^-_{J,\theta,\mu,\lambda,\lambda^*} \text{ if } n < \omega,\ \mathfrak{a}_i \in [\mathrm{Reg} \cap \lambda^+]^n \text{ for } i < \theta \text{ then } \{a \in J : \max \mathrm{pcf}(\bigcup_{i<a} \mathfrak{a}_i) \leq \lambda^*\} \text{ is generated (as an ideal) by some } \mathcal{P} \subseteq J \text{ such that}$$

$$\kappa \in \bigcup_{i<\theta} \mathfrak{a}_i \Rightarrow \kappa > \left|\left\{a \in \mathcal{P} : \bigvee_{i\in a} \kappa \in \mathfrak{a}_i\right\}\right|.$$

3) Instead $\otimes^J_{\theta,\mu,\lambda,\lambda^*}$ in 4.1 we can define a game

$\otimes_{\theta,\mu,\lambda,\lambda^*}[\mathfrak{D}]$	First player has no winning strategy in the game defined below
$GM'_{\theta,\mu,\lambda,\lambda^*}[\mathfrak{D}]$	The play lasts ω moves, in the n-th move:

first player chooses $\bar{\lambda}^n = \langle \lambda^n_i : i \in A_n\rangle$, $A_0 = \theta$, $\bigwedge_{m<n} A_m \subseteq A_n$, $\bigwedge_{m<n} \bigwedge_{i\in A_n} \lambda^n_i < \lambda^m_i$, $\lambda^n_i = \mathrm{cf}(\lambda^n_i) \in (\mu, \lambda]$ and
second player chooses an ideal J_n on A_n, $J_n \subseteq \{a \subseteq A_n : \max \mathrm{pcf}\{\lambda^n_i : i \in a\} \leq \lambda^*\}$, J_n generated by $\leq \mu$ sets.
In the end (clearly $\bigcap_{n<\omega} A_n = \emptyset$) they produce the ideal J, the one generated by $\{a \subseteq \theta : \text{for some } n, a \subseteq A_n \setminus A_{n+1} \text{and } a \in J_n\}$.
Second player wins if $J \in \mathfrak{D}$.

Definition 4.3. *Assume $\bar{J} = \langle J_\ell : \ell < 3\rangle$, where $J_0 \subseteq J_1 \subseteq J_2 \subseteq \mathcal{P}(\theta)$, each J_ℓ is downward closed (usually is an ideal); we let $J^+_\ell =: \mathcal{P}(\theta) \setminus J_\ell$.*

1) $\mathrm{dcf}_{\bar{J}}(\lambda, <\mu) = \min\Big\{|\mathcal{F}| : \mathcal{F}$ is a family of functions each with domain from J_1^+ and range included in $[\lambda]^{<(1+\mu)}$ such that:
$(*)_{\mathcal{F}}$ for every $b \in J_2^+$ and $f \in {}^b\lambda$ for some $a \in J_1^+$ and $g \in \mathcal{F} \cap {}^a\lambda$ we have $(\forall^{J_0} i \in a)$ $(i \in b \;\&\; g(i) \in f(i))$
i.e. $\{i : i \in a$ and $i \notin b \vee g(i) \notin f(i)\} \in J_0\Big\}$

2) $\mathrm{ecf}_{\bar{J}}(\lambda, < \mu) = \min\Big\{|\mathcal{F}| : \mathcal{F}$ is a family of functions each with domain from J_1^+ and range included in $[\lambda]^{<1+\mu}$ such that:
$(*)_{\mathcal{F}}$ for every $b \in J_2^+$ and $f \in {}^b\lambda$ for some $a \in J_1^+$ and $g \in \mathcal{F}$ we have $(\exists^{J_0} i \in a)(i \in b \;\&\; g(i) \in f(i))$
i.e. $\{i : i \in a, i \in b$ and $g(i) \in f(i)\} \notin J_0\Big\}$

3) Let x be d or e. Now $\mathrm{xcf}_{\bar{J}}(\lambda, < \mu^+)$ is written $\mathrm{xcf}_{\bar{J}}(\lambda, \leq \mu)$ or $\mathrm{xcf}_{\bar{J}}(\lambda, \mu)$. Also $\mathrm{xcf}_{\bar{J}}(\lambda, 1)$ is written $\mathrm{xcf}_{\bar{J}}(\lambda)$. Also $\mathrm{xcf}_{\langle [\lambda]^{<\sigma}, [\lambda]^{<\theta}, [\lambda]^{<\chi}\rangle}(\lambda, < \mu)$ is written $\mathrm{xcf}_{\chi,\theta,\sigma}(\lambda, < \mu)$, etc.

Definition 4.4. $\P_{\lambda,\mu,\theta}$ =: $\min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\theta$ *is such that for every* $A \in [\lambda]^\mu$ *for some* $a \in \mathcal{P}$ *we have* $a \subseteq A\}$. *If* $\mu = \lambda$ *we may omit it. Let* $\P_\lambda$ *mean* $\P_{\lambda,\aleph_0}$ *and* $\P = \P_{\aleph_1}$.

Claim 4.5. 1) $\P_{\lambda,\mu,\theta} = \mathrm{dcf}_{\langle \{\emptyset\}, [\lambda]^{<\theta}, [\lambda]^{<\mu}\rangle}(\lambda)$ when $\theta \leq \mu \leq \lambda$.
2) $\mathrm{ecf}_{\mu,\theta,\sigma}(\lambda, \theta) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^\theta$ and for every $A \in [\lambda]^\mu$ for some $a \in \mathcal{P}$ we have $|A \cap a| \geq \sigma\}$.

Proof. Read the definitions. $\square_{4.5}$

Now we can phrase the analog of 4.1 for dcf.

Lemma 4.6. *Assume* $\theta < \mu < \lambda \leq \lambda^*$, $\bar{J} = \langle J_0, J_1, J_2\rangle$ *is an increasing sequence of ideals on* θ *and assume*

$\otimes^J_{\theta,\mu,\lambda,\lambda^*}$ *if* $n < \omega$, $\mathfrak{a}_i \in [\mathrm{Reg} \cap \lambda^+ \backslash \mu^+]^n$ *for* $i < \theta$ *then* $\{a/J_\kappa : a \in J_2$ *and* $\max \mathrm{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*\} \subseteq \mathcal{P}(\theta)/J_2$ *has a dense subset of cardinality* $\leq \mu$.[2]

<u>Then</u> *there is a set* H *such that*

[2] See 4.2(3).

(*a*) *H a st of partial functions from* θ *to* $[\lambda]^{\leq\mu}$
(*b*) $|H| \leq \lambda^*$
(*c*) *for every function* $g : \theta \to \lambda$ *we can find* h *and* $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i < \theta \rangle$ *such that*
  (*i*) $\mathfrak{a}_i$ *is a finite set of regular cardinals from* $(\mu, \lambda]$
  (*ii*) h *is a function from* θ *to* $[\lambda]^{\leq\mu}$ *such that* $i < \theta \Rightarrow g(i) \in h(i)$
  (*iii*) *for any* $n < \omega$ *and* $a \in J_2^+$*:*
  if $(\forall i \in a)[\mathfrak{a}_i| \leq n]$ *and* $\max \mathrm{pcf}(\bigcup_{i \in a} \mathfrak{a}_i) \leq \lambda^*$ *then for some* $b \in J_1^+$ *such that* $a \subseteq b$ *mod* J *we have* $h \restriction b \in H$.

Claim 4.7. Assume:

$(*)_0$ $\aleph_0 < \aleph_{\alpha(*)} \leq \text{⫯}$
$(*)_1$ $\aleph_{\alpha(*)} < \aleph_{\omega_2}$ or at least

$$\mathrm{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \text{⫯}$$

$(*)_2$ $\mathfrak{a} \subseteq \mathrm{Reg} \cap \text{⫯} \backslash \aleph_{\alpha(*)+1}$ & $|\mathfrak{a}| \leq \aleph_0 \Rightarrow |\mathrm{pcf}(\mathfrak{a})| \leq \aleph_{\alpha(*)}$
$(*)_3$ if $\lambda_i \in (\aleph_1, \text{⫯}) \cap \mathrm{Reg}$ for $i < \omega_1$ then for some $a \in [\omega_1]^{\aleph_1}$, for every $b \in [a]^{\aleph_0}$ we have $\max \mathrm{pcf}(\{\lambda_i : i \in b\}) \leq \text{⫯}$.

Then $\text{⫯}_{\text{⫯}} = \text{⫯}$.

Remark 4.8. 1) This means that the conclusion holds except when some dubious statements on pcf, ones which have high consistency strength (or are inconsistent) and ⫯ is somewhat large.

2) There are obvious monotonicity properties and $\mathrm{ecf}_{\bar{J}}(\lambda, < \mu) \leq \mathrm{dcf}_{\bar{J}}(\lambda, < \mu)$.

Proof. Let $\theta = \aleph_1, \mu = \aleph_{\alpha(*)}, \lambda = \text{⫯}, \lambda^* = \text{⫯}, J = [\omega_1]^{\leq\aleph_0}$. Apply 4.1. The assumption $\otimes^J_{\theta,\mu,\lambda,\lambda} = \otimes^J_{\aleph_1,\aleph_{\alpha(*)},\text{⫯},\text{⫯}}$ holds by $(*)_2$. So let $H \subseteq \{h : h$ is a function from $\aleph_1$ to $[\text{⫯}]^{\mu}\}$, $|H| = \lambda^* = \lambda$ be as in the conclusion there. Let $\chi = \beth_7^+$, $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, $|\mathfrak{B}| = \lambda$, $\lambda + 1 \subseteq \mathfrak{B}$, $H \in \mathfrak{B}$. We want to show $\mathcal{P} = \mathfrak{B} \cap [\lambda]^{\aleph_0}$ exemplifies $\text{⫯}_\lambda = \lambda$. So assume $g : \theta \to \lambda$, hence there are h, $\langle\langle \lambda_i^n : n < n_i \rangle : i < \omega_1\rangle$, as there. Let n^* be such that $B_0 =: \{i < \omega_1 : n_i = n^*\}$ is uncountable. By using n times $(*)_3$ we can find an uncountable $B \subseteq B_0 (\subseteq \omega_1)$ such that

$(*)$ $a \subseteq B$ & $a \in J \Rightarrow \max \mathrm{pcf}\{\lambda_i^n : n < n^*, i \in a\} \leq \lambda$.

So for every $a \in [B]^{\aleph_0}$, for some $b \in [\omega_1]^{\aleph_0}$ we have $a \subseteq b$ and $h \restriction b \in H \subseteq \mathfrak{B}$.

Let for a set $b \in \mathcal{H}(\chi)$ of ordinals, f_b be the $<^*_\chi$-first one- to-one function from $|b|$ onto b. Let $g'(i) = f^{-1}_{h(i)}(g(i))$, so g' is a function from $\theta = \aleph_1$ to $\mu = \aleph_{\alpha(*)}$ (as $|h(i)| \leq \mu$). Now $\mathrm{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \text{⫯}$ so $\mathrm{cov}(\aleph_{\alpha(*)}, \aleph_2, \aleph_2, \aleph_1) \leq \lambda$ (the only property of $\alpha(*)$ we use) so there is $\mathcal{P}' \subseteq [\aleph_{\alpha(*)}]^{\leq\aleph_1}$, $|\mathcal{P}'| \leq \lambda$ exemplifying this and without loss of generality $\mathcal{P}' \subseteq \mathfrak{B}$ & $\mathcal{P}' \in \mathfrak{B}$. So the set $\{g'(i) : i \in B\}$ is included in a countable union of members of $\mathcal{P}'$, so for some $Y \in \mathcal{P}'$ (so $Y \in [\aleph_{\alpha(*)}]^{\leq\aleph_1}$ and $Y \in \mathfrak{B}$) we have $B^* =: \{i \in B : g'(i) \in Y\}$ is uncountable.

Define h':

$$\mathrm{Dom}(h') = \omega_1, h'(i) = \{\alpha \in h(i) : f_{h(i)}(\alpha) \in Y\}.$$

So h' is a function from ω_1 to $[\lambda]^{\leq\aleph_1}$ (as $|Y| \leq \aleph_1$) and $i \in B^* \Rightarrow g(i) \in h'(i)$; remember $a \in [B^*]^{\leq\aleph_0} \Rightarrow (\exists b)[a \subseteq b \subseteq \omega_1 \ \&\ h \restriction b \in \mathfrak{B}]$.

Let $Z =: \{(i, f^{-1}_{h'(i)}(g(i))) : i \in B\}$, it is a subset of $\omega_1 \times \omega_1$ of cardinality λ, but ⫯ $= \lambda, \lambda + 1 \subseteq \mathfrak{B}$, so for some infinite $z \in \mathfrak{B}$ we have $z \subseteq Z$. Let $z_0 = \{i < \omega_1 : \bigvee_j (i, j) \in z\}$, so $z_0 \in \mathfrak{B}$, $z_0 \in [\omega_1]^{\aleph_0}$ and even $z_0 \subseteq B^*$, hence $h' \restriction z_0 \in \mathfrak{B}$. So as $h' \restriction z_0 \in \mathfrak{B}$ and $\{(i, f^{-1}_{h'(i)}(g(i))) : i \in z_0\} = z \in \mathfrak{B}$ also $g \restriction z_0 \in \mathfrak{B}$, so $\mathrm{Rang}(g \restriction z_0) \in B$, so $\mathrm{Rang}(g \restriction z_0) \in \mathcal{P}$ and we are done. $\square_{4.7}$

Definition 4.9.

$$St^3_{\lambda,\kappa} = \min\Big\{|\mathcal{P}| : \begin{array}{ll} (a) & \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) & (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \textit{ infinite}) \\ (c) & \mathcal{P} \textit{ is } AD \textit{ which means } a \neq b \in \mathcal{P} \Rightarrow a \cap b \textit{ finite} \end{array}\Big\}$$

(the main case $\kappa = \aleph_1$).

Definition 4.10.

$$St^4_{\lambda,\kappa} = \min\Big\{|\mathcal{P}| : \begin{array}{ll} (a) & \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) & (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \textit{ infinite}) \\ (c) & \sup\{otp(a) : a \in \mathcal{P}\} < \omega_1 \end{array}\Big\}.$$

Definition 4.11.

$$St^5_{\lambda,\kappa} = \min\Big\{\mathcal{P} : \begin{array}{ll} (a) & \mathcal{P} \subseteq [\lambda]^{\aleph_0} \\ (b) & (\forall A \in [\lambda]^\kappa)(\exists b \in \mathcal{P})(b \cap A \textit{ infinite}) \\ (c) & \textit{the } BA \textit{ of subsets of } \lambda \textit{ which } \mathcal{P} \textit{ and the singletons} \\ & \quad \textit{generate is superatomic of rank } < \omega_1 \end{array}\Big\}.$$

Fact 4.12. $\mathrm{dcf}_{\kappa,\aleph_0,\aleph_0}(\lambda) \leq St^\ell_{\lambda,\kappa}$.

* * *

Claim 4.13. 1) Given $\lambda \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$, the following cardinals are equal for $k < \omega, k > 0$:

(a) $\mathrm{ecf}_{\kappa,\aleph_0,\aleph_0}(\lambda)$

$(b)_k$ $\min\{|\mathcal{F}|$:

(i) $\mathcal{F}$ is a family of partial functions f from λ to k

(ii) $f \in \mathcal{F} \Rightarrow |\mathrm{Dom}(f)| = \aleph_0$,

(iii) $f \in \mathcal{F}, \ell < k \Rightarrow f^{-1}(\{\ell\})$ is infinite

(iv) if $\langle A_0, \ldots, A_{k-1}\rangle$ are pairwise disjoint subsets of λ each of cardinality κ <u>then</u> for some $f \in \mathcal{F}$ we have $\ell < k \Rightarrow f^{-1}(\{\ell\}) \cap A_\ell$ is infinite$\}$

$(c)_k$ like $(b)_k$ replacing (iii), (iv) by

$(iii)^+$ if $\langle \alpha_{\varepsilon,\ell} : \varepsilon < \kappa \text{ and } \ell < k\rangle$ is a sequence of ordinals, with no repetitions then for infinitely many $\varepsilon < \kappa$, for each $\ell < k$, $f(\alpha_{\varepsilon,\ell}) = \ell$ (so $\alpha_{\varepsilon,\ell} \in \mathrm{Dom}(f)$).

Proof. Let $\lambda^b_\kappa, \lambda^c_k$ be the cardinal from $(b)_k$, $(c)_k$ respectively and λ^* the cardinal from (a). Clearly $\lambda^b_k \le \lambda^b_{k+1}, \lambda^c_k \le \lambda^c_{k+1}, \lambda^b_k \le \lambda^c_k, \lambda^c_1 = \lambda^b_1 = \lambda^*$. So it suffices to prove $\lambda^c_k \le \lambda^*$, assume $\mathcal{P}$ exemplifies $\lambda^* = \mathrm{ecf}_{\kappa,\aleph_0,\aleph_0}(\lambda)$ as phrased in 4.5 (2). If $\lambda^* \ge 2^{\aleph_0}$ let $\mathcal{F}^* = \{f :$ for some $a \in \mathcal{P}$, f is a function from a to k such that $\ell < k \Rightarrow |f^{-1}(\{\ell\})| = \aleph_0\}$ clearly it exemplifies $\lambda^c_k \le |\mathcal{F}^*| = 2^{\aleph_0} \times \lambda^* = \lambda^*$. So assume $\lambda^* < 2^{\aleph_0}$ and let $\bar\eta = \langle \eta_i : i < \lambda^*\rangle$ be a sequence of pairwise distinct members of ${}^\omega 2$. Let $\bar g = \langle g_\ell : \ell < k\rangle$ be such that: $g_\ell : \lambda \to \lambda$ and $(\forall \alpha_0 \dots \alpha_{k-1} < \lambda)(\exists \beta < \lambda)[\bigwedge_{\ell<k} g_\ell(\beta) = \alpha_\ell]$. Let $\mathcal{P}' = \{a^{\bar g} : a \in \mathcal{P}\}$ where $a^{\bar g} = \{g_\ell(x) : \ell < k, x \in a\}$ and let $\mathcal{F} = \{f_{b,h} : b \in \mathcal{P}$ and for some n we have $h \in {}^{({}^n 2)}k$ and $\ell < k \Rightarrow |f^{-1}_{b,h}(\{\ell\})| = \aleph_0\}$ where $f_{b,h}$ is the function with domain $b^{\bar g}$ and $f_{b,h}(i) = h(\eta_i \restriction n)$ where $h \in {}^{({}^n 2)}k$.

Clearly $\mathcal{F}$ has the right cardinality and form. Let us show that it satisfies the main requirement: let $\langle A_0, \dots, A_{k-1}\rangle$ be a sequence of subsets of λ each of cardinality κ. Let $A_\ell = \{\gamma_{\ell,\varepsilon} : \varepsilon < \kappa\}$ (no repetition). Let $\gamma_\varepsilon < \lambda$ be such that $\bigwedge_{\ell<k} g_\ell(\gamma_\varepsilon) = \gamma_{\ell,\varepsilon}$. For each ε for some $n(\varepsilon)$ we have: $\langle \eta_{\gamma_{\ell,\varepsilon}} \restriction n(\varepsilon) : \ell < k\rangle$ is with no repetitions. As $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ without loss of generality for some $\bar\nu = \langle \nu_\ell : \ell < k\rangle$ and $n(*)$ we have $\varepsilon < \kappa \Rightarrow \eta_{\gamma_{\ell,\varepsilon}} \restriction n(*) = \nu_\ell$ and $\varepsilon < \kappa \Rightarrow n(\varepsilon) = n(*)$. Now by the choice of $\mathcal{P}$ for some $a \in \mathcal{P}$, $W = \{\varepsilon : \gamma_\varepsilon \in a\}$ is infinite. Let $b = a^{\bar g}$, let $h : {}^{n(*)}2 \to k$ be such that $h(\nu_\ell) = \ell$, now $f_{b,h} \in \mathcal{F}$ is as required. $\square_{4.13}$

Claim 4.14. Assume

$(*)_1$ $\daleth_{\lambda,\kappa,\aleph_0} = \lambda$ (e.g. $\lambda = \Sigma\lambda_n$ where $\lambda_{n+1} \ge \mathrm{dcf}_{\kappa,\aleph_0,\aleph_0}(\lambda_n, \aleph_0)$ for $n < \omega$, $\lambda_0 \ge \kappa$) and $\kappa = \mathrm{cf}(\kappa) > \aleph_0$.

<u>Then</u> there is a Boolean Algebra B of cardinality λ into which the free Boolean algebra generated by λ elements can be embedded <u>but</u> such that there is no homomorphism from B onto the free Boolean Algebra generated by κ elements.

Remark 4.15. 1) So we can find quite many pairs $\lambda = \sum_{n<\omega} \lambda_n$ and κ as required in 4.14 (or 4.15). E.g. any $\lambda = \lambda_n > \beth_\omega$ and $\kappa \in [\beth_\omega, \lambda] \cap \mathrm{Reg}$ is large enough $\kappa = \mathrm{cf}(\kappa) \le \beth_\omega$ is as required by [Sh 460].
2) On the problem see Fuchino, Shelah, Soukup [FShS 543], the proof is similar.
3) From the proof we can strengthen the last phrase in the conclusion to "no homomorphism from B into $Fr(\kappa)$ with range of cardinality κ". Similarly in 4.17.

Proof. Let $\mathcal{F}_n = \{f^n_\alpha : \alpha < \lambda\}$ be as guaranteed by clause $(b)_2$ of 4.13. Without loss of generality $\alpha < \lambda \wedge \ell < 2 \Rightarrow (\exists^{\aleph_0} i)(f^n_\alpha(i) = \ell)$. So let $\mathrm{Dom}(f^n_\alpha) = \{j^n_{\alpha,k,\ell}, \ell < 2, k < \omega\}$ with no repetitions such that $f^n_\alpha(j^n_{\alpha,k,\ell}) = \ell$.
Remember the variety of Boolean rings has the operations $x \cup y$, $x \cap y$, $x - y$ and constant 0 (but no 1 and no $-x$), so any ideal of a Boolean algebra is a Boolean

ring and if the ideal is maximal, the Boolean algebra is definable in the Boolean ring.

Let B_0 be the Boolean ring freely generated by $\{x^0_i : i < \lambda\}$. Let B_1 be the Boolean ring generated by $B_0 \cup \{x^1_i : i < \lambda\}$ freely except:

(a) the equations which holds in B_0
(b) $x^1_\alpha \cap \sigma^{0,0}_{\alpha,k} = 0$ where $\sigma^{0,0}_{\alpha,k} = x^0_{j^n_{\alpha,k,0}} - \bigcup_{m<k} x^0_{j^n_{\alpha,m,1}}$
(c) $x^1_\alpha \cap \sigma^{0,1}_{\alpha,k} = \sigma^{0,1}_{\alpha,k}$ where $\sigma^{0,1}_{\alpha,k} = x^0_{j^n_{\alpha,k,1}} - \bigcup_{m<k} x^0_{j^n_{\alpha,m,0}}$.

Similarly let B_{n+1} be the Boolean ring generated by $B_n \cup \{x^{n+1}_\alpha : \alpha < \lambda\}$ freely except

(a) the equations which holds in B_n
(b) $x^{n+1}_{2\alpha} \cap \sigma^{n,0}_{\alpha,k} = 0$ where $\sigma^{n,0}_{\alpha,k} = x^n_{\beta^n_{\alpha,k,0}} - \bigcup_{m<k} x^n_{\beta^n_{\beta,m,1}}$
(c) $x^{n+1}_\alpha \cap \sigma^{n,0}_{\alpha,k} = \sigma^{n,0}_{\alpha,k}$ where $\sigma^{n,0}_{\alpha,k} = x^n_{j^n_{\alpha,k,1}} - \bigcup_{m<k} x^n_{j^n_{\alpha,m,0}}$

Now $B_\omega = \bigcup_{n<\omega} B_n$ is a Boolean ring and let B be the Boolean algebra for which B_ω is a maximal ideal. Assume f is a homomorphism from B onto $Fr(\kappa)$, the Boolean algebra freely generated say by $\{z_i : i < \kappa\}$. Now B is generated by $\{x^n_\alpha : n < \omega, \alpha < \lambda_n\}$. So as f is onto, for some n, for every $\zeta < \kappa$ for some α, $f(x^n_\alpha) \notin \langle z_\varepsilon : \varepsilon < \zeta\rangle_{Fr(\kappa)}$. By the Δ-system lemma, we can find a stationary $S \subseteq \kappa$, Boolean term $\sigma_1 = \sigma_1(x_0, \dots, x_{n(*)-1}), m(*) < n(*)$, ordinals $\varepsilon(0) < \dots < \varepsilon(m(*) - 1) < \min(S)$, and for each $\zeta \in S$, ordinals $\varepsilon(m(*), \zeta) < \dots < \varepsilon(n(*) - 1, \zeta)$ all in the interval $[\zeta, \min(S \backslash (\zeta + 1))]$ and $\alpha_\zeta < \lambda_n$ such that $f(x^n_{\alpha_\zeta}) = \sigma_1(z_{\varepsilon(0)}, \dots, z_{\varepsilon(m(*)-1)}, z_{\varepsilon(m(*),\zeta)}, \dots, z_{\varepsilon(n(*)-1,\zeta)})$, where all the $n(*)$ variables are needed in the term σ_1.

Let $S = \{\zeta(i) : i < \kappa\}$ with $\zeta(i)$ increasing in i, let h be a homomorphism from $\mathrm{Fr}(\kappa)$ to the two element Boolean Algebra such that $h(f(x^n_{\alpha_{2\zeta+\ell}})) = \ell$ for $\ell = 0, 1$ (exist as $\langle f(x^n_{\alpha_\zeta}) : \zeta < \omega\rangle$ is independent. Let $g = h \circ f$ and f^n_α as guaranteed by $(*)_1$ for g and $A_\ell = \{\alpha_{2\zeta+\ell} : \zeta < \kappa\}$. So for $\ell = 0, 1$ the sets $W_\ell =: \{\alpha_{\zeta(2i+\ell)} : \alpha_{\zeta(2i+\ell)} \in \mathrm{Dom}(f^n_\alpha)\}$ are infinite and $f^n_\alpha \restriction W_\ell$ is constantly ℓ. So $z^* = f(x^{n+1}_\alpha) \in Fr(\kappa)$ satisfies

$(*)_0$ $k < \omega_0 \Rightarrow Fr(\kappa) \models z^* \cap f(\sigma^{n,0}_{\alpha,k}) = 0$
$(*)_1$ $k < \omega_1 \Rightarrow Fr(\kappa) \models z^* \cap f(\sigma^{n,1}_{\alpha,k}) = f(\sigma^{n,1}_{\alpha,k})$.

But for some finite $u \subseteq \kappa$, $z^* \in \langle z_\gamma : \gamma \in u\rangle_{\mathrm{Fr}(\kappa)}$, so there is $\alpha_{\zeta(2i_0)} \in W_0$, such that u is disjoint to $\{\varepsilon(m(*), \zeta(2i_0)), \dots, \varepsilon(n(*)-1, \zeta(2i_0+1))\}$ and there is $\alpha_{\zeta(2i_1+1)} \in W_1$ such that u is disjoint to $\{\varepsilon(m(*), \zeta(2i_1 + 1)), \dots, \varepsilon(n(*) - 1, \zeta(2i_1 + 1))\}$. For them $(*)_0$, $(*)_1$ gives a contradiction.

So B is a Boolean algebra, of cardinality $\leq \lambda$ (as it is generated by $\{x^n_\alpha : \alpha < \lambda_n, n < \omega\}$), we can embed into it the free Boolean algebra with λ generators $\{x^n_{2\alpha+1} : \alpha < \lambda_n, n < \omega\}$) and B is with no homomorphism onto the free Boolean algebra with κ generators. $\square_{4.14}$

Definition 4.16. *Let B^{fcf}_μ be the Boolean Algebra of finite and cofinite subsets of μ.*

Claim 4.17. Assume $\lambda_{n+1} \geq \mathrm{ecf}_{\kappa,\aleph_0,\aleph_0}(\lambda_n, \aleph_0)$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $\lambda = \sum_n \lambda_n$. Then there is a Boolean Algebra B of cardinality λ into which B^{fcf}_λ can be embedded but such that there is no homomorphism from B onto B^{fcf}_κ.

Proof. Let

$$\mathcal{P}_n = \{a^n_\alpha : \alpha < \lambda_{n+1}\} \subseteq [\lambda_n]^{\aleph_0}$$

exemplifies $\lambda_{n+1} \geq \mathrm{ecf}_{\kappa,\aleph_0,\aleph_0}(\lambda_n, \aleph_0)$ by 4.5(2). We define by induction on n a countable subset x^n_α of $(n+1) \times \lambda$ for each $\alpha < \lambda_n$. For $n = 0$ let $x^0_\alpha = \{(0, \alpha)\}$. For $n + 1$ let

$$x^{n+1}_{2\alpha} = \{(n+1, 2\alpha)\} \cup \bigcup_{\beta \in a^n_\alpha} x^n_\beta \text{ and } x^{n+1}_{2\alpha+1} = \{(n+1, 2\alpha+1)\}.$$

Let B be the Boolean Algebra of subsets of $\omega \times \lambda$ generated by $\{x^n_\alpha : \alpha < \lambda_n, n < \omega\}$.

Clearly $|B| \leq \sum_n \lambda_n = \lambda$, also $\{x^n_{2\alpha+1} : \alpha < \lambda, n < \omega\}$ generate a subalgebra isomorphic to B^{fcf}_λ hence $|B| \geq \lambda$ (so $|B| = \lambda$) and B^{fcf}_λ can be embedded into B.

Lastly, suppose g is a homomorphism from B onto B^{fcf}_κ. Let $z_\zeta =: \{\zeta\} \in B^{\mathrm{fcf}}_\kappa$. For each $\zeta < \kappa$ for some $n(\zeta) < \omega, \alpha(\zeta) < \lambda_n$ we have $g(x^{n(\zeta)}_{\alpha(\zeta)}) \notin \langle z_\zeta : \varepsilon < \zeta\rangle_{B^{\mathrm{fcf}}_\kappa}$, so for some stationary $S \subseteq \kappa, [\zeta \in S \Rightarrow n(\zeta) = n(*)]$. If $g(x^n_\alpha) \in B^{fcf}_\kappa$ is infinite then $\{g(x) : x \in B \text{ and } g(x) \leq g(x^n_\alpha)\} = \{g(x \cap x^n_\alpha) : x \in B\}$ is countable so g is not onto B, a contradiction. So possibly shrinking S without loss of generality $\langle g(x^{n(*)}_{\alpha(\zeta)}) : \zeta \in S\rangle$ is a Δ-system of finite subsets of κ with heart called w.

For some $\beta < \lambda_{n+1}$ the set $u = \{\zeta \in S : \alpha(\zeta) \in a^{n+1}_\beta\}$ is infinite, clearly $\zeta \in u \Rightarrow x^n_{\alpha(\zeta)} \leq x^{n+1}_{2\beta}$ hence $g(x^{n(*)}_{\alpha(\zeta)}) \leq g(a^{n+1}_{2\beta})$ hence $g(x^{n+1}_{2\beta})$ is infinite hence it is co-finite, contradicting an earier statement. $\square_{4.17}$

Definition 4.18.

$$St^6_{\lambda,\kappa} = \min\Big\{|\mathcal{P}| : \textit{there is } \mathcal{P} \subseteq [\lambda]^{\aleph_0} \textit{such that}$$

(i) $\mathcal{P}$ is $\aleph_2$-free i.e. if $a_i \in \mathcal{P}$ for $i < i^* < \aleph_2$ are disjoint then for some finite $b_i \subseteq a_i$ the sets $\langle a_i \backslash b_i : i < i^*\rangle$ are pairwise disjoint

(ii) for every $f : \kappa \to \lambda$ for some $a \in \mathcal{P}$

$$(\exists^\infty \alpha < \kappa)(f(\alpha) \in a)\Big\}.$$

Claim 4.19. Assume $\lambda_{n+1} \leq St^6_{\lambda_n,\kappa}$ for $n < \omega, \lambda = \sum_{n<\omega} \lambda_n$. Then there is a Boolean Algebra B as in the previous claim which is superatomic.

Proof. Like the previous claim.

5. More on free subsets and pcf

Claim 5.1. Assume $\mathrm{IND}(\langle J^{\mathrm{bd}}_{\lambda_\varepsilon} : \varepsilon < \varepsilon(*)\rangle)$ with λ_ε increasing and $\lambda = \sum_{\varepsilon<\varepsilon(*)} \lambda_\varepsilon$. If $\mu > \lambda$ and $\theta_i \in \mathrm{Reg} \cap \mu \backslash \lambda$ for $i < \lambda$, then for some ε we can find $\mathfrak{c} \subseteq \mu \cap \mathrm{pcf}\{\theta_i : i < \lambda\}$ and $\mathfrak{b}_\tau \in J_{\leq\tau}[\{\theta_i : i < \lambda\}]$ for $\tau \in \mathfrak{c}$ such that

(∗) there is no $a \in [\lambda]^{\lambda_\varepsilon}$ such that $[\tau \in \mathfrak{c} \Rightarrow (\forall^{\lambda_\varepsilon} i \in a)\theta_i \notin \mathfrak{b}_\tau]$.

Proof. Easy from the definition.

Claim 5.2. 1) For every $\mathfrak{a} \subseteq \mathrm{Reg}$, $\lambda \leq |\mathfrak{a}| < \min(\mathfrak{a})$ for some $\mathfrak{b} \subseteq \mathfrak{a}$, $\kappa < |\mathfrak{b}| \leq \lambda$, $\Pi\mathfrak{b}/[\mathfrak{b}]^{\leq\kappa}$ has true cofinality (so if λ is minimal for this κ, this holds for any $\kappa' < \lambda$).
2) If $\mathrm{IND}(\langle J^{\mathrm{bd}}_{\lambda_n} : n < \omega\rangle)$, $\lambda_n < \lambda_{n+1}$, λ_n regular, $|\mathfrak{a}| < \lambda_0$, $|\mathfrak{a}| < \min(\mathfrak{a})$ and

$$\mu = \sup_{n<\omega} \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda_n]^{|\mathfrak{a}|^+} \text{ and } (\forall A \in [\lambda_n]^{\lambda_n})(\exists B \in \mathcal{P})[B \subseteq A]\}$$

then $|\mathrm{pcf}(\mathfrak{a})| \leq \mu$.
3) Assume $\sigma < \theta < \lambda_n$, (for $n < \omega$) $\mathrm{IND}(\langle J_n : n < \omega\rangle)$, J_n an ideal on λ_n and μ satisfies: $\bigwedge_n \mu > \lambda_n$ and (we can guess filters which are $(< \theta)$-based).

$(*)_{\mu,J_n}$ there is a set $\mathcal{E}$, $|\mathcal{E}| \leq \mu$, each member of $\mathcal{E}$ is an ideal on some bounded subset of κ_n such that:

⊗ if $Y \in J_n^+$ (so $Y \subseteq \lambda_n$), and I is a $(< \theta)$-based σ-complete ideal on Y generated by $\leq \mu$ sets then for some $I' \in \mathcal{E}$, we have $(\mathrm{Dom}(I') \cap Y \in I^+$ and $\mathrm{Dom}(I') \backslash Y \in I'$ and $I' \restriction (Y \cap \mathrm{Dom}(I')) \supseteq I \restriction (Y \cap \mathrm{Dom}(I'))$.

If $|\mathfrak{a}| \leq \theta$ and $\Sigma\{\lambda_n : n < \omega\} < \mathrm{Min}(\mathfrak{a})$ and $\mathfrak{a} \subseteq \mathrm{Reg}$ then $(\mathrm{pcf}_{\sigma\text{-complete}}(\mathfrak{a})) \leq \mu$.

Proof. 1) Straight, e.g. if $\langle \mathfrak{b}_\theta : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ is a generating set let $\mathfrak{b} = \mathfrak{b}_\theta$, θ minimal such that $|\mathfrak{b}_\theta| > \kappa$.
2) Suppose not. Let $\langle \tau_{i+1} : i < \mu^+\rangle$ be the first μ^+ members of $\mathrm{pcf}(\mathfrak{a})$ listed in increasing order. Let $\tau_\delta = \bigcup_{i<\delta} \tau_{1+i}$ for limit $\delta \leq \mu^+$. For each limit $\delta < \mu^+$ for some $n = n_\delta < \omega$, τ_δ is $\{J^{\mathrm{bd}}_{\lambda_n}\}$-inaccessible (by 3.16). So for some $n(*) < \omega$, $\{\delta < \mu^+ : n_\delta = n(*)\}$ is stationary, hence

(∗) for no $\theta_\alpha \in \mathrm{Reg} \cap \tau_{\mu^+}$ for $\alpha < \lambda_{n(*)}$, do we have

$$\prod_{\alpha<\lambda_{n(*)}} \theta_\alpha / J^{bd}_{\lambda_{n(*)}} \text{ is } \tau_{\mu^+}\text{-directed.}$$

By [Sh 420, §1] we can find $\langle C_\alpha : \alpha \in S\rangle$, $S \subseteq \mu^+$, $C_\alpha \subseteq \alpha$, $\mathrm{otp}(C_\alpha) \leq \lambda_{n(*)}$, $[\beta \in C_\alpha \Rightarrow \beta \in S \;\&\; C_\beta = C_\alpha \cap \beta]$ and $\mathrm{otp}(C_\alpha) = \lambda_{n(*)} \Rightarrow \alpha = \sup(C_\alpha)$ and $\{\alpha \in S : \mathrm{otp}(C_\alpha) = \lambda_{n(*)}\}$ is stationary. Now we imitate [Sh 400, §2].
3) Similar to 2). $\square_{5.2}$

6. Odds and ends

As in [Sh 430, §6] this section is dedicated to things I forgot to say. We repeat and elaborate older things [Sh 430, 6.6D,6.6E,6.6F], [Sh 410, 3.7].

Claim 6.1. Suppose D is a σ-complete filter on $\theta = \mathrm{cf}(\theta)$ such that $[\alpha < \theta \Rightarrow \theta \backslash \alpha \in D]$, σ is regular $> \kappa^+ + |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar{\beta} = \langle \beta^\alpha_\epsilon : \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \neq \emptyset \bmod D$ there is $\langle \beta^*_\epsilon : \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:

(*a*) $\epsilon \in \kappa \backslash w \Rightarrow \sigma \leq \mathrm{cf}(\beta^*_\epsilon) \leq \theta$,

(*b*) $B =: \{\alpha \in X :$ if $\epsilon \in w$ then $\beta^\alpha_\epsilon = \beta^*_\epsilon$ and: if $\epsilon \in \kappa \backslash w$ then β^α_ϵ is $< \beta^*_\epsilon$ but $\beta^\alpha_\varepsilon > \sup\{\beta^*_\zeta : \zeta < \kappa, \beta^*_\zeta < \beta^*_\epsilon\}\}$ is $\neq \emptyset \bmod D$

(*c*) if $\beta'_\epsilon < \beta^*_\epsilon$ for $\epsilon \in \kappa \backslash w$ then $\{\alpha \in B :$ if $\epsilon \in \kappa \backslash w$ then $B'_\epsilon < \beta^\alpha_\epsilon\} \neq \emptyset \bmod D$.

Remark 6.2. 1) Of course, we can replace κ by any set of this cardinality.

2) May look at [Sh 620], §7, there more is said concerning 6.1.

Proof. Let $f_\alpha : \kappa \to \mathrm{Ord}$ be $f_\alpha(i) = \beta^\alpha_i$. For notational simplicity $\chi \in \lambda$.

Let χ be large enough. We choose by induction on $i < \sigma$, a model N_i such that:

$N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$;
$\|N_i\| \leq 2^\kappa + |i|^\kappa$;
$2^\kappa \subseteq N_0$;
$\kappa, \sigma, \theta, X \in N_0$, $\langle f_\alpha : \alpha < \theta \rangle \in N_0$;
$i < j \Rightarrow N_i \prec N_j$;
$\langle N_j : j \leq i \rangle \in N_{i+1}$;
N_i increasing continuous.

Let $\delta_i =: \min(\cap\{B : B \in N_i \cap D\})$, now δ_i is well defined (as D is σ-complete and $\sigma > 2^\kappa + |i|^\kappa \geq \|N_i\|$, hence the intersection is in D). Now $\delta_i \geq \sup(\theta \cap N_i)$. As $\alpha \in \theta \cap N_i \Rightarrow \theta \backslash \alpha \in D \cap N_i$ and as $\delta_i \in N_{i+1}$ (as $\{N_i, D, \theta\} \in N_{i+1}$) clearly $\langle \delta_i : i < \sigma \rangle$ is strictly increasing. We define for $i < \sigma$, a function $g_i \in {}^\kappa\chi$ by

$$g_i(\zeta) = \min(N_i \cap \chi \backslash f_{\delta_i}(\zeta))$$

(it is well defined as $f_{\delta_i}(\zeta) < \bigcup_{\alpha<\theta}(f_\alpha(i) + 1) \in N_0 \prec N_i$). Clearly $E =: \{\alpha < \sigma : N_\alpha \cap \sigma = \alpha\}$ is a club of σ, and as $(\forall \alpha < \sigma)[|\alpha|^\kappa < \sigma]$ clearly $\alpha < \beta \in E$ & $a \subseteq N_\alpha$ & $|a| \leq \kappa \Rightarrow a \in N_\beta$. Now $i \in E$, $\mathrm{cf}(i) = \kappa^+$ implies $N_i = \bigcup_{j<i} N_j$ and $\mathrm{Rang}(g_i) \subseteq \bigcup_{j<i} N_j$ hence $\bigvee_{j<i}[\mathrm{Rang}(g_i) \subseteq N_j]$; but by the previous sentence every subset of N_j of cardinality $\leq \kappa$ belongs to N_i, hence $g_i \in \bigcup_{j<i} N_j$. So by Fodor Lemma for some stationary subset S of $\{i \in E : \mathrm{cf}(i) = \kappa^+\}$ and some $g^* : \kappa \to \mathrm{Ord}$ and some $u \subseteq \kappa$ and some $i(*) < \sigma$ we have: $[i \in S \Rightarrow g_i = g^*]$, $(\forall i \in S)(\forall \zeta < \kappa)[f_{\delta_i}(\zeta) = g^*(\zeta) \Leftrightarrow \zeta \notin u]$ and $g^* \in N_{i(*)}$; note $u \in N_0 \subseteq N_{i(*)}$ as $u \subseteq \kappa$ and we can assume $i(*) < \min(S)$ and $i(*) \in E$.

Let $w =: \kappa\backslash u$ and $\beta^*_i =: g^*(i)$ for $i < \kappa$ now we show that $w, \langle\beta^*_i : i < \kappa\rangle$ are as required.

Clause (b):

The set B is defined from: $\langle\beta^*_i : i < \kappa\rangle$ and w and $\bar{f} = \langle f_\alpha : \alpha < \theta\rangle$. As all of them belong to $N_{i(*)}$ clearly $B \in N_{i(*)}$, so if $B = \emptyset$ mod D then $(\theta\backslash B) \in D \cap N_{i(*)}$ hence $\zeta \in S \Rightarrow \delta_\zeta \in \theta\backslash B \Rightarrow \delta_\zeta \notin B$; but $\delta_\zeta \in B$ by the definition of B, g^*, g_ζ, S.

Clause (a):

If $\epsilon \in \kappa\backslash w$, $\mathrm{cf}(\beta^*_\epsilon) > \theta$ then $\gamma^*_\epsilon =: \sup\{f_\alpha(\epsilon) : \alpha < \theta$ and $f_\alpha(\epsilon) < \beta^*_\epsilon\}$ is $< \beta^*_\epsilon$ and it belongs to $N_{i(*)}$ (as ϵ, $\langle f_\alpha : \alpha < \lambda\rangle$ and β^*_ϵ belongs to $N_{i(*)}$) and for any $\zeta \in S$ we get a contradiction to $g_\zeta(\epsilon) = \beta^*_\zeta$.

Clearly $\zeta \in \kappa\backslash w \Rightarrow \mathrm{cf}[g^*(\zeta)] \geq \sigma$ as otherwise $g^*(\zeta) = \sup(N_i \cap g^*(\zeta))$ (as $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ and $N_{i(*)} \cap \sigma = i(*)$ because $i(*) \in E$) and easy contradiction.

Clause (c):

If there is $\bar{\beta}' = \langle\beta'_\epsilon \in u\rangle$ contradicting clause (c), then there is such a sequence defined from B, $\langle f_\alpha : \alpha < \theta\rangle$, u, w, $\langle\beta^*_i : i < \kappa\rangle$, just use the $<^*_\chi$-first one, hence without loss of generality $\bar{\beta}' \in N_{i(*)}$, so for any $\zeta \in S$ we get a contradiction. $\square_{6.1}$

Observation 6.3. If $|\mathfrak{a}| < \min(\mathfrak{a})$, $H \subseteq \Pi\mathfrak{a}$, $|H| = \theta = \mathrm{cf}(\theta) \notin \mathrm{pcf}(\mathfrak{a})$ and also $\theta > \sup(\theta \cap \mathrm{pcf}(\mathfrak{a}))$ then for some $g \in \Pi\mathfrak{a}$, the set $H_g =: \{f \in H : f < g\}$ has cardinality θ; in fact H is the union of $\leq \sup(\theta \cap \mathrm{pcf}(\mathfrak{a}))$ sets of the form H_g.

Proof. Why? This is as $\Pi\mathfrak{a}\,/J_{<\theta}[\mathfrak{a}]$ is $\min(\mathrm{pcf}(\mathfrak{a}))\backslash\theta)$-directed and the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $< \theta$ sets.

In details, let $\langle\mathfrak{b}_\sigma[\mathfrak{a}] : \sigma \in \mathrm{pcf}(\mathfrak{a})\rangle$ be a generating sequence for $\mathfrak{a}$ (exists by [Sh:g, Ch. VIII,2.6]).

For $\sigma \in \mathrm{pcf}\mathfrak{a}$ let $f^\sigma_\alpha \in \Pi\mathfrak{a}$ for $\alpha < \sigma$ be such that $\langle f^\sigma_\alpha : \alpha < \sigma\rangle$ is $<_{J_\theta[\mathfrak{a}]}$-increasing and cofinal and moreover $\{f^\sigma_\alpha \restriction \mathfrak{b}_\sigma[\mathfrak{a}] : \alpha < \sigma\}$ is cofinal in $\Pi\mathfrak{b}_\sigma[\mathfrak{a}]$ (where $J_\sigma[\mathfrak{a}] = J_{<\sigma}[\mathfrak{a}] + \mathfrak{b}_\sigma[\mathfrak{a}]$), exists as $(\Pi\mathfrak{b}_\sigma, [\mathfrak{a}], <)$ has cofinality σ by [Sh:g, Ch. II,3.1].

Now as $\Pi\mathfrak{a}\,/J_{<\theta}[\mathfrak{a}]$ is $\min(\mathrm{pcf}(\mathfrak{a})\backslash\theta^+)$-directed (as $J_{<\theta}[\mathfrak{a}] = J_{<\theta^+}[\mathfrak{a}] = J_{<\min(\mathrm{pcf}(\mathfrak{a})\backslash\theta^+)}[\mathfrak{a}]$ and by [Sh:g, Ch. I,1.5] there is $g \in \Pi\mathfrak{a}$ such that: $h \in H \Rightarrow h > g$ mod $J_{<\theta}[\mathfrak{a}]$; hence for each $h \in H$ for some finite $\Theta(h) \subseteq \theta \cap \mathrm{pcf}(\mathfrak{a})$ we have

$$\{\sigma \in \mathfrak{a} : h(\sigma) \geq g(\sigma)\} \subseteq \bigcup\{\mathfrak{b}_\sigma[\mathfrak{a}] : \sigma \in \Theta(h)\}.$$

Also for every $\sigma \in \mathrm{pcf}(\mathfrak{a})$ we can find $\alpha_\sigma(h)$ (for $\sigma \in \mathrm{pcf}(\mathfrak{a})$, $h \in H$) such that $h \restriction \mathfrak{b}_\sigma[\mathfrak{a}] < f^\sigma_{\alpha_\sigma(h)} \restriction \mathfrak{b}_\sigma[\mathfrak{a}]$. So $h < \max(\{g, f^\sigma_{\alpha_\sigma(h)} : \sigma \in \Theta(h)\})$. Let

$$G =: \big\{\max\{g, f^{\sigma_1}_{\alpha_1}, \ldots, f^{\sigma_n}_{\alpha_n}\} : n < \omega, \{\sigma_1, \ldots, \sigma_n\} \subseteq \theta \cap \mathrm{pcf}(\mathfrak{a}) \text{ and } \alpha_1 < \sigma_1, \ldots, \alpha_n < \sigma_n\big\}$$

it has cardinality $\leq \aleph_0 + \sup(\theta \cap \mathrm{pcf}(\mathfrak{a})) < \theta$ and $\theta = \mathrm{cf}(\theta) = |H|$ and $\forall h \in H \exists g' \in G(h < g)$.

So for some $g^* \in G$ the set $\{h \in H : h < g^*\}$ has cardinality θ as required. $\square_{6.3}$

We comment to [Sh 410, 3.7] (which solve a problem from Gerlits Hajnal Szentmiklossy [GHS]).

Claim 6.4. 1) Suppose γ^* and $i^* = i(*)$ are ordinals and $\bar{\chi} = \langle \chi_i : i < i^* \rangle$ is a sequence of infinite cardinals; so of course we can find n, $\bar{w} = \langle w_\ell : \ell < n \rangle$, $\bar{\kappa} = \langle \kappa_\ell : \ell < n \rangle$, $\bar{\sigma} = \langle \sigma_\ell : \ell < n \rangle$ such that $\bar{w}$ is a partition of i^*, $|w_\ell| = \kappa_\ell$, $\bigwedge_{i \in w_\ell} \chi_i \leq \sigma_\ell$, and

$$(\forall \chi < \sigma_\ell)(\exists^{\kappa_\ell} i)(i \in w_\ell \ \& \ \chi < \chi_i \leq \sigma_\ell)$$

and $\bar{\kappa}$ is strictly increasing, $\bar{\sigma}$ is strictly decreasing, in fact $\bar{w}, \bar{\kappa}, \bar{\sigma}$ are unique for our given $i^*, \bar{\chi}$.
Then the following are equivalent

$(A)_{\bar{\chi},\gamma^*}$ there are $f_\alpha \in \prod_{i < i(*)} \chi_i$ for $\alpha < \gamma^*$ satisfying $\alpha < \beta \Rightarrow (\exists i < i^*)[f_\alpha(i) < f_\beta(i)]$

$(B)_{\bar{\chi},\gamma^*}$ for some $\ell < n$ we have: $2^{\kappa_\ell} \geq |\gamma^*|$ or for every regular $\mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+$ for some singular $\lambda^* \leq \sigma_\ell$ we have

$(*)$ $\mathrm{cf}(\lambda^*) \leq \kappa_\ell$, $\lambda^* > 2^{\kappa_\ell}$ and $\mathrm{pp}^+(\lambda^*) > \mu_1$

2) If

$\otimes$ $(\forall \mu_1)(\mu_1 = \mathrm{cf}(\mu_1) \leq |\gamma^*| \rightarrow (\exists \mu_2)[\mu_2 = \mathrm{cf}(\mu_2) \ \& \ \mu_1 \leq \mu_2 \leq |\gamma^*| \ \& \ (\forall \alpha < \mu_2)|\alpha|^{\aleph_0} < \mu_2])$,

then in part (1), $(B)_{\bar{\chi},\gamma^*}$ the demand $(*)$ on λ^* can be replaced by

$(*)'$ $\mathrm{cf}(\lambda^*) \leq \kappa_\ell$, $\lambda^* > 2^{\kappa_\ell}$ and $(\forall \mu < \lambda^*)(\mu^{\kappa_\ell} < \lambda^*)$ and $(\lambda^*)^{\kappa_\ell} \geq \mu_1$.

Now we call it $(B)'_{\bar{\chi},\gamma^*}$ (so if $\otimes$ then $(A)_{\bar{\chi},\gamma^*} \Leftrightarrow (B)_{\bar{\chi},\gamma^*} \Leftrightarrow (B)'_{\bar{\chi},\gamma^*}$).

3) If $\otimes$ from part (2) holds, then also $(A)_{\bar{\chi},\gamma^*} \Leftrightarrow (B)''_{\bar{\chi},\gamma^*} \Leftrightarrow (B)^+_{\bar{\chi},\gamma^*}$ where

$(B)''_{\bar{\chi},\gamma^*}$ $|\gamma^*| \leq \max\{(\sigma_\ell)^{\kappa_\ell} : \ell < n\}$

$(B)^+$ for some $\ell < n$ we have: $2^{\kappa_\ell} \geq |\gamma^*|$ or for every regular $\mu_1 \in (\gamma^* + 1) \setminus \sigma_\ell^+$ for some singular $\lambda^* \leq \sigma_\ell$ we have

$(*)^+$ $\mathrm{cf}(\lambda^*) \leq \kappa_\ell$, $\lambda^* > 2^{\kappa_\ell}$ and $(\forall \mu < \lambda^*)(\mu^{\kappa_\ell} < \lambda^*)$ and $\mathrm{pp}^+(\lambda^*) > \mu_1$.

4) In part (2), (3) instead $\otimes$ we may let $\lambda_0 = \max\{2^{\kappa_\ell} : \ell < n\}$, $\lambda_1 = |\gamma^*|$, demand

$\oplus_{\lambda_0,\lambda_1}$ if $\lambda_0 < \mu \leq \lambda_1$, $\mathrm{cf}(\mu) = \aleph_0$ and $(\forall \lambda < \mu)(|\lambda|^{\aleph_0} < \mu)$ then $\mathrm{pp}(\mu) =^+ \mu^{\aleph_0}$.

Remark. 1) On $\oplus_{\lambda_0,\lambda_1}$ from 6.4(4), see [Sh 430],§1.
2) Note that we could in 6.4(1) demand

$$(\forall \chi < \sigma_\ell)(\exists^{\kappa_\ell} i)(i \in w_\ell \ \& \ \chi \leq \chi_1 < \sigma_\ell)$$

and can allow χ_i (hence σ_i) to be any ordinal, and even let κ_i be ordinal so the demand is $(\forall \alpha < \sigma_\ell)[\kappa_\ell = \mathrm{otp}\{i \in w_\ell : \alpha \leq \chi_i < \sigma_\ell\}]$. This causes no serious change.

Proof. 1) $(B)_{\bar\chi,\gamma^*} \Rightarrow (A)_{\bar\chi,\gamma^*}$.

Let $\ell < n$ exemplifies $(B)_{\bar\chi,\gamma^*}$ so there are $f'_\alpha \in {}^{(\kappa_\ell)}(\sigma_\ell)$ for $\alpha < \gamma^*$ such that $\alpha < \beta < \gamma^* \Rightarrow (\exists j < \kappa_\ell)[f'_\alpha(j) < f'_\beta(j)]$.
[Why? Easy by cases.]

Let $h : w_\ell \to \kappa_\ell$ be such that:

$$(\forall j < \kappa_\ell)(\forall \sigma < \sigma_\ell)(\exists^{\kappa_\ell} i)[i \in w_\ell \ \&\ \sigma < \chi_i \le \sigma_\ell \ \&\ h(i) = j].$$

Let $f_\alpha \in \prod_{i<i(*)} \chi_i$ be: $f_\alpha(i) = f'_\alpha(h(i))$ if $i \in w_\ell$, $f'_\alpha(h(i)) < \chi_i$ and $f_\alpha(i) = 0$ otherwise. So if $\alpha < \beta < \gamma^*$ then for some $j < \kappa_\ell$, $f'_\alpha(j) < f'_\beta(j) < \sigma_\ell$ so for some $i \in w_\ell$ we have: $h(i) = j \ \&\ \chi_i > f'_\beta(j)$. So $f_\alpha(i) \le f'_\alpha(j) < f'_\beta(j) = f_\beta(i)$ as required.

$(A)_{\bar\chi,\gamma^*} \Rightarrow (B)_{\bar\chi,\gamma^*}$

Assume this fails, note that clearly $\gamma_1 < \gamma_2 \ \&\ (A)_{\bar\chi,\gamma_2} \Rightarrow (A)_{\bar\chi,\gamma_1}$ and $\gamma_1 < \gamma_2 \ \&\ (B)_{\bar\chi,\gamma_2} \Rightarrow (B)_{\bar\chi,\gamma_1}$. Without loss of generality γ^* is minimal (for our $\bar\chi$) for which this fails; so γ^* is minimal such that $(B)_{\bar\chi,\gamma^*}$ fails. Inspecting $(B)_{\bar\chi,\gamma^*}$, as n is finite, clearly γ^* is a regular cardinal, call it θ. By renaming we can assume that i^* is a cardinal. Now let $\langle f_\alpha : \alpha < \theta\rangle$ exemplifies $(A)_{\bar\chi,\theta}$; now apply 6.1 above with i^*, $\langle f_\alpha : \alpha < \theta\rangle$, D^{cb}_θ the filter of cobounded subsets of θ and $\max_{\ell<n}(2^{\kappa_\ell})^+ = (2^{|i^*|})^+$ here standing for κ, $\left\langle \langle \beta^\alpha_i : i < \kappa\rangle : \alpha < \theta \right\rangle$, D, σ there.

So we get w, $\langle \beta^*_i : i < i^*\rangle$, B as there, and let $\mathfrak{a} = \{\mathrm{cf}(\beta^*_i) : i \in i^* \backslash w\}$, so $\mathfrak{a} \subseteq \mathrm{Reg} \cap (\max_{\ell<n} \sigma_\ell)\backslash(2^{|i^*|})^+$ (see clause (a) of 6.1) and $|\mathfrak{a}| \le |i^*|$. Now if $\theta \le \max \mathrm{pcf}(\mathfrak{a})$ then for some ℓ, $\theta \le \max \mathrm{pcf}\{\mathrm{cf}(\beta^*_i) : i \in w_\ell \backslash w\}$, and so $(B)_{\bar\chi,\theta}$ does not fail, contradicting an earlier assumption. So $\theta > \max \mathrm{pcf}(\mathfrak{a})$, so there is a cofinal $H \subseteq \prod_{i\in(i^*\backslash w)} \beta^*_i$ of cardinality $< \theta$, so there are $h_\alpha \in H$ such that $f_\alpha \restriction (i^*\backslash w) < h_\alpha$ but $|B| = \theta > |H|$ (by the choice of D^{cb}_θ as D) so for some $h^* \in H$ the set $B_1 = \{\alpha \in B : h_\alpha = h^*\}$ is unbounded in θ. By clause (c) of the conclusion of 6.1 for some $\alpha \in B$ we have $i \in i^*\backslash w \Rightarrow h^*(i) < f_\alpha(i)$. Choose $\beta \in B_1 \backslash(\alpha+1)$, so $\alpha < \beta$ are in B, hence $f_\alpha \restriction w = f_\beta \restriction w$ and $i \in i^*\backslash w \Rightarrow f_\beta(i) < h^*(i) < f_\alpha(i)$, so $\bigwedge_i f_\beta(i) \le f_\alpha(i)$, a contradiction to "$\langle f_\gamma : \gamma < \theta\rangle$ exemplifies $(A)_{\bar\chi,\theta}$".

2), 3) Clearly $(B)^+_{\bar\chi,\gamma^*} \Rightarrow (B)_{\bar\chi,\gamma^*} \Rightarrow (B)''_{\bar\chi,\gamma^*}$ by checking. So we should just prove the following two implications:

$(B)''_{\bar\chi,\gamma^*} \Rightarrow (B)'_{\bar\chi,\gamma^*}$

We can assume $|\gamma^*| > 2^{\kappa_\ell}$ for $\ell < n$ (otherwise the conclusion is trivial). We know by $(B)''_{\bar\chi,\gamma^*}$ that for some ℓ, $(\sigma_\ell)^{\kappa_\ell} \ge |\gamma^*|$. Let us check now $(B)'_{\bar\chi,\gamma^*}$ so let a regular $\mu_1 \in (\gamma^*+1)\backslash\sigma^+_\ell$ be given. So $(\sigma_\ell)^{\kappa_\ell} \ge |\gamma^*| \ge \mu_1$ hence $\lambda =: \min\{\lambda : \lambda^{\kappa_\ell} \ge \mu_1\}$ is $\le \sigma_\ell$ but is $> 2^{\kappa_\ell}$, hence it is singular, $\mathrm{cf}(\lambda) \le \kappa_\ell$ and $(\forall \alpha < \lambda)(|\alpha|^{\kappa_\ell} < \lambda)$, i.e. as required in $(*)'$.

$(B)'_{\bar\chi,\gamma^*} \Rightarrow (B)^+_{\bar\chi,\gamma^*}$

Again we can assume $|\gamma^*| > 2^{\kappa_\ell}$ for $\ell < n$. Let us check $(B)_{\bar\chi,\gamma^*}$, so let a regular $\mu_1 \in (\gamma^*+1)\backslash\sigma^+_\ell$ be given. As we are assuming $\otimes$ from 6.4(2) there is $\mu_2 \in \text{Reg} \cap (\gamma^* + 1)\backslash\mu_1$ such that $(\forall\alpha < \mu_2)(|\alpha|^{\aleph_0} < \mu_2)$. Apply $(B)'_{\bar{\chi},\gamma^*}$ for μ_2 and get λ_2^* as in $(*)$ for μ_2 instead of μ_1. Clearly $(\lambda_2^*)^{\kappa_\ell} \geq \mu_2$ and let $\lambda^* = \min\{\lambda : \lambda^{\kappa_\ell} \geq \mu_2\}$, so $\lambda^* \leq \lambda_2^* \leq \sigma_\ell$, $\lambda^* > 2^{\kappa_\ell}$ and $(\forall\mu < \lambda^*)[\mu^{\kappa_\ell} < \lambda^*]$, and clearly $\text{cf}(\lambda^*) \leq \kappa_\ell$, so $(\lambda^*)^{\kappa_\ell} = (\lambda^*)^{\text{cf}(\lambda)}$.

By the choice of μ_2 necessarily $\text{cf}(\lambda^*) > \aleph_0$ (otherwise $\mu_2 \leq (\lambda^*)^{\kappa_\ell} = (\lambda^*)^{\aleph_0} < \mu_2$). By [Sh:g], (see 6.5 below), $\text{pp}(\lambda^*) =^+ (\lambda^*)^{\text{cf}(\lambda^*)} = (\lambda^*)^{\kappa_\ell}$ as required in $(*)^+$.

4) The only place we use the assumption $\otimes$ in the proof of part (2), (3) was in choosing μ_2 in the proof of $(B)'_{\bar{\chi},\gamma^*} \Rightarrow (B)^+_{\bar{\chi},\gamma^*}$ and the use of its property is to show $\text{cf}(\lambda^*) > \aleph_0$ (to be able to use [Sh:g, Ch. VIII],§1) but we can use instead $\oplus_{\lambda_0,\lambda_1}$. $\square_{6.4}$

Remember that by [Sh:g]:

Observation 6.5. If μ is singular, $\text{cf}(\mu) > \aleph_0$ and $\alpha < \mu \Rightarrow |\alpha|^{\text{cf}(\mu)} < \mu$ <u>then</u> $\mu^{\text{cf}(\mu)} =^+ \text{pp}(\mu)$.

Proof. By [Sh 430, 3.5], [Sh:g, CH.VIII,1.8], [Sh:g, Ch. II,5.6]. $\square_{6.5}$

Definition 6.6. *1) For $F \subseteq {}^{\delta}\text{Ord}$, we say F is free$^\ell$ when we can find $\zeta_f < \delta$ for $f \in F$ such that:*

(a) *if $\ell = 1$ then*

$$f \neq g \in F \ \&\ \zeta = \max\{\zeta_f, \zeta_g\} \Rightarrow f \restriction \zeta \neq g \restriction \zeta$$

(b) *if $\ell = 2$ then*

$$f \neq g \in F \ \&\ \delta > \zeta \geq \max\{\zeta_f, \zeta_g\} \Rightarrow f(\zeta) \neq g(\zeta)$$

(c) *if $\ell = 3$ then*

$$f \neq g \in F \ \&\ \delta > \zeta > \varepsilon = \max\{\zeta_f, \zeta_g\} \ \&\ f(\varepsilon) \leq g(\varepsilon) \Rightarrow f(\zeta) \leq g(\zeta)$$

(d) *if $\ell = 4$ then for $f, g \in F$, $f \leq g$ (i.e. $\zeta < \delta \Rightarrow f(\zeta) \leq g(\zeta)$) or $g \leq f$*

(e) *if $\ell = 5$ then for some ζ and h we have*

$$f \in F \Rightarrow f \restriction \zeta = h$$

$$\{f(\zeta) : f \in F\} \text{ is with no repetition.}$$

2) Let free$^{\ell,m}$ means free$^\ell$ and free^m. For J an ideal on δ we write J-free$^\ell$ if $\zeta_f < \delta$ is replaced by $s_f \in J$, that is there are $s_f \in J$ for $f \in F$ such that:

(a) if $\ell = 1$ then

$$f \neq g \in F \ \&\ s = s_f \cup s_g \Rightarrow f \restriction s \neq g \restriction s$$

(b) if $\ell = 2$ *then*

$$f \neq g \in F \ \&\ \zeta \in \delta \backslash s_f \backslash s_g \Rightarrow f(\zeta) \neq g(\zeta)$$

(c) if $\ell = 3$ *then*

$$f \neq g \in F \ \&\ \{\xi, \zeta\} \subseteq \delta \backslash s_f \backslash s_g \Rightarrow [f(\xi) \leq g(\varepsilon) \equiv f(\zeta) \leq g(\zeta)]$$

(d) if $\ell = 4$ *then*

$$f, g \in F \Rightarrow [f \leq g \vee g \leq f].$$

Definition 6.7. *For* $F \subseteq {}^{\delta}Ord$*:*

1) We say F *is* μ*-free*^{x} *if every* $F' \in [F]^{\mu}$ *is free*^{x}*.*
2) We say F *is* (μ, κ)*-free*^{x} *if every* $F' \in [F]^{\mu}$ *there is* $F'' \in [F']^{\kappa}$ *which is free*^{x}*.*

Fact 6.8. 1) "F is free$^{\ell}$ implies F is free^m" when (ℓ, m) is one of (2,1), (4,3), (5,1).
2) Similarly for μ-free^x and (μ, κ)-free^x.

Proof. Straight.

On 6.9 see [Sh:g, Ch. II],§1, [Sh:g, Ch. II],§3, [Sh 282], [Sh:g, Ch. II,4.10], Shelah Zapletal [ShZa 561].

Claim 6.9. 1) If $J^{\mathrm{bd}}_{\delta} \subseteq J$, J an ideal on δ, $\prod_{i<\delta} \lambda_i / J$ is λ^+-directed, $\langle \lambda_i : i < \delta \rangle$ an increasing sequence of regulars $> \delta$ with limit μ, $\mu < \lambda = \mathrm{cf}(\lambda)$, then there are regulars $\lambda'_i < \lambda_i$ with $\mathrm{tlim}_J \langle \lambda'_i : i < \delta \rangle = \mu$ and $f_\alpha \in \prod_{i<\delta} \lambda'_i$ for $\alpha < \lambda$ such that $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$-increasing cofinal in $\prod_{i<\delta} \lambda'_i$ and $\{f_\alpha : \alpha < \lambda\}$ is $\mu^+ - J$-free^{1,2,3}.
2) Assume $\mathfrak{a}$ is a set of regular $> |\mathfrak{a}|$ with no last element, J an ideal on $\mathfrak{a}$ extending $J^{\mathrm{bd}}_{\mathfrak{a}}$ and $\mathfrak{c} = \{\theta \in \mathfrak{a} : \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})\}$ and $\lambda = \max \mathrm{pcf}(\mathfrak{a})$. Then there is $\langle f_\alpha : \alpha < \lambda \rangle$ cofinal in $\Pi \mathfrak{a}$, $<_J$-increasing, such that:

(∗) if $\theta \in \mathfrak{c}$ then $\{f_\alpha \restriction (\theta \cap \mathfrak{a}) : \alpha < \lambda\}$ has cardinality $< \theta$.

3) If $\mu > \mu_0 \geq \kappa \geq \mathrm{cf}(\mu)$, $\lambda = \mu^+$ (or just $\mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}^+_{\kappa}(\mu)$) then for some $\mathfrak{a} \subseteq (\mu_0, \mu) \cap \mathrm{Reg}$ we have $|\mathfrak{a}| \leq \kappa$, $[\theta \in \mathfrak{a} \Rightarrow \max \mathrm{pcf}(\theta \cap \mathfrak{a}) < \theta]$ and $\lambda = \max \mathrm{pcf}(\mathfrak{a})$ (if $[\alpha < \mu \Rightarrow |\alpha|^{\kappa} < \mu]$ we can have $\mu = \sup(\mathfrak{a})$, $\mathrm{otp}(\mathfrak{a}) = \mathrm{cf}(\mu)$ (so part (2) is not empty)).
4) If J is an ideal on $\mathfrak{a}$, $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$-increasing $<_J$-cofinal in $\Pi \mathfrak{a}$ and J is generated by $< \min(\mathfrak{a})$ sets (as an ideal) then for every $A \in [\lambda]^{\lambda}$ for some $\mathfrak{d} \in J$ we have: for every $g \in \Pi \mathfrak{a}$ for λ ordinals $\alpha \in A$, $g \restriction (\mathfrak{a} \backslash \mathfrak{d}) < f_\alpha \restriction (\mathfrak{a} \backslash \mathfrak{d})$. Hence $\bar{f}$ is $(\lambda, \min(\mathfrak{a})) - J$-free$^{\{2,3\}}$.
5) Assuming $|\mathfrak{a}| < \min(\mathfrak{a})$, $\lambda = \mathrm{tcf}(\Pi \mathfrak{a}, <_{J^{\mathrm{bd}}_{\mathfrak{a}}})$, $\mathfrak{c} = \{\theta \in \mathfrak{a} : \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})\}$ is unbounded in $\mathfrak{a}$ and $\langle f_\alpha : \alpha < \lambda \rangle$ is as in part (2). Then not only for each $\theta \in \mathfrak{a}$ the family $\{f_\alpha : \alpha < \lambda\}$ is (λ, θ)-free$^{\{2,3\}}$, but for any $A \in [\lambda]^{\lambda}$ for every large enough $\theta \in \mathfrak{c}$ there is $B \in [A]^{\theta}$ such that $\langle f_\alpha \restriction (\theta \cap \mathfrak{a}) : \alpha \in B \rangle$ is constant and $\langle f_\alpha \restriction (\mathfrak{a} \backslash \theta) : \alpha \in B \rangle$ is strictly increasing.
6) Assume $\lambda = \mathrm{tcf}(\Pi \mathfrak{a}, <_J)$, $J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq J$, $\lambda > \mu = \sup(\mathfrak{a})$ and $\langle f_\alpha : \alpha \in \lambda \rangle$ is $<_J$-increasing and $<_J$-cofinal in $\Pi \mathfrak{a}$. Then

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(a) if $\kappa \le \mu^+$ and[3] $\{\delta < \lambda : \text{cf}(\delta) < \kappa\} \in I[\lambda]$, <u>then</u> for some $A \in [\lambda]^\lambda$ we have $\{f_\alpha : \alpha \in A\}$ is $(< \kappa)$-free^{2,3}, also we can find $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ as above with $A = \lambda$, such that $\bar{f}$ is ^{b}continuous (see [Sh:g, Ch. I,§3])

(b) if $\kappa = \text{cf}(\kappa) < \mu^+$ and $\{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ belongs to $I[\lambda]$ <u>then</u> $\bar{f}$ is (κ, κ)-free^{2,3}.

7) Assume $\langle \mu_i : i \le \kappa \rangle$ is an increasing continuous sequence of singulars $> \kappa, \kappa = \text{cf}(\kappa) > \aleph_0, \theta \in \text{Reg} \cap \mu_0 \backslash \kappa^+$ and[4] $\{\delta < \mu_i^+ : \text{cf}(\delta) = \theta\} \in I[\mu_i^+]$ for $i = \kappa$. <u>Then</u> some $\bar{f} = \langle f_\alpha : \alpha < \mu_\kappa^+ \rangle$ is $<_{J_C^{\text{bd}}}$-increasing and cofinal in $\prod_{i<\kappa} \mu_i^+$ and is (θ, θ)-free^{2,3}. If we demand in addition $\{\delta < \mu_\kappa^+ : \text{cf}(\delta) \le \theta\} \in I[\mu_\kappa^+]$ <u>then</u> $\bar{f}$ is $< \theta^+$-free^{2,3}.

Proof. 1) By [Sh:g, Ch. II,§1].
2) By [Sh:g, Ch. II,3.5].
3) By [Sh:g, Ch. II,§3].
4) Can prove as in [Sh 282]. Or as in 6.1, as below.

Let $A \in [\lambda]^\lambda$ and let $\{\mathfrak{d}_\zeta : \zeta < \zeta^*\} \subseteq J$ be a family generating J closed under finite union such that $\zeta^* < \lambda_0$. We shall prove that for some $\zeta < \zeta^*$ we have $(\forall g \in \Pi\mathfrak{a})(\exists^\lambda \alpha \in A)(g \restriction (\mathfrak{a} \backslash \mathfrak{d}_\zeta) < f_\alpha \restriction (\mathfrak{a} \backslash \mathfrak{d}_\zeta))$. If not, for each $\zeta < \zeta^*$ some $g_\zeta \in \Pi\mathfrak{a}$ and $\alpha_\zeta < \lambda$ exemplifies it, i.e. $\alpha \in A \backslash \alpha_\zeta \Rightarrow \neg(g \restriction (\mathfrak{a} \backslash \mathfrak{d}_\zeta) < f_\alpha \restriction (\mathfrak{a} \backslash \mathfrak{d}_\zeta))$. So defined the function $g \in \Pi\mathfrak{a}$ by $g(\theta) =: \sup[\{g_\zeta(\theta) : \zeta < \zeta^*\} \cup \{f_{\alpha^*}(\theta) + 1\}]$ is well defined where we let $\alpha^* = \sup\{\alpha_\zeta : \zeta < \zeta^*\} < \lambda$. Now $\{f_\alpha : \alpha < \lambda\}$ is $<_J$-increasing and $<_J$-cofinal in $\Pi\mathfrak{a}$ so for some $\alpha \in A \backslash (\alpha^* + 1)$ we have $g <_J f_\alpha$, so for some $\zeta < \zeta^*$ we have $\{\theta \in \mathfrak{a} : \neg(g(\theta) < f_\alpha(\theta))\} \subseteq \mathfrak{d}_\zeta$, so for $\mathfrak{d}_\zeta$, the pair (g_ζ, α_ζ) is not as required, a contradiction.
5), 6, 7) Left to the reader. $\square_{6.9}$

Definition 6.10. *We say* $(\Pi\mathfrak{a}, <_J)$ *is* x*-free*^{y} *if there is a* $<_J$*-increasing* $<_J$*-cofinal* $\langle f_\alpha : \alpha < \lambda \rangle$ *from* $\Pi\mathfrak{a}$ *which is* x*-free*^{y}.

Fact 6.11. If $y \in \{1, 2, 3, \{2, 3\}\}$, $x \in \{\mu - J, (\mu, \theta) - J\}$ and $(\Pi\mathfrak{a}, <_J)$ is x-free^y and $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$-increasing $<_J$-cofinal in $\Pi\mathfrak{a}$ <u>then</u> for some $A \in [\lambda]^\lambda$ the set $\{f_\alpha : \alpha \in A\}$ is x-free^y.

Proof. Straight.

Question 6.12. Let $\mu > \kappa \ge \text{cf}(\mu)$. For how many $\mathfrak{a} \subseteq \text{Reg} \cap \mu$ such that $\mu = \sup(\mathfrak{a})$, $J_{\mathfrak{a}}^{\text{bd}} \subseteq J$, J an ideal on $\mathfrak{a}$, $\lambda = \text{tcf}(\Pi\mathfrak{a}, <_J)$, is $(\Pi\mathfrak{a}, <_J)$ not μ-free?

The proof of [Sh:g, Ch. IX,3.5] has a gap (in the reference to [Sh:g, Ch. IX,3.3A]). What we know is only

Lemma 6.13. *1) Assume* $\lambda > \theta > \text{cf}(\lambda) \ge \sigma = \text{cf}(\sigma) > \aleph_0$. <u>*Then*</u> $\text{cov}(\lambda, \lambda, \theta, \sigma) =^+ \sup \bigcup\{\text{pcf}_{\Gamma(\theta,\sigma), J_{\mathfrak{a}}^{bd}}(\mathfrak{a}) : \mathfrak{a} \subseteq \text{Reg} \cap \lambda, \lambda = \sup(\mathfrak{a}), |\mathfrak{a}| < \min(\mathfrak{a})\}$ *where*

$$\text{pcf}_{\Gamma(\theta,\sigma),J}(\mathfrak{a}) = \Big\{\text{tcf}(\Pi\mathfrak{a}, <_I) : I \textit{ an ideal on } \mathfrak{a} \textit{ extending } J, \text{tcf}(\Pi\mathfrak{a}, <_I) \textit{ well defined, } I \textit{ is } \sigma\textit{-complete and for some } \mathfrak{b} \in I, |\mathfrak{a} \backslash \mathfrak{b}| < \theta\Big\}$$

[3] Note: if λ is a successor of regular, $\lambda > \kappa^+$ then this holds.
[4] Note: if μ_i is regular $> \theta^+$ then this holds.

(the $=^+$ means that if the left side is regular then the supremum in the right side is obtained).
2) If in addition $(\forall \mu < \lambda)(\mathrm{cov}(\mu, \theta, \theta, \sigma) < \lambda)$, $(\theta, \sigma$ *regular)* <u>*then*</u>

$$\mathrm{cov}(\lambda, \lambda, \theta, \sigma) =^+ \mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda).$$

3) So for $\mathcal{Y} = \mathcal{Y}_\mu$, $Eq = Eq_\mu$ *be as in [*Sh 420, §*3*-§*5], * $\mathrm{cf}(\mu) = \sigma > \aleph_0$ *for simplicity,* $\mu > \theta > \mathrm{cf}(\mu)$.
If $\mathfrak{a} \subseteq \mathrm{Reg}\backslash\mu, |\mathfrak{a}| < \mu, |\mathfrak{a}| < \min(\mathfrak{a}), \lambda$ *inaccessible,* $J = J^{\mathrm{bd}}_{\mathfrak{a}}$, $\lambda = \sup \mathrm{pcf}_{\Gamma(\theta,\sigma), J^{\mathrm{bd}}}(\mathfrak{a})$*, then we can find* $e \in Eq, \bar\lambda = \langle \lambda_x : x \in \mathcal{Y}_\mu/e\rangle$ *and* $D \in FIL(\mathcal{Y}_\mu)$ *such that:*

$$\lambda = \mathrm{tcf}(\Pi\bar\lambda/D)$$
$$\lim_D(\bar\lambda) = \mu$$
$$\lambda_x = \mathrm{cf}(\lambda_x).$$

Proof. As in [Sh 410],§1 (replacing normal by σ-complete) or make the following changes in the proof of [Sh:g, Ch. IX,3.5]: $\|N^x_k\| = \mu_k, \mu_k + 1 \subseteq N^x_k$.

Claim 6.14. 1) Assume

- (i) λ is inaccessible
- (ii) $|\mathfrak{a}|^+ < \min(\mathfrak{a})$,
- (iii) $\mu =: \sup(\mathfrak{a}) < \lambda$
- (iv) $R \subseteq \lambda \cap \mathrm{Reg}\backslash\mu, |R| = \lambda$,
- (v) for $\tau \in R, \mathfrak{b}_\tau \subseteq \mathfrak{a}, \sup(\mathfrak{b}_\tau) = \mu, J_\tau$ an ideal on $\mathfrak{b}_\tau$ including $J^{\mathrm{bd}}_{\mathfrak{b}_\tau}, \tau = \mathrm{tcf}(\Pi\mathfrak{b}_\tau, <_{J_\tau})$
- (vi) $\lambda \notin \mathrm{pcf}(\mathfrak{a})$

<u>Then</u> for some $\langle \lambda_\theta : \theta \in \mathfrak{a} \rangle$ we have

- (a) $\lambda_\theta = \mathrm{cf}(\lambda_\theta) < \theta, \lambda_\theta > |\mathfrak{a}|$
- (b) $\lim_{J_\tau} \langle \lambda_\theta : \theta \in b_\theta \rangle = \mu$
- (c) $\lambda = \mathrm{tcf}(\prod_{\theta\in\mathfrak{a}} \lambda_\theta, <_{J_{<\lambda}[\mathfrak{a}]})$ so $\lambda = \max \mathrm{pcf}\{\lambda_\theta : \theta \in \mathfrak{a}\}$
- (d) $R' =: \{\min \mathrm{pcf}_{J_\tau}(\prod_{\theta\in\mathfrak{b}_\tau} \lambda_\theta) : \tau \in R\}$ is[5] unbounded in λ.

So $\mathfrak{a}' =: \{\lambda_\theta : \theta \in \mathfrak{a}\}, R', \lambda, \mu$ satisfy clauses (i),(ii), (iv), (v) and
$(iii)^-$ $|\mathfrak{a}'|^+ \le \min(\mathfrak{a}')$
$(vi)^*$ $\max \mathrm{pcf}(\mathfrak{a}') = \lambda$.

2) Assume $(i), (ii), (iii)^-, (iv), (v), (vi)^*$ are satisfied by $\mathfrak{a}, R, \mathfrak{b}_\tau, J_\tau$ (for $\tau \in R$). <u>Then</u> for some $f_\tau \in \Pi\mathfrak{b}_\tau$ for $\tau \in R$ we have:

$(*)$ for every $g \in \Pi\mathfrak{a}$ for some τ we have $g \restriction \mathfrak{b}_\tau <_{J_\tau} f_\tau$.

[5] Recall $\mathrm{pcf}_J(\Pi\{\lambda_x : x \in X\})$, J an ideal on X is $\{\mathrm{tcf}(\Pi\lambda_*/D) : D$ an ultrafilter on X disjoint to the ideal $J\}$.

Proof. 1) By [Sh:g, Ch. II,1.5A] we can find $\langle \lambda_\theta : \theta \in \mathfrak{a} \rangle$ satisfying (a), (b), (c). If (d) fails, choose for each $\tau \in R$, $\mathfrak{b}'_\tau \in J^+_\tau$ such that $\chi_\tau = \text{tcf}(\prod_{\theta \in \mathfrak{b}'_\tau} \lambda_\theta, <_{J_\tau \restriction \mathfrak{b}'_\tau})$ is well defined and equal to min $\text{pcf}_{J_\tau}(\prod_{\theta \in \mathfrak{b}_\tau} \lambda_\theta, <_{J_\tau})$. So $\{\chi_\tau : \tau \in R\}$ is bounded in λ hence for some χ, $R' = \{\tau \in R : \chi_\tau = \chi\}$ is unbounded in λ. Hence we can find $\zeta^* < |\mathfrak{a}|^+$ and $\tau_\zeta \in R'$ for $\zeta < \zeta^*$ such that $\lambda \leq \max \text{pcf}\{\tau_\zeta : \zeta < \zeta^*\}$ (why? choose by induction on $\zeta < |\mathfrak{a}|^+$, $\tau_\zeta \in R'$, $\tau_\zeta > \max \text{pcf}\{\tau_\varepsilon : \varepsilon < \zeta\}$ and use localization). Let D_ζ be an ultrafilter on $\mathfrak{a}$ such that $\mathfrak{b}'_{\tau_\zeta} \in D_\zeta$, $J_{\tau_\zeta} \cap D_\zeta = \emptyset$, so $\text{tcf}(\Pi\mathfrak{a}, <_{D_\zeta}) = \tau_\zeta$, and let E be an ultrafilter on $\{\tau_\zeta : \zeta < \zeta^*\}$ such that $\text{tcf}(\prod_\zeta \tau_\zeta, <_E) \geq \lambda$. Let $D = \{\mathfrak{c} \subseteq \mathfrak{a} : \{\zeta : \mathfrak{c} \in D_\zeta\} \in E\}$. By [Sh:g, Ch. I,1.10,1.11] we conclude $\text{tcf}(\Pi\mathfrak{a}, <_D) \geq \lambda$ hence $D \cap J_{<\lambda}[\mathfrak{a}] = \emptyset$. By clause (c) clearly $\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D$ has true cofinality λ. Also letting $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \chi \notin \text{pcf}\{\lambda_\theta : \theta \in \mathfrak{b}\}$, it is an ideal such that for any ultrafilter D' on $\mathfrak{a}$ we have $\chi = \text{tcf}(\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D') = \chi \Leftrightarrow D \cap J = \emptyset$, hence in particular $\zeta < \zeta^* \Rightarrow D_\zeta \cap J = \emptyset$ hence by the choice of D we have $D \cap J = \emptyset$ hence $\text{tcf}(\prod_{\theta \in \mathfrak{a}} \lambda_\theta / D) = \chi < \lambda$ contradicting an earlier sentence.

2) Let $\langle f^*_\alpha : \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing and cofinal. Let $f_\alpha = f^*_\alpha \restriction \mathfrak{b}_\alpha$. $\square_{6.14}$

Concerning [Sh 430, 3.1] we comment

Theorem 6.15. *1) Assume $\lambda > \theta \geq \kappa > \aleph_0$ are regular and*

$(*)_{\theta,\kappa}$ *if* $\mathfrak{a} \subseteq \text{Reg} \cap \lambda \backslash \theta$ *and* $|\mathfrak{a}| < \theta$ *then there are* $\zeta^* < \lambda$ *and* $\mathfrak{b}_\zeta \in J_{<\lambda}[\mathfrak{a}]$ *for* $\zeta < \zeta^*$ *such that for every* $\mathfrak{b} \in [\mathfrak{a}]^{<\kappa}$ *for some* $\zeta < \zeta^*$, $\mathfrak{b} \subseteq \mathfrak{b}_\zeta$.

Then the following conditions are equivalent:

$(A) = (A)_{\lambda,\theta,\kappa}$ *for every* $\mu < \lambda$ *we have* $\text{cov}(\mu, \theta, \kappa, 2) < \lambda$

$(B) = (B)_{\lambda,\theta,\kappa}$ *if* $\mu < \lambda$ *and* $a_\alpha \in [\mu]^{<\kappa}$ *for* $\alpha < \lambda$ *then for some* $W \subseteq \lambda$ *of cardinality* λ *we have* $|\bigcup_{\alpha \in W} a_\alpha| < \theta$

$(C) = (C)_{\lambda,\theta,\kappa}$ *if* a_α *is a set of cardinality* $< \kappa$ *for* $\alpha < \lambda$ *and* $W_0 \subseteq \{\delta < \lambda : \text{cf}(\delta) \geq \kappa\}$ *then for some stationary* $W \subseteq W_0$ *and set* b *of cardinality* $< \theta$ *we have* $\langle a_\alpha \backslash b : \alpha \in W \rangle$ *is a sequence of pairwise disjoint sets.*

2) If $\lambda > \theta_1 > \theta_2 \geq \kappa > \aleph_0$ *where* λ, κ *are regular,* $(A)_{\lambda,\theta,\kappa} \Leftrightarrow (B)_{\lambda,\theta_1,\kappa}$ *and* $cov(\theta_1, \theta_2, \kappa, 2) < \lambda$ *then* $(A)_{\lambda,\theta_2,\kappa} \Leftrightarrow (B)_{\lambda,\theta_2,\kappa}$ *(so if for some* θ_1, $(*)_{\theta_1,\kappa}$, $\lambda > \theta_1 > \theta_2 = cf(\theta) \geq \kappa$ *and* $cov(\theta_1, \theta, \kappa, 2) < \lambda$ *then the conclusions holds).*

Proof. 1) Read the proof of [Sh 430, 3.1] (which was written in a way appropriate to this generalization), but defining the M_n, $\langle N^n_\zeta : \zeta < \theta \rangle$, we omit clause (d), that is, $N^n_\zeta \in \mathfrak{A}_{\delta(*)}$ and instead demand

$(d)'$ for each n we can find $\mathcal{P}_n \subseteq [\theta]^{<\theta}$ such that $(\forall a \in [\theta]^{<\kappa})(\exists b \in \mathcal{P})(a \subseteq b)$ and $\langle N^n_b : b \in \mathcal{P}_n \rangle$ such that $N^n_b \prec \mathfrak{B}_{\delta(*)}$, $N^n_\zeta = \bigcup\{N^n_b : b \in \mathcal{P}_n, b \subseteq \zeta\}$ and $b_1 \subseteq b_2 \Rightarrow N^n_{b_1} \prec N^n_{b_2}$ and $f_n \restriction (\text{Reg} \cap N^n_b) \in \mathfrak{B}$ for $b \in \mathcal{P}_n$.

2) Left to the reader.

Question 6.16. Can we in [Sh 430, 4.2](1) weaken clause (β) in the conclusion to "$\lambda_x > \mu_0$ for D-almost all $x \in \mathcal{Y}/e$" then we can weaken the hypothesis [Sh 420, 6.1C] (was stated in [Sh 430], earlier version clear).

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