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On ultraproducts of Boolean algebras and irr

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Abstract.

§1. Consistent inequality

[We prove the consistency of $\text{irr}(\prod_{i < \kappa} B_i / D) < \prod_{i < \kappa} \text{irr}(B_i) / D$ where D is an ultrafilter on κ and each B_i is a Boolean algebra and $\text{irr}(B)$ is the maximal size of irredundant subsets of a Boolean algebra B , see full definition in the text. This solves the last problem, 35, of this form from Monk's list of problems in [M2]. The solution applies to many other properties, e.g. Souslinity.]

§2. Consistency for small cardinals

[We get similar results with $\kappa = \aleph_1$ (easily we cannot have it for $\kappa = \aleph_0$) and Boolean algebras B_i ($i < \kappa$) of cardinality $< \beth_{\omega_1}$.]
 This article continues Magidor Shelah [MgSh 433] and Shelah Spinas [ShSi 677], but does not rely on them: see [M2] for the background.

§1. Consistent inequality

Definition 1.1. Assume $\mu < \lambda$, λ is strongly inaccessible Mahlo. Let $B^* = B_\lambda$ be the Boolean algebra freely generated by $\{x_\alpha : \alpha < \lambda\}$ and for $u \subseteq \lambda$ let B_u be the subalgebra of B^* generated by $\{x_\alpha : \alpha \in u\}$.

1) We define a forcing notion $\mathbb{Q} = \mathbb{Q}_{\mu, \lambda}^1$ as follows:

$p \in \mathbb{Q}$ iff: p has the form (w^p, B^p) , we may write $(w[p], B[p])$ for typographical reasons, satisfying:

- (i) $w^p = w[p] \subseteq \lambda$
- (ii) $B^p = B[p]$ is a Boolean algebra of the form $B_{w[p]} / I^p$ where $I^p = I[p]$ is an ideal of $B_{w[p]}$, so B^p is generated by $\{x_\alpha / I^p : \alpha \in w^p\}$
- (iii) $x_\alpha / I^p \notin \langle \{x_\beta / I^p : \beta \in w^p \cap \alpha\} \rangle_{B[p]}$, equivalently $x_\alpha \notin \langle \{x_\beta : \beta \in w^p \cap \alpha\} \cup I^p \rangle_{B_{w[p]}}$
- (iv) for every strongly inaccessible $\chi \in (\mu, \lambda]$ we have $|w^p \cap \chi| < \chi$.

The order is given by $p \leq q$ iff $w^p \subseteq w^q$ and $I^p = I^q \cap B_{w[q]}$, so, abusing notation, we pretend that $B^p \subseteq B^q$, not distinguishing sometimes x_α from $x_\alpha / I^p \in B^p$ or (see below) from x_α / I in B .

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2) We define $\underline{I} = \cup\{I^p : p \in \underline{G}_{\mu,\kappa}^1\}$ and \underline{B} is defined as B_λ/\underline{I} .

Claim 1.2. For $\mu < \lambda$ as in Definition 1.1, the forcing notion $\mathbb{Q}_{\mu,\lambda}^1$ is μ^+ -complete (hence, adds no new subsets to μ), has cardinality λ , satisfies the λ -c.c., collapse no cardinal, changes no cofinality, so cardinal arithmetic which holds after the forcing is clear.

Proof. Like the proof of the same facts for Easton forcing.

Claim 1.3. For the forcing $\mathbb{Q} = \mathbb{Q}_{\mu,\lambda}^1$ with μ, λ as in Definition 1.1 we have

- 1) $\Vdash_{\mathbb{Q}}$ “ \underline{B} is a Boolean Algebra generated by $\{x_\alpha : \alpha < \lambda\}$ such that $\alpha < \lambda \Rightarrow x_\alpha \notin \langle \{x_\beta : \beta < \alpha\} \rangle_B$, so $|\underline{B}| = \lambda$ and $\lambda = \cup\{w^p : p \in \underline{G}_{\mu,\lambda}^1\}$ ”.
- 2) $\Vdash_{\mathbb{Q}}$ “ $\text{irr}^+(\underline{B}) = \lambda = \text{irr}(\underline{B})$ ”, see Definition 1.4 below.
- 3) $\Vdash_{\mathbb{Q}}$ “if $y_\beta \in \underline{B}$ for $\beta < \lambda$ then for some $\beta_0 < \beta_1 < \beta_2 < \lambda$ we have $\underline{B} \models y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$ ”.
- 4) Let B^* be a finite Boolean algebra generated by $\{a^*, b^*, y_0^*, \dots, y_{n(*)}^*\}$ such that $y_m^* \notin \langle \{y_\ell^* : \ell < m\} \cup \{a^*, b^*\} \rangle_{B^*}$ and $0 < a^* < y_m^* < b^* < 1$ for $m \in \{0, \dots, n(*)\}$.

Then it is forced, ($\Vdash_{\mathbb{Q}_{\mu,\lambda}^1}$) that:

$\square_B^{\lambda, n(*)}$ if $y_\beta \in \underline{B}$ for $\beta < \lambda$ and $\beta \neq \gamma \Rightarrow y_\beta \neq y_\gamma$ then we can find

a, b in \underline{B} satisfying $0 < a < b < 1$ and

$\beta_0 < \dots < \beta_{n(*)} < \lambda$ such that

(α) $\underline{B} \models “a < y_{\beta_\ell} < b”$

(β) there is an embedding f of B^* into \underline{B} mapping a^* to a , b^* to b and y_ℓ to $y_{\beta_\ell}^*$ for $\ell = 0, \dots, n(*)$.

Recalling

Definition 1.4. For a Boolean algebra B let:

- 1) $X \subseteq B$ is called *irredundant*, if no $x \in X$ belongs to the subalgebra $\langle X \setminus \{x\} \rangle_B$ of B generated by $X \setminus \{x\}$.
- 2) $\text{irr}^+(B) = \cup\{|X|^+ : X \subseteq B \text{ is irredundent}\}$.
- 3) $\text{irr}(B) = \cup\{|X| : X \subseteq B \text{ is irredundent}\}$ so $\text{irr}(B)$ is $\text{irr}^+(B)$ if the latter is a limit cardinal and is the predecessor of $\text{irr}^+(B)$ if the later is a successor cardinal.

Remark. Concerning 1.3, for the case $\kappa = \aleph_1$ see Rubin [Ru83], generally see [Sh 128], [Sh:e].

Proof of 1.3. 1) Should be clear.

2) Clearly for every $\chi < \lambda$ and $p \in \mathbb{Q}_{\mu,\lambda}^1$ we can find an $\alpha < \lambda$ such that $\alpha > \chi$ and $w^p \cap [\alpha, \alpha + \chi) = \emptyset$, hence we can find a q such that $p \leq q \in \mathbb{Q}_{\mu,\lambda}^1$ and $w^q = w^p \cup [\alpha, \alpha + \chi)$ and in B^q the set $\{x_\beta : \beta \in [\alpha, \alpha + \chi)\}$ is independent, hence $q \Vdash “\text{irr}^+(\underline{B}) > \chi”$. So we get $\Vdash “\text{irr}^+(B) \geq \lambda$. To prove equality use part (3).

3) Assume toward contradiction that $p \Vdash “\langle y_\beta : \beta < \lambda \rangle$ is a counterexample”. We can find for each $\beta < \lambda$ a quadruple $(p_\beta, n_\beta, \langle \alpha_{\beta,\ell} : \ell < n_\beta \rangle, \sigma_\beta)$ such that:

- (i) $p \leq p_\beta \in \mathbb{Q}_{\mu,\lambda}^1$
- (ii) $n_\beta < \omega$
- (iii) $\alpha_{\beta,\ell} \in w^{p_\beta}$ increasing with ℓ
- (iv) $\sigma_\beta(x_0, \dots, x_{n_\beta-1})$ is a Boolean term
- (v) $p_\beta \Vdash$ “in \mathcal{B} we have $y_\beta = \sigma_\beta(x_{\alpha_{\beta,0}}, x_{\alpha_{\beta,1}}, \dots, x_{\alpha_{\beta,n_\beta-1}})$ ”. Call the righthand side y_β , so by part (1), without loss of generality, $\{\alpha_{\beta,\ell} : \ell < n_\beta\} \subseteq w^{p_\beta}$ hence y_β is a member of $B_{w[p_\beta]}$.

So we can choose a stationary $S \subseteq \{\chi : \chi \text{ strongly inaccessible, } \mu < \chi < \lambda\}$ and $n, \sigma, m, \langle \alpha_\ell : \ell < m \rangle, w, r$ such that for every $\beta \in S$ we have: $n_\beta = n, \sigma_\beta = \sigma, \ell < m \Rightarrow \alpha_{\beta,\ell} = \alpha_\ell, \ell \in [m, n] \Rightarrow \alpha_{\beta,\ell} \geq \beta$ and $w^{p_\beta} \cap \beta = w$. Without loss of generality also $\alpha < \beta \in S \Rightarrow w^{p_\alpha} \subseteq \beta$. Without loss of generality

- ⊗ for β_0, β_1 in S the mapping $F_{\beta_0, \beta_1} = \text{id}_w \cup \{(\langle \alpha_{\beta_0, \ell}, \alpha_{\beta_1, \ell} \rangle : \ell < n)\}$ induces an isomorphism g_{β_1, β_0} from the Boolean algebra $\langle \{x_\gamma : \gamma \in w\} \cup \{x_{\beta_0, \ell} : \ell < n\} \rangle_{B[p_{\beta_0}]}$ onto the Boolean algebra $\langle \{x_\gamma : \gamma \in w\} \cup \{x_{\beta_1, \ell} : \ell < n\} \rangle_{B[p_{\beta_1}]}$ that is g_{β_1, β_0} maps x_γ to x_γ for $\gamma \in w$ and maps $x_{\beta_0, \ell}$ to $x_{\beta_1, \ell}$ for $\ell < n$.

Choose in S three ordinals $\beta_0 < \beta_1 < \beta_2$ and we define $q \in \mathbb{Q}_{\mu,\lambda}^1$ such that $w^q = w[p_{\beta_0}] \cup w[p_{\beta_1}] \cup w[p_{\beta_2}]$ and B^q is the Boolean algebra generated by $\{x_\alpha : \alpha \in w[p_{\beta_0}] \cup w[p_{\beta_1}] \cup w[p_{\beta_2}]\}$ freely except the equations which hold in p_{β_ℓ} for each $\ell = 0, 1, 2$ and the equation $y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$, in other words I^q is the ideal of B_{w^q} generated by $I[p_{\beta_0}] \cup I[p_{\beta_1}] \cup I[p_{\beta_2}] \cup \{y_{\beta_1} \cap y_{\beta_2} - y_{\beta_0}, y_{\beta_0} - y_{\beta_1} \cap y_{\beta_2}\}$. We should prove that $q \in \mathbb{Q}_{\mu,\lambda}^1$ and $I[q] \cap B_{w[p_{\beta_\ell}]} = I[p_{\beta_\ell}]$ for $\ell = 0, 1, 2$ (the rest: $p_{\beta_\ell} \leq q$ hence $p \leq q$ and $q \Vdash$ “ $y_{\beta_\ell} = y_{\beta_\ell}$ for $\ell = 0, 1, 2$ and $y_{\beta_1} \cap y_{\beta_2} = y_{\beta_0}$ ” should be clear).

Let B_0 be the trivial Boolean algebra $\{0, 1\}$.

For $w \subseteq \lambda$ and $f \in {}^w 2$ let \hat{f} be the unique homomorphism from the Boolean algebra B_w freely generated by $\{x_\alpha : \alpha \in w\}$ to $\{0, 1\}$ such that $\alpha \in w \Rightarrow \hat{f}(x_\alpha) = f(\alpha)$. For $p^* \in \mathbb{Q}_{\mu,\lambda}^1$ let $\mathcal{F}[p^*] = \{f : f \in ({}^{w^{p^*}} 2) \text{ and } \{x_\alpha : f(\alpha) = 1\} \cup \{-x_\alpha : f(\alpha) = 0\} \text{ generates an ultrafilter of } B[p^*]\}$. For each $f \in \mathcal{F}[p^*]$ let $f^{[p^*]}$ be the homomorphism from $B[p^*]$ to B_0 induced by f , i.e., $f^{[p^*]}(x_\alpha) = f(\alpha)$ for every $\alpha \in w^{p^*}$. Clearly $\mathcal{F}[p^*]$ gives all the information on p^* . Define $u = w^{p_{\beta_0}} \cup w^{p_{\beta_1}} \cup w^{p_{\beta_2}}$ and let

$$\begin{aligned} \mathcal{F} &= \{f : f \in {}^u 2, \text{ and } \ell \leq 2 \Rightarrow f \upharpoonright w[p_{\beta_\ell}] \in \mathcal{F}[p_{\beta_\ell}] \text{ and} \\ &\quad B_0 \models \text{“} \hat{f}(\sigma(\langle x_{\beta_1, \ell} : \ell < n \rangle)) \cap \hat{f}(\sigma(\langle x_{\beta_2, \ell} : \ell < n \rangle)) \\ &\quad = \hat{f}(\sigma(\langle x_{\beta_0, \ell} : \ell < n \rangle)) \text{”}\}. \end{aligned}$$

We need to show that \mathcal{F} is rich enough, clearly $\otimes_1 + \otimes_2$ below suffice.

- ⊗₁ if $\ell \in \{0, 1, 2\}$ and $f_\ell \in \mathcal{F}[p_{\beta_\ell}]$ then there is an $f \in \mathcal{F}$ extending f_ℓ .

[Why? For $m = 0, 1, 2$ let p'_{β_m} be such that $B[p'_{\beta_m}]$ is the subalgebra of $B[p_{\beta_m}]$ generated by $\{x_\gamma : \gamma \in w[p_{\beta_m}]\}$ and $\gamma < \beta_m \vee \gamma \in \{\alpha_{\beta_m, 0}, \dots, \alpha_{\beta_m, n-1}\}$. We define for $m = 0, 1, 2$ a homomorphism g_m from $B[p'_{\beta_m}]$ to B_0 such that:

$\gamma \in w \Rightarrow g_m(x_\gamma) = f_\ell(\gamma)$ and $\gamma = \beta_{m,k} \Rightarrow g_m(x_\gamma) = f_\ell(\beta_{\ell,k})$. This is possible by \otimes and let h_m be chosen as follows: it is $f_\ell^{[p_{\beta_\ell}]}$ if $\ell = m$ and it is chosen as any homomorphism from $B[p_{\beta_m}]$ to B_0 extending g_m if $m \in \{0, 1, 2\} \setminus \{\ell\}$, as $B[p_{\beta_m}]$ is a subalgebra of $B[p_{\beta_m}]$ this clearly exists. Let $f_m \in {}^w[p_{\beta_\ell}]2$ for $m = 0, 1, 2$ be $f_m(\gamma) = h_m(x_\gamma)$; for $m = \ell$ the definitions are compatible; i.e., the definition of f_ℓ we have just given and the old one. Finally, let $f = f_0 \cup f_1 \cup f_2$. This is clearly a well defined function; now of the three conditions in the definition of \mathcal{F} , the first holds by the definition of u , the second by the choice of the h_m 's and the third by the choice of the g_m 's, it is easy to see $f_\ell \subseteq f \in \mathcal{F}$.]

\otimes_2 if $\ell \in \{0, 1, 2\}$, $\alpha \in w[p_{\beta_\ell}]$ then there are $f', f'' \in \mathcal{F}$ such that $f'(\alpha) \neq f''(\alpha)$ but $f' \upharpoonright (\alpha \cap u) = f'' \upharpoonright (\alpha \cap u)$.

[Why? As $p_{\beta_\ell} \in \mathbb{Q}_{\mu,\lambda}^1$ we can find $f'_\ell, f''_\ell \in \mathcal{F}[p_{\beta_\ell}]$ such that $f'_\ell(\alpha) \neq f''_\ell(\alpha)$ but $f'_\ell \upharpoonright (\alpha \cap w[p_{\beta_\ell}]) = f''_\ell \upharpoonright (\alpha \cap w[p_{\beta_\ell}])$. Now for $m \in \{0, 1, 2, \} \setminus \{\ell\}$ recalling \otimes above there are $f'_m \in \mathcal{F}[p_{\beta_m}]$ which extends $f'_\ell \circ F_{\beta_\ell, \beta_m}$ and $f''_m \in \mathcal{F}[p_{\beta_m}]$ which extends $f''_\ell \circ F_{\beta_\ell, \beta_m}$ in both cases this is shown as in the proof of \otimes_1 . If $\ell = 0$, let $f' = f'_0 \cup f'_1 \cup f'_2 \in \mathcal{F}$ and let $f'' = f''_0 \cup f''_1 \cup f''_2 \in \mathcal{F}$; both memberships hold as in the proof of \otimes_1 and we are done. Also if $\alpha < \beta_\ell$ (so $\alpha \in w = \bigcap_{m \leq 2} w[p_{\beta_m}]$)

the same proof works. So assume $\ell \neq 0$, $\alpha \notin w = \bigcap_{m \leq 2} w[p_{\beta_m}]$. If $(f'_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) =$

$(f''_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell})$ let $f' = f'_0 \cup f'_1 \cup f'_2$, $f'' = f''_\ell \cup (f' \upharpoonright (w[p_{\beta_0}] \cup w[p_{\beta_{3-\ell}]])$, clearly O.K. So without loss of generality assume $(f'_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) = 0$, $(f''_\ell)^{[p_{\beta_\ell}]}(y_{\beta_\ell}) = 1$, $\ell \in \{1, 2\}$ and $\alpha \in w[p_{\beta_\ell}] \setminus w[p_{\beta_0}]$; and then choose $f' = f'_0 \cup f'_1 \cup f'_2$ as above and $f'' = f''_\ell \cup (f' \upharpoonright (w[p_{\beta_0}] \cup w[p_{\beta_{3-\ell}]])$. Now check; the main point is that as $\hat{f}'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat{f}'_0(y_{\beta_0})$ we have $B_0 \models \hat{f}''(y_{\beta_1}) \cap \hat{f}''(y_{\beta_2}) = \hat{f}''(y_{\beta_\ell}) \cap \hat{f}''(y_{\beta_{3-\ell}}) = \hat{f}''_\ell(y_{\beta_\ell}) \cap \hat{f}'(y_{\beta_{3-\ell}}) = 1_{B_0} \cap \hat{f}'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat{f}'_{3-\ell}(y_{\beta_{3-\ell}}) = \hat{f}'_0(y_{\beta_0}) = \hat{f}''(y_{\beta_0})$.

4) The proof is similar to that of the previous part (with a, b now in $p_{\beta_\ell} \upharpoonright \beta_\ell!$). $\square_{1.3}$

Claim 1.5. 1) If $\mathbb{Q} = \mathbb{Q}_{\mu,\lambda}^1 * \mathbb{Q}^2$ and $\Vdash_{\mathbb{Q}_{\mu,\lambda}^1}$ “ \mathbb{Q}^2 satisfies the $(\lambda, 3)$ -Knaster condition (see below)”, then $\Vdash_{\mathbb{Q}}$ “ $\text{irr}^+(B) = \lambda$ ”.

2) If in \mathbf{V} the condition $\square_B^{\lambda, n(*)}$ from 1.3(4) holds and the forcing notion \mathbb{Q} satisfies the $(\lambda, n^* + 1)$ -Knaster condition then also in $\mathbf{V}^{\mathbb{Q}}$ the condition $\square_B^{\lambda, n(*)}$ holds. Hence if $\mathbb{Q} = \mathbb{Q}_{\mu,\lambda}^1 * \mathbb{Q}^2$ and $\Vdash_{\mathbb{Q}_{\lambda,\mu}}$ “ $\square_B^{\lambda, n(*)}$ holds (see 1.3(4))” and $\Vdash_{\mathbb{Q}_{\lambda,\mu}^1}$ “ \mathbb{Q}^2 satisfies the $(\lambda, n(*) + 1)$ -Knaster condition” then $\Vdash_{\mathbb{Q}_{\lambda,\mu}^1 * \mathbb{Q}^2}$ “ $\square_B^{\lambda, n(*)}$ ”.

3) In part (1) we even get the conclusion of Claim 1.3(3).

Definition 1.6. 1) The λ -Knaster condition says that among any λ members there is a set of λ members which are pairwise compatible. Recall that it is preserved by composition.

2) For $n^* \leq \omega$, the (λ, n^*) -Knaster condition says that among any λ member there is a set of λ such that any $< 1 + n^*$ of them have a common upper bound.

Proof of 1.5. 1), 3) Clearly it suffices to prove (3).

This follows immediately by 1.3(3), in fact, just such \mathbb{Q}^2 preserves the properties mentioned there.

2) Similarly using 1.3(4). □_{1.5}

Theorem 1.7. *Suppose*

(a) \mathbf{V} satisfies GCH above μ (for simplicity)

(b) κ is measurable, $\kappa < \chi < \mu$

(c) μ is supercompact, Laver indestructible, more explicitly,

(*) for some $h_\ell : \mu \rightarrow \mathcal{H}(\mu)$, (for $\ell = 0, 1$) we have for every $(< \mu)$ -directed complete forcing \mathbb{Q} , cardinal $\theta \geq \mu$ and \mathbb{Q} -name \underline{x} of a subset of θ , there is in $\mathbf{V}[G_{\mathbb{Q}}]$ a normal ultrafilter \mathcal{D} on $[\theta]^{<\mu}$ such that

$$\prod_{a \in [\theta]^{<\mu}} (h_1(a \cap \mu), h_2(a \cap \mu)) / \mathcal{D} \cong (\theta, \underline{x}[G_{\mathbb{Q}}])$$

(d) $\lambda > \mu$ is strongly inaccessible, Mahlo and λ^* is such that $\lambda^* = (\lambda^*)^\mu \geq \lambda$

(e) D^* is a normal ultrafilter on κ .

Then for some forcing notion \mathbb{P} we have, in $\mathbf{V}^{\mathbb{P}}$:

(α) forcing with \mathbb{P} collapse no cardinal of \mathbf{V} except those in the interval (μ^+, λ)

(β) forcing with \mathbb{P} adds no subsets to χ , preserves “ μ is strong limit” and makes $2^\mu = \lambda^*$

(γ) μ is strong limit of cofinality κ and $\langle \mu_i : i < \kappa \rangle$ is an increasing continuous sequence of strong limit cardinals with limit μ

(δ) for each $i < \kappa$, $\mu_i < \lambda_i \leq \lambda_i^* = (\lambda_i^*)^{\mu_i} = 2^{\mu_i}$ and we let $\mu_\kappa = \mu$, $\lambda_\kappa = \lambda$, $\lambda_\kappa^* = \lambda^*$

(ϵ) for each $i \leq \kappa$ we have: B_i is a Boolean algebra of cardinality λ_i and $\text{irr}^+(B_i) = \lambda_i$

(ζ) for $i < \kappa$, λ_i is a Mahlo cardinal which is weakly inaccessible, but

(η) $\lambda = \lambda_\kappa$ is μ^{++} (this in $\mathbf{V}^{\mathbb{P}}$)

(θ) $B = B_\kappa$ is isomorphic to $\prod_{i < \kappa} B_i / D^*$, hence

$$\boxtimes \text{irr}^+(B) = \lambda = \mu^{++} \text{ so } \text{irr}(B) = \mu^+ \text{ whereas } \text{irr}(B_i) = \text{irr}^+(B_i) = \lambda_i \text{ and } \prod_{i < \kappa} \lambda_i / D^* = \lambda, \text{ so } \text{irr}\left(\prod_{i < \kappa} B_i / D^*\right) < \prod_{i < \kappa} \text{irr}(B_i) / D^*.$$

Proof. Let $\mathbb{Q}_1 = \mathbb{Q}_{\mu, \lambda}^1$ and B be from 1.2, let and for $Z \subseteq \lambda^*$ let $\mathbb{Q}_{2,Z}$ be $\{f : f \text{ a partial function from } Z \text{ to } \{0, 1\} \text{ with domain of cardinality } < \mu\}$ ordered by inclusion, let $\mathbb{Q}_2 = \mathbb{Q}_{2, \lambda^*}$ and let $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$. Let $G = G_1 \times G_2 \subseteq \mathbb{Q}$ be generic over \mathbf{V} and let $\mathbf{V}_0 = \mathbf{V}$, $\mathbf{V}_1 = \mathbf{V}[G_1]$ and $\mathbf{V}_2 = \mathbf{V}[G] = \mathbf{V}_1[G_2]$.

□₀ In \mathbf{V}_2 , $\underline{B}[G_1]$ is a Boolean algebra of cardinality λ with $\text{irr}^+(B) = \lambda$ and, for notational simplicity, with a set of elements λ .

[Why? In \mathbf{V}_1 , $\underline{B}[G_1]$ is like that by 1.3. Now as in \mathbf{V}_1 , \mathbb{Q}_2 satisfies the (λ, n) -Knaster for every n hence clearly by 1.5 we are done.]

In \mathbf{V}_2 we have $2^\mu = \lambda^*$ and the cardinal μ is still supercompact, hence it is well known that

\boxtimes_1 for every $Y \subseteq 2^\mu$ for some normal ultrafilter \mathcal{D} on μ and $\bar{Y} = \langle Y_i : i < \mu \rangle$, $Y_i \subseteq 2^{|i|}$ we have \bar{Y}/\mathcal{D} is Y (i.e. $\bar{Y}/\mathcal{D} \in \mathbf{V}_2^\mu/\mathcal{D}$ and in the Mostowski Collapse of $\mathbf{V}_2^\mu/\mathcal{D}$ the element \bar{Y}/\mathcal{D} is mapped to Y), hence $(2^\mu, Y, \mu, <)$ is isomorphic to $\prod_{i < \mu} (2^{|i|}, Y_i, i, <)/\mathcal{D}$.

Again it is well known and follows from \boxtimes_1 that there is a sequence $\bar{\mathcal{D}}^0 = \langle \mathcal{D}_\zeta^0 : \zeta < (2^\mu)^+ \rangle$ of normal (fine) ultrafilters on μ satisfying: for each $\zeta < (2^\mu)^+$ the sequence $\bar{\mathcal{D}}^0 \upharpoonright \zeta$ belongs to (the Mostowski collapse of) $\mathbf{V}_2^\mu/\mathcal{D}_\zeta^0$. In \mathbf{V}_2 we can code $\bar{B} = \bar{B}[G_1]$ and $\mathcal{P}(\mu)$ and $\bar{\mathcal{D}}^0 \upharpoonright \kappa$ as a subset Y of $2^\mu = \lambda^*$ and get \mathcal{D}, \bar{Y} as in \boxtimes_1 hence for some set $A \in \mathcal{D}$ of strongly inaccessible cardinals $> \chi$ there is a sequence $\langle (\mu_i, \lambda_i, B_i, \lambda_i^*) : i \in A \rangle$ such that:

(*)₁ for $i \in A$ we have $i = \mu_i < \lambda_i \leq \lambda_i^* = (\lambda_i^*)^{\mu_i} < \mu$, λ_i is weakly inaccessible, Mahlo, B_i is a Boolean algebra generated by $\{x_\alpha : \alpha < \lambda_i\}$, $x_\alpha \notin \langle \{x_\beta : \beta < \alpha\} \rangle_{B_i}$, $\text{irr}^+(B_i) = \lambda_i$ and, for notational simplicity, its sets of elements is λ_i

(*)₂ B is isomorphic to $\prod_{i \in A} B_i/\mathcal{D}$ and $(\lambda^*, <) \cong \prod_{i \in A} (\lambda_i^*, <)/\mathcal{D}$.

For $i \in \mu \setminus A$ choose $\mu_i, \lambda_i, \lambda_i^*, B_i$ such that (*)₁ holds such that $\mu_i \geq i$; why are there such λ_i, B_i ? Just e.g. use $\lambda_{\text{Min}(A \setminus i)}$, $B_{\text{Min}(A \setminus i)}$.

Let $\mathcal{D}_i = \mathcal{D}_i^0$ for $i < \kappa$ and \mathcal{D}_κ be the \mathcal{D} as above. So \mathcal{D}_i (for $i \leq \kappa$) is a normal ultrafilter on μ and we have $i < j \leq \kappa \Rightarrow \mathcal{D}_i \in \mathbf{V}_2^\mu/\mathcal{D}_j$, that is, there is a sequence $\bar{g} = \langle g_{i,j} : i < j \leq \kappa \rangle$ satisfying $g_{i,j} \in {}^\mu(\mathcal{H}(\mu))$ such that \mathcal{D}_i is (the Mostowski collapse of) $g_{i,j}/\mathcal{D}_j \in \mathbf{V}_2^\mu/\mathcal{D}_j$.

All this was in $\mathbf{V}_2 = \mathbf{V}[G]$. So we have \mathbb{Q} -names $\bar{g} = \langle g_{i,j} : i < j \leq \kappa \rangle$, $\bar{\mathcal{D}} = \langle \mathcal{D}_i : i \leq \kappa \rangle$ and $\langle (\mu_i, \lambda_i, B_i, \lambda_i^*) : i < \mu \rangle$. As $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$, \mathbb{Q}_2 satisfies the μ^+ -c.c. and \mathbb{Q}_1 is μ^+ -complete without loss of generality \bar{g} is a \mathbb{Q}_2 -name and \bar{g} is from $\mathbf{V}[G_2]$. Hence without loss of generality \bar{g} and similarly $\langle (\mu_i, \lambda_i, B_i, \lambda_i^*) : i < \mu \rangle$ belong to $\mathbf{V}[G_{2,Z}]$ where $G_{2,Z} = G_2 \cap \mathbb{Q}_{2,Z}$, for some $Z \in [\lambda^*]^{\leq \mu}$; $\text{wlog } Z = \emptyset$ as we could have forced first with $\{f \in \mathbb{Q}_2 : \text{Dom}(f) \subseteq Z\}$. Let $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ be (the \mathbb{Q} -name of the) Magidor forcing for $(\bar{\mathcal{D}}, \bar{g})$ (see [Mg4]). Let $\langle \mu_i : i < \kappa \rangle$ be the $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ -name of the increasing continuous κ -sequence converging to μ which the forcing adds and we can restrict ourselves to the case $\mu_0 > \chi$. Clearly clauses $(\alpha) - (\zeta)$ in the conclusion hold for $\mathbb{P} = \mathbb{Q} * \mathbb{P}(\bar{\mathcal{D}}, \bar{g})$. Now

\boxtimes_2 in \mathbf{V}_2 , if $p \in \mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ and $p \Vdash \prod_{i < \kappa} \lambda_{\mu_i}$ then there are q , an extension of p in $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ and $f \in \prod_{i \in A^*} \lambda_i$ such that $q \Vdash_{\mathbb{P}(\bar{\mathcal{D}}, \bar{g})} \{i < \kappa : f(i) = f(\mu_i)\} \in D^*$.

[Why? By the properties of $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ there are a pure extension q_0 of p in $\mathbb{P}(\bar{\mathcal{D}}, \bar{g})$ and/or sequence $\langle u_i : i < \kappa \rangle$ such that above q_0 we have: $f(i)$ depends just on $\langle \mu_j : j \in u_i \cup \{i\} \rangle$ where $u_i \subseteq i$ is finite. As D^* is a normal ultrafilter on κ , for

some $a^* \in D^*$ and a finite $u \subseteq \kappa$ we have $i \in a^* \Rightarrow u_i = u$. So there is a q such that $\mathbb{P}(\bar{D}, \bar{g}) \models q_0 \leq q$ and $q \Vdash \text{“}\mu_j = \mu_j^*\text{”}$ for $j \in u$, and so f is well defined.]

Let $G_3 \subseteq \mathbb{P}(\bar{D}, \bar{g})$ be generic over \mathbf{V}_2 and $\mathbf{V}_3 = \mathbf{V}_2[G_3]$ and let $\mu_i = \mu_i[G_3]$ so really $\langle \mu_i : i < \kappa \rangle$ is generic for $\mathbb{P}(\bar{D}, \bar{g})$. Now we shall show that:

⊠₃ in $\mathbf{V}_3 = \mathbf{V}_2[G_3]$ we have

$$B \cong \prod_{i < \kappa} B_{\mu_i}/D^*.$$

[Why? In \mathbf{V}_2 , by $(*)_2$ above there is an isomorphism F from B onto $\prod_{i < \mu} B_i/D = \prod_{i \in A^*} B_i/D_\kappa$, so let $F(x) = f_x/D_\kappa$ with $f_x \in \prod_{i \in A^*} \lambda_i$ for $x \in B$, i.e. $x \in \lambda$.

In \mathbf{V}_3 let $f'_x \in \prod_{i < \kappa} \lambda_{\mu_i}$ be defined by $f'_x(i) = f_x(\mu_i)$ and we define a

function F' from B , i.e. from λ to $\prod_{i < \kappa} B_{\mu_i}/D^*$ by $F'(x) = f'_x/D^*$. Now

$Y \in \mathcal{D} \Rightarrow \{i < \kappa : \mu_i \in Y\} = \kappa \bmod J_\kappa^{bd}$ by the definition of $\mathbb{P}(\bar{D}, \bar{g})$, so as F is one to one also F' is, and F' commute with the Boolean operations as F does; lastly F' is onto by ⊠₂.]

⊠₄ if $i < \kappa$ then $\mathcal{H}(\mu_{i+1})^{\mathbf{V}_3}$ is the same as $\mathcal{H}(\mu_{i+1})^{\mathbf{V}_0^{\mathbb{P}^i}}$, for some μ_i -centered forcing notion from $\mathcal{H}(\mu_{i+1})$ (hence this forcing notion is λ_{μ_i} -Knaster).

[Why? Note that $\mathcal{H}(\mu_j)^{\mathbf{V}_2} = \mathcal{H}(\mu_j)^{\mathbf{V}_0}$ for $j \leq \kappa$. Also for each $i < \kappa$ in \mathbf{V}_0 there are \mathcal{D}^i_j , a normal ultrafilter on μ_i such that $(\bar{D}^i, \bar{g}^i) = (\langle \mathcal{D}^i_j : j \leq i \rangle, \langle g_{j_1, j_2} \upharpoonright \mu_i : j_1 < j_2 \leq i \rangle) \in \mathbf{V}$ is as above, i.e. $j_1 < j_2 \leq i \Rightarrow \mathcal{D}^i_{j_1} = g_{j_1, j_2}/\mathcal{D}^i_{j_2} \in \mathbf{V}^{\mu_i}/\mathcal{D}^i_{j_2}$, $g^i_{j_1, j_2} \in \mu_i(\mathcal{H}(\mu_i))$ so $\mathbb{P}(\bar{D}^i, \bar{g}^i)$ is as in [Mg4], and for some $G_{3,i} \subseteq \mathbb{P}(\langle \mathcal{D}^i_j : j \leq i \rangle, \langle g_{j_1, j_2} \upharpoonright \mu_i : j_1 \leq \mu_2 \leq i \rangle)$ generic over \mathbf{V}_0 (equivalently over \mathbf{V}_2) we have $G_{3,i} \in \mathbf{V}_3$ and $\mathcal{H}(\mu_{i+1})^{\mathbf{V}_3} = \mathcal{H}(\mu_{i+1})^{\mathbf{V}_2[G_{3,i}]} = \mathcal{H}(\mu_{i+1})^{\mathbf{V}_0[G_{3,i}]}$. See [Mg4]. As $\mathbb{P}(\bar{D}^i, \bar{g}^i)$ is μ_i -centered, clearly ⊠₄ follows.]

So obviously (by 1.5)

⊠₅ in \mathbf{V}_3 , for each $i < \kappa$ we have B_{μ_i} is a Boolean algebra of cardinality λ_{μ_i} , $\text{irr}^+(B_{\mu_i}) = \lambda_{\mu_i}$, λ_{μ_i} is weakly Mahlo.

Also in $\mathbf{V}[G_1]$, the forcing notion \mathbb{Q}_2 satisfies the λ -Knaster condition and in $\mathbf{V}_2 = \mathbf{V}[G_1, G_2]$, the forcing notion $\mathbb{P}(\bar{D}, \bar{g})$ from [Mg4] is μ -centered hence satisfies the λ -Knaster hence

⊠₆ in \mathbf{V}_3 , B is a Boolean Algebra of cardinality λ , a Mahlo cardinal and $\text{irr}^+(B) = \lambda$.

Now let $\mathbb{R} = \text{Levy}(\mu^+, < \lambda)^{\mathbf{V}} = \{f \in \mathbf{V} : \text{Dom}(f) \subseteq \{(\alpha, \gamma) : \alpha < \lambda, \gamma < \mu^+\}, |\text{Dom}(f)| \leq \mu \text{ and for } \gamma < \alpha, \text{ we have } f(\alpha, \gamma) < 1 + \alpha\}$, ordered by inclusion. Clearly \mathbb{R} satisfies the λ -Knaster condition, is μ^+ -complete in \mathbf{V} and also in \mathbf{V}_1 . Let $G_{\mathbb{R}} \subseteq \mathbb{R}$ be generic over \mathbf{V}_1 . Now in $\mathbf{V}[G_1, G_{\mathbb{R}}]$, the forcing notion \mathbb{Q}_2 has the same definition and same properties. Also (as in [MgSh 433], [ShSi 677]), in

$\mathbf{V}[G_1, G_2, G_{\mathbb{R}}]$ the $\mathcal{D}_i (i \leq \kappa)$ are still normal ultrafilters on μ and the definition of $\mathbb{P}(\bar{D}, \bar{g})$ gives the same forcing notion with the same properties and add the same family of subsets to κ (as $\mathcal{P}(\kappa)^{\mathbf{V}[G_1, G_2]} = \mathcal{P}(\kappa)^{\mathbf{V}[G_1, G_2, G_{\mathbb{R}}]}$).

So $G_{\mathbb{R}}$ is a subset of \mathbb{R} generic over $\mathbf{V}[G_1, G_2, G_3]$. Also in $\mathbf{V}[G_1, G_2]$, \mathbb{R} satisfies the λ -Knaster condition and in $\mathbf{V}[G_1, G_2, G_{\mathbb{R}}]$, $\mathbb{P}(\bar{D}, \bar{g})$ is μ -centered hence satisfies the λ -Knaster condition. Let $\mathbf{V}_4 = \mathbf{V}_3[G_{\mathbb{R}}]$, so in \mathbf{V}_4 all the conclusions above holds but $\lambda = \mu^{++}$ hence $\text{irr}(B) = \mu^+$ whereas $\text{irr}^+(B)$ remains $\lambda = \mu^{++}$. So we are done. $\square_{1.7}$

Claim 1.8. 1) In the Theorem 1.7 we can replace

“a Boolean algebra B of cardinality λ , $\text{irr}^+(B) = \lambda$ ” by e.g. “a λ -Souslin tree”

The “ λ strongly inaccessible Mahlo” is needed just for applying 1.3, etc., but for $\prod_{i < \kappa} B_i/D^* \cong B$ it is not needed (any model M , with universe $\subseteq \lambda$ is O.K.)

2) We can apply the proof above to the proof in [Sh 128] hence to theories of cardinality $< \mu$ for simplicity in logics with Magidor Malitz quantifiers.

Proof. Similar to 1.7. $\square_{1.7}$

§2. Consistency for small cardinals

Theorem 2.1 generalizes 1.7 in some ways. First D^* , instead of being a normal ultrafilter on κ is just a normal filter which is large in an appropriate sense so later it will be applied to the case $\kappa = \aleph_1$ (after a suitable preliminary forcing). Second, we deal with a general model and properties. Thirdly, the forcing makes μ to \beth_κ (and more)

Theorem 2.1. *Suppose*

- (a) \mathbf{V} satisfies GCH for every $\mu' \geq \mu$ (for simplicity)
- (b) κ is regular uncountable, $\aleph_0 \leq \theta \leq \kappa < \chi < \mu < \vartheta < \lambda \leq \lambda^* = (\lambda^*)^\mu$, say $\vartheta = \mu^+$
- (c) μ is supercompact, Laver indestructibly or just indestructibly λ^* -hypermeasure (generally on such indestructibility see [GiSh 344], on the amount of hypermeasurable needed here see Gitik Magidor [GM])
- (d) D^* is a filter on κ including the clubs and if f is a pressing down function on κ then for some $u \in [\kappa]^{<\theta}$ we have $\{\delta < \kappa : f(\delta) \in u\} \in D^*$
- (e) \mathbb{Q}_1 is a ($< \mu$)-directed complete forcing, $|\mathbb{Q}_1| \leq \lambda^*$ and $\Vdash_{\mathbb{Q}_1}$ “ \underline{M} is a model with universe λ and vocabulary $\tau \in \mathcal{H}(\chi)$ ”
- (f) \mathbb{R} is a μ^{++} -complete forcing notion of cardinality $\leq \lambda^*$
- (g) \mathbb{Q}_2 is the forcing of adding λ^* μ -Cohen subsets to μ and $\mathbb{Q} = \mathbb{Q}_1 \times \mathbb{Q}_2$.

Then for some forcing notion \mathbb{P} we have $\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R} \triangleleft \mathbb{P}$ and in $\mathbf{V}^{\mathbb{P}}$:

- (α) the forcing with \mathbb{P} collapse no cardinal except those collapsed by $\mathbb{Q}_1 \times \mathbb{R}$, in fact $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R})$ is ϑ^- -centered; i.e., $\mu^{+\alpha}$ -centered if $\vartheta = \mu^{+\alpha+1}$

- (β) forcing with \mathbb{P} add no subset of χ , forcing with $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R})$ satisfies $\boxtimes_{\kappa, \mu, \vartheta, \lambda, \lambda^*}^+$ from Definition 2.2 below as witnessed by $\langle \mu_i : i < \kappa \rangle$
- (γ) $\mu_i = \mu_i[G_{\mathbb{P}}]$, μ is strong limit of cofinality κ and $\langle \mu_i : i < \kappa \rangle$ is an increasing continuous sequence of strong limit singulars with limit μ (and $\mathcal{H}(\mu_{i+1})$ satisfies a parallel of the statement \boxtimes_4 from the proof of 1.7),
- (δ) for each $i < \kappa$ we have $\mu_i < \lambda_i \leq \lambda_i^* = (\lambda_i^*)^{\mu_i}$ and $\mu_\kappa = \mu$, $\lambda_\kappa = \lambda$, $\lambda_\kappa^* = \lambda^*$ and $(\mu_i, \lambda_i, \lambda_i^*)$ is quite similar to $(\mu, \lambda, \lambda^*)$ (see proof), more specifically: in some intermediate universe \mathbf{V}_1 , for some normal ultrafilter \mathcal{D} on μ and $F, F_* : \mu \rightarrow \mu$ we have $\prod_{i < \mu} (F(i), <)/\mathcal{D} \cong (\lambda, <)$, $\lambda_i = F(\mu_i)$ and $\prod_{i < \mu} (F_*(i), <)/\mathcal{D} \cong (\lambda^*, <)$ and $F_*(\mu_i) = \lambda_i^*$ and we have $\bar{M} = \langle M_i : i < \mu \rangle$ and M_i a model with universe λ_i and vocabulary τ ; and $\prod_{i < \mu} M_i/\mathcal{D} \cong M$
- (ε) for $i < \kappa$ we have $2^{\mu_i} = \lambda_i^*$ and $2^{\lambda_i^*} = \mu_{i+1}$
- (ζ) $\prod_{i < \kappa} M_{\mu_i}/D^*$ is isomorphic to M if D^* is a normal ultrafilter, in fact, $\{(f(\mu_i) : i < \kappa)/D^* : f \in \mathbf{V}_1 \text{ and } f \in \prod_{i < \mu} F(i)\}$ is the universe of $\prod_{i < \kappa} M_{\mu_i}/D^*$
- (η) for every $f \in \prod_{i < \kappa} M_i/D^*$ we can in \mathbf{V}_1 find $\varepsilon(f) < \theta$ and $g_{f,\varepsilon} \in \prod_{i < \mu} F(i)$ for $\varepsilon < \varepsilon(f)$ such that $\{i < \kappa : \bigvee_{\varepsilon < \varepsilon(f)} f(i) = g_{f,\varepsilon}(\mu_i)\} \in D^*$
- (θ) $\prod_{i < \kappa} (\lambda_i, <)/D^*$ is λ -like linear ordering (not necessarily well ordering as possibly $\theta > \aleph_0$) if $\Vdash_{\mathbb{Q}}$ “ $\text{cf}(\lambda) > \kappa$ and $\alpha < \lambda \rightarrow |\alpha|^\kappa < \lambda$ ”
- (ι) if D^* is a normal ultrafilter, $\mathbb{Q}_1 = \mathbb{Q}_{\mu, \lambda}^1$ (of 1.1) and $\mathbb{R} = \text{Levy}(\mu, < \lambda)$, then the conclusion on irr in 1.7 holds.

Definition 2.2. 1) We say $\boxtimes_{\gamma, \mu, \vartheta, \lambda^*}(\mathbb{Q})$ or we say \mathbb{Q} satisfies $\boxtimes_{\gamma, \mu, \vartheta, \lambda^*}$ (as witnessed by $(\bar{\mu}, \mathcal{D})$) if:

- (i) \mathbb{Q} is a forcing notion of cardinality $\leq \lambda^*$
- (ii) \mathbb{Q} satisfies the ϑ -c.c.
- (iii) \mathbb{Q} (i.e. forcing with \mathbb{Q}) add a sequence $\langle \mu_i : i < \gamma \rangle$ of cardinals $< \mu$, strongly inaccessible in \mathbf{V} , strong limit in $\mathbf{V}^{\mathbb{Q}}$
- (iv) $\Vdash_{\mathbb{Q}}$ “ μ_i ($i < \gamma$) is increasing continuous”
- (v) \mathcal{D} is a normal ultrafilter on μ
- (vi) for every $p \in \mathbb{Q}$ for some $\beta < \gamma$ for every $A \in \mathcal{D}$ there is q satisfying $p \leq q \in \mathbb{Q}$ such that $q \Vdash$ “ $\{\mu_i : \beta < i < \gamma\} \subseteq A$ ”
- (vii) if γ is a limit ordinal then $\Vdash_{\mathbb{Q}}$ “ $\mu = \bigcup_{i < \gamma} \mu_i$ ”
- (viii) in $\mathbf{V}^{\mathbb{Q}}$ we have $2^\mu = \lambda^*$ and μ is strong limit.

2) We say $\boxtimes_{\gamma, \mu, \vartheta, \lambda^*}^+$ or we say \mathbb{Q} satisfies $\boxtimes_{\gamma, \mu, \vartheta, \lambda^*}^+$ (as witnessed by $(\bar{\mu}, f_\theta, f_{\lambda^*})$) if:

- (a) \mathbb{Q} satisfies $\boxtimes_{\gamma, \mu, \vartheta, \lambda^*}$ as witnessed by $\bar{\mu} = \langle \mu_i : i < \gamma \rangle$
- (b) if $G \subseteq \mathbb{Q}$ is generic over \mathbf{V} then for every $\beta < \gamma$ we have $\mathcal{H}(\mu_{\beta+1})^{\mathbf{V}^{\mathbb{Q}}}$ is gotten from $\mathcal{H}(\mu_{\beta+1})^{\mathbf{V}}$ by a forcing $\mathbb{Q}_{\beta+1}$ which is like \mathbb{Q} with (β, μ_β) here standing for (γ, μ) there.

Proof. Like the proof of 1.7 but we use [GM] instead of [Mg4]; note that $\vartheta = \mu^{+3}$ comes from making the forcing μ^{+3} -c.c. So the pure decision of $\mathbb{P}(\bar{D}, \bar{g})$ is changed accordingly. Of course, the change in the assumption on D^* also has some influence.

□_{2.1}

So we get e.g.

Conclusion 2.3. Assume \mathbf{V} satisfies ZFC + μ is supercompact + “ $\lambda > \mu$ is strong inaccessible”.

1) For some forcing extension \mathbf{V}^* , for some ultrafilter D^* on ω_1 there is $\langle \lambda_i : i < \omega_1 \rangle$ such that:

- (i) for $i < \omega_1$, λ_i is weakly inaccessible $< \beth_{\omega_1}$
- (ii) $\lambda = \beth_{\omega_1}^{++}$
- (iii) the linear order $\prod_{i < \omega_1} (\lambda_i, <)/D^*$ is λ -like,
- (iv) λ_i is first weakly inaccessible $> \beth_i$.

2) In part (1) we have: for some sequence $\langle B_i : i < \omega_1 \rangle$ of Boolean algebras, each of cardinality $< \beth_{\omega_1}$ we have $\text{Length}(\prod_{i < \omega_1} B_i/D^*) < \prod_{i < \omega_1} \text{Length}(B_i)/D^*$.

3) If λ in \mathbf{V} , $\lambda > \mu$ is Mahlo, replace (iv) by (iv)' and we can demand in addition that for some sequence $\langle B_i : i < \omega_1 \rangle$ of Boolean algebra, $|B_i| = \text{irr}(B_i) = \lambda_i$ we have $\text{irr}(\prod_{i < \omega_1} B_i/D^*) = \beth_{\omega_1}^+ < \prod_{i < \omega_1} \lambda_i/D^*$ where

(iv)' λ_i is the first weakly inaccessible Mahlo cardinal $> \beth_i$.

Proof. 1) We start getting by forcing using a forcing notion from $\mathcal{H}(\mu)$ (see [Sh:f, Ch.XVI,2.5,p.793] and history there) a normal filter D^0 on ω_1 such that $\mathcal{P}(\omega_1)/D^*$ is layered¹ and $\diamond_{\aleph_1} + 2^{\aleph_1} = \aleph_2$. Hence (see [FMSH 252] and history there) there is an ultrafilter D^* on ω_1 extending D as required in 2.1 clause (d) for $\kappa = \theta = \aleph_1$, that is: if $g \in {}^{\omega_1}\omega_1$ is pressing down on some member of D^* then for some $\alpha < \omega_1$, $\{\beta < \omega_1 : g(\beta) < \alpha\} \in D^*$. Next by forcing with some \aleph_2 -complete μ -c.c. forcing notion of cardinality μ , we get Laver indestructibility (by [L]). Now apply 2.1 with $\kappa = \theta = \aleph_1$, $\mathbb{R} = \text{Levy}(\mu^+, < \lambda)$, \mathbb{Q}_1 trivial or for part (3) as in 1.1 recall that λ is strongly inaccessible Mahlo. Note that easily in $\mathbf{V}^{\mathbb{P}}$, $(\forall \lambda_1 < \lambda)$

¹ it means that this Boolean algebra is $\bigcup_{i < \omega_2} B_i^*$, B_i^* is a Boolean algebra of cardinality \aleph_1 , increasing continuous with i , and $\text{cf}(i) = \aleph_1 \Rightarrow B_i < \mathcal{P}(\omega_1)/D^*$

($\lambda_1^{\aleph_0} < \lambda$). The main new point is clause (iii) which follows by clause (η) of the conclusion of 2.1 and the previous sentence; see the proof of part (3).

2) The proofs in [MgSh 433] applies also in our changed circumstances.

3) But for irr the problem seems more involved. We use 2.5 below instead of 1.3 and note that \mathbb{Q}_2, \mathbb{R} and the Gitik Magidor forcing $\mathbb{P}/(\mathbb{Q}_1 \times \mathbb{Q}_2 \times \mathbb{R})$ though not fully preserving $(*)_{\lambda, < \mu, \underline{B}}$ of 2.5 below it still preserves enough as we now prove.

So in $\mathbf{V}^{\mathbb{P}}$ let $f_\alpha/D^* \in \prod_{i < \kappa} B_{\mu_i}/D^*$ so $f_\alpha \in \prod_{i < \kappa} B_{\mu_i}$ for $\alpha < \lambda$. By clause (η) of 2.1

for each α we can find in \mathbf{V}_2 a sequence $\langle g_{\alpha,n} : n < \omega \rangle$ satisfying $g_{\alpha,n} \in \prod_{i < \mu} B_i$

such that $\{i < \omega_1 : (\exists n)(f_\alpha(i) = g_{\alpha,n}(\mu_i))\} \in D^*$. Without loss of generality we have $A_{\alpha,n} = A_n$ where $A_{\alpha,n} = \{i < \omega_1 : f_\alpha(i) = g_{\alpha,n}(\mu_i)\}$, as $2^{\aleph_1} < \beth_{\omega_1} < \lambda = \text{cf}(\lambda)$. Now in \mathbf{V}_1 , there is an isomorphism \mathbf{j} from $\prod_{i < \mu} B_i/D$ onto B , so $\mathbf{j}(g_{\alpha,n}/D) \in$

B . In $\mathbf{V}_2[G_{\mathbb{R}}]$ we apply $(*)_{\lambda, \aleph_0, B}$ of 2.5 and find $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \lambda$ such that $n < \omega \Rightarrow B \models \mathbf{j}(g_{\beta_0,n}/D) = \sigma(\mathbf{j}(g_{\beta_0,n}/D), \mathbf{j}(g_{\beta_1,n}/D), \mathbf{j}(g_{\beta_2,n}/D), \mathbf{j}(g_{\beta_3,n}/D))$ where

σ is the Boolean term $\sigma^*(x_0, x_1, x_2) = (x_0 \cap x_1) \cup (x_0 \cap x_2) \cup (x_1 \cap x_2)$. Hence

$$Y_n =^{df} \{\zeta < \mu : B_\zeta \models g_{\beta_0,n}(\zeta) = \sigma^*(g_{\beta_1,n}(\zeta), g_{\beta_2,n}(\zeta), g_{\beta_3,n}(\zeta))\} \in \mathcal{D}$$

hence $Y = \bigcap_{n < \omega} Y_n \in \mathcal{D}$ hence for some $i^* < \kappa$, $(\forall i)[i^* \leq i < \kappa \rightarrow \mu_i \in Y]$

but $\mu_i \in Y \Rightarrow (\forall n < \omega)[B_{\mu_i} \models g_{\beta_0,n}(\mu_i) = \sigma(g_{\beta_1,n}(\zeta), g_{\beta_2,n}(\zeta), g_{\beta_3,n}(\zeta))]$. As $A_{\beta_\ell,n} = A_n$ we are done. $\square_{2.3}$

Remark 2.4. 1) In 2.3(1), (2) without loss of generality \beth_{ω_1} is the limit of the first ω_1 (weakly) inaccessible.

2) In 2.3(3) without loss of generality \beth_{ω_1} is the limit of the first ω_1 Mahlo (weakly) inaccessible. Can we omit Mahlo?

3) Of course, 2.3 is just one extreme variant.

4) If we would like to replace in 2.3, \aleph_1 by $\kappa = \kappa^{<\kappa} > \aleph_1$, we can use [FMSH 252], hence higher large cardinals.

Claim 2.5. 1) For $\mathbb{Q} = \mathbb{Q}_{\mu,\lambda}^1$, \underline{B} as in 1.3 we have, for $\tau < \mu$ it is forced ($\Vdash_{\mathbb{Q}_{\mu,\lambda}^1}$) that:

$(*)_{\lambda, \tau, \underline{B}}$ if $y_{\alpha,\varepsilon} \in \underline{B}$ for $\alpha < \lambda$, $\varepsilon < \tau$ then for some $\beta_0 < \beta_1 < \beta_2 < \beta_3$ we have $\varepsilon < \tau \Rightarrow y_{\beta_0,\varepsilon} = \sigma^*(y_{\beta_1,\varepsilon}, y_{\beta_2,\varepsilon}, y_{\beta_3,\varepsilon})$ where $\sigma^*(y_1, y_2, y_3) = (y_1 \cap y_2) \cup (y_1 \cap y_3) \cup (y_2 \cap y_3)$.

2) If B is a Boolean algebra, $\tau < \lambda$ and \mathbb{Q}^* is τ^+ -complete (or just do not add new τ -sequence of ordinals $< |B|$) and satisfies the $(\lambda, 4)$ -Knaster property (i.e. among any λ conditions there are λ , any three of them has a common upper bound), then forcing by \mathbb{Q}^* preserve $(*)_{\lambda, \tau, B}$.

Proof. 1) As in the proof of 1.3, again the point is checking $(*)_{\lambda, \tau, \underline{B}}$ so let $p \Vdash \langle y_{\beta,\varepsilon} : \beta < \lambda, \varepsilon < \tau \rangle$ be a counterexample". For each $\alpha < \lambda$ choose p_α such that $p \leq p_\alpha$ and $p_\alpha \Vdash "y_{\alpha,\varepsilon} = y_{\alpha,\varepsilon}"$ for $\varepsilon < \tau$ and without loss of generality $y_{\alpha,\varepsilon} \in$

$B_{w[p_\alpha]}$ and choose $\alpha_{\beta,\zeta} \in w[p_\beta]$ for $\zeta < \zeta_\beta$ such that $y_{\beta,\varepsilon} \in \{\{x_\gamma : \gamma \in \{\alpha_{\beta,\varepsilon} : \varepsilon < \zeta_\beta\}\} \cap B_{w[p_{\beta_\ell}]}\}$ for some $\zeta_\beta < \tau^+$ with $\alpha_{\beta,\varepsilon}$ increasing with ε , and let $\xi_\beta \leq \zeta_\beta$ be such that $(\forall \varepsilon)[\alpha_{\beta,\varepsilon} < \beta \equiv \varepsilon < \xi_\beta]$. Let $y_{\beta,\varepsilon} = \sigma_{\beta,\varepsilon}(\dots, x_{\alpha_{\beta,\varepsilon}}, \dots)_{\varepsilon < \zeta_\beta}$ (so the term $\sigma_{\beta,\varepsilon}$ uses only finitely many of its variables). We choose S, w, r , etc., as in the proof there with $\xi \leq \zeta, \langle \alpha_\varepsilon : \varepsilon < \xi \rangle, \langle \sigma_\varepsilon : \varepsilon < \tau \rangle$ replacing $m \leq n, \langle \alpha_\ell : \ell < m \rangle, \sigma$.

We choose $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S and it is enough to find $q \in \mathbb{Q}_{\mu,\lambda}^1$ such that $\ell < 4 \Rightarrow p_{\beta_\ell} \leq q$ and $q \Vdash \text{“}y_{\beta_0,\varepsilon} = \sigma(y_{\beta_1,\varepsilon}, y_{\beta_2,\varepsilon}, y_{\beta_3,\varepsilon}) \text{ for } \varepsilon < \tau\text{”}$. We define $u = \bigcup_{\ell < 4} w[p_{\beta_\ell}]$ and \mathcal{F} as there, i.e.,

$$\{f : f \in {}^u 2, f \upharpoonright w[p_{\beta_\ell}] \in \mathcal{F}[p_{\beta_\ell}] \text{ for } \ell < 4 \text{ and for some } \ell \in \{1, 2, 3\} \text{ we have } m \in \{0, 1, 2, 3\} \setminus \{\ell\} \zeta < \mu \Rightarrow f(x_{\alpha_{\beta_0,\zeta}}) = f(x_{\alpha_{\beta_m,\zeta}})\}.$$

Now check.

2) Straightforward. □_{2.5}

Another case is, e.g.

Claim 2.6. We can replace in all the results above $\text{irr}(B)$ by $\text{h-cof}(B)$.

Remark. Recall $\text{h-cof}^+(B) = \cup\{|Y|^+ : Y \subseteq B \text{ and } Y = \{a_\alpha : \alpha < |Y|\}$ satisfies $\alpha < \beta \Rightarrow \neg(a_\beta \leq a_\alpha)$, see [M2, Th.18.1,p.226].

Proof. Similar just easier.

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