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LARGE INDECOMPOSABLE MINIMAL GROUPS

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Abstract

Assuming $V = L$ we prove that there exist indecomposable almost-free minimal groups of size λ for every regular cardinal λ below the first weakly compact cardinal. This is to say that there are indecomposable almost-free torsion-free abelian groups of cardinality λ which are isomorphic to all of their finite index subgroups.

1. Introduction

The concept of *minimality* has been studied in various contexts and in different manners. In general group theory, Robinson and Timm [11] called such groups *hc-groups*, the terminology deriving from considerations of connectedness of manifolds. Moreover, the related concept of *quasi-minimality* was first introduced in a topological context (by, for example, Ginsburg and Sands [4]): a topological space is called quasi-minimal if it is homeomorphic to all its subspaces of the same cardinality. An abelian group G is said to be *quasi-minimal (purely quasi-minimal, directly quasi-minimal, minimal)* if G is isomorphic to its subgroups (pure subgroups, direct summands, subgroups of finite index, respectively) of the same cardinality as G. A comprehensive study of all these notions can be found in a Ph.D. thesis by Ó hÓgáin [10] (see also [5, 6]). At present the quasi-minimal groups can be completely characterized while a complete characterization of the purely quasi-minimal groups can only be achieved assuming additional set theory, for example, the generalized continuum hypothesis (GCH) (see [6]). The situation for directly quasi-minimal groups is even more open and can be shown to be undecidable in ZFC (see [6]). Minimal torsion groups and minimal mixed groups were investigated in [5], while [10] takes care of the minimal torsion-free abelian groups. Besides nice examples of characterizations of large classes of minimal torsion-free groups, it was shown in [10] that there exist indecomposable minimal torsion-free abelian groups of size not exceeding the continuum $2^{\aleph_0}$. However, it was an open question whether one can construct arbitrarily large indecomposable minimal groups.

In this paper we show that in Gödel's constructible universe L there are indecomposable almost-free minimal abelian groups of size λ for any regular uncountable cardinal λ below the first weakly compact cardinal. The ideas follow those of [9] but are considerably easier.

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All groups under consideration are abelian. For notations and basic facts, we refer to [3] for abelian groups and [2] or [7] for set-theory.

2. Minimal groups

In this chapter we state some of the basic facts and results about minimal abelian groups. All of it can be found in [10] which is the most comprehensive work on aspects of minimality in abelian group theory (see also [6]). A subgroup H of an abelian group G is of finite index in G if the quotient G/H is finite.

Definition 2.1 An abelian group G is called (p)-minimal (p a prime) if whenever G/H is finite (isomorphic to $\mathbb{Z}/p\mathbb{Z}$) for some subgroup H of G, then $G \cong H$.

Clearly, divisible groups are minimal and it is easily established that an abelian group G is minimal if and only if its reduced part is minimal (see [10, Chapter III, Theorem 1.4]). Moreover, if G is a mixed minimal abelian group, then both its torsion part $t(G)$ and the quotient $G/t(G)$ are minimal [10, Chapter IV, Theorem 6.1]. However, the converse implication is known in some very special cases only, for example, if G splits or $t(G)$ is a p-group and $G/t(G) \cong \mathbb{Q}$. Thus most of the work is concentrated on the torsion and torsion-free cases.

The first lemma reduces the question of minimality to p-minimality for every prime p. Note that (p)-minimality is preserved under isomorphism as is easily checked (see also [10, Chapter III, Theorem 1.6]).

Lemma 2.2 *A torsion-free abelian group G is minimal if and only if G is p-minimal for every prime p.*

Proof. Let G be a torsion-free group. One implication is obvious, hence assume that G is p-minimal for every prime p. Let $H \subseteq G$ be such that $|G/H| = n$ for some positive integer n. Let $n = \prod_{k \leq m} p_k$ be the unique factorization of n into a product of (not necessarily distinct) primes. We induct on the number of primes m. If $m = 1$ then there is nothing to prove by assumption. Hence assume that $m \geq 2$. Write $n = pn'$ and choose an element $x \in G \backslash H$ such that $x + H$ has order p in G/H. Put $H' = \langle H, x \rangle$, then $|G/H'| = n'$ and thus $G \cong H'$ by induction. Finally, $H'/H \cong \mathbb{Z}/p\mathbb{Z}$ shows that $G \cong H' \cong H$.

In [10, Chapter IV] a complete characterization of the torsion minimal groups was obtained. If G is an abelian p-group, then we denote by $f_n(G)$ the nth Ulm invariant of G (see [3, Chapter VI, p. 154]).

Theorem 2.3 *Let G be a reduced torsion abelian group. Then the following hold.*

(i) *G is minimal if and only if all its primary components are minimal.*
(ii) *If G is a p-group, then G is minimal if and only if $f_n(G)$ is infinite for all $n < \omega$ or there exists some $m < \omega$ such that $f_n(G)$ is infinite for all $n < m$ and $f_n(G) = 0$ for all $n \geq m$.*

As a consequence a reduced p-group G is minimal if and only if $G/p^\omega G$ is minimal. Moreover, direct sums of minimal p-groups are minimal, while direct summands of minimal p-groups need not be minimal again.

In the torsion-free case one of the most useful results obtained in [10, Chapter V, Theorems 2.1, and 2.3] is the following.

THEOREM 2.4 *If G is a torsion-free algebraically-compact group, then G is minimal. Moreover, every torsion-free group G of p-rank at most 1 for all primes p is minimal.*

Thus all pure subgroups of the p-adic integers J_p and their sum $\bigoplus_p J_p$ and product $\prod_p J_p$, respectively, are minimal, which establishes the existence of indecomposable minimal groups of size up to the continuum in ZFC. Motivated by this result Ó hÓgáin raised the question whether or not there are arbitrarily large indecomposable minimal groups in ZFC (see [10, page 96]). We shall show in this paper that at least in Gödel's universe there are such groups, but let us first state some more results from [10]. Although any free group is minimal, complete decomposability is not sufficient to ensure minimality, as the example [10, p. 108] shows. A characterization of the minimal completely decomposable groups of finite rank was established in [10, Chapter V, Theorem 4.17] and later extended to infinite rank by the second author [12, Theorem 2.5].

THEOREM 2.5 *If $G = \bigoplus_{i\in I} G_i$ is completely decomposable with each G_i of rank 1, then G is minimal if and only if*

(i) *there is no countable ascending chain of types $\{type(G_{i_n}) : n \in \omega, i_n \in I\}$ with q-rank$(G_{i_n}) = 1$ for some prime q and all $n \in \omega$;*
(ii) *p-rank(G_i) + p-rank$(G_j) \leq 1$ for all primes p and pairs G_i, G_j of incomparable type.*

If the completely decomposable group is of infinite rank, then we obtain another characterization, but we need in addition to the p-rank condition that G is cofinitely hereditarily separable. This follows from a more general theorem on the minimal separable groups which can be characterized in the following way [10, Chapter V, Theorem 4.20].

THEOREM 2.6 *A torsion-free separable group G is minimal if and only if every finite rank summand of G is minimal and G is cofinitely hereditarily separable, that is, every finite index subgroup of G is separable.*

This interesting result shows that a direct summand of a minimal separable group is again minimal and that Whitehead groups, that is, groups satisfying $\mathrm{Ext}(G, \mathbb{Z}) = 0$, are minimal. In general even every coseparable group is minimal by [10, Chapter V, Proposition 4.31] (a coseparable group is a torsion-free group G such that $\mathrm{Ext}(G, \mathbb{Z})$ is torsion-free). However, we have the following surprising example [10, Chapter V, Theorem 4.34].

EXAMPLE 2.7 The Baer Specker group $P = \prod_{\aleph_0} \mathbb{Z}$ is separable and homogeneous of type $\mathbb{Z}$ but not minimal.

3. Our prediction principle

Let us recall some basic prediction principles from set-theory. Let λ be an uncountable regular cardinal and S a stationary subset of λ. We have the following principles.

PREDICTION PRINCIPLES 3.1 *By $\diamondsuit_\lambda(S)$ and $\diamondsuit^+_\lambda(S)$ we denote the following.*

$\diamondsuit_\lambda(S)$: *There is a family of functions* $\{h_\delta : \delta \to \delta \mid \delta \in S\}$ *such that for every function* $h : \lambda \to \lambda$, *the set* $\{\delta \in S \mid h\restriction_\delta = h_\delta\}$ *is stationary in* λ.

$\diamondsuit^+_\lambda(S)$: *There exists a sequence* $\{S_\alpha \mid \alpha \in S\}$ *of subsets of the powerset* $\mathbb{P}(\alpha)$ *of* α *such that* $|S_\alpha| \leq \alpha$ *and for every* $X \subseteq \lambda$ *there is a closed and unbounded subset* $C \subseteq \lambda$ *such that* $X \cap \alpha \in S_\alpha$ *and* $C \cap \alpha \in S_\alpha$ *for all* $\alpha \in C \cap S$.

Both principles are well-known prediction principles found by Jensen (see [1] or [7, 8]). The first was used to prove the existence of Souslin trees, the latter to show the existence of Kurepa trees under $V = L$. Note that $\diamondsuit^+_\lambda(S)$ holds for every stationary set S in case λ is an (infinite) successor cardinal or strongly inaccessible. For all other regular uncountable cardinals we have $\diamondsuit^+_\lambda(S)$ at least for $S = \{\alpha \in \lambda \mid cf(\alpha) = \aleph_0\}$.

For the convenience of the reader we state a different combined version of $\diamondsuit_\lambda(S)$ and $\diamondsuit^+_\lambda(S)$ and call it $\diamondsuit^{(+)}_\lambda(S)$.

THE $\diamondsuit^{(+)}_\lambda(S)$ PRINCIPLE 3.2 *There is a sequence* $\bar{P} = \langle P_\delta \mid \delta \in S\rangle$ *of sets and a sequence of free groups* $\langle M_\delta : \delta \in S\rangle$ *such that*

(i) $P_\delta \subseteq \{(A, C) \mid A \subseteq \delta, C \subseteq \delta$ *closed* $\}$;

(ii) $|P_\delta| \leq |\delta|$;

(iii) *for every* $A \subseteq \lambda$ *there is a closed unbounded subset* $C \subseteq \lambda$ *such that* $(A \cap \delta, C \cap \delta) \in P_\delta$ *for all* $\delta \in C \cap S$;

(iv) *if* M *is a* λ*-free group of size* λ *and* $M = M_1 \oplus M_2$, *then* $M_1\restriction_\delta = M_\delta$ *for stationarily many* $\delta \in S$.

We first show that $\diamondsuit^{(+)}_\lambda(S)$ follows from $\diamondsuit_\lambda(S)$ and $\diamondsuit^+_\lambda(S)$.

THEOREM 3.3 *Assume* $\diamondsuit_\lambda(S)$ *and* $\diamondsuit^+_\lambda(S)$ *hold for a regular uncountable cardinal* λ *and a stationary subset* S *of* λ. *Then also* $\diamondsuit^{(+)}_\lambda(S)$ *holds. In particular,* $(V = L)$ *implies that for every regular uncountable cardinal* λ *there is a stationary subset* $S \subseteq \lambda$ *such that* $\diamondsuit^{(+)}_\lambda(S)$ *holds.*

Proof. It is easy and by now standard to see that $\diamondsuit_\lambda(S)$ implies the existence of the family of groups $\langle M_\delta : \delta \in S\rangle$, so we leave the details to the reader. Now, let the sequence $\{S_\delta : \delta \in S\}$ be given by $\diamondsuit^+_\lambda(S)$. We define $P_\delta = \{(A, C) \mid A, C \in S_\delta$ and C is closed in $\delta\}$ for $\delta \in S$. By definition, (i) and (ii) follow since $|S_\delta| \leq \delta$ for all $\delta \in S$. Finally, if $A \subseteq \lambda$, then $\diamondsuit^+_\lambda(S)$ implies the existence of some cub $C \subseteq \lambda$ such that $A \cap \delta \in S_\delta$ and $C \cap \delta \in S_\delta$ for all $\delta \in C$. Clearly, $C \cap \delta$ is still closed in δ but not necessarily unbounded. Thus (iii) follows.

Let us explain why the principle is useful to construct indecomposable minimal groups. We shall construct our indecomposable minimal group G as the union of an ascending chain $\langle G_\alpha : \alpha < \lambda\rangle$ of free groups of smaller cardinality than λ. In order to ensure that the resulting group is minimal we will consider any finite index subgroup H of G. Then there is a cub C such that the pairs $(H \cap \delta, \delta \cap C)$ were guessed by $\diamondsuit^{(+)}_\lambda(S)$ (iii) sufficiently often. We may assume that $G_\alpha \not\subseteq H$ for all $\alpha \in C$, and this will be used to extend a partial isomorphism of G_α onto $H \cap G_\alpha$ (for $\alpha \in C$) inductively to one of G and H. At the same time we shall use the diamond functions coming from $\diamondsuit^{(+)}_\lambda(S)$ (iv) to ensure indecomposability. To do so we introduce the following convenient notation: Assume $\diamondsuit^{(+)}_\lambda(S)$ holds. If G is an abelian group with $\delta = |G| < \lambda$ and $G = K_1 \oplus K_2$ is a non-trivial direct decomposition of G such that $K_1 \cong M_\delta$, then we say that this decomposition was guessed by $\diamondsuit^{(+)}_\lambda(S)$.

We conclude this section with a few remarks on large cardinals. Recall that λ is *strongly inaccessible* if it is an uncountable regular strong limit cardinal, where κ is said to be a *strong limit cardinal* if $2^\gamma < \kappa$ for every $\gamma < \kappa$. The existence of strongly inaccessible cardinals cannot be shown to be relatively consistent with ZFC but on the other hand no one has yet proved that their existence is inconsistent. Moreover, a cardinal κ is *weakly compact* if it is strongly inaccessible and for every κ-complete Boolean subalgebra B of the powerset of κ which is of cardinality κ, every κ-complete filter on B is contained in a κ-complete ultrafilter on B ([**2**, p. 29]). For instance, measurable cardinals are weakly compact, and there are κ weakly compact cardinals below a measurable cardinal κ. Last but not least, there are κ strongly inaccessible cardinals below a weakly compact cardinal κ.

4. The main theorem

In this section we shall prove the following main theorem.

MAIN THEOREM 4.1 ($V = L$) Let λ be a regular uncountable cardinal below the first weakly compact cardinal. Then there exists an almost-free indecomposable minimal abelian group of cardinality λ.

The proof of our main theorem will use the principle $\diamondsuit_\lambda^{(+)}(S)$, and for technical reasons we shall restrict ourselves to strongly inaccessible cardinals λ. This will make some cardinal arguments more transparent at some places by the choice of the stationary set S (see below). However, this is not a serious restriction as we shall point out in the proof.

From now on let λ be a fixed strongly inaccessible cardinal below the first weakly compact cardinal and let

$$S \subseteq \{\delta < \lambda \mid \delta \text{ a singular strong limit ordinal and } \delta = \beth_\delta\}$$

be stationary, co-stationary and non-reflecting such that for every $\gamma \in S$ we have $\mathrm{cf}(\gamma) = \aleph_0$ (for the existence see, for example, [**2**, Theorem VI 3.2]). Moreover, assume that $\diamondsuit_\lambda^{(+)}(S)$ holds. We will construct by induction on $\alpha < \lambda$ an increasing chain of torsion-free abelian groups G_α ($\alpha < \lambda$) with the following properties.

(T1) G_α is free abelian with $|G_\alpha| = \beth_\alpha$.
(T2) $\{G_\alpha \mid \alpha < \lambda\}$ is increasing continuous.
(T3) If $\alpha > \delta \notin S$, then G_α/G_δ is free of rank $\beth_\alpha$.
(T4) If $\alpha = \delta + 1$ and $\delta \in S$, then G_α/G_δ is divisible.
(T5) There is a sequence of maps $\langle f_{(A,C)} : (A, C) \in \bar{P}_\alpha \rangle$ where $\bar{P}_\alpha = \{(A, C) \in P_\alpha \mid A \leq G_\alpha, G_\alpha/A \cong \mathbb{Z}/p\mathbb{Z}$ and $G_{\min(C)}/(A \cap G_{\min(C)}) \cong \mathbb{Z}/p\mathbb{Z}$ for some prime $p = p_{(A,C)}\}$ such that
  (a) if $(A, C) \in \bar{P}_\alpha$, then $f_{(A,C)}$ is an isomorphism from G_α onto A;
  (b) if $(A, C) \in \bar{P}_\alpha$ and $\alpha \geq \beta > \min(C)$ (hence $(A \cap \beta, C \cap \beta) \in \bar{P}_\beta$), then $f_{(A\cap\beta,C\cap\beta)} \subseteq f_{(A,C)}$ as functions; moreover, if $\gamma = \sup\{\beta \mid \beta \in I\}$ for some $I \subseteq (\min(C), \alpha]$, then $f_{(A\cap\gamma,C\cap\gamma)} = \bigcup_{\beta\in I} f_{(A\cap\beta,C\cap\beta)}$;
  (c) if $(A, C) \in \bar{P}_\alpha$, $\beta < \alpha$, $\beta \notin S$, then $f_{(A,C)}\restriction_{G_{\beta+1}}$ is an endomorphism of $G_{\beta+1}$ mapping G_β onto $G_\beta \cap A$.
(T6) If $\alpha = \delta + 1$, $\delta \in S$ and $G_\delta = G_\delta^1 \oplus G_\delta^2$ is a non-trivial decomposition of G_δ which was guessed by $\diamondsuit_\lambda^{(+)}(S)$ such that $\delta = \sup\{\gamma < \delta \mid \gamma \notin S, G_\gamma = (G_\gamma \cap G_\delta^1) \oplus (G_\gamma \cap G_\delta^2)$ *non-trivial*$\}$, then there is no $g \in \mathrm{End}(G_\alpha)$ such that $g\restriction_{G_\delta^1} = id_{G_\delta^1}$ and $g\restriction_{G_\delta^2} = 0$.

Let us first remark a few things: the idea is to construct the desired group along with partial isomorphisms $f_{(A,C)} : G_\alpha \to A$ such that in the end, taking a finite index subgroup $\tilde{A}$ of $G = \bigcup_{\alpha<\lambda} G_\alpha$ we shall obtain an isomorphism $f : G \to \tilde{A}$ as the union of the partial isomorphisms $f_{(\tilde{A}\cap\beta, C\cap\beta)}$ for β in some cub. The reason we can do this is the principle $\diamondsuit_\lambda^{(+)}(S)$ that predicts the subgroup $\tilde{A}$ sufficiently often. Moreover, along the construction we will also 'kill' undesired projections onto direct summands of G, so that G becomes indecomposable.

Concerning the stated properties of our chain of groups G_α, let us mention that (T5)(b) ensures continuity of the partial isomorphisms $f_{(A,C)}$ and that for $(A, C) \in \bar{P}_\alpha$ and $\beta > \min(C)$, $p = p_{(A,C)}$ we have

$$\mathbb{Z}/p\mathbb{Z} \cong G_{\min(C)}/(A \cap G_{\min(C)}) \hookrightarrow G_\beta/(A \cap G_\beta) \hookrightarrow G_\alpha/A \cong \mathbb{Z}/p\mathbb{Z},$$

and hence $(A \cap \beta, C \cap \beta) \in \bar{P}_\beta$ as claimed in (T5)(b). Note that the sets $\bar{P}_\alpha$ are defined — not like in other constructions — after we have constructed the group G_α. Thus the difference is that the guessing is not made at the beginning anymore but in the middle of the construction, and hence depends on the chain members. However, it does not matter whether we know what to do at stage $\alpha + 1$ right from the beginning or just at stage α. Last but not least we would like to remark that on the following pages it will happen many times that groups are identified with their universe, that is, a group G_α will be identified with the cardinal $|G_\alpha|$.

Let us now prove that once we have constructed the chain of groups G_α $(\alpha < \lambda)$, the union $G = \bigcup_{\alpha<\lambda} G_\alpha$ is an indecomposable minimal group of cardinality λ. The construction itself shall be carried out in the next section.

THEOREM 4.2 ($\diamondsuit_\lambda^{(+)}(S)$) *Let G_α $(\alpha < \lambda)$ be as above, then $G = \bigcup_{\alpha<\lambda} G_\alpha$ is an almost-free indecomposable minimal group of cardinality λ.*

For the proof we need two simple results on filtrations of torsion-free abelian groups.

LEMMA 4.3 *Let K be a group with $\lambda = |K|$ regular uncountable and a λ-filtration $K = \bigcup_{\alpha<\lambda} K_\alpha (K_0 = \{0\})$. If $K = H_1 \oplus H_2$, then the set $C = \{\alpha < \lambda \mid K_\alpha = (K_\alpha \cap H_1) \oplus K_\alpha \cap H_2)\}$ is closed and unbounded in λ.*

Proof. The proof is straightforward and thus left to the reader.

LEMMA 4.4 *Let K and C be as in Lemma 4.3 for some decomposition $K = H_1 \oplus H_2$. If S is stationary and co-stationary in λ, then $D = \{\delta < \lambda \mid \delta = \sup\{\gamma < \delta \mid \gamma \in C\backslash S\}\}$ is closed and unbounded in λ.*

Proof. Certainly $0 \in D$, and it is immediate to see that D is closed in λ because C is closed. To prove that D is unbounded let $\delta < \lambda$. We have to show that there is a $\delta \leq \delta' \in D$. Since C is unbounded in λ and S is co-stationary, there is a $\delta \leq \delta_1 \in C\backslash S$. Continuing this way we obtain an increasing sequence $\delta_n \in C\backslash S$ $(n \in \omega)$ of ordinals less than λ. Then $\delta \leq \delta' = \sup\{\delta_n \mid n \in \omega\} \in D$ is as required.

Proof of Theorem 4.2 Clearly G is a torsion-free abelian group of cardinality λ and it is almost-free since all layers G_α are free. First we show that G is indecomposable. Assume that $G = G_1 \oplus G_2$ is a non-trivial decomposition of G and choose a projection $\varphi \in \mathrm{End}(G)$ onto G_1, that is, $\varphi\restriction_{G_1} = \mathrm{id}_{G_1}$

and $\varphi\restriction_{G_2} = 0$. Then

$$D = \{\alpha < \lambda \mid \alpha = \sup\{\gamma < \alpha \mid \gamma \in C\backslash S\}\} = \{\alpha < \lambda \mid \alpha = \sup\{\gamma < \alpha \mid \gamma \notin S, G_\gamma = (G_\gamma \cap G_1) \oplus (G_\gamma \cap G_2)\}\}$$

is a closed and unbounded subset of λ by Lemma 4.4. Let $\delta_1 \in D \cap S$ and choose $\delta_1 < \delta_2 \in D\backslash S$ such that $\varphi\restriction_{G_{\delta_2}} \in \mathrm{End}(G_{\delta_2})$, which is possible since S is co-stationary and the set $\{\delta < \lambda \mid \varphi\restriction_{G_\delta} \in \mathrm{End}(G_\delta)\}$ is a closed and unbounded subset of λ. Moreover, by $\diamondsuit_\lambda^{(+)}(S)$ (iv) we may assume that the decomposition $G_{\delta_1} = (G_{\delta_1} \cap G_1) \oplus (G_{\delta_1} \cap G_2)$ was guessed by $\diamondsuit_\lambda^{(+)}(S)$. By (T3) the quotient $G_{\delta_2}/G_{\delta_1+1}$ is free, and hence there exists a projection $\psi \in \mathrm{End}(G_{\delta_2})$ such that $\psi\restriction_{G_{\delta_1+1}} = \mathrm{id}_{G_{\delta_1+1}}$. Let $g = \varphi\restriction_{G_{\delta_2}} \psi : G_{\delta_2} \to G_{\delta_1+1}$. Now $g\restriction_{G_{\delta_1+1}}$ is a forbidden homomorphism by (T6), a contradiction. Thus G is indecomposable.

It remains to show that G is minimal. As mentioned earlier we will now without loss of generality identify groups with their cardinality several times. Let $H \subset G$ such that $G/H \cong \mathbb{Z}/p\mathbb{Z}$. By Lemma 2.2 it is enough to prove that $H \cong G$. Let $\alpha^* = \min\{\alpha < \lambda \mid G_\alpha \nsubseteq H\}$. Note that α^* exists because $H \neq G$. By $\diamondsuit_\lambda^{(+)}(S)$ there is a closed and unbounded subset $C \subseteq \lambda$ such that for $\delta \in C \cap S$ we have $(H \cap G_\delta, C \cap \delta) \in P_\delta$ (here we use that for $\delta \in S$ we have $\delta = \beth_\delta = |G_\delta|$). We may assume that $C \subseteq \lambda\backslash(\alpha^* + 1)$. We claim that $(H \cap G_\delta, C \cap \delta) \in \bar{P}_\delta$. Since $\delta > \alpha^* + 1$ we obtain

$$G_\delta/(H \cap G_\delta) \cong (G_\delta + H)/H \neq 0,$$

and thus

$$0 \neq G_\delta/(H \cap G_\delta) \subseteq G/H \cong \mathbb{Z}/p\mathbb{Z}.$$

Therefore $G_\delta/(H \cap G_\delta) \cong \mathbb{Z}/p\mathbb{Z}$ for every $\delta \in C \cap S$ and hence $(H \cap G_\delta, C \cap \delta) \in \bar{P}_\delta$. Applying (T5)(a) we obtain that $f_{(H\cap G_\alpha, C\cap\alpha)}$ is an isomorphism from G_α onto $H \cap G_\alpha$ for every $\alpha \in C \cap S$. Since $\lambda = \sup(C \cap S)$ we get by (T5)(b) that $f = \bigcup_{\alpha\in C\cap S} f_{(H\cap G_\alpha, C\cap\alpha)}$ is a well-defined isomorphism from G onto H. Thus G is a minimal group and this finishes the proof.

5. The construction

It remains to prove that we can inductively define the groups G_α $(\alpha < \lambda)$ satisfying (T1) to (T6) under the assumption of $\diamondsuit_\lambda^{(+)}(S)$.

Let us first introduce some convenient terminology. If $(A, C) \in \bar{P}_\alpha$ and $\alpha \in S$, then we require that $G_\alpha/A \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p. This prime is denoted by $p_{(A,C)}$; however, note that necessarily we will obtain $p_{(A\cap\beta, C\cap\beta)} = p_{(A,C)}$ if $\beta > \min(C)$. Furthermore, if $\alpha \notin S$, then (T3) says that $G_{\alpha+1}/G_\alpha$ is free of rank $\beth_{\alpha+1}$, hence we may write $G_{\alpha+1} = G_\alpha \oplus \bigoplus_{i<\beth_{\alpha+1}} \mathbb{Z}x_i^\alpha$. Using this terminology we will, in addition to (T1) to (T6), carry on some more facts in the induction, namely:

- (T7) if $(A, C) \in \bar{P}_\alpha$ let $y^l_{(A,C)}$ $(l < p_{(A,C)})$ be representatives of the quotient G_α/A such that $y^l_{(A\cap\beta, C\cap\beta)} = y^l_{(A,C)}$ for all $l < p_{(A,C)}$ and $\beta > \min(C)$ (note that this is possible by identifying $G_\beta/(A \cap G_\beta)$ with $(A + G_\beta)/A$);
- (T8) if $\alpha \notin S$, $(A, C) \in \bar{P}_{\alpha+1}$, then there is a permutation $\pi = \pi_{(A\cap\alpha, C\cap\alpha)}$ of $\beth_{\alpha+1}$ such that $f_{(A,C)}(x_i^\alpha) \in \{x^\alpha_{\pi(i)} + y^l_{(A,C)} \mid l < p_{(A,C)}\}$ for all $i < \beth_{\alpha+1}$ (note that this is possible since for every $i \in \beth_{\alpha+1}$ there is $l < p_{(A,C)}$ such that $x_i^\alpha + y^l_{(A,C)} \in A$ as $G_{\alpha+1}/A \cong \mathbb{Z}/p_{(A,C)}\mathbb{Z}$ and $f_{(A,C)}$ is an isomorphism from $G_{\alpha+1}$ onto A);

(T9) if $\alpha \notin S$, $\alpha < \alpha_1$, and $(A_i, C_i) \in \bar{P}_{\alpha_1}$ for $i < i^*$ such that $\langle (A_i \cap \alpha, C_i \cap \alpha) \mid i < i^* \rangle$ is without repetitions, then $\langle \pi_{(A_i \cap \alpha, C_i \cap \alpha)} \mid i < i^* \rangle$ acts freely on $\beth_{\alpha+1}$.

We would like to emphasize that the permutation in (T9) depends only on $(A \cap \alpha, C \cap \alpha)$ and not on the pair (A, C).

We now distinguish four cases for α in the induction, namely $\alpha = 0$, α a limit ordinal, $\alpha = \delta + 1$ and $\delta \notin S$, $\alpha = \delta + 1$ and $\delta \in S$.

Case A: $\alpha = 0$.

Letting $G_0 = \{0\}$ there is nothing to prove. Note that $\bar{P}_0 = \emptyset$ and that (T8) and (T9) will be satisfied later in the construction.

Case B: α a limit ordinal.

Assume that G_β and $f_{(A,C)}$ for $(A, C) \in \bar{P}_\beta$ have been defined for $\beta < \alpha$ satisfying (T1)–(T9). We let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Since S is non-reflecting it is easily established that G_α is a free abelian group of cardinality $\beth_\alpha$ such that G_α / G_δ is free for all $\delta \notin S$, $\delta < \alpha$ (see, for instance, the proof of [2, Theorem VII 1.3]). Thus (T1)–(T3) hold. Moreover, (T4) and (T6) hold trivially since α is a limit ordinal. Let $(A, C) \in \bar{P}_\alpha$ and put $f_{(A,C)} = \bigcup_{\min(C)<\beta<\alpha} f_{(A\cap\beta, C\cap\beta)}$. Then $f_{(A,C)}$ is an isomorphism from G_α onto A and (T5)(a)–(c) are easily established. Furthermore, put $y^l_{(A,C)} = y^l_{(A\cap\beta,C\cap\beta)}$ for some $\beta > \min(C)$ and $l < p_{(A,C)} = p_{(A\cap\beta,C\cap\beta)}$, so (T7) holds. Finally, $\pi_{(A,C)}$ will be constructed and (T8) and (T9) will be proved in the next case for δ a limit ordinal.

Case C: $\alpha = \delta + 1$ and $\delta \notin S$.

Let G_δ and $f_{(A,C)}$ be defined for $(A, C) \in \bar{P}_\delta$ and $\delta \notin S$. We choose independent elements x^δ_i for $i < \beth_\alpha$ and put $G_\alpha = G_\delta \oplus \bigoplus_{i<\beth_\alpha} \mathbb{Z} x^\delta_i$. Then G_α is free of cardinality $\beth_\alpha$. Since G_α / G_δ is free and by assumption G_δ / G_β is free for all $\beta < \delta$, $\beta \notin S$, it follows that (T1)–(T3) hold. Moreover, since $\delta \notin S$ there is nothing to prove for (T4) and (T6). Let $(A, C) \in \bar{P}_\alpha$. Then A is a subgroup of G_α of index p. If $(A \cap \delta, C \cap \delta) \in \bar{P}_\delta$, then let $y^l_{(A,C)} = y^l_{(A\cap\delta,C\cap\delta)}$ for $l < p_{(A,C)}$ and otherwise choose any set of representatives of G_α / A. Then (T7) holds. Now, $A/(A \cap G_\delta)$ embeds into G_α / G_δ which is free by construction. Thus $A = (A \cap G_\delta) \oplus F$ for some free group F of cardinality $\beth_\alpha$. Note that $|G_\alpha| = \beth_\alpha$ and $|G_\delta| = \beth_\delta$. Thus it would be easy to define an isomorphism $f_{(A,C)} : G_\alpha \to A$ extending $f_{(A\cap\delta,C\cap\delta)}$ (if it exists, that is $(A \cap \delta, C \cap \delta) \in \bar{P}_\delta$) by taking the direct sum of the isomorphism $f_{(A\cap\delta,C\cap\delta)}$ and any isomorphism between the two free groups F and $\bigoplus_{i<\beth_\alpha} \mathbb{Z} x^\delta_i$. (If $(A \cap \delta, C \cap \delta) \notin \bar{P}_\delta$, then take $f_{(A,C)}$ to be any isomorphism between G_α and A as free groups.) This would take care of (T5). However, we also want to ensure (T8) and (T9) and thus we have to choose the isomorphism $f_{(A,C)}$ more carefully. We choose a system of permutations $\langle \pi_{(A'\cap\delta,C'\cap\delta)} : (A', C') \in \bar{P}_\alpha \rangle$ without repetitions which acts freely on $\beth_\alpha$. Then (T9) is satisfied. For our given $(A, C) \in \bar{P}_\alpha$ and $\pi_{(A\cap\delta,C\cap\delta)}$ we choose for every $i \in \beth_\alpha$ one of the representatives $y^l_{(A,C)}$, say $y^{l,i}_{(A,C)}$, such that $x^\delta_{\pi_{(A\cap\delta,C\cap\delta)}(i)} + y^{l,i}_{(A,C)} \in F$; This is possible since F is of finite index in $\bigoplus_{i<\beth_\alpha} \mathbb{Z} x^\delta_i$. Then the elements $\{x^\delta_{\pi_{(A\cap\delta,C\cap\delta)}(i)} + y^{l,i}_{(A,C)} : i < \beth_\alpha\}$ form a basis of F. (Note that the $y^{l,i}_{(A,C)}$ are from G_δ). Hence we may define $f_{(A,C)}(x^\delta_i) = x^\delta_{\pi_{(A\cap\delta,C\cap\delta)}(i)} + y^{l,i}_{(A,C)}$ and obtain an isomorphism from $\bigoplus_{i<\beth_\alpha} \mathbb{Z} x^\delta_i$ onto F. Extending $f_{(A\cap\delta,C\cap\delta)}$ in the obvious way by this isomorphism provides $f_{(A,C)}$ such that (T7) and (T8) are satisfied.

Case D: $\alpha = \delta + 1$ and $\delta \in S$.

As $\delta \in S$, hence $\mathrm{cf}(\delta) = \omega$, we may choose a sequence of ordinals $\beta_n \in \delta \setminus S$ such that $\lim_{n\in\omega} \beta_n = \delta$. Moreover, we choose elements $i_n \in \beth_{\beta_n+1}$ for all $n \in \omega$. Finally, let $\langle k_n : n \in \omega \rangle$ be a sequence of integers such that

(1) $k_n > 1$,
(2) k_n divides k_{n+1},
(3) k_n does not divide $\prod_{r<n} k_r$,
(4) for all $p \in \Pi$ there is $m \in \omega$ such that p divides k_m

for all $n \in \omega$. For instance, one can choose $k_n = (n!)!$. In order to ensure (T5) we need to define G_α in such a way that all isomorphisms $f_{(A\cap\delta, C\cap\delta)}$ for $(A, C) \in \bar{P}_\alpha$ lift to isomorphisms from G_α onto A. Since we also require that the quotient G_α / G_δ is divisible, the lifting must be unique. Note that this is possible since for $(A, C), (A', C') \in \bar{P}_\alpha$ with $(A \cap \delta, C \cap \delta) = (A' \cap \delta, C' \cap \delta)$ we necessarily have $A = A'$ because $(A + A')/A \cong A/(A \cap A')$ is divisible and bounded at the same time, hence zero.

Let $F_\delta = \left\langle \sigma_{(A,C)} : (A, C) \in \bar{P}_\delta \right\rangle$ be the non-commutative free group generated by the independent elements $\sigma_{(A,C)}$ for $(A, C) \in \bar{P}_\delta$. Clearly, every element $\sigma = \prod_{(A,C)\in\bar{P}_\delta} \sigma_{(A,C)} \in F_\delta$ corresponds to an endomorphism f_σ of G_δ by composition of the corresponding $f_{(A,C)}$, for example, $f_{\sigma_{(A,C)}} = f_{(A,C)}$ and $f_\sigma = f_{(A_1,C_1)} \circ f_{(A_2,C_2)} \circ \cdots \circ f_{(A_n,C_n)}$ if $\sigma = \sigma_{(A_1,C_1)} \sigma_{(A_2,C_2)} \cdots \sigma_{(A_n,C_n)}$. We now define

$$G_\alpha = \left\langle G_\delta, z_n^\sigma : \sigma \in F_\delta, n \in \omega \right\rangle$$

freely generated by G_δ and the elements z_n^σ for $n \in \omega$ and $\sigma \in F_\delta$ modulo the following relations:

$$k_n z_{n+1}^\sigma = z_n^\sigma - f_\sigma(x_{i_n}^{\beta_n}) \qquad (*)$$

for $\sigma \in F_\delta$ and $n \in \omega$. It is easy to check that G_α is a free abelian group with the following properties. (Recall that for $\delta \in S$ we have $\delta = \beth_\delta = |G_\delta|$)

(1) $G_\delta \subseteq G_\alpha$;
(2) $|G_\alpha| = \beth_\alpha$;
(3) G_α / G_{β_n} is free for $n \in \omega$.

In order to see (1) and (3) we will use clauses (T8) and (T9) (compare also [2]). First note that G_α is generated by the elements $\{x_i^\gamma : i < \beth_{\gamma+1}, \gamma \notin S, \gamma < \alpha\}$ and $\{z_n^\sigma : n \in \omega, \sigma \in F_\gamma, \gamma \in S, \gamma < \alpha\}$. We fix $n \in \omega$ and will define a projection π from G_α onto G_{β_n}. Let $\pi\restriction_{G_{\beta_n}} = \mathrm{id}\restriction_{G_{\beta_n}}$. Furthermore, if $\beta_n \leq \gamma < \alpha$ we define

- $\pi(x_i^\gamma) = 0$ for all $i < \beth_{\gamma+1}$ if $\gamma \notin S$ and
- $\pi(z_m^\sigma) = 0$ for all $m \geq n$ and $\sigma \in F_\gamma$ if $\gamma \in S$.

Finally, if $\beta_n \leq \gamma \in S$ and $m < n$, then we define $\pi(z_m^\sigma)$ by downward induction on m for $\sigma \in F_\gamma$. Here is an example: for simplicity we will assume that $\sigma = \sigma_{(A,C)}$; then

$$\pi(k_{n-1} z_n^\sigma) = \pi(z_{n-1}^\sigma) - \pi(f_\sigma(x_{i_{n-1}}^{\beta_{n-1}})),$$

and hence

$$0 = \pi(z_{n-1}^\sigma) - \pi\left(x_{\pi_\sigma(i_{n-1})}^{\beta_{n-1}} + y_\sigma^{l, i_{n-1}}\right).$$

Consequently, $\pi(z^\sigma_{n-1}) = x^{\beta_{n-1}}_{\pi_\sigma(i_{n-1})} + \pi(y^{l,i_{n-1}}_\sigma)$ is well-defined. Continuing this way we are able to determine the remaining values of $\pi(z^\sigma_m)$ for $m < n$.

Straightforward calculations show that π is a projection, hence G_{β_n} is a direct summand of G_α. The corresponding complement must be free since the $f_{(A,C)}(x^\gamma_i)$ are independent for fixed (A, C) by clause (T8) and also for different $(A, C) \neq (A', C')$ by clause (T9).

Thus (T1)–(T4) hold, (T7) is easily established and there is nothing to show in (T8) while (T9) holds by induction hypothesis. It remains to prove (T5).

Let $(A, C) \in \bar{P}_\alpha$. If $(A \cap \delta, C \cap \delta) \notin \bar{P}_\delta$, then choose any isomorphism between G_α and A as free abelian groups. If $(A \cap \delta, C \cap \delta) \in \bar{P}_\delta$, then we claim that $f_{(A\cap\delta,C\cap\delta)}$ extends uniquely to some $f_{(A,C)}$ from G_α onto A. We first show that for $\sigma \in F_\delta$ we have that

$$z^\sigma_n \in A \quad \text{for all } n \in \omega \quad \textit{if } \sigma = \sigma'\sigma_{(A\cap\delta,C\cap\delta)}$$

for some $\sigma' \in F_\delta$. Since we write maps from the right, $\sigma = \sigma'\sigma_{(A\cap\delta,C\cap\delta)}$ implies that f_σ maps G_δ into A. Thus, A being of finite index $p_{(A,C)}$ in G_α, we choose n large enough such that $p_{(A,C)}$ divides k_m for all $m \geq n$. Then the equation $(*)$ implies that $z^\sigma_m = k_m z^\sigma_{m+1} + f_\sigma(x^{\beta_m}_{i_m}) \in A$ for all $m \geq n$ and by downward induction we also obtain that $z^\sigma_m \in A$ for all $m < n$. We now define $f_{(A,C)}$ by putting

- $f_{(A,C)}\restriction_{G_\delta} = f_{(A\cap\delta,C\cap\delta)}$,
- $f_{(A,C)}(z^\sigma_n) = z_n^{\sigma\sigma_{(A\cap\delta,C\cap\delta)}}$

for all $n \in \omega$ and $\sigma \in F_\delta$. By the above it follows that $f_{(A,C)}$ is well defined and maps G_α into A and is certainly an isomorphism since multiplication by $\sigma_{(A\cap\delta,C\cap\delta)}$ is an isomorphism of the free group F_δ.

It remains to ensure (T6). Therefore let $\bar{t} = \langle \beta^{\bar{t}}_n, i^{\bar{t}}_n, k^{\bar{t}}_n : n \in \omega\rangle$ be defined as above, that is the $\beta^{\bar{t}}_n$ form a sequence of ordinals from $\delta\backslash S$ converging to δ, the $i^{\bar{t}}_n$ are elements from $\beth_{\beta_n+1}$ for all $n \in \omega$ and the $k^{\bar{t}}_n$ form a sequence of integers with properties (1)–(4). Let $G^{\bar{t}}_\alpha$ be the group that satisfies the relations in $(*)$. Formally, the generators $z^{\bar{t},\sigma}_n$ of $G^{\bar{t}}_\alpha$ and also the entries of $\bar{t}$ depend on $\bar{t}$; however, for simplicity we will suppress the index $\bar{t}$ and it will be clear from the context in which group $G^{\bar{t}}_\alpha$ we are working. The aim is now to prove that for some sequence $\bar{t}$ the corresponding group $G^{\bar{t}}_\alpha$ has the desired property (T6). By way of contradiction let us assume that $G_\delta = G^1_\delta \oplus G^2_\delta$ is a non-trivial decomposition of G_δ which was guessed by $\diamondsuit^{(+)}_\lambda(S)$ and assume that $\delta = \sup\{\gamma < \delta | \gamma \notin S, G_\gamma = (G_\gamma \cap G^1_\delta) \oplus (G_\gamma \cap G^2_\delta) \text{ non-trivial}\}$. Let $\varphi : G_\delta \to G^1_\delta$ be the projection of G_δ onto G^1_δ. We claim that there is a choice of $\bar{t}$ such that φ does not lift to an endomorphism of $G^{\bar{t}}_\alpha$. Note that a possible extension of φ to $G^{\bar{t}}_\alpha$ has to be unique since $G^{\bar{t}}_\alpha / G^{\bar{t}}_\delta$ is divisible. Assume not, so for any sequence $\bar{t}$ there is an extension $\varphi^{\bar{t}}$ of φ to $G^{\bar{t}}_\alpha$. If G^2_δ is contained in some G_{β_m} for some m, then we may assume without loss of generality that $G^2_\delta = G_{\beta_m}$ and since G_δ / G_{β_m} is free, this case is easily handled (see, for instance, the book by Eklof and Mekler [2]). Thus assume that $G^2_\delta \not\subseteq G_{\beta_m}$ for all m. Moreover, we may assume that G_{β_m} is fully invariant under φ for all m. In fact, we only consider those $\bar{t}$ with this property. Let $\bar{t}$ be fixed but arbitrary and for simplicity let us denote $\varphi^{\bar{t}}$ by φ. Moreover, let σ^* be the unit element in F_δ and abbreviate $z^{\sigma^*}_n$ by z_n for $n \in \omega$. Then

$$\varphi(z_0) = \sum_{l<l^*} b_l z^{\sigma_l}_{n_l} + x^* \tag{$**$}$$

for some $b_l \in \mathbb{Z}$, $n_l, l^* \in \mathbb{N}$ and $\sigma_l \in F_\delta$, $x^* \in G_\delta$. By the relations from $(*)$ we may assume without loss of generality that $n_l = n^*$ for some fixed integer n^*. Moreover, combining some of the summands in the above representation we may also assume that $\langle \sigma_l : l < l^* \rangle$ is without repetitions.

Now choose a finite subset $P \subseteq \bar{P}_\delta$ such that $\{\sigma_l : l < l^*\}$ is contained in $\langle \sigma_{(A,C)} : (A, C) \in P \rangle$. Moreover, let $m^* \in \omega$ be such that

(i) $\langle (A \cap \beta_{m^*}, C \cap \beta_{m^*}) : (A, C) \in P \rangle$ is without repetitions;
(ii) $m^* > \sum_{l<l^*} |b_l|$;
(iii) $x^* \in G_{\beta_{m^*}}$;
(iv) $m^* > n^*$;
(v) $y^l_{(A,C)} \in G_{\beta_{m^*}}$ for $l < p_{(A,C)}$ and $(A, C) \in P$.

Note that the choice of m^* is possible since $\sup_{n\in\omega} \beta_n = \delta$. We fix $m > m^*$ and calculate the equation from $(**)$ modulo $(G_{\beta_m} + k_m G_\alpha)$. Recall that $x^{\beta_m}_{i_m} \in G_{\beta_m+1} \setminus G_{\beta_m}$ and that k_m does not divide $\prod_{r<m} k_r$; hence we have

$$\begin{aligned} z_0 &= x^{\beta_0}_{i_0} + k_0 z_1 \\ &= x^{\beta_0}_{i_0} + k_0 x^{\beta_1}_{i_1} + k_0 k_1 z_2 \\ &= x^{\beta_0}_{i_0} + k_0 x^{\beta_1}_{i_1} + k_0 k_1 x^{\beta_2}_{i_2} + k_0 k_1 k_2 z_3 \\ &= x^{\beta_0}_{i_0} + k_0 x^{\beta_1}_{i_1} + k_0 k_1 x^{\beta_2}_{i_2} + \cdots + \left(\prod_{r<m} k_r\right) x^{\beta_m}_{i_m} + \left(\prod_{r\le m} k_r\right) z_{m+1} \\ &\equiv \left(\prod_{r<m} k_r\right) x^{\beta_m}_{i_m} \bmod (G_{\beta_m} + k_m G_\alpha). \end{aligned}$$

Applying φ we obtain that

$$\varphi(z_0) \equiv \left(\prod_{r<m} k_r\right) \varphi(x^{\beta_m}_{i_m}) \bmod (G_{\beta_m} + k_m G_\alpha).$$

Similarly, using clause (T8), we get that for every $l < l^*$

$$z^{\sigma_l}_{n^*} \equiv \left(\prod_{r=n^*}^{m-1} k_r\right) x^{\beta_m}_{i^*_l} \bmod (G_{\beta_m} + k_m G_\alpha)$$

holds for some sequence $\langle i^*_l : l < l^* \rangle$ without repetitions. Consequently,

$$\left(\prod_{r<m} k_r\right) \varphi(x^{\beta_m}_{i_m}) \equiv \sum_{l<l^*} b_l \left(\prod_{r=n^*}^{m-1} k_r\right) x^{\beta_m}_{i^*_l} \bmod (G_{\beta_m} + k_m G_\alpha).$$

We would like to remark that $\varphi(x^{\beta_m}_{i_m})$ does not depend on the chosen $\bar{\imath}$. Moreover, the sequence $\langle \sigma_l, b_l : l < l^* \rangle$ also does not depend on the integer m. However, given m and $\varphi(x^{\beta_m}_{i_m})$ we may reconstruct uniquely the sequence of data $\langle \sigma_l, b_l, i^*_l : l < l^* \rangle$.

We conclude by a pigeon hole argument and variation of $\bar{t}$ that for some suitable c_l we have

$$\varphi(x_i^\beta) \equiv \sum_{l<l^*} c_l f_{\sigma_l}(x_i^\beta) \bmod G_{\beta_{m^*}}$$

for all $\beta > \beta_{m^*}$ and $i \in \beth_{\beta+1}$. Since $G_{\rho+1}/G_\rho$ is divisible for $\rho \in (\delta \backslash \beta_{m^*}) \cap S$ we conclude by induction that also

$$\varphi(y) \equiv \sum_{l<l^*} c_l f_{\sigma_l}(y) \bmod G_{\beta_{m^*}}$$

for all $y \in G_\delta$. We now apply φ^2 and use that φ is a projection, hence $\varphi^2 = \varphi$. It follows that

$$\varphi^2(x_i^\beta) = \sum_{l_1<l^*} \sum_{l_2<l^*} c_{l_1} c_{l_2} f_{\sigma_{l_1}} f_{\sigma_{l_2}}(x_i^\beta) = \sum_{l<l^*} c_l f_{\sigma_l}(x_i^\beta)$$

Thus $\sigma_{l_1}\sigma_{l_2} \in \{\sigma_l : l < l^*\}$, and therefore the set $\{\sigma_l : l < l^*\}$ is a finite subgroup of the free group F_γ. Consequently, we must have $l^* = 1$ and $\sigma_0 = 1_F$ and either $c_0 = 1$ or $c_0 = -1$. However, since φ is a projection, the case $c_0 = -1$ does not occur, so we have

$$\varphi(y) \equiv y \bmod G_{\beta_{m^*}}$$

for all $y \in G_\delta$. Hence $G_\delta^2 \subseteq G_{\beta_{m^*}}$. But this was assumed not to be the case. Thus there must have been some $\bar{t}$ such that φ does not have an extension. This finishes the proof.

By standard arguments we immediately obtain the following theorem.

THEOREM 5.1 ($V = L$) *Let λ be a regular uncountable cardinal below the first weakly compact cardinal. Then there exist 2^λ almost-free non-isomorphic indecomposable minimal groups of cardinality λ.*

Proof. Let E be the stationary non-reflecting co-stationary set as above. It is well known that one can divide E into 2^λ disjoint subsets E_κ of the same kind. For each E_κ we can construct an almost-free indecomposable minimal group G_κ. By construction, the Γ-invariant of G_κ is the equivalence class of E_κ. Hence these groups are all non-isomorphic.

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