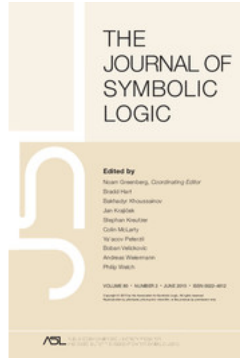
Sh:840



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MODEL THEORY WITHOUT CHOICE? CATEGORICITY

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Abstract. We prove Łos conjecture = Morley theorem in ZF, with the same characterization, i.e., of first order countable theories categorical in $\aleph_\alpha$ for some (equivalently for every ordinal) $\alpha > 0$. Another central result here in this context is: the number of models of a countable first order T of cardinality $\aleph_\alpha$ is either $\geq |\alpha|$ for every α or it has a small upper bound (independent of α close to $\beth_2$).

Annotated Content

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§0. Introduction. I have known for long that there is no interesting model theory without (the axiom of) choice, not an exciting question anyhow as we all know that AC is true. This work is dedicated to a try to refute this opinion, i.e., this work throws some light on this in the contrary direction: Theorem 0.2 seriously, Theorem 3.14, (the parallel of the ZFC theorem 0.3) in a stronger way.

Lately, I have continued my work on pcf without full choice (see [Sh:835], earlier [Sh:497], [Sh:E38], later [Sh:938]) and saw that with suitable "reasonable" weak version of (the axiom of) choice essentially we can redo all [Sh:c] (for first order classes with well ordered vocabulary; see 6.3).

Then it seems reasonable to see if older established version suffices, say ZF+$\mathrm{DC}_{\aleph_0}$. We first consider Łoś conjecture which can be phrased (why only $\aleph_\alpha$'s and not other powers? see below)

0.1. The choiceless Łoś Conjecture. For a countable (first order theory) T:

$(*)_1$ T is categorical in $\aleph_\alpha$ for (at least) one ordinal $\alpha > 0$ iff

$(*)_2$ T is categorical in $\aleph_\beta$ for every ordinal $\beta > 0$.

In §1 we shall show that the Morley's proof works exactly when there is an uncountable well ordered set of reals. In §2 we give a new proof which works always (under ZF); it used Hrushovski [Hr89d], so:

0.2. Theorem. [ZF] *For any countable T we have: $(*)_1$ of 0.1 iff $(*)_2$ of 0.1 iff T is $\aleph_0$-stable with no two cardinal models.*[1]

Note that though we have been ready enough to use ZF + $\mathrm{DC}_{\aleph_0}$ in fact we solve the problem in ZF.

A theorem from [Sh:c] is

0.3. Theorem. [ZFC] *For a countable complete (first order theory) T, one of the following occurs:*

(a) $\dot{I}(\aleph_\alpha, T) \geq |\alpha|$ *for every ordinal* α,

(b) $\dot{I}(\aleph_\alpha, T) \leq \beth_2$ *for every ordinal* α *(and can analyze this case: either* $\dot{I}(\aleph_\alpha, T) = 1$ *for every* α *or* $\dot{I}(\aleph_\alpha, T) = \mathrm{Min}\{\beth_2, 2^{\aleph_\alpha}\}$ *for every* $\alpha > 0$*).*

We shall prove a similar theorem in ZF in 3.14.

Thirdly, we consider an old conjecture from Morley [Mo65]: if a complete (first order) T is categorical in the cardinal $\lambda, \lambda = |T| > \aleph_0$ then T is a definitional extension of some $T' \subseteq T$ of smaller cardinality. The conjecture actually says that T is not really of cardinality λ. This was proved in ZFC. Keisler [Ke71a] proved it when $|T| < 2^{\aleph_0}$. By [Sh:4] it holds if $|T|^{\aleph_0} = |T|$. It is fully proven in [Sh:c, IX, 1.19, pg. 491]). The old proof which goes by division to three cases is helpful but

[1] That is, for no model M of T, formula $\varphi(x, \bar{y}) \in \mathbb{L}_{\tau(T)}$ and sequence $\bar{a} \in {}^{\ell g(\bar{y})}M$, do we have $\aleph_0 \leq |\varphi(M, \bar{a})| < \|M\|$.

not sufficient. Without choice (but note that λ is an $\aleph$) the case T superstable (or just $\kappa_r(T) < \lambda$) has really a similar proof. The other two cases, T is unstable and T stable with large $\kappa_r(T)$, are not. Here in §4 it is partially confirmed, e.g., when λ is regular, the proofs are different though related.

In §7 we deal with power of non-well orderable sets, in §5 we deal with consistency results and in §6 we look what occurs to classical theorems of model theory.

We may consider isomorphism after appropriate forcing. Baldwin–Laskowski–Shelah [BLSh:464], Laskowski–Shelah [LwSh:518] deal with the question "does T or even $\mathrm{PC}(T_1, T)$ have non-isomorphic models which become isomorphic after some c.c.c. forcing?" But this turns out to be very different and does not seem related to the work here.

However, the following definition 0.4 suggests a problem which is closely related but it may be easier to find examples of such objects, so called below "cardinal cases" with "not so nice behaviour" than to find forcing extension of $\mathbf{V}$ which satisfies ZF + a failure of some hopeful theorem.

0.4. DEFINITION. (1) A cardinal case is a pair $(\lambda, \mathbf{P})$ where λ is a cardinal and $\mathbf{P}$ is a family of forcing notions.

(2) A cardinal$^+$ case is a triple $(\lambda, \mathbf{P}, <)$ such that λ is a cardinal, $\mathbf{P}$ a family of forcing notions and $<$ a partial order on $\mathbf{P}$ such that $\mathbb{P}_1 \leq \mathbb{P}_2 \Rightarrow \mathbb{P}_1 \lessdot \mathbb{P}_2$ so if we omit $<$ we mean $\lessdot$.

(3) We say that a theory T or more generally a (definition, absolute enough, of a) class $\mathfrak{K}$ of models is categorical in the cardinal case $(\lambda, \mathbf{P})$ when: for every $M_1, M_2 \in \mathfrak{K}_\lambda$ (i.e., $\in \mathfrak{K}$ of cardinality λ), for some $\mathbb{P} \in \mathbf{P}$ we have $\Vdash_{\mathbb{P}}$ "$M_1 \cong M_2$".

(4) We say that a theory T or more generally a (definition, absolute enough, of a) class $\mathfrak{K}$ of models is categorical in the cardinal$^+$ case $(\lambda, \mathbf{P}, <)$ when for any $\mathbb{P} \in \mathbf{P}$, in $\mathbf{V}^{\mathbb{P}}$ we have: if $M_1, M_2 \in \mathfrak{K}_\lambda^{\mathbf{V}[\mathbb{P}]}$, then for some $\mathbb{P}' \in \mathbf{P}$ satisfying $\mathbb{P} \leq \mathbb{P}'$ we have $\Vdash_{\mathbb{P}'/\mathbb{P}}$ "$M_1 \cong M_2$".

(5) Similarly uncategorical, has/does not have μ pairwise non-isomorphic models, etc.

(6) We may replace cardinal by power.

0.5. QUESTION. Characterize countable (complete first order) T which may be categorical in some uncountable power (say in some forcing extension of $\mathbf{L}[T]$). See on this §7.

This work may be continued in [Sh:F701].

We thank Udi Hrushovski for various comments and pointing out that Łoś conjecture proof is over after 2.12 as Kueker conjecture is known in the relevant case (in earlier versions the proof (of the choiceless Łoś conjecture) was more interesting and longer). We thank Moti Gitik for a discussion of the consistency results and for pointing out 5.5).

Lately, I have learned that Truss and his students were pursuing the connection between universes with restricted choice and model theory by a different guiding line: using model theory to throw light on the arithmetic of Dedekind finite powers, works in this direction are Agatha Walczak–Typke [WT05], [WT07]. Very interesting, does not interact with the present investigation, but may be relevant to Question 0.5.

Recall

0.6. Definition. A cardinal is the power of some well ordered set (so an $\aleph$ or a natural number).

In [Sh:F701] we may deal with theories in a vocabulary which is not well ordered.

0.7. Convention. If not said otherwise

(a) T is a first order theory in a vocabulary $\tau \subseteq \mathbf{L}$,
(b) T is complete,
(c) T is infinite,
(d) if T is countable for simplicity $\tau, T \subseteq \mathscr{H}(\aleph_0)$ (for notational simplicity).

This is justified by

0.8. *Observation.* Assume τ is a countable vocabulary and T is a first order theory in τ, i.e., $T \subseteq \mathbb{L}_\tau$.

(1) There is a vocabulary $\tau' \subseteq \mathscr{H}(\aleph_0)$ ($\subseteq \mathbf{L}_\omega$) and first order theory T' in τ' (so $T' \subseteq \mathbb{L}_\tau \subseteq \mathscr{H}(\aleph_0)$) such that for every cardinal λ, T is categorical in λ iff T' is categorical in λ (and even $\dot{I}(\lambda, T) = \dot{I}(\lambda, T')$, similarly for power and the parallel of 0.9 below).
(2) If T is categorical in some cardinal λ then $T \cup \{(\exists^{\geq n} x)(x = x) : n < \omega\}$ is complete.

0.9. *Observation.* Assume τ is a vocabulary which can be well ordered (i.e., $|\tau| \in$ Card).

There is a vocabulary $\tau' \in \mathbf{L}$ (or even $\tau' \in \mathbf{L}_{|\tau|^+}$) and a function f from $\mathbb{L}(\tau')$ onto $\mathbb{L}(\tau)$ (note that $\mathbb{L}(\tau') \subseteq \mathbf{L}_{|\tau|^+}$) mapping predicates/functions symbols to predicate/function symbols respectively with the same arity such that:

$\boxtimes_1$ f maps the set of (complete) first order theories in $\mathbb{L}_\tau$ onto the set of (complete) first order theories in $\mathbb{L}_{\tau'}$ (really this is the derived map, $\hat{f}$)

$\boxtimes_2$ for some definable class $\mathbf{F}$ which is a function, $\mathbf{F}$ maps the class of τ-models onto the class of τ'-models such that

  (a) $\mathbf{F}$ is one to one onto and $\mathrm{Th}(\mathbf{F}(M)) = \hat{f}(\mathrm{Th}(M))$,
  (b) $\mathbf{F}$ preserves isomorphisms and non-isomorphisms,
  (c) $\mathbf{F}$ preserves $M \subseteq N, M \prec N$,
  (d) if $\mathbf{F}(M) = M'$ then for every sentence $\psi \in \mathbb{L}(\tau')$, $M' \models \psi \Leftrightarrow M \models f(\psi)$ where $\hat{f}(\psi)$ is defined in $\boxtimes_1$,
  (e) $\mathbf{F}$ preserves power, so equality and inequality of powers (hence for any theory $T \subseteq \mathbb{L}_\tau$, letting $T' = \hat{f}(T)$, for any set X, $|\{M/\cong : M \in \mathrm{Mod}_T$ has power $|X|\}| = |\{M'/\cong : M' \in \mathrm{Mod}_{T'}$ has power $|X|\}|$.

We shall use absoluteness freely recalling the main variant.

0.10. Definition. (1) We say $\varphi(\bar{x})$ is upward ZFC-absolute <u>when</u>: if $\mathbf{V}_1 \subseteq \mathbf{V}_2$ (are transitive classes containing the class Ord of ordinals, both models of ZFC) and $\bar{a} \in \mathbf{V}_1$ <u>then</u> $\mathbf{V}_1 \models \varphi(\bar{a}) \Rightarrow \mathbf{V}_2 \models \varphi(\bar{a})$.

(2) Replacing upward by downward mean we use $\Leftarrow$; omitting upward mean we use $\Leftrightarrow$. Similarly for version ZFC′ of ZFC (e.g., ZF + DC); but absolute means ZFC-absolute.

0.11. Convention. (1) If not said otherwise, for a theory T belonging to $\mathbf{L}[Y_0]$, $Y_0 \subseteq \mathrm{Ord}$, saying "$T$ satisfies Pr", ("Pr" stands for "Property") we mean "for some $Y_1 \subseteq \mathrm{Ord}$ for every $Y_2 \subseteq \mathrm{Ord}$, T satisfies Pr in $\mathbf{L}[Y_0, Y_1, Y_2]$".
(2) But "T categorical in λ" always means in $\mathbf{V}$.

Recall

0.12. Definition. (1) $\theta(A) = \mathrm{Min}\{\alpha$: there is no function from A onto $\alpha\}$.
(2) $\Upsilon(A) = \mathrm{Min}\{\alpha$: there is no one-to-one function from α into $A\}$.

0.13. Definition. (1) If $T \subseteq \mathbb{L}(\tau)$, Γ is a set of types in $\mathbb{L}(\tau)$, i.e., each $p \in \Gamma$ is an m-type for some m, then $\mathrm{EC}(T, \Gamma)$ is the class of τ-models M of T which omits every $p(\bar{x}) \in \Gamma$.
(2) If $T \subseteq \mathbb{L}(\tau)$ is complete, $T \subseteq T_1 \subseteq \mathbb{L}(\tau_1)$ and $\tau \subseteq \tau_1$ then $\mathrm{PC}(T_1, T)$ is the class of τ-reducts of models M_1 of T_1. Similarly for a set Γ of types in $\mathbb{L}(\tau_1)$ let $\mathrm{PC}(T, T_1, \Gamma)$ be the class of τ-reducts of models $M \in \mathrm{EC}(T_1, \Gamma)$.

We shall use Ehrenfeucht–Mostowski models.

0.14. Definition. (1) Φ is proper for linear orders when:
(a) for some vocabulary $\tau = \tau_\Phi = \tau(\Phi)$, Φ is an ω-sequence, the n-th element a complete quantifier free n-type in the vocabulary τ,
(b) for every linear order I there is a τ-model M denoted by $\mathrm{EM}(I, \Phi)$, generated by $\{a_t : t \in I\}$ such that $s \neq t \Rightarrow a_s \neq a_t$ for $s, t \in I$ and $\langle a_{t_0}, \dots, a_{t_{n-1}}\rangle$ realizes the quantifier free n-type from clause (a) whenever $n < \omega$ and $t_0 <_I \cdots <_I t_{n-1}$; so really M is determined only up to isomorphism but we may ignore this and use $I_1 \subseteq J_1 \Rightarrow \mathrm{EM}(I_1, \Phi) \subseteq \mathrm{EM}(I_2, \Phi)$. We call $\langle a_t : t \in I\rangle$ "the" skeleton of M; of course "the" is an abuse of notation as it is not necessarily unique.
(2) If $\tau \subseteq \tau(\Phi)$ then we let $\mathrm{EM}_\tau(I, \Phi)$ be the τ-reduct of $\mathrm{EM}(I, \Phi)$.
(3) For first order T, let $\Upsilon^{\mathrm{or}}_\kappa[T]$ be the class of Φ proper for linear orders such that
(a) $\tau_T \subseteq \tau_\Phi$ and τ_Φ has cardinality $\leq \kappa$,
(b) for any linear order I the model $\mathrm{EM}(I, \Phi)$ if I is well-orderable then this model has cardinality $|\tau(\Phi)| + |I|$ and we have $\mathrm{EM}_{\tau(T)}(I, \Phi) \in K$,
(c) for any linear orders $I \subseteq J$ we have $\mathrm{EM}_{\tau(T)}(I, \Phi) \prec \mathrm{EM}_{\tau(T)}(J, \Phi)$.
(4) We may use Skeleton $\langle \bar{a}_t : t \in I\rangle$ with $\alpha = \ell g(\bar{a}_t)$ constant but in the definition of "$\Phi \in \Upsilon^{\mathrm{or}}_\kappa[T]$ we add $\alpha < \kappa^+$. Alternatively $\bar{a}_t = \langle F_i^{\mathrm{EM}(I,\Phi)}(a_t) : i < \alpha\rangle$, where $F_i \in \tau_\Phi$ are unary function symbols. We use Φ, Ψ only for such objects. Let $\Upsilon^{\mathrm{or}}_T = \Upsilon^{\mathrm{or}}_{|T|+\aleph_0}[T]$.

§1. Morley's proof revisited. The main theorem of this section is 1.1. The proof is just adapting Morley's proof in ZFC. We shall use 0.8(2) and convention 0.7 freely.

1.1. Theorem. [ZF + there is an uncountable well ordered set of reals] *The following conditions on a countable (first order) T are equivalent*:
(A) *T is categorical in some cardinal $\aleph_\alpha > \aleph_0$, in* $\mathbf{V}$, *of course*
(B) *T is categorical in every cardinal $\aleph_\beta > \aleph_0$, in* $\mathbf{V}$, *of course*
(C) *T is (in $\mathbf{L}[T]$), totally transcendental (i.e., $\aleph_0$-stable) with no two cardinal models (i.e., for no model M of T and formula $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ do we*

have $\aleph_0 \leq |\varphi(M, \bar{a})| < \|M\|$ *and* $\|M\|$ *is a cardinal, i.e., the set of elements of* M *is well-orderable hence its power is a cardinal*),

(D) *if* $\mathbf{V}' \subseteq \mathbf{V}$ *is a transitive class extending* $\mathbf{L}$, $T \in \mathbf{V}'$ *and* $\mathbf{V}'$ *satisfies* ZFC *then the conditions in* (C) *hold*,

(E) *for some* $\mathbf{V}'$ *clause* (D) *holds.*

PROOF. By 0.7 or better 0.8(2) without loss of generality T is complete, $T \subseteq \mathscr{H}(\aleph_0)$. Trivially (B) $\Rightarrow$ (A). Next (A) $\Rightarrow$ (C) by claims 1.2, 1.3 below. Lastly, $(C) \Rightarrow$ (A) by 1.4 below and (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E) holds by absoluteness. $\square_{1.1}$

1.2. CLAIM. [ZF + $\exists$ a set of $\aleph_1$ reals] *If* T *is* (*countable*) *and in* $\mathbf{L}[T]$ *the theory* T *is not* $\aleph_0$*-stable and* $\lambda > \aleph_0$ *then* T *is not categorical in* λ.

PROOF. In $\mathbf{L}[T]$ we can find E.M. models, i.e., $\Phi \in \Upsilon_T^{\text{or}}$ such that $\tau(\Phi)$ is countable, extends $\tau = \tau_T$ and $\text{EM}_\tau(I, \Phi)$ is a model of T (of cardinality λ) for every linear order I (of cardinality λ) and let $M_1 = \text{EM}_\tau(\lambda, \Phi)$ and without loss of generality the universe of M_1 is λ.

In $\mathbf{V}$ let $\bar{\eta} = \langle \eta_\alpha : \alpha < \omega_1 \rangle$ be a sequence of pairwise distinct reals. In $\mathbf{L}[T]$ there is a countable model M_0 of T with $\mathbf{S}(M_0)$ uncountable so containing a perfect set. Hence also in $\mathbf{L}[T, \bar{\eta}]$, M_0 is a countable model of τ with $\mathbf{S}(M_0)$ containing a perfect set, hence there is (in $\mathbf{L}[T, \bar{\eta}]$) a model M_2 of T of cardinality λ (λ is still an uncountable cardinal in $\mathbf{L}[T, \bar{\eta}]$) such that $M_0 \prec M_2$ and there is a sequence $\langle a_i : i < \omega_1 \rangle$, $a_i \in M_2$ realizes $p_i \in \mathbf{S}(M_0)$ with $\langle p_i : i < \omega_1^{\mathbf{V}} \rangle$ pairwise distinct. Without loss of generality the universe of M_2 is λ.

Clearly even in $\mathbf{V}$, the model M_1 satisfies "if $A \subseteq M_1$ is countable then the set $\{\text{tp}(a, A, M_1) : a \in M_1\}$ is countable" whereas M_2 fails this; hence the models M_1, M_2 have universe λ, are models of T and are not isomorphic, so we are done. $\square_{1.2}$

1.3. CLAIM. *Assume* T *is countable* $\aleph_0$*-stable and has a two cardinal model* (*in* $\mathbf{L}[T]$, *but both are absolute*).

Then T *is not categorical in* λ, *in fact*, $\dot{I}(\aleph_\alpha, T) \geq |\alpha|$ *for every ordinal* α.

PROOF. So in $\mathbf{L}(T)$ it has a model M_1 and a finite sequence $\bar{a} \in {}^{\ell g(\bar{y})}(M_1)$ and a formula $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ such that $\aleph_0 \leq |\varphi(M_1, \bar{a})| < \|M_1\|$. If $\aleph_\beta < \lambda$, working in $\mathbf{L}[T]$ without loss of generality $|\varphi(M_1, \bar{a})| = \aleph_\beta$, $\|M_1\| = \lambda$ (by [Sh:3]) and the universe of M_1 is λ. By the $\aleph_0$-stability T has (in $\mathbf{L}[T]$) a saturated model M_2 of cardinality λ, so without loss of generality the universe of M_2 is λ. So $\bar{a}' \in {}^{\ell g(\bar{y})}(M_2) \Rightarrow |\varphi(M_2, \bar{a}')| \notin [\aleph_0, \lambda)$. Clearly even in $\mathbf{V}$, M_1, M_2 are models of T of cardinality λ and are not isomorphic. In fact for every $\aleph_\beta \leq \lambda$, T has an $\aleph_\beta$-saturated not $\aleph_{\beta+1}$-saturated model M_β of cardinality λ such that $|\varphi(M_\beta, \bar{a}_\beta)| = \aleph_\beta$ for some $\bar{a} \in {}^{\ell g(\bar{y})}(M_\beta)$. So $\min\{|\varphi((M_\beta, \bar{b})| + \aleph_0 : \bar{b} \in {}^{\ell g(\bar{y})}(M_\beta)|\} = \aleph_\beta$, hence $\mathbf{V} \models M_\beta \not\cong M_\gamma$ when $\aleph_\beta < \aleph_\gamma \leq \lambda$ so also the second phrase in the conclusion of the claim holds and even $\dot{I}(\aleph_\alpha, T) \geq |\alpha + 1|$. $\square_{1.3}$

1.4. CLAIM. *Assume* T *is countable* $\aleph_0$*-stable with no two cardinal models even just in* $\mathbf{L}[T]$ *and* $\lambda > \aleph_0$. *Then* T *is categorical in* λ.

PROOF. Let M_1, M_2 be models of T of cardinality λ, without loss of generality both have universe λ, clearly $\mathbf{L}[T, M_1, M_2]$ is a model of ZFC and by absoluteness T still satisfies the assumption of 1.4 in it, and M_1, M_2 are (also in it) uncountable models of T of the same uncountable cardinality in this universe. But by 1.1 being a theorem of ZFC clearly M_1, M_2 are isomorphic in $\mathbf{L}[T, M_1, M_2]$, hence in $\mathbf{V}$. $\square_{1.4}$

§2. Stability and categoricity. Our aim in this section is the categoricity spectrum for countable T (i.e., Th. 2.1), but in the claims leading to the proof we do not assume countability. Note that the absoluteness of various properties is easier for countable T.

2.1. Theorem. [ZF] *For countable T, clauses* (A), (B), (C), (D), (E) *of Theorem* 1.1 *are equivalent.*

2.2. *Observation.* (1) If T is unstable so has the order property, say as witnessed by $\varphi(\bar{x}, \bar{y})$ and, of course, $\tau = \tau_T \subseteq \mathbf{L}$ then for some $\Phi \in \mathbf{L}[T]$

⊛ (a) Φ is proper for linear orders,

(b) $\tau \subseteq \tau_\Phi$ and for every linear order I, $\mathrm{EM}_\tau(I, \Phi)$ is a model of T with skeleton $\langle \bar{a}_t : t \in I\rangle, \ell g(\bar{a}_\tau) = \ell g(\bar{x}) = \ell g(\bar{y})$,

(c) $\mathrm{EM}_\tau(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\mathrm{if}(s<t)}$,

(d) $\tau(\Phi) \subseteq \mathbf{L}$ and $|\tau(\Phi)| = |T|$ (if $\tau(T) \in \mathbf{L}$, without loss of generality $\tau(\Phi) \in \mathbf{L}$) and without loss of generality $\mathbf{L}[T] \models |\tau(\Phi)| = |T|$.

(2) It follows that if I is well orderable then the universe of $\mathrm{EM}(I, \Phi)$ is well orderable so it is of cardinality $|I| + |T| + \aleph_0$ hence we can assume it has this cardinal as its universe.

Proof. (1) By [Sh:c].

(2) Follows. $\square_{2.2}$

Our first aim is to derive stability from categoricity, for diversion we give some versions.

2.3. Claim. *Let Φ be as in 2.2. Then $M_1 \not\cong M_2$ when κ_1, κ_2 are regular uncountable cardinals ($> |T|$) and for some $A \subseteq \mathrm{Ord}$, in $\mathbf{L}[A]$*

⊛ (a) $M_\ell = \mathrm{EM}_\tau(I_\ell, \Phi)$ *in* $\mathbf{L}[A]$, (*so* $T, \Phi \in \mathbf{L}[A], I_\ell \in \mathbf{L}[A]$) *for* $\ell = 1, 2$,

(b) $\bar{s}^1 = \langle s^1_\alpha : \alpha < \kappa_1\rangle$ *is increasing in* I_1, $\bar{t}^1 = \langle t^1_\alpha : \alpha < \kappa_2\rangle$ *is decreasing in* I_1 (*in* $\mathbf{L}[A]$),

(c) $\alpha < \kappa_1 \wedge \beta < \kappa_2 \Rightarrow s^1_\alpha <_{I_1} t^1_\beta$ *but* $\neg(\exists s \in I_1)[(\forall \alpha < \kappa_1)(s^1_\alpha < r) \wedge (\forall \beta < \kappa_2)(s < t^1_\beta)]$,

(d) *in I_2 there is no pair of sequences like $\bar{s}^1, \bar{t}^1$,*

(e) *also in the inverse of I_2, there is no such pair,*

(f) [*only for simplicity, implies* (d) + (e)] *I_2 is $\cong I_2 \times \mathbb{Q}$ ordered lexicographically.*

Proof. Without loss of generality the universes of M_1, M_2 are ordinals, and toward contradiction assume f is an isomorphism from M_1 onto M_2. We can work in $\mathbf{L}[A, M_1, M_2, f]$ which is a model of ZFC, so easy to contradict (as in [Sh:12], see detailed proof showing more in 3.2). $\square_{2.3}$

2.4. *Conclusion* ([ZF + $|T|^+$ is regular]). If T is categorical in some cardinal $\lambda > |T|$, then T is stable (in $\mathbf{L}[T]$).

A fuller version is

2.5. Claim. *$M_1 \not\cong M_2$ when for some $\lambda > |T|$ we have:*

⊛ (a) $M_\ell = \mathrm{EM}_\tau(I_\ell, \Phi)$ *where T, Φ are as in 2.2 so $\Phi \in \mathbf{L}[T]$,*

(b) $I_1 = \lambda \times \mathbb{Q}$ *ordered lexicographically*,

(c) $I_2 = \sum_{\alpha \leq \lambda} I^2_\alpha \in \mathbf{L}[T]$ *where* I^2_α *is isomorphic to* $\alpha + \alpha^*$ (α^* *the inverse of* α) *or just*

(c^-) I_2 *is a linear order of cardinality* λ *such that for every limit ordinal* $\delta \leq |T|^+$, I_2 *has an interval isomorphic to* $\delta + \delta^*$,

(d) I_1, I_2 *has cardinality* λ.

PROOF. Let $\theta = |T|$ in $\mathbf{L}[T]$ and $\theta_1 = (\theta^+)^{\mathbf{V}}$. Without loss of generality M_ℓ has universe λ, assume toward contradiction that $M_1 \cong M_2$ let f be an isomorphism from M_1 onto M_2 and consider the universe $\mathbf{L}[T, M_1, M_2, f]$. In this universe θ_1 may be singular but is still a cardinal so $\delta =: (\theta^+)^{\mathbf{L}[T, M_1, M_2, f]}$ is necessarily $\leq (|\theta|^+)^{\mathbf{V}}$ hence I_2 has an interval isomorphic to $\delta + \delta^*$. Now we continue as in 2.3 (see details in 3.2). $\square_{2.5}$

2.6. *Conclusion.* If T is categorical in the cardinal $\lambda > |T|$, <u>then</u> T is stable.

2.7. DISCUSSION. (1) We may like to have many models. So for T unstable if there are α regular cardinals $\leq \lambda$ we can get a set of pairwise non-isomorphic models of T of cardinality λ indexed by $|\mathscr{P}(\alpha)|$.

It is not clear what, e.g., we can get in $\aleph_1$. As 2.5 indicate it is hard to have few models, i.e., to have such universe (see more in §3); but for our present purpose all this is peripheral, as we have gotten two.

On uni-dimensional see [Sh:c, V, Definition 2.2, pg. 241] and [Sh:c, V. Theorem 2.10, p. 246].

2.8. DEFINITION. A stable theory T is uni-dimensional <u>if</u> there are no $M \models T$ and two infinite indiscernible sets in M which are orthogonal.

2.9. CLAIM. *Assume T is stable (in* $\mathbf{L}[T]$, *anyhow this is* Z^-*-absolute).* <u>*Then*</u> *for every* $\lambda > |T|$, T *has a model* $M \in \mathbf{L}[T]$ *of cardinality* λ *such that*:

$\odot$ *in* M *there are no two (infinite) indiscernible non-trivial sets each of cardinality* $\geq |T|^+$ *which are orthogonal.*

PROOF. We work in $\mathbf{L}[T]$ or $\mathbf{L}[T, Y]$, $Y \subseteq \mathrm{Ord}$ and let $\kappa = |T|^{\mathbf{L}[T]}$ and $\partial = \theta^{\mathbf{V}}(\mathscr{P}(\kappa))$. Let μ be large enough (e.g., $\beth((2^\partial)^+)$, i.e., the $(2^\partial)^+$-th beth), let $\mathfrak{C}$ be a μ^+-saturated model of T. Let $\mathbf{I} = \{a_i : i < \mu\} \subseteq \mathfrak{C}$ be an infinite indiscernible set of cardinality μ and minimal, i.e., $\mathrm{Av}(\mathbf{I}, \cup\mathbf{I})$ is a minimal type. Let $M_1 \prec \mathfrak{C}$ be κ^+-prime over $\mathbf{I}$.

More specifically

$\circledast$ $\langle A_\varepsilon : \varepsilon \leq \kappa^+\rangle$ is an increasing sequence of subsets of $M_1, A_{\kappa^+} = M_1, A_0 = \{a_i : i < \mu\}$ and $\bar{B} = \langle B_a : a \in M_1\rangle$, satisfies $[a \in A_0 \Rightarrow B_a = \{a\}]$ and if $a \in A_{\varepsilon+1} \backslash A_\varepsilon$ then $B_a \subseteq A_\varepsilon$ and $\mathrm{tp}(a, B_a, \mathfrak{C}) \vdash \mathrm{tp}(a, A_{\varepsilon+i} \backslash \{a\}, \mathfrak{C})$ and $|B_a| \leq \kappa$ and without loss of generality $B_a = \{b_{a,j} : j < \kappa\}$ and $\zeta < \varepsilon \Rightarrow B_a \cap A_{\zeta+1} \nsubseteq A_\zeta$ and $a' \in B_a \Rightarrow B_{a'} \subseteq B_a$.

Expand M_1 to M_2 by adding $P^{M_2} = \mathbf{I}, <^{M_2} = \{(a_i, a_j) : i < j < \mu\}, E^{M_2} = \{(b_1, b_2)$: for some $\varepsilon < \kappa^+$ we have $b_1 \in A_\varepsilon \wedge b_2 \in (A_{\varepsilon+1} \backslash A_\varepsilon)\}, F_j^{M_2}(a) = b_{a,j}$ for $j < \kappa$, (hence $a \in A_{\varepsilon+1} \backslash A_\varepsilon \wedge \zeta < \varepsilon \Rightarrow c\ell_{M_2}\{a\} \cap A_{\zeta+1} \backslash A_\zeta \neq \emptyset$) and add Skolem functions, still $\tau(M_2) \in \mathbf{L}[T]$ has cardinality $\kappa = |T|^{\mathbf{L}[T]} = (|T| + \aleph_0)^{\mathbf{L}[T]}$.

Now (as in the proof of the omitting type theorem, see e.g., [Sh:c, VII, §5]) we can find $\langle \mathbf{I}_n : n < \omega \rangle, \mathbf{I}_n \subseteq \mathbf{I}$ is an n-indiscernible sequence in M_2 of cardinality $> 2^\partial$ and

$(*)_1$ if for $n < \omega. M_2 \models a_0^n < \cdots < a_{n-1}^n$ and $\ell < n \Rightarrow a_\ell^n \in \mathbf{I}_n$ then $p_n = \mathrm{tp}(\langle a_0^n, \dots, a_{n-1}^n \rangle, \emptyset, M_2) = \mathrm{tp}(\langle a_0^{n+1}, \dots, a_{n-1}^{n+1} \rangle, \emptyset, M_2)$.

Let $\mathbf{I}_n = \{a_\alpha : \alpha \in \mathscr{U}_n\}$ and note that

$\boxtimes_1$ $\langle \bar{\sigma}(a_{i(\alpha,0)}, \dots, a_{i(\alpha,m-1)}) : \alpha \in Z \rangle$ is an indiscernible set in M_1 (equivalently in $\mathfrak{C}$) when:
  (a) $2m \leq n < \omega$,
  (b) $Z \subseteq \mathscr{U}_m$ is infinite,
  (c) $i(\alpha, \ell) \in \mathscr{U}_n$ is increasing with $\ell < m$ for $\alpha \in Z$,
  (d) for each $\ell, k < m$ and $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ from Z we have $i(\alpha_1, \ell) < i(\beta_1, k) \Leftrightarrow i(\alpha_2, \ell) < i(\beta_2, k)$,
  (e) $\bar{\sigma}(x_0, \dots, x_{m-1})$ is a finite sequence of $\tau(M_2)$-term,
  (f) all $\langle a_{i(\alpha,0)}, \dots, a_{i(\alpha,m-1)} \rangle$ for $\alpha \in Z$ realize the same type (equivalently of quantifier-free type) in M_2.

[Why? If $k \leq n$ and $j < \ell g(\bar{\sigma})$ then the truth values of $\sigma_j(a_{i_0}, \dots, a_{i_{k-1}}) \in A_\varepsilon$ for $i_0 < \cdots < i_{k-1} < \mu$ such that $a_{i_0}, \dots, a_{i_{k-1}} \in \mathbf{I}_n$ depend on σ only, we can prove this by induction on $\max\{\varepsilon_j : j < \ell g(\bar{\sigma})\}$, using the properties of the B_a's. By the properties of $\mathbf{F}^t_{\kappa}$-constructions[2] ([Sh:c, IV]) we are easily done.[3]]

Moreover

$\boxtimes_2$ in $\boxtimes_1$, the $\mathbb{L}(\tau_T)$-type of $\langle \bar{\sigma}(a_{i(\alpha,0)}, \dots, a_{i(\alpha,m-1)}) : \alpha \in Z \rangle$ depends just on $Z, \bar{\sigma}$ and the truth values in (d) from $\boxtimes_1$ and the types in (f) of $\boxtimes_1$ over $\mathrm{acl}_{M_1}(\emptyset)$.

[Why: Note that $\mathrm{acl}_{M_2}(\emptyset) \subseteq$ (the Skolem hull of $\emptyset$ in M_2) and $\mathbf{I} \cap \mathrm{acl}_{M_2}(\emptyset)$ is infinite.]

So we can find a $\tau(M_2)$-model M_3 generated by the indiscernible sequence $\langle b_\alpha : \alpha < \lambda \rangle$ such that for every $n < \omega$ and $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$, recalling $(*)_1$ we have $p_n = \mathrm{tp}(\langle b_{\alpha_0}, \dots, b_{\alpha_{n-1}} \rangle, \emptyset, M_3)$. Without loss of generality the Skolem hull of $\emptyset$ in M_3 is the same as in M_2. Let $M_4 = M_3 | \tau_T$. Clearly M_4 is a model of T of cardinality λ.

Now suppose that

$(*)_2$ in $\mathbf{V}$ we have $\mathbf{J} \subseteq M_4$ (or even $\mathbf{J} \subseteq {}^{\omega\geq}(M_4)$), is an indiscernible set of cardinality $\geq \kappa^+$ orthogonal to P^{M_3} which is an infinite indiscernible set in M_4 (this is absolute enough).

Let $\mathbf{J} \supseteq \{c_\alpha : \alpha < (\kappa^+)^{\mathbf{V}}\}$ with the c_α's pairwise distinct. Let $c_i = \sigma_i(b_{\alpha(i,0)}, \dots, b_{\alpha(i),n(\alpha)-1})$ where $\alpha(i,0) < \cdots < \alpha(i, n(i))$ (may be clearer in $\mathbf{L}[T, Y, \mathbf{J}]$).

So in $\mathbf{L}[T, Y, \mathbf{J}]$ for some $Z \subseteq (\kappa^+)^{\mathbf{V}}$ of cardinality $\geq (\kappa^+)^{\mathbf{L}[T,Y,\mathbf{J}]}$ (so maybe $\mathbf{V} \models |Z| < \kappa^+$) we have $i \in Z \Rightarrow \sigma_i = \sigma_* \wedge n(i) = n(*)$, and the truth value of $\alpha(i_1, \ell_1) < \alpha(i_2, \ell_2)$ for $i_1 < i_2$ depend just on (ℓ_1, ℓ_2). In $\mathbf{L}[T, Y]$ for each $n \geq 2n(*)$, we can find $a_{i,\ell}^n \in \mathbf{I}_n$ for $i < \partial, \ell < n(*)$ such that $M_2 \models a_{i_1,\ell_1}^n < a_{i_2,\ell_2}^n \Leftrightarrow \alpha(0, \ell_1) < \alpha(1, \ell_2)$ for every $i_1 < i_2 < \partial$ and $M_2 \models a_{i,0}^n < a_{i,1}^n < \cdots < a_{i,n(*)-1}^n$ for $i < \partial$. By $\boxtimes_1$ we know that $\langle \sigma_*(a_{i,0}^n, \dots, a_{i,n(*)-1}^n) : i < \partial \rangle$ is an indiscernible set in M_1 hence an indiscernible set over $\mathrm{acl}_{M_2}(\emptyset)$. By $\boxtimes_2$, its type over $\mathrm{acl}_{M_2}(\emptyset)$

[2]Instead we can use the conclusion derived in $\boxtimes_2$.

[3]This is not the end of the proof, we still need to show another indiscernible set does not exist.

does not depend on n when $n \geq 2n(*)$. As $|\mathbf{I}_n| > \partial > (2^{|T|})^{\mathbf{L}[T,Y,\mathbf{J}]}$, we easily get, see [Sh:c, Ch. V, 2.5, pg. 244] that this indiscernible set is not orthogonal to the indiscernible set $\{a_i : i < \mu\}$. Also easily letting $f : Z \to \partial$ be one to one order preserving, the type which $\langle c_i : i \in Z\rangle$ realizes over $\mathrm{acl}_{M_2}(\emptyset)$ in M_3 is the same as the type of $\langle \sigma_*(a^n_{f(i),0}, \dots) : i \in Z\rangle$ realized in M_2 over $\mathrm{acl}_{M_2}(\emptyset)$ for $n \geq 2n(*) + 1$, as for formulas with $\leq m$ variable we consider $n > m$. As $\mathbf{J}$ was chosen to be indiscernible orthogonal to $\mathrm{tp}(a^n_{0,0}, \mathrm{acl}_{M_2}(\emptyset), M_1)$, i.e., to the indiscernible set P^{M_4}, we get a contradiction. So there is no $\mathbf{J}$ as in $(*)_2$. As $\mathbf{I}$ is minimal, it follows that in $\mathbf{V}$, if for $\ell = 1, 2$ the set $\mathbf{J}_\ell \subseteq {}^{n(\ell)}(M_4)$ is indiscernible of cardinality $\geq \theta^+$ then $\mathbf{J}_1, \mathbf{J}_2$ are not orthogonal to $\mathbf{I}$ hence $\mathbf{J}_1, \mathbf{J}_2$ not orthogonal (e.g., works in $\mathbf{L}[T, Y, \mathbf{J}_1, \mathbf{J}_2]$).

But this says that M_4 is a model as required in the conclusion of 2.9. $\square_{2.9}$

2.10. *Remark.* (1) By $\mathbf{F}^f_{\aleph_0}$-constructions (see [Sh:c, IV]) we can get models with peculiar properties.
(2) On absoluteness see 3.1.
(3) In fact by [Sh:300f, §1], we can assume that $\langle \sigma(b_0, \dots, b_{m-1}) : m < n, b_0 <^{M_2} \cdots <^{M_2} b_{m-1}$ are from $\mathbf{I}_n\rangle$ where $\sigma = \sigma(x_0, \dots, x_{m-1})$ and σ is a $\tau(M_2)$-term, is (fully) indiscernible in the model $M_2 | \tau_T$, i.e., in M_1, see definition there. But the argument above is simpler.

2.11. *Conclusion.* If T is stable and categorical in $\lambda > |T|$ then (in $\mathbf{L}[T, Y]$ where $Y \subseteq \mathrm{Ord}$):

⊠ (a) T is uni-dimensional,
(b) T is superstable,
(c) T has no two cardinal models,
(d) $D(T)$ has cardinality $\leq |T|$; moreover $D(T) \in \mathbf{L}[T]$ and $\mathbf{L}[T] \models |D(T)| \leq |T|$.

Proof. Assume clause (a) fails and we shall produce two models of cardinality (and universe) λ. The first N_1 is from 2.9. The second is a model N_2 such that there are indiscernible $\mathbf{I}, \mathbf{J} \subseteq N_1$ (or ${}^{\omega>}(N_1)$) of cardinality λ which are orthogonal; this contradicts the categoricity hence clause (a).

The superstability, i.e., clause (b) follows from clause (a) by Hrushovski [Hr89d].

Clause (c), no two cardinal models follows from clause (a) by [Sh:c, V, §6].

Now $|D(T)| \leq |T|$ (clause (d)) is trivial as otherwise we have two models M_1, M_2 of T of cardinality λ such that some $p \in D(T)$ is realized in one but not the other (i.e., first choose $M_1 \in \mathbf{L}[T]$ realizing $\leq |T|$ types. Clearly $\{p \in D(T) : p$ is realized in $M\}$ is a well ordered set so by the assumption we can choose $p \in D(T)$ not realized in M_1 and lastly choose M_2 realizing p). $\square_{2.11}$

2.12. Claim. *If clauses* (b), (c), (d) *of* 2.11 *hold and T is categorical in $\lambda > |T|$ then*:

(e) *any model M of T of cardinality μ, for any $\mu > |T|$ is $\aleph_0$-saturated.*

Proof. Assume clause (e) fails as exemplified by M and we shall get contradiction to clause (c) of 2.11, so without loss of generality the universe of M is μ.

For any $Y \subseteq \mathrm{Ord}$ working in $\mathbf{L}[T, Y, M]$ we can find $\bar{a} \subseteq M$ and formula $\varphi(x, \bar{y}) \in \mathbb{L}_{\tau(T)}$ such that $\varphi(x, \bar{a})$ is a weakly minimal formula in M, existence as in [Sh:31]; note that "every model M of T with universe $\lambda > |T|$ is totally saturated, i.e., $A \subseteq M$, $|A| < \lambda$, $p \in \mathbf{S}_\varepsilon(A) \Rightarrow y$ is realized in M" can be proved as we proved $\boxtimes(0)$, in fact follows from "T" has no two cardinal model. Let $N_0 \prec M$ be of cardinality $|T|$ such that $\bar{a} \subseteq N_0$ and $N_0 \in \mathbf{L}[T, M]$.

Case 1: $\{p \in \mathbf{S}(N_0): p$ is realized in $M\}$ has power $> |T|$ in $\mathbf{V}$.

But as $|M|$ is an ordinal this set is well ordered so the proof of 1.2 applies contradicting categoricity in λ and we get more than needed.

Case 2: Not Case 1 but there is a finite $A \subseteq M$ such that $\bar{a} \subseteq A$ and $p \in \mathbf{S}(A), \varphi(x, \bar{a}) \in p$ and the type p is omitted by M.

As in [Sh:31] (using "not Case 1" here instead "T stable in $|T|$" there) we can find (in $\mathbf{L}[T, M]$) a model M' such that $M \prec M', M'$ omits the type p and $\|M'\| \geq \lambda$, so by DLST (= the downward Lowenheim–Skolem–Tarksi theorem) some $N_1 \prec M'$ has cardinality λ and is not $\aleph_0$-saturated. Hence for some complete type $p(\bar{x}, \bar{y}) \in D(T)^{\mathbf{L}[T, M]}$, for some $\bar{b} \in {}^{\ell g(\bar{y})}(N_1)$, the model N_1 omits the type $p(\bar{x}, \bar{b})$ which is a type, i.e., finitely satisfiable in N_1.

By clause (d) of $\boxtimes$ of 2.11 we have $|D(T)| \leq |T|$ in $\mathbf{L}[T, Y]$ and $D(T)$ is included in $\mathbf{L}[T, Y]$. So in $\mathbf{L}[T, Y]$, for every finite $A \subseteq N \models T$, $\mathbf{S}(A, N)^{\mathbf{V}}$ is the same as $\mathbf{S}(A, N)$ computed in $\mathbf{L}[T, Y]$ and is there of cardinality $\leq |D(T)|$ hence absolute. So in $\mathbf{L}[T, Y]$ we can find a model N_2 of T of cardinality λ which is $\aleph_0$-saturated.

[Alternatively to this, we can choose a model N_2 of cardinality λ such that: if $\bar{b}' \in {}^{\ell g(\bar{y})}N_2$ realizes $\mathrm{tp}(\bar{b}', \emptyset, M')$ then for some $\bar{a} \in {}^{\ell g(\bar{x})}N_2$ the sequence $\bar{a}'{}^\frown\bar{b}'$ realizes $p(\bar{x}, \bar{y})$.]

By the previous paragraphs this is a contradiction to categoricity.

Case 3: Neither Case 1 nor Case 2.

Subcase A: T countable.

Let N_1 be such that

$\circledast$ (a) $N_1 \prec M$ is countable,
(b) $\bar{a} \subseteq N_1$,
(c) if $\bar{a} \subseteq A \subseteq N_1, A$ finite and p is a non-algebraic type satisfying $\varphi(x, \bar{a}) \in p \in \mathbf{S}(A, M)$ then p is realized in N_1

(possible as by clause (d) of $\boxtimes$ of 2.11 the set $D(T)$ is countable and "neither case 1 nor case 2").

Let N_2 be a countable saturated model of T such that $N_1 \prec N_2$. We can build an elementary embedding f (still working in $\mathbf{L}[T, M]$) from N_1 into N_2 such that $f(\varphi(N_1, \bar{a})) = (\varphi(N_2, \bar{a}))$. This contradicts clause (c) of $\boxtimes$ of 2.11.

The last subcase is not needed for this section's main theorem 2.1, (but is needed for 2.11).

Subcase B: T uncountable.

So possibly increasing $Y \subseteq \mathrm{Ord}$, in $\mathbf{L}[T, Y]$ we have two models M_1, M_2 of T, M_1 is $\aleph_0$-saturated, M_2 is not but $\varphi(x, \bar{a}), M_2$ fails cases 1 and 2; we work in $\mathbf{L}[T, Y]$. Let $\ell g(\bar{a}) = n$ and $T^+ \in \mathbf{L}[T, Y]$ be the first order theory in the vocabulary $\tau^+ = \tau_T \cup \{c_\ell: \ell < n\} \cup \{P\}$ where c_ℓ an individual constant, P a unary predicate such that $M^+ = (M, c_0^{M^+}, \dots, c_{n-1}^{M^+}, P^{M^+})$ is a model of T^+ iff $M = M^+|\tau$ is a model of $T, \varphi(M, c_0^{M^+}, \dots, c_{n-1}^{M^+})$ is infinite and $\subseteq P^{M^+}$ and $M|P^{M^+} \prec M$. As T is uni-dimensional (more specifically clause (c) of $\boxtimes$ of 2.11 + [Sh:c, V, §6]) T^+ is inconsistent, hence for some finite $\tau' \subseteq \tau, T^+ \cap \mathbb{L}(\tau' \cup$

$\{c_0, \dots, c_{n-1}, P\}$) is inconsistent. Now choose $\bar{a}_1 \in {}^n(M_1)$ realizing $\mathrm{tp}(\bar{a}, \emptyset, M_2)$, let $\bar{a}_2 = \bar{a}$ and let χ be large enough, $\mathfrak{B} \prec (\mathscr{H}(\chi)^{\mathbf{L}[T,Y]}, \in)$ be countable such that $\{M_1, M_2, \tau', \bar{a}_1, \bar{a}_2\} \in \mathfrak{B}$; recall that we are working in $\mathbf{L}[T, Y]$. Now replacing M_1, M_2 by $(M_1|\tau') \cap \mathfrak{B}, (M_2|\tau') \cap \mathfrak{B}$ we get a contradiction as in Subcase A. $\square_{2.12}$

Proof of Theorem 2.1. By 0.8(2) and 0.9 without loss of generality T is complete, $T \subseteq \mathscr{H}(\aleph_0)$. Trivially (B) $\Rightarrow$ (A), by 1.4 we have (C) $\Rightarrow$ (B) and by absoluteness (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E), so it suffices to prove (C) assuming (A). By 2.6 the theory T is stable hence the assumption of 2.11 holds hence its conclusion, i.e., $\boxtimes$ of 2.11 holds whenever $Y \subseteq \mathrm{Ord}$, in particular $D(T) \in \mathbf{L}[T]$. So by 2.12 we can conclude: every model of T of cardinality $\lambda > \aleph_0$ is $\aleph_0$-saturated (in $\mathbf{V}$ or, equivalently, in $\mathbf{L}[T, M]$ when M has universe λ). If T is $\aleph_0$-stable use 1.3. So we can assume T is not $\aleph_0$-stable but is superstable (recall clause (b) of $\boxtimes$ of 2.11) hence T is not categorical in $\aleph_0$ (even has $\geq \aleph_0$ non-isomorphic models, by a theorem of Lachlan, see, e.g., [Sh:c]), in any $\mathbf{L}[T, Y]$. So by Kueker conjecture (proved by Buechler [Be84] for T superstable and by Hrushovski [Hr89] for stable T), we get contradiction. $\square_{2.1}$

2.13. *Remark.* See more in [Sh:F701] about T which is categorical in the cardinal $\lambda > |T|$, T not categorical in some $\mu > |T|$.

§3. A dichotomy for $\dot{I}(\aleph_\alpha, T)$: bounded or $\geq |\alpha|$.

Our aim is to understand the lower part of the family of functions $\dot{I}(\lambda, T)$, T countable: either $(\forall \alpha)\dot{I}(\aleph_\alpha, T) \geq |\alpha|$ or $\dot{I}(\aleph_\alpha, T)$ is constant and not too large (for α not too small), see 3.14. For completeness we give a full proof of 3.2.

We need here absoluteness between models of the form $\mathbf{L}[Y]$ and this may fail for "$\kappa(T) > \kappa$", "T stable uni-dimensional". But usually more is true.

3.1. *Observation.* (1) "T is first order", "$\tau_T \subseteq \mathbf{L}_\omega$", "$\tau_T \subseteq \mathbf{L}$", "$T$ is complete" are Z^--absolute.

(2) For T (not necessarily $\in \mathbf{L}$ but, below we can omit DC if we consider only universes in which τ_T well orderable, our standard assumption) which is complete:

(a) "T is stable" is Z^--absolute,

(b) "T is superstable" is (Z^-+ DC)-absolute (and downward Z^--absolute; Z^--absolute if $\tau(T) \subseteq \mathbf{L}$),

(c) "T totally transcendental" is (Z^- + DC)-absolute (and downward Z^--absolute; Z^--absolute if $T \subseteq \mathbf{L}$); "T is $\aleph_0$-stable, $\tau(T) \subseteq \mathbf{L}$" is Z^--absolute,

(d) the appropriate ranks are (Z^- + DC)-absolute (Z^--absolute if $T \subseteq \mathbf{L}$) as the rank of $\{\varphi(x, \bar{a})\}$ in M depend just on $T, \varphi(x, \bar{y})$ and $\mathrm{tp}(\bar{a}, \emptyset, M)$,

(e) "M a model of T and $\mathbf{I}, \mathbf{J} \subseteq M$ (or ${}^{\omega>}M$) are infinite indiscernible sets, and $\mathbf{I}, \mathbf{J}$ are orthogonal and where T is stable", is Z^--absolute,

(f) "T is stable not uni-dimensional" is upward Z^--absolute,

(g) for countable T, "T is stable not uni-dimensional" is Z^--absolute when $T \subseteq \mathbf{L}$,

(h) "T is countable stable with the OTOP (omitting type order property, see 3.7 below)" is Z^--absolute,

(i) "M is primary over A, M a model of the (complete) stable theory T" is upward Z^--absolute.

PROOF. E.g.,

(2) *Clause* (b):

This just asks if the tree $\mathcal{T}$ has an ω-branch where the n-th level of $\mathcal{T}$ is the set of sequence $\langle \varphi_\ell(x, \bar{y}_\ell) : \ell < n \rangle$ such that for every m, $\{\varphi(x_\eta, \bar{y}_\nu)^{\text{if}(\nu \triangleleft \eta)} : \ell < n, \nu \in {}^\ell m, \eta \in {}^n m\}$ is consistent with T.

Clause (e): Recall that a definition (the one we choose here) is

$(*)_1$ Av$(\mathbf{I}, \mathbf{I} \cup \mathbf{J})$ and Av$(\mathbf{J}, \mathbf{I} \cup \mathbf{J})$ are weakly orthogonal types which is equivalent to

$(*)_2$ for every $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{b} \in {}^{\ell g(\bar{z})}(\mathbf{I} \cup \mathbf{J})$ for some $\psi_\ell(x, \bar{z}_\ell) \in \mathbb{L}(\tau_T)$, $\bar{c}_\ell \in {}^{\ell g(\bar{z}_\ell)}(\mathbf{I} \cup \mathbf{J})$ such that $\psi_\ell(\bar{x}, \bar{c}_\ell)$ is satisfied by infinitely many $\bar{a} \in \mathbf{I}$ if $\ell = 1$, $\bar{a} \in \mathbf{J}$ if $\ell = 2$ and truth value $\mathbf{t}$ we have $M \models (\forall \bar{x}, \bar{y})[\psi_1(\bar{x}, \bar{c}_1) \wedge \psi_2(\bar{y}, \bar{c}_2) \rightarrow \varphi(\bar{x}, \bar{y}, \bar{c})^{\mathbf{t}}]$.

Clause (h):

We just ask for the existence of the $\Phi \in \Upsilon_T^{\text{or}}$ so with τ_Φ countable $\supseteq \tau_T$ and type $p(\bar{x}, \bar{y}, \bar{z})$ from $D(T)$ such that $(\ell g(\bar{y}) = \ell g(\bar{z})$ and) for any linear order I, which is well orderable EM$_\tau(I, \Phi)$ is a model of T of cardinality $|T| + |I|$ and $p(\bar{x}, \bar{a}_s, \bar{a}_t)$ is realized in it iff $s <_I t$ (so O.K. for stable T). $\square_{3.1}$

3.2. CLAIM. *If T is unstable and $|T| = \aleph_{\beta_*} < \aleph_\alpha = \lambda$ <u>then</u> $\dot{I}(\lambda, T) \geq |\alpha - \beta_*|$.*

PROOF. In $\mathbf{L}[T]$, let Φ be as in 2.2 such that for every a linear order I we have $s, t \in I \Rightarrow \text{EM}(I, \Phi) \models \varphi[\bar{a}_s, \bar{a}_t]^{\text{if}(s<t)}$, where, of course, $\varphi(\bar{x}, \bar{z}) \in \mathbb{L}(\tau(T))$.

First, we define for $\gamma \leq \aleph_\alpha$

$$J_\gamma =: \gamma + (\gamma)^*.$$

We can specify: the set of members of J_γ is $\{(\gamma, \ell, \zeta) : \ell \in \{0, 1\}, \zeta < \gamma\}$ and $(\gamma, \ell_1, \zeta_1) < (\gamma, \ell_2, \zeta_2)$ iff $\ell_1 = 0 \wedge \ell_2 = 1$ or $\ell_1 = \ell_2 = 0 \wedge \zeta_1 < \zeta_2$ or $\ell_1 = \ell_2 = 1 \wedge \zeta_1 > \zeta_2$.

Second, for $\beta \in [\beta_*, \alpha]$ let $J^\beta = \sum_{\gamma \leq \aleph_\beta} J_\gamma + J_\infty$ where $J_\infty = (\aleph_\alpha + 1) \times \mathbb{Q}$ ordered lexicographically.

Third, let $M_1^\beta = \text{EM}(J^\beta, \Phi)$.

Lastly, $M^\beta := M_1^\beta \restriction \tau_T$ is clearly a model of T of cardinality $\aleph_\alpha$. We like to "recover", "define" $\aleph_\beta$ from $M^\beta / \cong$ at least when $\beta \geq \beta_*$. This is sufficient as the sequence $\langle M_\beta : \beta \in [\beta_*, \alpha] \rangle$ exists (in fact in $\mathbf{L}[T]$). We shall continue after stating 3.3.

DISCUSSION. (1) In ZFC we could recover from the isomorphism types, stationary subsets modulo the club filter so as we get $2^{\aleph_\alpha}$; if, e.g., $\aleph_\alpha$ is regular and there are $2^{\aleph_\alpha}$ subsets of $\aleph_\alpha$ any two with a stationary difference we get $\dot{I}(\aleph_\alpha, T) = 2^{\aleph_\alpha}$. But here (ZF) the stationary subsets of a regular uncountable λ may form an ultrafilter or all uncountable cardinals are singulars.

(2) More than 3.3 is true in $\mathbf{L}[T, Y]$; EM(J, Φ) satisfies $\otimes_\theta$ iff J has a (θ, θ)-cut (provided J has no $(1, \theta), (\theta, 1), (0, \theta), (\theta, 0)$ cuts), see below.

(3) See more in 3.6 on OTOP.

(4) Of course, we can prove theorems saying e.g.: if $\aleph_\alpha > |T|$ is regular, T unstable then $\dot{I}(\aleph_\alpha, T) \geq |\mathcal{P}(\aleph_\alpha)/(\text{the club filter on } \aleph_\alpha)|$.

3.3. SUBCLAIM. *If $J = J^\beta$, $M = M^\beta$ are as above and $Y \subseteq$ Ord satisfies $M \in \mathbf{L}[T, Y]$ <u>then</u> in $\mathbf{L}[T, Y]$ for any regular cardinal θ (of $\mathbf{L}[T, Y]$)*

$(*)$ *$\otimes_\theta \Leftrightarrow \theta > \aleph_\beta$ where*

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$\otimes_\theta$ *if p is a set of Δ-formulas with parameters from M of cardinality θ where*

$$\Delta =: \{\varphi(\bar{x}, \bar{z}_1) \wedge \neg\varphi(\bar{x}, \bar{z}_2)\}$$

and any subset q of p of cardinality $< \theta$ is realized in M then some $q \subseteq p$, $|q| = \theta$ is realized in M.

PROOF OF CLAIM 3.2 FROM THE SUBCLAIM 3.3. Why does this subclaim help us to prove the Theorem? Assume $\beta_* \leq \beta_1 < \beta_2 \leq \alpha$ and we consider M^{β_1}, M^{β_2} as above and toward a contradiction we assume that there is an isomorphism f from M^{β_1} onto M^{β_2}.

Let $Y \subseteq$ Ord code $T, M^{\beta_1}, M^{\beta_2}$ and f. So $\mathbf{L}[T, M^{\beta_1}, Y] = \mathbf{L}[Y] = \mathbf{L}[T, M^{\beta_2}, Y]$. In this universe let θ the first cardinal greater than the ordinal $> \aleph^{\mathbf{V}}_{\beta_1}$ so $\aleph^{\mathbf{V}}_{\beta_1} < \theta \leq \aleph^{\mathbf{V}}_{\beta_1+1} \leq \aleph_{\beta_2}$.

QUESTION. Why we cannot prove that $\theta = \aleph^{\mathbf{V}}_{\beta_1+1}$? As possibly $\mathbf{L}[Y] \models \aleph^{\mathbf{V}}_{\beta_1+1}$ is singular or just a limit cardinal.

NOTE. Maybe every $\mathbf{L}[Y]$-cardinal from $(\aleph^{\mathbf{V}}_{\beta_1}, \aleph^{\mathbf{V}}_{\beta_1+1})$ have cofinality $\aleph_0$ in $\mathbf{V}$!

But in $\mathbf{L}[Y], \aleph^{\mathbf{V}}_\beta, \aleph^{\mathbf{V}}_{\beta+1}$ are still cardinals so the successor of $\aleph^{\mathbf{V}}_\beta$ in $\mathbf{L}[Y]$ is $\leq \aleph^{\mathbf{V}}_{\beta+1}$ but in $\mathbf{L}[Y]$ this successor, θ, is regular. (In $\mathbf{V}, \theta$ may not be a cardinal at all). In $\mathbf{L}[T]$ there are many possibilities for θ (it was defined from Y!) and we have built M^β_1 before knowing who they will be in $\mathbf{L}[Y]$ so

$$\theta > \aleph_{\beta_1} \Leftrightarrow M^{\beta_1} \models \otimes_\theta \Leftrightarrow M^{\beta_2} \models \otimes_\theta \Leftrightarrow \theta > \aleph_{\beta_2}$$

(the first $\Leftrightarrow$ by $(*)$ of the subclaim and the second $\Leftrightarrow$ as f is an isomorphism)

but $\aleph_{\beta_1} < \theta \leq \aleph_{\beta_2}$; contradiction.

PROOF OF THE SUBCLAIM 3.3. I.e., in $\mathbf{L}[T, Y]$ we have to prove:

$(*)$ $[\otimes_\theta \Leftrightarrow \theta > \aleph_\beta]$.

First we will prove:

$(*)_1$ $\theta \leq \aleph_\beta \Rightarrow \neg\otimes_\theta$.

By the choice of $J = J^\beta$ clearly J_θ is an interval of J so let

$$p =: \{\varphi(\bar{x}, \bar{a}_{(\theta,1,i)} \wedge \neg\varphi(\bar{x}, \bar{a}_{(\theta,0,i)}) : i < \theta\}.$$

Let $q \subseteq p, |q| < \theta$ now as θ is regular (in $\mathbf{L}[T, Y]$) for some $j < \theta$ we have

$$q \subseteq p_j = \{\varphi(\bar{x}, \bar{a}_{(\theta,1,i)}) \wedge \neg\varphi(\bar{x}, \bar{a}_{(\theta,0,i)}) : i < j\}.$$

We have a natural candidate for a sequence realizing q: the sequence $\bar{a}_{(\theta,1,j)}$. Now

$$i < j \Rightarrow (\theta, 1, j) <_{J_\theta} (\theta, 1, i) \Rightarrow M \models \varphi[\bar{a}_{(\theta,1,j)}, \bar{a}_{(\theta,1,i)}].$$
$$i < j \Rightarrow (\theta, 0, i) <_{J_\theta} (\theta, 1, j) \Rightarrow M \models \neg\varphi[\bar{a}_{(\theta,1,j)}, \bar{a}_{(\theta,0,i)}].$$

So we have proved that every $q \subseteq p, |q| < \theta$ is realized in the model. Secondly, we need to show:

$\otimes$ no $\bar{a} \in M$ satisfies θ of formulas from p.

Assume toward contradiction that $\bar{a}$ is a counterexample.

So we can find $n < \omega$, a finite sequence of terms $\bar{\sigma}(\bar{x}_0, \dots, \bar{x}_{n-1})$ from $\tau(\Phi)$ and $t_0 <_J t_1 <_J \cdots <_J t_{n-1}$ such that $\bar{a} = \bar{\sigma}(\bar{a}_{t_0}, \dots, \bar{a}_{t_{n-1}})$. Now for each ℓ for some $i_\ell < \theta, t_\ell$ is not in the interval $((\theta, 0, i_\ell), (\theta, 1, i_\ell))_J$.

Let:

$$j^* = \max[\{i_\ell + 1 : \ell < n\} \cup \{1\}].$$

Now consider $\varphi(\bar{x}, \bar{a}_{(\theta,1,j)} \wedge \neg\varphi(\bar{x}, \bar{a}_{(\theta,0,j)}))$ for $j \in [j^*, \theta)$. So $t_\ell <_J (\theta, 1, j) \equiv t_\ell <_J (\theta, 0, j)$ for $\ell = 0, \ldots, n-1$ hence $M \models \varphi[\bar{\sigma}(\bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}}), \bar{a}_{(\theta,1,j)}] \Leftrightarrow M \models \varphi[\bar{\sigma}(\bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}}), \bar{a}_{(\theta,0,j)}]$. So $\bar{a} = \sigma(\bar{a}_{t_0}, \ldots, \bar{a}_{t_{n-1}})$ fail the j-th formula from p for $j \in [j^*, \theta)$. So p really exemplifies the $\neg\otimes_\theta$. So we have proved $(*)_1$ which is one implication of the Subclaim.

Now we will prove:

$(*)_2$ if $\mathbf{L}[T, Y] \models$ "θ is regular $> \aleph_\beta$" then $\otimes_\theta$.

So let $p = \{\varphi(\bar{x}, \bar{a}_i) \wedge \neg\varphi(\bar{x}, \bar{b}_i) : i < \theta\} \in \mathbf{L}[T, Y]$ be given. For $j < \theta$ let

$$p_j =: \{\varphi(\bar{x}, \bar{a}_i) \wedge \neg\varphi(\bar{x}, \bar{b}_i) : i < j\}.$$

So some $\bar{c}_j \in M$ realizes it and let $(\bar{a}_i, \bar{b}_i, \bar{c}_i) = \langle \bar{\sigma}_i^k(a_{t_0^i}, \ldots, a_{t_{n_i-1}^i}) : k = 0, 1, 2\rangle$ where $\bar{\sigma}_i^k$ is a finite sequence of terms from $\tau(\Phi)$ and $J \models t_0^i < t_1^i < \cdots < t_{n_i-1}^i$; note that we can make $\langle t_\ell^i : \ell < n_i\rangle$ not to depend on k because we can add dummy variables.

As $\tau(\Phi)$ is of cardinality $< \theta = \mathrm{cf}(\theta)$ (in $\mathbf{L}[T, Y]$), for some σ_*^k, n_* the set $S = \{i : \sigma_i^k = \sigma_*^k$ for $k = 0, 1, 2$ and $n_i = n_*\}$ is unbounded in θ.

Recall

$$J^\beta = \sum_{\gamma \leq \aleph_\beta} J_\gamma + (\aleph_\alpha + 1) \times \mathbb{Q}.$$

So for some $m_i \leq n_*$

$$t_\ell^i \in \sum_{\gamma \leq \aleph_\beta} J_\gamma \Leftrightarrow \ell < m_i$$

shrinking S without loss of generality $i \in S \Rightarrow m_i = m_*$.

<u>Now</u> $\mathbf{L}[T, Y] \models$ "$|\sum_{\gamma \leq \aleph_\beta} J_\gamma| \leq \sum_{\gamma \leq \aleph_\beta} |J_\gamma| = \sum_{\gamma \leq \aleph_\beta} (|\gamma| + \aleph_0) \leq \aleph_\beta < \theta = \mathrm{cf}(\theta)$".

So without loss of generality

$\circledast_1$ $\ell < m_* \Rightarrow t_\ell^i = t_\ell^*$ for $i \in S$ and for $\ell \in [m_*, n_*)$ let $t_\ell^i = (\varepsilon_\ell^i, q_\ell^i)$ where $q_\ell^i \in \mathbb{Q}$.

Clearly for q_ℓ^i there are $\aleph_0$ possibilities so without loss of generality, for each $\ell \in [m_*, n_*)$

$\circledast_2^\ell$ $q_\ell^i = q_\ell^*$ for $i \in S$,

$\circledast_3^\ell$ $\langle \varepsilon_\ell^i : i \in S\rangle$ is <u>constant</u> say ε_ℓ^* <u>or is strictly increasing</u> with limit ε_ℓ^* and is strictly increasing iff $\ell \in u$

so without loss of generality

$\circledast_4$ (i) if $\ell_1 \neq \ell_2$ are in the interval $[m_*, n_*)$ and $\varepsilon_{\ell_1}^* < \varepsilon_{\ell_2}^*$ then $i, j \in S \Rightarrow \varepsilon_{\ell_1}^i \leq \varepsilon_{\ell_1}^* < \varepsilon_{\ell_2}^j$,

(ii) if $\ell_1 \neq \ell_2 \in [m_*, n_*)$ and $\varepsilon_{\ell_1}^* = \varepsilon_{\ell_2}^* \wedge \ell_1 \in u \wedge \ell_2 \notin u$ and $i < j$ are in S then $\varepsilon_{\ell_1}^i < \varepsilon_{\ell_2}^j$ (follows).

We choose $t_0 <_J t_1 <_J \cdots <_J t_{n-1}$ which satisfies

$\circledast_5$ (a) if $\ell < m_*$ then $t_\ell = t^*_\ell$,

(b) if $\ell \in [m_*, n_*)$ and $\langle t^i_\ell : i \in S\rangle$ is constant then $t_\ell = t^*_\ell$,

(c) if $\ell \in [m_*, n_*)$, $\langle t^i_\ell : i \in S\rangle$ is not constant (i.e., $\ell \in u$) then: (recall that $\langle q^i_\ell : i \in S\rangle$ is constantly q^*_ℓ, $\langle \varepsilon^i_\ell : i \in S\rangle$ is strictly increasing with limit ε^*_ℓ) we choose $t_\ell = (\varepsilon_\ell, q_\ell)$ such that $\varepsilon_\ell = \varepsilon^*_\ell$. $q_\ell = \min(\{0\} \cup \{q^*_k : k \in [m_*, n_*)\}) - n^* + \ell$ (the computation is in $\mathbb{Q}$!)

Hence

$\circledast_6$ (α) q_ℓ is $< q^*_k$ for every $k \in [m_*, n_*)$ when $\ell \in u$,

(β) if $\varepsilon^*_\ell = \varepsilon^*_k$ and $\ell, k \in u$ then $q_\ell < q_k \equiv \ell < k$.

<u>Now</u> note that:

$\circledast_7$ for $\varepsilon < \zeta < \theta$ from S, in J the quantifier free types of $\langle t^\varepsilon_\ell : \ell < n_*\rangle^\frown\langle t_\ell : \ell < n_*\rangle$ and $\langle t^\varepsilon_\ell : \ell < n_*\rangle^\frown\langle t^\zeta_\ell : \ell < n_*\rangle$ are equal [all the shrinking was done for this].

Now for $\varepsilon < \zeta$ from S, by the original choice above $M^\beta \models \varphi[\bar{c}_\zeta, \bar{a}_\varepsilon] \wedge \neg\varphi[\bar{c}_\zeta, \bar{b}_\varepsilon]$ that is: $M^\beta_1 \models \varphi[\bar{\sigma}^0_*(a_{t^\zeta_0}, \dots), \bar{\sigma}^1_\varepsilon(a_{t^\varepsilon_0}, \dots)] \wedge \neg\varphi[\bar{\sigma}^0_*(a_{t^\zeta_0}, \dots), \bar{\sigma}^2_\varepsilon(a_{t^\varepsilon_0}, \dots)]$.

By the last sentence and $\circledast_7$ + indiscernibility of $\langle \bar{a}_t : t \in J\rangle$ in M^β_1 we have $M \models \varphi[\bar{\sigma}^0_*(a_{t_0}, \dots), \bar{\sigma}^1_\varepsilon(a_{t^\varepsilon_0}, \dots)] \wedge \neg\varphi[\bar{\sigma}^0_*(a_{t_0}, \dots), \bar{\sigma}^2_\varepsilon(a_{t^\varepsilon_0}, \dots)]$.

Let $\bar{c} = \bar{\sigma}^0_*(a_{t_0}, \dots)$ in M^β_1-sense, so $\varepsilon \in S \Rightarrow M \models \varphi[\bar{c}, \bar{a}_\varepsilon] \wedge \neg\varphi[\bar{c}, \bar{b}_\varepsilon]$. Hence $\{\varphi(\bar{x}, \bar{a}_\varepsilon) \wedge \neg\varphi(\bar{x}, \bar{b}_\varepsilon) : \varepsilon \in S\}$ is realized in M^β and $\mathbf{L}[T, Y] \models$ "$|S| = \theta$" as promised. $\square_{3.2}$

3.4. Claim. *If T is stable not uni-dimensional, $|T| = \aleph_\beta < \aleph_\alpha = \lambda$ <u>then</u> $\dot{I}(\lambda, T) \geq |\alpha - \beta|$.*

Proof. As in 2.9; if $\gamma \in [\beta, \alpha]$ <u>then</u> there is a model M of T of cardinality λ such that M satisfies $(*)_\gamma$ but not $\beta \leq \gamma_1 < \gamma \Rightarrow \neg(*)_{\gamma_1}$ where

$(*)_\gamma$ if $\mathbf{I}, \mathbf{J} \subseteq M$ are infinite orthogonal indiscernible sets and $|\mathbf{I}| = \lambda$ then $|\mathbf{J}| \leq \aleph_\gamma$. $\square_{3.4}$

3.5. *Conclusion.* If $\lambda = \aleph_\alpha > \aleph_\beta = |T|$ and $\dot{I}(\lambda, T) < |\alpha - \beta|$ <u>then</u> (in $\mathbf{L}[T, Y]$ when $Y \subseteq \mathrm{Ord}$)

$\boxtimes_T$ (a) T is stable and uni-dimensional,

(b) T is superstable,

(c) T has no two cardinal models,

(d) $D(T)$ has cardinality $\leq |T|$ or cardinality $< |\alpha - \beta|$.

Proof. T is stable by 3.2 and uni-dimensional by 3.4 so clause (a) holds. This implies clause (c), see [Sh:c, V,§6]. Clause (d) is trivial by now and clause (b) follows from clause (a) by Hrushovski [Hr89d]. $\square_{3.5}$

3.6. Claim. *In 3.5 we can add to $\boxtimes_T$ also clause* (e) *and if T is countable also clause* (f) *where*

$\boxtimes_T$ (e) *T fails the* OTOP (*see* [Sh:c, XII, Def. 4.1, pg. 608] *or* 3.7(1) *below*)

$\boxtimes_T$ (f) *T has the prime existence property* (*see* [Sh:c, XII, Def. 4.2, pg. 608] *or* 3.7(2) *below*) *hence for $\mathfrak{C}_T$ a model of T with universe $|\mathfrak{C}_T| \subseteq \mathbf{L}$:*
for any non-forking tree $\langle N_\eta : \eta \in \mathscr{T}\rangle$ of models $N_\eta \prec \mathfrak{C}_T$, there is a prime (even primary, i.e., $\mathbf{F}^t_{\aleph_0}$-primary) model $N \prec \mathfrak{C}_T$ over $\cup\{N_\eta : \eta \in \mathscr{T}\}$, it is unique up to isomorphism over $\cup\{N_\eta : \eta \in \mathscr{T}\}$.

PROOF. Clause (e) holds exactly as for stability, i.e., as in 3.2 only the formulas $\varphi(\bar{x}, \bar{y})$ are not first order but of the form $(\exists \bar{z}) \bigwedge_n \varphi_n(\bar{z}, \bar{x}, \bar{y})$, where each φ_n is first order. Clause (f) follows by [Sh:c, XII], i.e., it holds in any $\mathbf{L}[T, Y]$ which suffices. $\square_{3.6}$

3.7. DEFINITION. (1) T has OTOP if for some type $p = p(\bar{x}, \bar{y}, \bar{z})$ in $\mathbb{L}(\tau_T)$ the theory T has it for p, which means that for every λ for some model M of T with well ordered universe and $\bar{b}_\alpha \in {}^{\ell g(\bar{y})}M, \bar{c}_\alpha \in {}^{\ell g(\bar{z})}M$, for $\alpha, \beta < \lambda$ we have: for any $\alpha, \beta < \lambda$ the model M realizes the type $p(\bar{x}, \bar{b}_\alpha, \bar{c}_\beta)$ iff $\alpha < \beta$.

(2) T has the prime existence property <u>when</u> for every triple (M_0, M_1, M_2) in stable amalgamation in a model $\mathfrak{C}_T$ of T such that $|\mathfrak{C}_T|$ is well orderable (so $M_\ell \prec \mathfrak{C}_T$), the set of isolated types is dense in $\mathbf{S}^m(M_1 \cup M_2)$ for every m.

3.8. CLAIM. [T countable] *We can add clause* (g) *below to* $\boxtimes_T$ *from* 3.5 + 3.6:

$\boxtimes_T$ (g) *if clause* (A) <u>*then*</u> *for some* M' *clause* (B) *below holds* (*both in* $\mathbf{L}[T, Y]$) *where*

- (A) (α) $M_\emptyset \prec M_{\{i\}} \prec M^*$ *are countable models of* T *for* $i < \omega \times 2$,
  - (β) $(M_{\{i\}}, c)_{c \in M_\emptyset} \cong (M_{\{0\}}, c)_{c \in M_\emptyset}$ *for* $i < \omega$ *that is* $M_{\{i\}}$ *is isomorphic to* $M_{\{0\}}$ *over* $M_\emptyset$ *for* $i < \omega$,
  - (γ) $(M_{\{\omega+i\}}, c)_{c \in M_\emptyset} \cong (M_{\{\omega\}}, c)_{c \in M_\emptyset}$ *that is* $M_{\{\omega+i\}}$ *is isomorphic to* $M_{\{\omega\}}$ *over* $M_\emptyset$ *for* $i < \omega$,
  - (δ) $\{M_{\{i\}} : i < \omega \times 2\}$ *is independent over* $M_\emptyset$ *inside* M^*,
  - (ε) M^* *is prime over* $\cup\{M_{\{i\}} : i < \omega \times 2\}$.
- (B) (α) $M_\emptyset \prec M' \prec M^*$,
  - (β) $(M', c)_{c \in M_\emptyset} \cong (M_{\{0\}}, c)_{c \in M_\emptyset}$,
  - (γ) $\langle M_{\{i\}} : i < \omega \rangle^\frown \langle M' \rangle$ *is independent over* $M_\emptyset$.

3.9. *Remark.* (1) We can formulate (B) closer to $\circledast_6$ inside the proof of 3.10.

(2) We can omit "T countable" but then have to change Y with the same proof.

(3) We know more on T's satisfying $\boxtimes_T$ of 3.5 by Laskowski [Las88] and Hart–Hrushovski–Laskowski [HHL00].

PROOF. Note that $|T| = \aleph_0$ and choose the ordinals $\beta_* = \beta(*), \alpha_* = \alpha(*)$ such that $\beta_* = 0, \lambda = \aleph_{\alpha_*}$; most of the proof we do not use $\beta_* = 0$ but we use $\boxtimes_T$(a)–(f).

We do more than is strictly necessary for the proof; we use $\odot_i$ to denote definitions, working in $\mathbf{L}[T, Y]$ if not said otherwise and $\mathfrak{C}_Y$ is a monster for T in $\mathbf{L}[T, Y]$:

$\odot_1$ (a) for a model $M \prec \mathfrak{C}_Y$ let $\mathbf{S}^{c,\theta}_Y(M) = \{\text{tp}(\bar{a}, M, N) : M \prec N \prec \mathfrak{C}_Y, \|N\| \leq \theta$ and $\bar{a}$ enumerates $N\}$, omitting θ means some θ,

(b) in this case we say N realizes $p = \text{tp}(\bar{a}, M, N)$,

(c) if $p = \text{tp}(\bar{a}, M, N)$ is as above, <u>then</u> we denote $|p| = \|N\|$,

$\odot_2$ for $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \zeta \rangle$ and $\bar{p} = \langle p_\varepsilon : \varepsilon < \zeta \rangle, p_i \in \mathbf{S}^c_Y(M)$, we say N is $(\bar{p}, \bar{\alpha})$-constructed over M when there is $\bar{M}$ such that

(a) $\bar{M} = \langle M_{\{i\}} : i < \alpha^\zeta \rangle$, where $\alpha^\varepsilon = \sum_{\xi < \varepsilon} \alpha_\xi$ for $\varepsilon \leq \zeta$,

(b) $M_{\{i\}}$ realizes p_ε if $i \in [\alpha^\varepsilon, \alpha^\varepsilon + \alpha_\varepsilon)$,

(c) $\langle M_{\{i\}} : i < \alpha^\zeta \rangle$ is independent over M,

(d) N is primary over $\bigcup_{i < \alpha^\zeta} M_{\{i\}}$,

$\odot_3$ we say N is $\bar{p}$-constructed over M if this holds for some $\bar{\alpha}$,

$\odot_4$ if $M \prec N \prec \mathfrak{C}_Y, p \in \mathbf{S}^c_Y(M)$ then we say q lifts p or (p, M) to N <u>when</u> $q \in \mathbf{S}^c_Y(N)$ and for some M_1, N_1 realizing p, q respectively, $\mathrm{tp}(M_1, N)$ does not fork over M and N_1 is primary over $N \cup M_1$,

$\odot_5$ for $M \prec \mathfrak{C}$ and $p_1, p_2 \in \mathbf{S}^c_Y(M)$ we say p_2 pushes p_1 (in $\mathbf{L}[T, Y]$) <u>when</u> for some ordinals α_1, α_2 there are $M'_{\{i\}}$ for $i < \alpha_1 + \alpha_2$ and $\bar{M}, M^*, M'$ satisfying

(a) M^* is $(\langle p_1, p_2\rangle, \langle \alpha_1, \alpha_2\rangle)$-constructed over M as witnessed by $\bar{M} = \langle M_{\{i\}}: i < \alpha_1 + \alpha_2\rangle$,

(b) $M \prec M' \prec M^*$,

(c) M' realizes p_1,

(d) $\langle M_{\{i\}}: i < \alpha_1\rangle^\frown\langle M'\rangle$ is independent over M,

$\odot_6$ (α) assume $p_\varepsilon, q_\varepsilon \in \mathbf{S}^c_Y(M)$ for $\varepsilon < \varepsilon(*)$; we say $(\bar{p}, \bar{\alpha})$ is equivalent to $(\bar{q}, \bar{\beta})$ when $\bar{\alpha} = \langle \alpha_\varepsilon : \varepsilon < \varepsilon(*)\rangle, \bar{\beta} = \langle \beta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ and there is M' which is both $(\bar{p}, \bar{\alpha})$-constructed over M and $(\bar{q}, \bar{\beta})$-constructed over M,

(β) we may write p instead of $\langle p\rangle, q$ instead of $\langle q\rangle$, and omitting $\bar{\alpha}, \bar{\beta}$ means "for some $\bar{\alpha}, \bar{\beta}$",

$\circledast_1$ if $p_1, p_2 \in \mathbf{S}^c_Y(M)$ and p_2 pushes p_1 <u>then</u> in $\odot_5$ without loss of generality $\alpha_1, \alpha_2 \leq \|M\| + |T| + |p_1| + |p_2|$,

[Why? By the DLST argument.]

$\odot_7$ (a) let $\mathrm{AP}^\theta_Y = \{(M, p_1, q_1)$: in $\mathbf{L}[T, Y], M \prec \mathfrak{C}_Y$ have cardinality $\leq \theta$ and $p_1, q_1 \in \mathbf{S}^c_Y(M)\}$,

(b) $\mathrm{AP}^\theta_Y \models$ "$(M_1, p_1, q_1) \leq (M_2, p_2, q_2)$" means that

(α) both triples are from AP^θ_Y,

(β) M_2 is (p_1, q_1)-constructed over M_1,

(γ) p_2, q_2 lift p_1, q_1 over M_2 respectively,

$\circledast_2$ if $\mathrm{AP}^\theta_Y \models$ "$(M_1, p_1, q_1) \leq (M_2, p_2, q_2)$" and q_2 pushes p_2 <u>then</u> q_1 pushes p_1,

[Why? Straight.]

$\circledast_3$ if $(M, p_1, q_1) \in \mathrm{AP}^{\aleph_{\beta(*)}}_Y$ and p_1 does not push q_1 <u>then</u> we can find μ_0, μ_1, M_*, p_2, q_2 and r such that

(a) $\mathrm{AP}^{\aleph_{\beta(*)}}_Y \models$ "$(M, p_1, q_1) \leq (M_*, p_2, q_2)$" hence by $\circledast_2$ the type p_2 does not push q_2, (this is the only point where we use "p_1 does not push q_1"),

(b) $\|M_*\| = \mu_0$,

(c) $r \in \mathbf{S}^{c,\mu_0}_Y(M_*)$ and $\aleph_{\beta(*)} \leq \mu_0 < \mu_1 < \lambda$,

(d) $(\langle p_2, q_2\rangle, \langle \lambda, \mu_1\rangle)$ is equivalent to $(\langle r\rangle, \langle \lambda\rangle)$, see $\odot_6$.

[Why? For every $\mu \in [\aleph_{\beta(*)}, \lambda)$ let N^μ be $(\langle p_1, q_1\rangle, \langle \lambda, \mu\rangle)$-constructed over M as witnessed by $\langle N_i : i < \lambda + \mu\rangle$.

As we are assuming that $\dot{I}(\lambda, T) < |\alpha_* - \beta_*|$, there are μ_0, μ_1 such that $|T| = \aleph_{\beta(*)} \leq \mu_0 < \mu_1 < \lambda$ and there is an isomorphism $f \in \mathbf{V}$ from N^{μ_0} onto N^{μ_1}; of course f is not necessarily from $\mathbf{L}[T, Y]$. We now work in $\mathbf{L}[T, Y, f]$ and in the end we use absoluteness (here we use "T countable").

Now by the DLST argument and properties of $\mathbf{F}^t_{\aleph_0}$-primary we can find (u_0, u_1, M^0, M^1) such that

$(*)_4$ (a) u_ℓ is a subset of $\lambda + \mu_\ell$ of cardinality μ_0 satisfying $|u_\ell \cap \lambda| = \mu_0 = |u_\ell \backslash \lambda|$ and $[\lambda, \lambda + \mu_0) \subseteq u_\ell$ for $\ell = 0, 1$,

(b) $M^\ell \prec N^{\mu_\ell}$ is primary over $M \cup \{N_i : i \in u_\ell\}$ for $\ell = 0, 1$,
(c) N^{μ_ℓ} is primary over $M^\ell \cup \{N_i : i \in (\lambda + \mu_\ell)\backslash u_\ell\}$ for $\ell = 0, 1$,
(d) f maps M^0 onto M^1.

For $i \in (\lambda + \mu_\ell)\backslash u_\ell$ let $N_{\ell,i} \prec N^{\mu_\ell}$ be primary over $M^\ell \cup N_{\{i\}}$ such that N^ℓ is primary over $\cup\{N_{\ell,j} : j \in (\lambda + \mu_\ell)\backslash u_\ell\}$; clearly

$(*)_5$ for $\ell = 0, 1$
(a) $M^\ell \prec N_{\ell,i} \prec N^\ell$,
(b) $(N_{\ell,i}, c)_{c \in M^\ell} \cong (N_{\ell,j}, c)_{c \in M^\ell}$ when $i, j \in \lambda\backslash u_\ell$ or $i, j \in (\lambda + \mu_\ell)\backslash\lambda\backslash u_\ell$,
(c) $\langle N_{\ell,i} : i \in (\lambda + \mu_\ell)\backslash u_\ell\rangle$ is independent over M^ℓ and N^ℓ is primary over their union.

Choose $\gamma_1 \in \lambda\backslash u_1, \gamma_2 \in [\lambda, \lambda + \mu_1)\backslash u_1$, so $(M^1, \mathrm{tp}(N_{1,\gamma_1}, M^1_\emptyset), \mathrm{tp}(N_{1,\gamma_2}, M^1_\emptyset))$ can serve as (M_*, p_1, q_1) and r is $f(\mathrm{tp}(M_{0,\gamma}, M^0))$ for any $\gamma \in \lambda\backslash u_0$.

So we have finished proving $\circledast_3$.]

$\circledast_4$ assume $\bar p, \bar q$ are sequences of members of $\mathbf{S}^c_Y(M)$ and $(\bar p, \bar\alpha), (\bar q, \bar\beta)$ are equivalent and $M \prec N$ and $p'_\varepsilon \in \mathbf{S}^c_Y(N)$ lift p_ε for $\varepsilon < \ell g(\bar p)$ and $q'_\varepsilon \in \mathbf{S}^c_Y(N)$ lift q_ε for $\varepsilon < \ell g(\bar q)$ <u>then</u> $(\langle p'_\varepsilon : \varepsilon < \ell g(\bar p)\rangle, \bar\alpha)$ and $(\langle q'_\varepsilon : \varepsilon < \ell g(q)\rangle, \bar\beta)$ are equivalent,

[Why? By properties of "primary".]

$\circledast_5$ if $p, q \in \mathbf{S}^{c,\theta}_Y(M)$ are equivalent then $(p, \theta), (q, \theta)$ are equivalent.

[Why? By DLST.]

Note

$\circledast_6$ in $\circledast_3$ we can conclude p_2, r are equivalent.

[Why? In clause (d) of $\circledast_3$, let N be the model and let a witness for N being (r, λ)-constructed be $\langle N^2_i : i < \lambda\rangle$ and for N being $(\langle p_2, q_2\rangle, \langle\lambda, \mu_1\rangle)$-constructed be $\langle N^*_i : i < \lambda + \mu_1\rangle$. Let $u_0 \subseteq \lambda, u_1 \subseteq \lambda + \mu_1$ be of cardinality $\mu_1, [\lambda, \lambda + \mu_1) \subseteq u_1$ and M'_* be such that:

$(*)_6$ (a) $M'_* \prec N$
(b) M'_* is primary over $\cup\{N^2_i : i \in u_1\}$,
(c) M'_* is primary over $\cup\{N^*_i : i \in u_2\}$,
(d) N is primary over $\cup\{N^2_i : i \in \lambda\backslash u_1\} \cup M'_*$,
(e) N is primary over $\cup\{N^*_i : i \in \lambda + \mu_1\backslash u_2\}$.

The liftings r', p'_2 of r, p_2 to M'_* are equivalent, so we "collapse" to cardinality μ_0 getting M''_* so M''_* is (r, μ_0)-constructed over M_* and (p_2, μ_0)-constructed over M_*. Then find liftings $r'', p''_2 \in \mathbf{S}^c_Y(M'_*)$ of r, p respectively, so r'', p''_2 are equivalent naturally but M''_*, M_* are isomorphic over M by an isomorphism mapping r'', p''_2 to r, p_2 so we get that r, p_2 are equivalent as required.]

$\circledast_7$ in $\circledast_3$ if M' is (p_2, μ_0)-constructed over M_* then q_2 is realized in M'.

[Why? Assume M' is a counterexample. Look again at the proof of $\circledast_3$, so M_* is M^1 there, and so M' is (p_2, λ)-constructed over $M^1 = M_*$ and N^1 is (p_2, λ)-constructed over $M^1 = M_*$, so by uniqueness of primary also in N^1 we cannot find $N' \prec N^1$ realizing q_2. But for any $\gamma \in [\lambda, \lambda + \mu_1]\backslash u_2$ the model $f^{-1}(N_{1,\gamma})$ contradict this.]

Now we can prove 3.8. Let $p, q \in \mathbf{S}^{c,\aleph_{\beta(*)}}_Y(M_\emptyset)$ be types which $M_{\{0\}}, M_{\{\omega\}}$ respectively realizes. Let $(M, p_1, q_1) = (M_\emptyset, p, q)$.

So by $\circledast_7$, there are $\mu_0 < \lambda$ and Y_1 and $M_2 \in \mathbf{L}[T, Y_1], M_2$ which is $(\langle p, q\rangle, \langle \mu_0, \mu_0\rangle)$-constructed over $M_\emptyset$ and p_2, q_2 lifting of p, q in $\mathbf{S}^{c,\mu_0}_{Y_1}(M_2)$ as there. So by DLST we can find such M_2', p_2', q_2' for the case $\mu_0 = \aleph_{\beta(*)}$, but this is absolute as $\beta(*) = 0$. Also it gives the required result, i.e., clause (B) of $\boxtimes_T$. $\square_{3.8}$

3.10. Theorem. [ZF] *If T is countable and $\boxtimes_T$ below holds, then (recalling 0.12(2)) in every cardinal $\mu \geq \Upsilon(\mathscr{P}(\omega))$ we have $\dot{I}(\mu, T)$ is $\leq |\mathscr{F}^*/E|$ where $\mathscr{F}^* = \{f : f$ a function from $\mathscr{P}(\omega)$ to $\omega + 1\}$, for some equivalence relation E on the set of those functions, where:*

$\boxtimes_T$ (a)–(d) *from* 3.5,
(e)–(f) *from* 3.6,
(g) *from* 3.8.

Remark. (1) Countability of T is not used (if we write $\mathscr{P}(|T|)$ instead of $\mathscr{P}(\omega)$), but the gain is not substantial. This applies to 3.11, 3.13, too.
(2) Fuller more accurate information is given in 3.13.

Proof. Let N be a model of T of cardinality μ so without loss of generality with universe μ, we work in $\mathbf{L}[T, N]$ and we shall analyze it. Now we first choose a countable $M_\emptyset \prec N_\ell$. As T is superstable, uni-dimensional we can find $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$ and $\bar{a} \in {}^{\ell g(\bar{y})}(M_\emptyset)$ such that $\varphi(x, \bar{a})$ is weakly minimal.

We can find $\langle a_\alpha : \alpha < \mu\rangle$ such that:

$\circledast_1$ (a) $a_\alpha \in \varphi(N, \bar{a}_\ell)\backslash M_\emptyset$,
(b) $\{a_\alpha : \alpha < \mu\}$ is independent in N over $M_\emptyset$ (in particular with no repetitions),
(c) modulo (a) + (b) the set $\{a_\alpha : \alpha < \mu\}$ is maximal hence
(d) $\varphi(N, \bar{a}) \subseteq \mathrm{acl}(M_\emptyset \cup \{a_\alpha : \alpha < \mu\})$.

Let $f \in \mathbf{L}[T, N]$ be a function from μ to μ such that $f(\alpha) \leq \alpha$ and $(\forall \beta < \mu)(\exists^\mu \alpha < \mu)(f(\alpha) = \beta)$. Now we try to choose $(M_{\{\alpha\}}, b_\alpha)$ by induction on $\alpha < \mu$ such that

$\circledast_2$ (a) $b_\alpha \in \varphi(N, \bar{a})$,
(b) $b_\alpha \notin \mathrm{acl}(M_\emptyset \cup \{b_\beta : \beta < \alpha\})$,
(c) $M_{\{\alpha\}} \prec N$ is $\mathbf{F}^c_{\aleph_0}$-primary over $M_\emptyset \cup \{b_\alpha\}$, see [Sh:c, IV],
(d) if $\alpha = 2\beta + 1$ and we can find $(M_{\{\alpha\}}, b_\alpha)$ satisfying (a) + (b) + (c) and $(M^{N_\alpha}_{\{\alpha\}}, c)_{c \in M_\emptyset} \cong (M_{\{f(\beta)\}}, c)_{c \in M_\emptyset}$ then $(M_{\{\alpha\}}, b_\alpha)$ satisfies this,
(e) if $\alpha = 2\beta$ and $\gamma_\alpha = \mathrm{Min}\{\gamma : a_\gamma \notin \mathrm{acl}(M_\emptyset \cup \{b_\varepsilon : \varepsilon < 2\beta\})$ then $b_\alpha = a_{\gamma_\alpha}$.

[Why can we can carry the induction? We can ignore clause (d) as if its hypothesis hold, then clause (e) is irrelevant, and this hypothesis says that we can fulfil clause (a), (b), (c), (d). Also if $\alpha = 2\beta + 1$ and the further assumption of (d) fail then we can act as in clause (e). Also in all cases by cardinality considerations recalling $|\varphi(M, \bar{a})| = \|M\|$ by (c) of $\boxtimes_T$ of 3.5 there is b_α satisfying clauses (a) + (b) and if clause (e)'s assumption holds, without loss of generality also its conclusion.

Let $B_\alpha = \mathrm{acl}_M(M_\emptyset \cup \{b_\alpha\})$. By the choice of $\varphi(x, \bar{a})$ if $\varphi(x, \bar{a}) \in p \in \mathbf{S}(B_\alpha, M)$ then <u>either</u> p forks over $\bar{a}$ hence is algebraic hence realized in B_α <u>or</u> p does not fork over $\bar{a}$ hence is finitely satisfiable in $M_\emptyset$. Let $\langle b_{\alpha,i} : i < i_\alpha\rangle$ be a maximal sequence of members of M such that for each i for some formula $\varphi(x, \bar{c}_{\alpha,i}) \in \mathrm{tp}(b_{\alpha,i}, B_\alpha \cup \{b_{\alpha,j} : j < i\})$ hence no extension in $\mathbf{S}(B_\alpha \cup \{b_{\alpha,j} : j < i\})$ forking over $\bar{c}_{\alpha,i}$. By [Sh:31] there is $M_{\{\alpha\}} \prec N$ with universe $B_\alpha \cup \{b_{\alpha,i} : i < i_\alpha\}$.

So we are done.]

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$\circledast_3$ $\langle b_\alpha : \alpha < \mu \rangle$ satisfies the requirements on $\langle a_\alpha : \alpha < \mu \rangle$.

[Why? Easy to check.]

So $\langle M_{\{\alpha\}} : \alpha < \mu \rangle$ is independent over $M_\emptyset$ inside N hence (by $\boxtimes_T$(f)) there is $N' \prec N$ primary over $\cup\{M_{\{\alpha\}} : \alpha < \mu\}$ and by $\circledast_1$(d) include $\varphi(N, \bar{a})$ hence by $\boxtimes_T$(c) we have $N' = N$.

We can find a set S and a partition $\langle I_t : t \in S \rangle$ of μ such that: for $\ell = 1, 2$ and $\alpha, \beta < \mu$ we have

$\circledast_4$ $(M_{\{\alpha\}}, a_\alpha, c)_{c \in M_\emptyset}$ is isomorphic to $(M_{\{\beta\}}, a_\beta, c)_{c \in M_\emptyset}$ iff $\bigvee_{t \in S} \{\alpha, \beta\} \subseteq I_t$.

Now how large can $|S|$ be? It is, in $\mathbf{V}, \leq |\mathscr{P}(\omega)|^{\mathbf{L}[T,N]} \leq |\mathscr{P}(\omega)|^{\mathbf{V}}$ (there is a function from a subset of $\mathscr{P}(\omega)$ onto this set). So $|S| < \theta(\mathscr{P}(\omega))$, but $\mathbf{L}[T, N] \models$ "ZFC + $|S| \leq 2^{\aleph_0} = |\mathscr{P}(\omega)|$" so there is a well ordering of $\mathscr{P}(\omega) \cap \mathbf{L}[T, N] = \mathscr{P}(\omega)^{\mathbf{L}[T,N]}, |S| \leq |\mathscr{P}(\omega)^{\mathbf{L}[T,N]}| < |S| < \Upsilon(\mathscr{P}(\omega))$ and $\mathbf{L}[T, N] \models$ "$|S| \leq 2^{\aleph_0}$". Now we shall prove:

$\circledast_5$ if $t \in S$ and $\mathbf{L}[T, N] \models$ "$\aleph_0 \leq |I_t| < \mu$" <u>then</u> for some $\alpha < \mu$ we have $(\forall s \in S)(|I_s \backslash \alpha| < \aleph_0)$ and $\mu < \Upsilon(\mathscr{P}(\omega))$.

Clearly $\circledast_5$ helps because "$\mu < \Upsilon(\mathscr{P}(\omega))$" contradict an assumption on μ.

Why $\circledast_5$ holds? Let $\alpha(*) = \mathrm{Min}(I_t)$, it is well defined as $I_t \neq \emptyset$ because "$\aleph_0 \leq |I_t|$" was assumed. Let $J = \{2\beta + 1 : f(\beta) = \alpha(*)\}$. If $2\beta + 1 \in J \Rightarrow 2\beta + 1 \in I_t$, then we get $|I_t| \geq |\{2\beta + 1 : f(\beta) = \alpha(*)\}| = \mu$ hence the assumption "$|I_t| < \mu$" is contradicted, so assume that $\alpha = 2\beta + 1 \in J \backslash I_t$. By clause (d) of $\circledast_2$ apply to $\alpha = 2\beta + 1$, we know that if (N', b) satisfies the demands on $(M_{\{\alpha\}}, b_\alpha)$ in clauses (a), (b), (c) (i.e., $b_\alpha \in \varphi(N, \bar{a}) \backslash \mathrm{acl}(M_\emptyset \cup \{b_\varepsilon : \varepsilon < \alpha\})$ and $N' \prec N$ is $\mathbf{F}^c_{\aleph_0}$-primary over $M_\emptyset \cup \{b\}$) then $(N', c)_{c \in M_\emptyset} \not\cong (M_{\{\alpha(*)\}}, c)_{c \in M_\emptyset}$. This implies that $I_t \subseteq \alpha$. As it is infinite, by $\boxtimes_T$(g) we get $(\forall s \in S)(|I_s \backslash \alpha| < \aleph_0)$ and recall $|\mu \backslash \alpha| = \mu$. So $\{\mathrm{Min}(I_s \backslash \alpha) : s \in S$ and $I_s \nsubseteq \alpha\}$ is a subset of μ of cardinality μ (working in $\mathbf{L}[T, N]$) and there is a one-to-one mapping from it into $\mathscr{P}(\omega)$ (using the isomorphism types of $(M_{\{\alpha\}}, c)_{c \in M_\emptyset}$). This gives $\mu < \Upsilon(\mathscr{P}(\omega))$. But this contradicts an assumption on μ.

So we know

$\circledast_6$ if I_t is infinite then it has cardinality μ.

Let $f = f_N = f_{N, M_\emptyset, \varphi(x, \bar{a})}$ be the partial function from $\mathscr{P}(\omega)$ into $\omega + 1$ defined as follows: if $t \in S$ and $\eta \in \mathscr{P}(\omega)$ codes[4] a model isomorphic to $(M_\alpha, c)_{c \in M_\emptyset}$ for $\alpha \in I_t$ then $f_N(\eta) = |I_t|$ if I_t is finite and $f_N(\eta) = \omega$ otherwise: of course, the choice of f_N is unique if we use the canonical well ordering of $\mathbf{L}[T, N]$ to make our choices in particular of $M_\emptyset, \varphi(x, \bar{a})$, but we could use "any such f" so increasing $\mathscr{F}_\mu$ below (and fix the coding).

Now in $\mathbf{V}$ for any model M of T of cardinality μ we define

$$\begin{aligned} \mathscr{F}_M = \{(f_N, N{\restriction}\omega, \varphi(x, \bar{a})) :\ & N \text{ is a model with universe } \mu \text{ isomorphic to } M \\ & \text{such that } N{\restriction}\omega \prec N \text{ so can serve as } M_\emptyset \\ & \text{and } \bar{a} \in {}^{\omega>}N \text{ and } \varphi(x, \bar{a}) \text{ is weakly minimal}\}, \end{aligned}$$

$$\mathbf{F}^* = \cup\{\mathscr{F}_M : M \text{ a model of } T \text{ of cardinality } \mu\}.$$

[4]See more details on this and similar points in the proof of 3.13.

Clearly

$\circledast_7$ (a) $\mathscr{F}_M$ depends just on $M/\cong$,

(b) if $\mathscr{F}_{M_1} \cap \mathscr{F}_{M_2} \neq \emptyset$ then $M_1 \approx M_2$ hence $\mathscr{F}_{M_1} = \mathscr{F}_{M_2}$ so there is an equivalence relation $E_{T,\mu}$ on a subset of $\mathscr{F}^*$ such that the $\mathscr{F}_M$'s are its equivalence classes,

(c) the number of models of T in μ up to isomorphism is equal to the number of $E_{T,\mu}$-equivalence classes.

So we are done. $\square_{3.10}$

3.11. Claim. [*T countable*] *The demand* $\boxtimes_T$ *from* 3.10 *is absolute* (*property of* T).

Proof. The new point is $\boxtimes_T(\mathrm{g})$ which should be clear. $\square_{3.11}$

3.12. *Remark.* (1) The proof of 3.10, 3.11 is really a particular case of "the number of special dimensions" from [Sh:c, XIII,§3] the number being here 1; see more on this Hrushovski Hart Laskowski [HHL00].

(2) The "primary over $\cup\{N_{\{\alpha\}}: \alpha\}$" is a special case of decompositions.

3.13. Theorem. *If T is countable and $\boxtimes_T$ from* 3.10 *holds <u>then</u>:*

(a) $\dot{I}(\mu, T)$ *is the same whenever* $\mu \geq \mu_* =: \theta(\mathscr{F}^*)$ *recalling* $\mathscr{F}^* = \{f: f$ *a function from* ${}^{\omega}2$ *to* ω *with* $\mathrm{supp}(f) = \{\eta: f(\eta) \neq 0\}$ *well orderable*$\}$.

Proof. We elaborate some parts done in passing in the proof of 3.10 (and add one point).

We can interpret $\eta \in {}^{\omega}2$ as a triple $(M_0, M_1, \varphi(x,\bar{a})) = (M_0^\eta, M_1^\eta, \varphi_\eta(x,\bar{a}_\eta))$ such that $M_0 \prec M_1$ are models of T, M_1 with universe ω, M_0 with universe $\{2n: n < \omega\}$ and $\varphi(x,\bar{a})$ a weakly minimal formula in M_0. So the equivalence relation E_1 is Σ^1_1 where $\eta E_1 \nu \Leftrightarrow [M_0^\eta = M_0^\nu, \varphi_\eta(x,\bar{a}_\eta) = \varphi_\nu(x,\bar{a}_\nu)$ and M_1^η, M_1^ν are isomorphic over $M_0^\eta = M_0^\nu]$ and E_0 a Borel equivalent relation where $\eta E_0 \nu \Leftrightarrow M_0^\eta = M_0^\nu$.

Let

$$\mathscr{P}_1 = \{A: A \subseteq {}^{\omega}2 \text{ is not empty, any two members are } E_0\text{-equivalent not } E_1 \text{ equivalent and } A \text{ is well orderable}\}.$$

Let

$$\mathscr{F} = \{f: \text{for some } A \in \mathscr{P}_1,\ f \text{ is a function from } A \text{ to } \omega + 1 \text{ such that } \omega \in \mathrm{Rang}(f)\}.$$

Let

$$\theta_* = \theta(\mathscr{F})(\leq \theta(\mathscr{F}^*)).$$

For N a model of T of cardinality $\geq \theta_*$ let $\mathscr{F}_N \subseteq \mathscr{F}$ be defined as in the proof of 3.10 but we can write f and not $(f, M_\emptyset, \varphi(x,\bar{a}))$ as $M_0, \varphi(x,\bar{a})$ are determined by Dom(f). Let

$$E^2_\mu = E^2_{T,\mu} = \{(f_1, f_2): \text{there is } N \in \mathrm{Mod}_{T,\mu} \text{ for which } f_1, f_2 \in \mathscr{F}_N\}.$$

(Recalling $\mathrm{Mod}_{T,\mu} = \{M: M$ is a model of T of cardinality $\mu\}$).

Now

$(*)_1$ if $\mu \geq \theta_*$ and $f \in \mathscr{F}$ then for some model N of T of cardinality μ we have $f \in \mathscr{F}_N$,

$(*)_2$ if $N_1 \cong N_2$ are from $\mathrm{Mod}_{T,\mu}$ and $\mu \geq \theta_*$ then $\mathscr{F}_{N_1} = \mathscr{F}_{N_2}$,
$(*)_3$ if $N_1, N_2 \in \mathrm{Mod}_{T,\mu}$, $\mu \geq \theta_*$ and $\mathscr{F}_{N_1} \cap \mathscr{F}_{N_2} \neq \emptyset$ then $N_1 \cong N_2$,
$(*)_4$ E^2_μ is an equivalence relation on $\mathscr{F}$,
$(*)_5$ $\mathscr{F}_N$ for N a model of T of cardinality $\geq \theta_*$ is an E_2-equivalence class,
$(*)_6$ E^2_μ is the same for all $\mu \geq \theta_*$.

[Why? Assume N^1, N^2 are models of T with universe μ, $f_\ell \in \mathscr{F}_{N_\ell}$ and let $N^i_\emptyset, a^\ell_\alpha, N^\ell_{\{\alpha\}} (\alpha < \mu)$ be as in the proof of 3.10 exemplifying this. Let $\theta_* \leq \mu_1 < \mu_2$. If $\mu = \mu_1$, $f_1 E^2_{T,\mu_2} f_2 \Rightarrow f_1 E^2_{T,\mu_1} f_2$ by the LST argument. The other direction, i.e., if $\mu = \mu_2$ is similar to the proof of 3.10, i.e., we blow up $\langle a_\alpha : \alpha \in I_t \rangle$ for some t (or every t) such that $|I_t| = \mu$ and continue as in 3.8.] $\square_{3.13}$

3.14. *Conclusion.* For every countable complete first order theory T, one of the following occurs

(A) for every α, $\dot{I}(\aleph_\alpha, T) \geq |\alpha|$, in fact there is a sequence $\langle M_\beta : \beta < \alpha \rangle$ of pairwise non-isomorphic models of T of cardinality $\aleph_\alpha$,
(B) for all $\mu \geq \mu_* =: \theta(\mathscr{F}^*)$ (which $\leq \theta({}^{\mathscr{P}(\omega)}\omega)$), $\dot{I}(\mu, T)$ is the same and has the form $\mathscr{F}^*/E$ for some equivalence relation E (see more in 3.14 and its proof).

3.15. Problem. [ZF] Give complete classification of $\dot{I}(\lambda, T)$ for T countable by the model theoretic properties of T and the set theoretic properties of the universe. But it may be wiser to make less fine distinctions.

3.16. Definition. (1) Let $|X| \precsim |Y|$ mean that $X = \emptyset$ or there is a function from Y onto X (so $|X| \leq |Y|$ implies this).
(2) Let $|X| \approx |Y|$ if $|X| \precsim |Y| \precsim |X|$ (so this weakens $|X| = |Y|$ and is an equivalence relation) and $|X|/\approx$ is called the essential power.

3.17. Thesis. It is most reasonable to interpret "determining $\dot{I}(\lambda, T)$" as finding $\dot{I}(\lambda, T)/\approx$ which is the essential power $|\{M/\cong : M$ a model of T with universe $\lambda\}|/\approx$.

3.18. Claim. *Assume $\boxtimes_T$ of 3.10 and T is countable.*
(1) *If T is $\aleph_0$-stable <u>then</u> $\dot{I}(\aleph_\alpha, T) = 1$ for every $\alpha > 0$.*
(2) *If $D(T)$ is uncountable and $\alpha > 0$ <u>then</u>:*
  (a) $|\{A \subseteq {}^\omega 2 : |A| \leq \aleph_\alpha\}| \leq \dot{I}(\aleph_\alpha, T)\}$,
  (b) *$\dot{I}(\aleph_\alpha, T)$ is $\precsim$-below $|\{A \subseteq {}^\omega 2 : |A| \leq \aleph_\alpha\}|$*
  (*note: $|A| \leq \aleph_\alpha \Rightarrow A$ is well ordered*).
(3) *If $D(T)$ is countable, T is not $\aleph_0$-stable and there is a set of $\aleph_1$ reals and $\alpha > 0$ <u>then</u>*

$$\dot{I}(\aleph_\alpha, T) \approx |\{A \subseteq {}^\omega 2 : |A| \leq \aleph_\alpha\}|.$$

Proof. As in [Sh:c]. (E.g., in (2) the first inequality holds as in $\mathbf{L}[T, Y]$ we can find countable complete $T_1 \supseteq T$ with Skolem functions $M_1 \models T_1$, $a_\eta \in {}^m M_1$ for $\eta \in {}^\omega 2$ and $b_n \in M_1$ for $n < \omega$ such that letting $\alpha = \omega$, $A = ({}^\omega 2)^{\mathbf{L}[T,Y]}$ we have

$(*)^\alpha_A$ (a) $\langle b_n : n < \alpha \rangle$ is a non-trivial indiscernible sequence in M_1 over $\{\bar{a}_\eta : \eta \in A\}$,
  (b) $\langle \mathrm{tp}(\bar{a}_\eta, \emptyset, M_1 \restriction \tau_T) : \eta \in {}^\omega 2 \rangle$ are pairwise distinct,
  (c) $\langle \bar{a}_\eta : \eta \in {}^\omega 2 \rangle$ is indiscernible in $(M_1, b_n)_{n<\omega}$ in the weak sense of [Sh:c, VII, §2],

(d) $\mathrm{tp}(\bar{a}_\eta, \emptyset, M_1{\restriction}\tau_T)$ is not realized in $M_1{\restriction}\mathrm{acl}(\{\bar{a}_\nu : \nu \in {}^\omega 2\backslash\{\eta\}\} \cup \{b_n : n < \omega\})$.

So in bigger universe this M_1 has a natural extension. So we can define M_1^+, $\langle a_\eta : \eta \in ({}^\omega 2)^{\mathbf{V}}\rangle$, $\langle b_\alpha : \alpha \in [\omega, \mu)\rangle$ naturally such that $(*)^\mu_{\mathscr{P}(\omega)}$ and define M_A for $A \subseteq {}^\omega 2$ as $\mathrm{Sk}(\{\bar{a}_\eta : \eta \in A\} \cup \{b_\alpha : \alpha < \mu\}, M_1)\}|\tau_T$; if A is well orderable then M_A has cardinality μ. $\square_{3.18}$

We now look at a well known example in our context.

3.19. Example. There is countable stable, not superstable T with $D(T)$ countable such that: if there are no sets of $\aleph_1$ reals then $\dot{I}(\aleph_\alpha, T)$ is "manageable"

(A) let G be an infinite abelian group, each element of order 2. So ${}^\omega G$ is also such a group. We define a model M:
 (a) its universe: $G \cup {}^\omega G$ (assuming $G \cap {}^\omega G = \emptyset$),
 (b) predicates $P^M = G, Q^M = {}^\omega G$,
 (c) the partial two-place function H_1^M which is the addition of G (you may add $x \notin G \wedge y \notin G \Rightarrow x +^M y = x$),
 (d) H_2^M is the addition on ${}^\omega G$ (coordinatewise),
 (e) a partial unary function F_n^M such that $\eta \in {}^\omega G \Rightarrow F_n^M(\eta) = \eta(n)$,
 (f) individual constants c_1, c_2 the zeroes of G and ${}^\omega G$ respectively,

(B) let $T = \mathrm{Th}(M)$. Let $K^* = \{N : N \models T$ and N omit $\{Q(x) \wedge Q(y) \wedge F_n(x) = F_n(y) \wedge x \neq y : n < \omega\}$,

(C) if $\langle M_t : t \in I\rangle$ is a sequence of models of T we can naturally define their sum $\oplus_{t\in I} M_t$. Clearly K^* is closed under sum (i.e., $|M| = P^M \cup Q^M$, $P^M = \{f : f$ is a function with domain I such that $f(t) \in P^{M_t}$ and $f(t)$ is the zero $c_1^{M_t}$ of the abelian group $(P^{M_t}, H_1^{M_t})$ for all but finitely many t's$\}$,
 $Q^M = \{g : g$ a function with domain $I, f(t) \in Q^{M_t}$ and for all but finitely many $t \in I$ we have $f(t) = c_1^{M_t}\}$
 (we ignore that for I finite, formally $P^M \cap Q^M \neq \emptyset\}$, etc.),

(D) (ZF) If M is a model from K^* of cardinality λ and λ is a $(< \lambda)$-free cardinal (see Definition 5.2 below) <u>then</u> $M = \bigoplus_{i<\lambda} M_i$ for some sequence $\langle M_i : i < \lambda\rangle$ such that $i < \lambda \Rightarrow \|M_i\| < \lambda$,

(E) in (D) if the cardinal λ is $(< \mu)$-free we can add $\|M_i\| < \mu$ (see Definition 5.2 below),

(F) (a) define for $M \models T$ a two-place relation E_M on M: $aE_Mb \Leftrightarrow (a = b) \vee (Q(a) \wedge Q(b) \wedge \bigwedge_n F_n(x) = F_n(y))$. It is an equivalence relation on M,
 (b) define M/E_M naturally,

(G) (a) $M/E_M \in K^*$ for any model M of T,
 (b) $M_1 \cong M_2 \Rightarrow M_1/E_{M_1} \cong M_2/E_{M_1}$,

(H) (a) if $M \models T$ and $a, b \in Q^M$ then $|a/E_M| = |b/E_M|$,
 (b) we can consider a/E_M (where $M \models T, a \in Q^M$) an abelian group $G_{a,M}$ with every element of order 2 except that the zero is not given,
 (c) if $a, b \in Q^M$ then $G_{a,M}, G_{b,M}$ as vector spaces over $\mathbb{Z}/2\mathbb{Z}$ without zero has the same dimension,
 (d) call this dimension $\lambda(M)$,

(I) if M_1, M_2 are models of T of cardinality $\aleph_\alpha$ then $M_1 \approx M_2$ iff $M_1/E_{M_2} \approx M_2/E_{M_2}$ and $\lambda(M_1) = \lambda(M_2)$ hence $\dot{I}(\aleph_\alpha, T) = \dot{I}(\leq \aleph_\alpha, K^*) \times |\omega + \alpha|$ where $\dot{I}(\leq \aleph_\alpha, K^*) = \Sigma\{\dot{I}(\aleph_\beta, K^*): \beta \leq \alpha\} = \Sigma\{\dot{I}(\aleph_\beta, K^*): \beta \leq \alpha, \aleph_\beta < \theta(\mathscr{P}, \omega)\}$. $\square_{3.19}$

§4. **On T categorical in $|T|$.** The ZFC parallel of 4.2 - 4.4 is the known "$|D(T)| < |T|$ implies T is the definitional extension of some $T' \subseteq T, |T'| < |T|$", see Keisler [Ke71a], which in Boolean algebra terms say "the number of ultrafilters of an infinite Boolean algebra B is $\geq |B|$".

4.1. Convention. For 4.2-4.6, T is first order (with τ_T not necessarily well-orderable).

4.2. Definition. For a first order T in the vocabulary $\tau = \tau_T$, usually for simplicity closed under deduction, we define the equivalence relation E_T on τ_T by

(a) *for predicates* $P_1, P_2 \in \tau$.
$P_1 E P_2$ iff: $P_1, P_2 \in \tau$ are predicates with the same arity and $(\forall \bar{x})(P_1(\bar{x}) \equiv P_2(\bar{x})) \in T$

(b) *for function symbols* $F_1, F_2 \in \tau$, e.g., individual constants.
$F_1 E F_2$ iff: $P_1, P_2 \in \tau$ are function symbols with the same arity and $(\forall \bar{x})(F_1(\bar{x}) = F_2(\bar{x})) \in T$

(c) no predicate $P \in \tau$ is E-equivalent to a function symbol $F \in \tau$.

4.3. Definition. Let $\tau = \tau_T, T$ a first order theory.

(1) The theory T is called reduced if E_T is the equality.
(2) Let τ/E_T be the vocabulary with predicates $P/E_T, P \in \tau$ a predicate with $\text{arity}(P/E_T) = \text{arity}_\tau(P)$ and similarly F/E_T.
(3) For a τ-model M of T we define $M^{[E_T]}$ naturally, i.e., $N = M^{[E_T]}$ iff they have the same universe, N is a (τ/E_T)-model, M is a τ-model $M \models T$ and $(R/E_T)^N = R^M$ for every predicate $R \in \tau(T)$ and $(F/E_T)^N = F^M$ for any function symbol $F \in \tau(T)$.
(4) For N a (τ/E_T)-model, $M = {}^{[E_T]}N$ is the τ-model such that $N = M^{[E_T]}$ if one exists.
(5) Let T/E_T be the set of $\psi \in \mathbb{L}(\tau(T)/E_T)$ such that if we replace any predicate R/E_T appearing in ψ by some $R' \in R/E_T$ and similarly for F/E_T, we get a sentence from $\{\psi \in \mathbb{L}(\tau_T): T \vdash \psi\}$, see 4.2.

4.4. *Observation.* [ZF] For every first order T (as in 4.2) in $\mathbb{L}(\tau_T)$

(a) E_T is an equivalence relation on τ,
(b) if M is a τ-model of T then $M^{[E_T]}$ is a uniquely determined (τ/E_T)-model of T/E and ${}^{[E_t]}(M^{[E_T]}) = M$,
(c) for every (τ/E)-model M of T/E_T, the τ-model ${}^{[E_t]}M$ uniquely determined and is a model of T and $({}^{[E_T]}M)^{[E_T]} = M$,
(d) T/E_T is a reduced first order theory.

See hopefully more on such T's in [Sh:F701].

4.5. *Hypothesis.* $\tau(T) \subseteq \mathbf{L}$ as usual.

4.6. Claim. [ZF] *If T is a complete first order theory in $\mathbb{L}(\tau)$ and T is reduced and $Y \subseteq \mathbf{L}$, then* $T \in \mathbf{L}[Y] \Rightarrow |D(T)|^{\mathbf{L}[T]} \geq |T|$.

PROOF. By the ZFC case (see Keisler [Ke71a]). $\square_{4.6}$

4.7. CLAIM. *If $T \subseteq \mathbf{L}$ is categorical in λ and $Y \in$ Ord* *then* *in $\mathbf{L}[T, Y]$ the following is impossible*

$\circledast$ (a) *T stable, $\lambda \geq |T| + \aleph_1 + \mu$,*

(b) *$M \prec \mathfrak{C}$ is $\mathbf{F}^f_{\aleph_0}$-primary over $\emptyset$, see* [Sh:c, IV],

(c) *$\bar{a}_i \in {}^n M$ for $i < \mu$,*

(d) *$\text{tp}(\bar{a}_\delta, \cup\{\bar{a}_i : i < \delta\})$ forks over $\cup\{a_j : j < \alpha\}$ whenever $\alpha < \delta < \mu, \delta$ a limit ordinal from S,*

(e) *every type over $\cup\{\bar{a}_i : i < \mu\}$ which is realized in M does not fork over some $\cup\{\bar{a}_i : i < \alpha\}$ for some $\alpha < \mu$,*

(f) *in $\mathbf{L}[T, Y]$ we have: μ regular uncountable, $S \subseteq \mu$ stationary.*

PROOF. Work in $\mathbf{L}[T, Y]$; without loss of generality M has cardinality λ, and toward contradiction assume $\circledast$ holds. By clause (b) there is $\bar{\mathbf{c}}$ such that

$(*)_1$ $\bar{\mathbf{c}} = \langle \bar{c}_i : i < i^* \rangle$,

$(*)_2$ $M = \cup\{\bar{c}_i : i < i^*\}$ and $\text{tp}(\bar{c}_i, \cup\{\bar{c}_j : j < i\})$ does not fork over some finite $B_i \subseteq \cup\{\bar{c}_j : j < i\}$ for each $i < i^*$.

So by the properties of non-forking (or of $\mathbf{F}^f_{\aleph_0}$-constructions, [Sh:c, IV]) without loss of generality we have ($i^* \geq \mu$ and) $\cup\{\bar{a}_i : i < \mu\} \subseteq \cup\{\bar{c}_j : j < \mu\}$. Hence for some club E of μ we have $\bar{a}_i \subseteq \bigcup_{j<\delta} \bar{c}_j \Leftrightarrow i < \delta$ for $i < \mu, \delta \in E$; clearly $\text{tp}(\bar{a}_\delta, \cup\{\bar{c}_i : i < \delta\})$ does not fork over some finite $C_\delta \subseteq \cup\{\bar{c}_j : j < \delta\}$. Hence there is stationary $S_1 \subseteq S \cap C$ such that $\delta \in S \Rightarrow C_\delta = C_*$, and let $\bar{c}$ list C_*.

By clause (e) of the assumption for some $\alpha_* < \mu$,

$(*)_2$ $\text{tp}(\bar{c}, \cup\{\bar{a}_i : i < \mu\})$ does not fork over $\cup\{\bar{a}_i : i < \alpha_*\}$

hence by the non-forking calculus

$(*)_3$ for $\delta \in S_1 \backslash (\alpha + 1)$ the type $\text{tp}(\bar{a}_\delta, \cup\{\bar{a}_i : i < \delta\})$ does not fork over $\cup\{\bar{a}_i : i < \alpha_*\}$.

By this contradicts clause (d) of the assumption. $\square_{4.7}$

4.8. CLAIM. *If T is stable, categorical in λ and $\lambda = |T| > \aleph_0$* *then*

Case (α): *if $(\exists Y \subseteq \text{Ord})(\aleph_1 = \aleph_1^{\mathbf{L}[T, Y]})$ then $\kappa_r(T) = \aleph_0$, i.e., T is superstable.*

Case (β): *if $(\forall Y \subseteq \text{Ord})(\aleph_1 > \aleph_1^{\mathbf{L}[T, Y]})$ then for every $Y \subseteq$ Ord we have $\mathbf{L}[T, Y] \models$ "$\kappa(T) < \aleph_1^{\mathbf{V}}$".*

PROOF. *Case* (α):

Assume the conclusion fails. Fix $Y \subseteq$ Ord such that $T \in \mathbf{L}[Y], \aleph_1 = \aleph_1^{\mathbf{L}[Y]}$ and $\mathfrak{C} = \mathfrak{C}_T \in \mathbf{L}[Y]$ is a χ-saturated (in $\mathbf{L}[Y]$) model of T and $\mathbf{L}[Y] \models \kappa(T) \geq \aleph_1$ where χ is large enough and regular in $\mathbf{L}[Y]$; and we shall work inside $\mathbf{L}[Y]$.

Let $\mu = \aleph_1^{\mathbf{L}[Y]} = \aleph_1^{\mathbf{V}}$. We can find $\langle \bar{a}_n : n < \omega \rangle, \bar{a}_n \in {}^{\omega>}\mathfrak{C}_Y$ and a type $p = \{\varphi_n(x, \bar{a}_n) : n < \omega\}$ such that $\varphi_n(x, \bar{a}_n)$ forks over $\cup\{\bar{a}_m : m < \omega\}$. Let $\langle \eta_i : i < \mu \rangle$ list ${}^{\omega>}(\mu)$ such that $\eta_i \triangleleft \eta_j \Rightarrow i < j$ and for every limit ordinal $\delta < \mu$ we have ${}^{\omega>}\delta = \{\eta_i : i < \delta\}$. We choose $\bar{\nu} = \langle \nu_\delta : \delta < \mu \text{ limit} \rangle$ such that ν_δ is increasing with limit δ.

We choose $\langle \bar{a}_\eta : \eta \in {}^{\omega>}\mu \rangle$ such that: if $\ell g(\eta) = n$ then $\hat{a}_{\eta \restriction 0} {}^\frown \bar{a}_{\eta \restriction 1} {}^\frown \cdots {}^\frown \bar{a}_\eta$ and $\bar{a}_0 {}^\frown \cdots {}^\frown \bar{a}_n$ realize the same type in $\mathfrak{C}$ and $\text{tp}(\bar{a}_\eta, \cup\{\bar{a}_\nu : \nu \in {}^{\omega>}\mu \text{ and } \neg(\eta \trianglelefteq \nu)\})$ does not fork over $\cup\{\bar{a}_{\eta \restriction k} : k < \ell g(\eta)\}$. For limit $\delta < \mu$ we choose b_δ which realizes

$\{\varphi_n(x, \bar{a}_{\nu_\delta \restriction n}) : n < \omega\}$ such that $\mathrm{tp}(b_\delta, \cup\{\bar{a}_\nu : \nu \in {}^{\omega>}\mu\} \cup \{b_{\delta'} : \delta' < \delta \text{ is limit}\})$ does not fork over $\cup\{\bar{a}_{\nu_\delta \restriction n} : n < \omega\}$.

Lastly, let a'_i be b_δ if $i = \delta$ and be $\bar{a}_{\eta_j}$ if $i = j + 1$, and be $<>$ if $i = 0$.

Let $M_1 \prec \mathfrak{C}$, $M_1 \in \mathbf{L}[Y]$ be a model of cardinality λ which is $\mathbf{F}^f_{\aleph_0}$-primary over $\emptyset$.

Let $M_2 \prec \mathfrak{C}_T$, $M_2 \in \mathbf{L}[Y]$ be $\mathbf{F}^f_{\aleph_0}$-primary over $\cup\{\bar{a}'_i : i < \mu\}$ of cardinality λ (see [Sh:c, IV]). Now by 4.7 for $\mu = \aleph_1$, the models M_1, M_2 are not isomorphic even in $\mathbf{L}[Y, Y_1]$ for any $Y_1 \subseteq \mathrm{Ord}$ (as $\aleph_1^{\mathbf{L}[Y, Y_1]} = \aleph_1^{\mathbf{L}[Y]} = \aleph_1^{\mathbf{V}}$), contradiction.

Case (β): Assume that the conclusion fails for Y. Clearly $\aleph_1^{\mathbf{V}}$ is a limit cardinal in $\mathbf{L}[T, Y']$ for every $Y' \subseteq \mathrm{Ord}$. So for every $\mu \in \mathrm{Card}^{\mathbf{L}[T,Y]} \cap \omega_1^{\mathbf{V}}$ we can find (in $\mathfrak{C}_T^{\mathbf{L}[T,Y]} \in \mathbf{L}[T, Y]$ chosen as above) a sequence $\bar{\mathbf{a}}_\mu = \langle \bar{a}_{\mu,i} : i < \mu\rangle$ such that $\bar{a}_{\mu,i} \in {}^{\omega>}\mathfrak{C}$ for $i < \mu$ and a type $p = \{\varphi_{\mu,i}(x, \bar{a}_{\mu,i}) : i < \mu\}$ in $\mathfrak{C}$ such that $\varphi_i(x, \bar{a}^\mu_i)$ forks over $\cup\{\bar{a}_{\mu,j} : j < i\}$ for every i. Choose by induction on $i < \mu$ an element $b^\mu_i \in \mathfrak{C}$ which realizes $\{\varphi_{\mu,j}(x, \bar{a}_{\mu,j}) : j < i\}$ but $\mathrm{tp}(b^\mu_i, \cup\{\bar{a}_{\mu,j} : j < \Gamma\} \cup \{b^\mu_j : j < i\})$ does not fork over $\cup\{\bar{a}_{\mu,j} : j < i\}$. Let $\bar{a}^\mu_i = \bar{a}_{\mu,i} {}^\frown \langle b^\mu_i\rangle$ so $\langle \bar{a}^\mu_i : i < \mu\rangle$ is as in clauses (c) + (d) of 4.7. Note that the function $(\mu, i) \mapsto \bar{a}^\mu_i$ belongs to $\mathbf{L}[T, Y]$. Without loss of generality $\{\bar{\mathbf{a}}_\mu : \mu \in \mathrm{Card}^{\mathbf{L}[T,Y]} \cap \omega_1^{\mathbf{V}}\}$ is independent over $\emptyset$ in $\mathfrak{C}_Y^{\mathbf{L}[T,Y]}$. In $\mathbf{L}[T, Y]$ let $M_1 \prec \mathfrak{C}_T^{\mathbf{L}[T,Y]}$ be of cardinality λ, $\mathbf{F}^f_{\aleph_0}$-primary over $\emptyset$. Let $M_2 \prec \mathfrak{C}_T^{\mathbf{L}[T,Y]}$ be of cardinality λ and $\mathbf{F}^f_{\aleph_0}$-primary over $\cup\{\bar{\mathbf{a}}_\mu : \mu \in \mathrm{Card}^{\mathbf{L}[T,Y]} \cap \omega_1^{\mathbf{V}}\}$. But T is categorical in λ so there is an isomorphism $\mathbf{f} \in \mathbf{V}$ from M_1 onto M_2 and now we shall work in $\mathbf{L}[T, Y, \mathbf{f}]$ and let $\mu_* = \aleph_1^{\mathbf{L}[T,Y]}$, clearly $\mu_* \in \mathrm{Reg}^{\mathbf{L}[T,Y]} \cap \omega_1^{\mathbf{V}}$ so $\bar{\mathbf{a}}_{\mu_*}$ is well defined. By the non-forking calculus, the statement ⊛ of 4.7 holds for μ_* so we are done. □ 4.8

4.9. *Remark.* Assume T is stable, (complete with infinite models of course), $\lambda = |T| \geq \aleph_\alpha > \aleph_0$ and for some $Y \subseteq \mathrm{Ord}$ we have $\mathbf{L}[Y] \models$ "$\kappa(T) > \aleph_\alpha$ or $\aleph_\alpha$ is a limit cardinal and $\kappa(T) \geq \aleph_\alpha$". Then $\dot{I}(\lambda, T) \geq |\alpha|$. The proof is similar.

4.10. Claim. *T is not categorical in $\lambda = |T| > \aleph_0$ when for some $Y \subseteq$* Ord:

⊛ (a) *T is stable,*

(b) $\mathbf{L}[T, Y] \models$ "$|D(T)| \geq \lambda = |T|$" (*holds if T is reduced, see* 4.6),

(c) *the conclusion of* 4.8 *holds,* (*or just for every $Y' \subseteq$* Ord *we have* $\lambda > \kappa(T)^{\mathbf{L}[Y', Y, T]}$).

Proof. Choose $Y \subseteq \mathrm{Ord}$ which exemplify the assumption of case (α) of 4.8 if it holds. In $\mathbf{L}[T, Y]$ letting $\kappa = \kappa(T)^{\mathbf{L}[T,Y]}$ let:

$(*)_1$ M_1 be $\mathbf{F}^a_\kappa$-constructible over $\emptyset$ of cardinality λ, i.e., for some sequence $\langle a_i, B_i : i < \lambda\rangle$ we have $M_1 = \{a_i : i < \lambda\}$ and $B_i \subseteq \{a_j : j < i\}$ has cardinality $< \kappa$ and $\mathrm{stp}(a_i, B_i) \vdash \mathrm{stp}(a_i, \{a_j : j < i\})$ (not necessarily $\mathbf{F}^a_\kappa$-saturated!),

$(*)_2$ M_2 be a model of T of cardinality λ with $\mathbf{I} \subseteq M_2$ indiscernible of cardinality λ.

[Why $(*)_2$ is possible? E.g., we can have $\|M_1\| = \lambda$ because $\mathbf{L}[T, Y] \models$ "$|D(T)| \geq \lambda$".]

So assume toward contradiction that M_1, M_2 are isomorphic, let $\mathbf{f} \colon M_1 \xrightarrow[\text{onto}]{\text{iso}} M_2$ be such an isomorphism and work in $\mathbf{L}[T, Y, \mathbf{f}]$. Now $\kappa(T)^{\mathbf{L}[T,Y,\mathbf{f}]}$ may be $> \kappa = \kappa(T)^{\mathbf{L}[T,Y]}$ and κ may be not a cardinality still the properties of M_1, M_2 from $(*)_1, (*)_2$ respectively holds in $\mathbf{L}[T, Y, \mathbf{f}]$ for $\kappa = \kappa(T)^{\mathbf{L}[T,Y,\mathbf{f}]}$. Now we can get a contradiction as in [Sh:c, IV]. □ 4.10

Putting together Claims 4.8, 4.10.

4.11. *Conclusion.* If T is stable in $\lambda = |T| \leq |D(T)|$ then T is not categorical in λ.

4.12. *Free Models.* Let T be complete and stable. $\mathfrak{C} = \mathfrak{C}_{Y,T}$ a monster for T in $\mathbf{L}[T, Y]$.

The proofs above (and actually [Sh:c]) suggest that we look more into free models.

4.13. Definition. (1) A model M of T, (a stable theory) is free when we can find a sequence $\langle a_i : i < \alpha \rangle$ enumerating M such that for each $i < \alpha$ the type $\text{tp}(a_i, \{a_j : j < i\}, M)$ does not fork over some finite subset say B_i.

(2) We call $\langle (A_i, a_i, B_i) : i < \alpha \rangle$ is a free representation of M where $A_i = \{a_j : j < i\}$.

Remark. So free is the same as being $\mathbf{F}^f_{\aleph_0}$-constructible over $\emptyset$.

4.14. Claim. *If $A \subseteq \mathfrak{C}, \lambda = |A|$ is singular and every $A' \subseteq A$ of cardinality $< \lambda$ is free then A is free.*

Proof. By compactness in singular ([Sh:54], [Sh:E18]). $\square_{4.14}$

§5. Consistency results. In spite of the evidence of §1, §4, without choice characterization for the number of non-isomorphic models is different then without choice. We look for consistency results for "there are few models in cases impossible by ZFC", in particular we ask (and give a partial answer):

5.1. Question. (1) Is it consistent with ZF that for some/many $\kappa > \aleph_0$ we have: every two strongly $\aleph_0$-homogeneous linear orders of cardinality κ, are isomorphic? (Add "κ singular or κ regular"; or add $\text{cf}(\kappa) = \aleph_0$.)

(2) Similarly is it consistent with ZF that

"if $M_1, M_2 \subseteq ({}^{\omega}\lambda, E_n)_{n<\omega}$ are strongly $\aleph_0$-homogeneous of cardinality κ then they are isomorphic".

(3) Instead categoricity proves the consistency of all models has nice descriptions, (see below):

Clearly 5.8 below proves that our use of elementary classes in the proof for stable, un-superstable T is necessary, that is we could not prove too good theorems on PC classes parallel to the ZFC case.

Toward 5.1(2) we consider:

5.2. Definition. (1) A cardinality λ is free or ω-sequence-free when every subset of ${}^{\omega}\lambda$ of cardinality λ is free, where

(2) A subset $A \subseteq {}^{\omega}\lambda$ is free when there is a one-to-one function $f : A \to {}^{\omega>}\lambda$ such that $\eta \in A \Rightarrow f(\eta) \lhd \eta$.

(3) A cardinal λ is $(< \mu)$-free when every subset of ${}^{\omega}\lambda$ of cardinality $\leq \lambda$ is $(< \mu)$-free where

(4) We say "$A \subseteq {}^{\omega}\lambda$ is $(< \mu)$-free if there is a function $f : A \to {}^{\omega>}\lambda$ which in some $\mathbf{L}[Y]$ is $(< \mu)$-to-one" and $\eta \in A \Rightarrow f(\eta) \lhd \eta$.

5.3. Question. (1) Is it consistent (with ZF) that for arbitrarily large μ, μ^+ is μ^+-free? ($\aleph_0$ always is).

(2) Is it consistent with ZF that all cardinals are free?

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5.4. Claim. [ZF + DC] *Let* $\kappa = \aleph_1$. *The following is a sufficient condition for* λ *being* $(<\kappa)$*-free (equivalently - free)*

$\boxdot_{\lambda,\mu}$ *for every* $A \subseteq \lambda$ *for some* $B \subseteq \lambda$ *we have*:

$(*)_1$ *if* $\mathbf{L}[A] \models$ "μ *is a cardinal* $< \lambda$ *but* $\geq \kappa$ *such that* $\mu < \mu^{\aleph_0}$, $\mu' = \mathrm{Min}\{\lambda, \mu^{\aleph_0}\}$" *then* $\mathbf{L}[A, B] \models$ "μ' *is an ordinal of cardinality* $\leq \mu$",

$(*)_2$ *if* $\mathbf{L}[A] \models$ "$\mu \leq \lambda$ *is regular uncountable* $\geq \kappa$ *and* $S = \{\delta < \lambda$: $\mathrm{cf}(\delta) = \aleph_0\}$" *then* $\mathbf{L}[A, B] \models$ "S *is a non-stationary subset of* μ".

Remark. (1) A condition for $\kappa > \aleph_1$ will be more complicated.
(2) $(<\aleph_1)$-free is equivalent to free (note that "in some $\mathbf{L}[Y]$" in Definition 5.2).

Proof. So assume that A is a subset of ${}^{\omega}\lambda$ of cardinality $\leq \lambda$.
Let

$$\Xi = \{(Y, f) \colon Y \subseteq \mathrm{Ord} \text{ and } f \in \mathbf{L}[Y] \text{ is a function from } A \text{ to } {}^{\omega>}\lambda \text{ such that } \eta \in A \Rightarrow f(\eta) \triangleleft \eta\}$$

and let $\bar{\mu}_{Y,f} = \langle \mu^{Y,f}_{\bar{\eta}} : \eta \in A\rangle$ be defined by $\mu^{Y,f}_{\eta} = |\{\eta' \in A \colon f(\eta') = f(\eta)\}|^{\mathbf{L}[f,Y]}$.
So the role of Y is in determining where we compute $\mu^{Y,f}_{\eta}$.
Now it suffices to prove

⊛ if $(Y, f) \in \Xi$ then there is $(Z, g) \in \Xi$ such that $\eta \in A \Rightarrow \bar{\mu}^{Y_1,f_1}_{\eta} < \mu^{Y,f}_{\eta} \vee \mu^{Y,f}_{\eta} < \kappa$.

[Why it suffices? If so by DC we can find $\langle (Y_n, f_n) \colon n < \omega\rangle \in \mathbf{V}$ such that $(Y_n, f_n) \in \Xi$ and

$(*)$ $\eta \in A \Rightarrow (\mu^{Y_n,f_n}_{\eta} > \mu^{Y_{n+1},f_{n+1}}_{\eta}) \vee (\mu^{Y_n,f_n}_{\eta} < \kappa)$.

Let $Y_* = \{\mathrm{cd}(\langle 1, n, \ell g(\eta)\rangle \frown \eta \frown \langle f_n(\eta)\rangle) \colon \eta \in {}^{\omega>}\lambda \text{ and } n < \omega\} \cup \{\mathrm{cd}(2, n, \alpha) \colon \alpha \in Y_n$ and $n < \omega\}$ where cd is a one-to-one definable function in $\mathbf{L}$ from ${}^{\omega>}\mathrm{Ord}$ into Ord.

Clearly $\langle f_n \colon n < \omega\rangle \in \mathbf{L}[Y_*]$ and define $h \colon A \to \omega$ by $h(\eta) = \mathrm{Min}\{n \colon \mu^{Y_n,f_n}_{\eta} < \kappa\}$, it clearly exists by ⊛.

Lastly, let $f \colon A \to {}^{\omega>}\lambda$ be defined by $f(\eta) = \eta \restriction \mathrm{pr}(h(\eta), \ell g(f_{h(\eta)}(\eta))$ where $\mathrm{pr}(n, m)$ is, e.g., $(n + m + 1)^2 + n$.

Now check.]

Proof of ⊛:. Let Z be like B is the claim's assumption with Y playing the roles of A; we work in $\mathbf{L}[Y, Z]$, without loss of generality $Y \in \mathbf{L}[Z]$. Let $\langle \eta_\alpha \colon \alpha < |A|\rangle$ list A with no repetitions. Let $\mathscr{U} = \{\alpha < |A| \colon$ for no $\beta < \alpha$ do we have $f(\eta_\beta) = f(\eta_\alpha)\}$ and let $\mu_\alpha = |\{\beta \colon f(\eta_\beta) = f(\eta_\alpha)\}$ for $\alpha \in \mathscr{U}$ and so $\langle \mu_\alpha \colon \alpha \in \mathscr{U}\rangle \in \mathbf{L}[Y]$.

In $\mathbf{L}[Y]$ let $\langle\langle \eta_{\alpha,\varepsilon} \colon \varepsilon < \mu_\alpha\rangle \colon \alpha \in \mathscr{U}\rangle$ be such that for each $\alpha \in \mathscr{U}$ the sequence $\langle \eta_{\alpha,\varepsilon} \colon \varepsilon < \mu_\alpha\rangle$ list $A_\alpha := \{\beta \colon f(\beta) = f(\alpha)\}$. Now

⊠ it suffices to prove that in $\mathbf{L}[Z]$, for every $\alpha \in \mathscr{U}$ there is $f_\alpha \colon A_\alpha \to {}^{\omega>}\lambda$ such that $\eta \in A_\alpha \Rightarrow |\{\nu \in A_\alpha \colon f_\alpha(\nu) = f_\alpha(\eta)\}| < \mu_\alpha$.

Note that in $\mathbf{L}[Z]$, μ_α is not necessary a cardinal, in this case $f_\alpha = f \restriction A_\alpha$ can serve! [Why? In $\mathbf{L}[Z]$ we can choose $\langle f_\alpha \colon \alpha \in \mathscr{U}\rangle$ in ⊛ and then put together f and $\cup\{f_\alpha \colon \alpha \in \mathscr{U}\}$ as above.]

The proof of the condition in ⊕ is by cases (on α):

Case 1: $\alpha \in \mathscr{U}$ and μ_α is not a cardinal in $\mathbf{L}[Z]$ or $\mu_\alpha < \kappa$.
Trivial.

Hence by clause (a) of the assumption

$(*)_2$ without loss of generality $\mathbf{L}[Z] \models$ "μ_α is a cardinality".

Case 2: In $\mathbf{L}[Z]$, μ_α is regular $> \kappa$.

Let $B_{\alpha,\varepsilon} = \{\eta_{\alpha,\zeta}(n) : n < \omega \text{ and } \zeta < \varepsilon\}$, so in $\mathbf{L}[Z]$, $\langle B_{\alpha,\varepsilon} : \varepsilon < \mu_\alpha\rangle$ is $\subseteq$-increasing continuous and let $C_{\alpha,0} = \{\delta < \lambda : \delta$ is a limit ordinal and for every $\varepsilon < \mu$ we have $\varepsilon < \delta$ <u>iff</u> for some $\zeta < \delta$, $\mathrm{Rang}(\eta_{\alpha,\varepsilon}) \subseteq B_{\alpha,\zeta}\}$.

In $\mathbf{L}[Z]$ there is a club $C_\alpha = \{\beta_\xi : \xi < \mu_\alpha\}$ of μ_α such that $\delta \in C_{.} = \mathrm{cf}(\delta)^{\mathbf{L}[Y]} > \aleph_0$ and $C_\alpha \subseteq C$ and $\beta_0 = 0$.

For $\varepsilon < \mu_\alpha$ let $\xi = \xi(\varepsilon)$ be maximal such that $\varepsilon \geq \beta_\xi$ and easily $\eta_{\alpha,\varepsilon} \notin {}^\omega(B_{\alpha,\beta_\varepsilon})$, and let $g(\eta_{\alpha,\varepsilon})$ be the shortest $\nu \trianglelefteq \eta_{\alpha,\varepsilon}$ which $\notin B_{\alpha,\beta_{\xi(\varepsilon)}}$.

Now check.

Case 3: $\mathrm{cf}^{\mathbf{L}[Z]}(\mu_\alpha) \geq \kappa$.

Similarly.

Case 4: $\mathrm{cf}^{\mathbf{L}[Z]}(\mu_\alpha)) = \aleph_0$.

Here we can find an increasing sequence $\langle B_n : n < \omega\rangle$ o subsets of λ of cardinality $< \mu$ such that $A_\alpha \subseteq \bigcup\limits_{n<\omega} {}^\omega(B_n)$.

So we can proceed as above. $\square_{5.4}$

DISCUSSION. Question 5.3 seems to me to call for iterating Radin forcing but for $\aleph_2$ there is a short cut. For this we quote.

5.5. THEOREM. *Assume* $ZF + \mathrm{DC} + AD$ *and* $\kappa = \aleph_1$. <u>*Then*</u>

$(*)_\kappa$ *for every* $A \subseteq \kappa$ *for some* $\eta \in {}^\omega 2$ *we have* $A \in \mathbf{L}[\eta]$ *and* $\eta^\#$ *(hence* $A^\#$*) exist.*

PROOF. Well known.

5.6. CLAIM. [ZF]

(1) *If* $\mathrm{DC} + AD + \kappa = \aleph_1$ *or just* $(*)_\kappa$ *from* 5.5 *holds,* <u>*then*</u> $\aleph_1$ *is free.*

(2) *Also* κ *is Ord-free (see Definition* 5.10 *below).*

PROOF. (1) We can easily check the criterion from 5.4 as for M a model with universe κ and vocabulary $\subseteq \mathbf{L}_\omega$, let $\eta \in {}^\omega 2$ be such that $M \in \mathbf{L}[\eta]$ and can work in $\mathbf{L}[\eta, \eta^\#]$.

(2) Easy, too. $\square_{5.6}$

5.7. *Observation.* [ZFC + DC] If $(*)_{\lambda,\partial}$ then $\boxdot_{\lambda,\partial}$ where

$(*)_{\lambda,\partial}$ for every $A \subseteq \lambda$ there is $B \subseteq \partial$ such that $A \in \mathbf{L}[B]$ and $B^\#$ exists (so $(*)_\kappa$ is $(*)_{\kappa,\aleph_0}$),

$\boxdot_{\lambda,\theta}$ every model M of cardinality λ with vocabulary of cardinality $\leq \partial$ (so τ_M well ordered) is isomorphic to a model of the form $\mathrm{EM}_\tau(\lambda, \Phi)$ for some template Φ with $|\tau_\Phi| \leq \partial$ (so τ_Φ well ordered).

Remark. This includes $(\lambda, <_\alpha)$ where $<_\alpha$ is a well order of λ of order type $\alpha \in [\lambda, \lambda^+]$.

5.8. CLAIM. *Assume* $T \subseteq T_1$ *are countable complete first order theories.*

(1) *If* T *is stable not superstable and* $\lambda > \aleph_0 + |T_1|$ *is not free (see Definition* 5.2*)* <u>*then*</u> $\mathrm{PC}(T_1, T)$ *is not categorical in* λ.

(2) *If* T *is unstable and* $\lambda > \aleph_0$ <u>*then*</u> $\mathrm{PC}(T_1, T)$ *is not categorical in* λ.

PROOF. Without loss of generality $T, T_1 \subseteq \mathbf{L}_\omega$.

(1) Working in $\mathbf{L}[T_1, T]$ we can find Φ proper for trees with $\omega + 1$ levels as in [Sh:c, VII], i.e., $\tau_\Phi \in \mathbf{L}[T_1, T]$, $\mathrm{EM}_{\tau(T_1)}(I, \Phi)$ a model of T_1 (e.g., for $I \subseteq {}^{\omega\geq}\lambda$) satisfying $\mathrm{EM}({}^{\omega\geq}\lambda, \Phi) \models \varphi_n(\bar a_\eta, \bar a_\nu)^{\mathrm{if}(\nu = \eta\restriction n)}$ when $\eta \in {}^\omega\lambda$, $\nu \in {}^n\lambda$.

Let $F: \lambda \to {}^{\omega}\lambda$ exemplify that λ is not free, i.e., its range is not free. Working in $\mathbf{L}[T, T_1, F]$ (so without loss of generality F is one to one), let $M_1 = \mathrm{EM}_\tau({}^1\lambda, \Phi), M_2 = \mathrm{EM}_\tau({}^{\omega>}\lambda \cup \mathrm{Rang}(F), \Phi)$ and assume toward contradiction that f is an isomorphism from M_1 onto M_2 and we shall work in $\mathbf{L}[T, T_1, F, f]$, in this universe let $\mathscr{U} \subseteq \lambda$ be of minimal cardinality such that $\{F(\alpha): \alpha \in \mathscr{U}\}$ is not free (in the same sense). By [Sh:52] (or [Sh:E18]), $|\mathscr{U}|$ is a regular uncountable cardinal, so by renaming without loss of generality $\mathscr{U} = \mu = \mathrm{cf}(\mu) > \aleph_0$. Let $W \subseteq \lambda, |W| = \mu, \{f(a_\alpha): \alpha \in \mathscr{U}\} \subseteq \mathrm{EM}({}^1W, \Phi)$ and let $\langle w_\alpha: \alpha < \mu\rangle$ be a filtration of W. Clearly M_1 satisfies $A \subseteq M_1 \wedge |A| < \mu \Rightarrow \mathbf{S}(A, M) = \{\mathrm{tp}(a, A, M): a \in M_1\}$ has cardinality $\leq |A| + \aleph_0 < \mu$. This holds in M_2 hence $(\forall\alpha < \mu)(\exists\beta < \mu)[\forall\gamma \in w_\alpha)[\{f(a_{F(\gamma)\restriction n}): n \leq \omega\} \subseteq \mathrm{EM}({}^1(w_\beta), \Phi)$. We continue as in [Sh:c, VIII, §2] and get contradiction.

(2) As in 2.5. □ 5.8

5.9. Claim. *Assume $\lambda > \aleph_0$ is a free cardinal.*

(1) *For $T = \mathrm{Th}({}^{\omega}\omega, E_n)_{n<\omega}, E_n = \{(\eta, \nu): \eta, \nu \in {}^{\omega}\omega, \eta\restriction n = \nu\restriction n)$, for some countable complete $T_1 \supseteq T$, $\mathrm{PC}(T_1, T)$ is categorical in λ (T_1 does not depend on λ).*

(2) *There is a countable complete stable not superstable T such that if $M \models T$, $\|M\| \leq \lambda$, <u>then</u> the isomorphism type of M is determined by two dimensions.*

Proof. (1) As in [Sh:100], T_1 will guarantee that for any $M \in \mathrm{PC}(T_1, T)$ we have:

$(*)_1$ if $a \in M$ then $\{b \in M: M \models bE_na$ for every $n < \omega\}$ has cardinality $\|M\|$,
$(*)_2$ if $a \in M, n < \omega$ then $\{b/E^M_{n+1}: b \in a/E_n\}$ has cardinality $\|M\|$.

So suppose $M_1, M_2 \in \mathrm{PC}(T_1, T)$ has universe λ and we work in $\mathbf{L}[T, M_1, M_2]$. There is $M'_\ell \cong M_\ell$ of cardinality λ and $A_\ell \subseteq {}^{\omega}\lambda, |A_\ell| = \lambda$ for $\ell = 1, 2$ such that

$(*)_3$ $|M'_\ell| = A_\ell \times \lambda, (\eta, \alpha)E_n(\nu, \beta)$ iff ($\eta, \nu \in A_\alpha, \alpha, \beta < \lambda$ and) $\eta\restriction n = \nu\restriction n$,
$(*)_4$ $\nu \in {}^{\omega>}\lambda \Rightarrow (\exists^\lambda\eta)(\nu \triangleleft \eta \in A_\ell)$.

By the assumption "λ is free" (see Definition 5.2) we can find $g_\ell: A_\ell \to \omega$ such that $\langle \eta\restriction g_\ell(\eta): \eta \in A_\ell\rangle$ is with no repetitions and we shall work in $\mathbf{L}[T_2, M_1, M_2, A_1, A_2, g_1, g_2]$. For $\kappa < \mu$ let $\mathscr{F}_\kappa$ be the family of functions h such that

$(*)^5_h$ (a) h is a partial one-to-one function from A_1 into A_2,
(b) $|\mathrm{Dom}(h)| = \kappa$,
(c) for $\eta_1, \eta_2 \in \mathrm{Dom}(h)$ and $n < \omega$ we have $\eta_1\restriction n = \eta_2\restriction n \Leftrightarrow h(\eta_1)\restriction n = h(\eta_2)\restriction n$,
(d) if $\ell \in \{1, 2\}$ and $\nu \in A_\ell$ and $(\forall n < \omega)(\exists\eta \in A_\ell)(\nu\restriction n = \eta\restriction n)$ then $\nu \in A_\ell$.

Let $A_\ell = \{\eta^\ell_\alpha: \alpha < \lambda\}$. It is easy to choose $h_\alpha \in \mathscr{F}_{\aleph_0+|\alpha|}$ by induction on α increasing continuous with α such that $\eta^1_\alpha \in \mathrm{Dom}(h_{\alpha+1}), \eta^2_\alpha \in \mathrm{Rang}(h_{\alpha+1})$.

(2) As in Example 3.19 using ω-power.

5.10. Definition. (1) We say λ is Ord-μ-free <u>when</u>:
for every linear order $M = (\lambda, <^M), \lambda$ for some $B \subseteq \lambda$ in $\mathbf{L}[A, B], M$ can be represented as $\cup\{M_i: i < \mu\}, M_i$ embeddable into $({}^n\lambda, <_{\mathrm{even}})$ where $\eta <_{\mathrm{even}} \nu \Leftrightarrow (\exists m < n)(m = \ell g(\eta) = \ell g(\nu) \wedge (\eta(m) \neq \nu(m)) \wedge (\eta(m) < \nu(m) \equiv m$ even) (see Laver [Lv71], [Sh:e, XII, §2]).

(2) If $\mu = \aleph_0$ we may omit it.

5.11. Claim. *If λ is* Ord-*free* *then any two strongly $\aleph_0$-homogeneous linear orders (see below) of cardinality λ of the same cofinality are isomorphic.*

5.12. Definition. I is a strongly $\aleph_0$-homogeneous if I is infinite dense isomorphic to any open interval and its interval.

Proof. See above.

§6. Comments on model theory in ZF. Before we comment on model theory without choice we write up the amount of absolute which holds.

6.1. *Observation.* Let T be countable complete first order theory, without loss of generality $\mathbb{L}_{\tau(T)} \subseteq \mathscr{H}(\aleph_0)$ (or if you like $\subseteq \omega$), so $T \subseteq \mathscr{H}(\aleph_0)$.

(1) "T is stable" is a Borel relation.

(2) "M is a countable model of T, $q(\bar{y}) \in \mathbf{S}^{<\omega}(M)$, $p(\bar{x}) \in \mathbf{S}^{<\omega}(M)$ and $M_\ell \prec M$ for $\ell = 0, 1, 2$ and for stable T, $M_1 \bigcup\limits_{M_0}^{M} M_2$, p does not fork over M_0, all coded naturally as a subsets of ω" are Borel.

(3) In part (2), "$p \perp q$" is Borel as well as "$p \underset{\mathrm{wk}}{\perp} q$" is Borel, also "$p \perp M_0$" by clause (e) of part (3A).

(3A) Let T^{eq} be T when we add predicates naming the equivalence classes so have a predicate $P_{\varphi(\bar{x},\bar{y})}$ equivalent to every $\varphi(\bar{x}) \in \mathbb{L}(\tau_T)$, ([Sh:c, III]) and $T^{\mathrm{eq}}_\forall$ be the universal part (pedantically the consequences of T^{eq}), so $\tau(T), \mathbb{L}(\tau(T^{\mathrm{eq}})), T^{\mathrm{eq}}, T^{\mathrm{eq}}_\forall$ are Borel definable from T. Also the following are Borel

- (a) A is a model of $T^{\mathrm{eq}}_\forall$ in this observation with universe $\subseteq \omega$ and we use A, A_ℓ to denote such models,
- (b) $A_1 \subseteq A_2$ are models of $T^{\mathrm{eq}}_\forall, A_2 = acl(A_1)$ (in any $M, A_2 \subseteq M \models T^{\mathrm{eq}}$), and computing such A_2 naturally defined,
- (c) $p \in \mathbf{S}^m(A), A$ a model of $T^{\mathrm{eq}}_\forall$; i.e., $\{\mathrm{tp}(\bar{a}, A, M) \colon A \subseteq M \models T^{\mathrm{eq}}, \bar{a} \in {}^r M\}$ we may write acl(A),
- (d) computing $p{\restriction}A_2$ from $A_1 \subseteq A_2$ and $p \in \mathbf{S}^m(A_2)$,
- (e) computing $R^m(p, \Delta, 2), R^m(p, \Delta, \aleph_0), \mathrm{rk}^m(p, \Delta, \aleph_0)$ for p an m-type over A, (a model of $T^{\mathrm{eq}}_\forall$), $\Delta \subseteq \mathbb{L}(\tau_T)$ finite),
- (f) $A_1 \subseteq A_2, p(\bar{x})$ an m-type over A_2 (in clause (c)'s sense) and $p(\bar{x})$ does not fork over A_2,
- (g) $A_1 \subseteq A_2, p(\bar{x})$ an m-type over A_2, the type $p(\bar{x})$ does not fork over A_2 and is stationary over A_1,
- (h) in (g) computing the unique extension $q \in \mathbf{S}^{\ell g(\bar{x})}(A_2)$ of $p(\bar{x})$ not forking over A_1 and $\mathrm{tp}(\bar{a}_0{}^\frown\bar{a}_1{}^\frown\bar{a}_1{}^\frown \dots A_2, M)$ when $A_2 \subseteq M \models T^{\mathrm{eq}}, \bar{a}_n$ realizes $p(\bar{x})$ in M and $p^{\omega}_{A_2} = \mathrm{tp}((\bar{a}_n, A_2 \cup \{\bar{a}_0, \dots, \bar{a}_{n-1}\}, M)$ does not fork over A_1,
- (i) $p_\ell(\bar{x}_\ell) \in \mathbf{S}^{m(\ell)}(A)$ for $\ell = 1, 2$ are weakly orthogonal,
- (j) for $A_\ell \subseteq A$, the stationary types $p_\ell(\bar{x}) \in \mathbf{S}^{m(\ell)}(A_\ell)$ for $\ell = 1, 2$ are orthogonal,

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(k) from $A \subseteq A_\ell, A_2$ such that $\mathrm{tp}(A_\ell, A)$ is stationary for $\ell = 1,2$ and computing $A', (f_\ell, A'_\ell)$ for $\ell = 1,2$ such that $A \subseteq A'$, f_ℓ an isomorphism from A_ℓ onto A'_ℓ over A, $A'_\ell \subseteq A$ for $\ell = 1,2$ and $A'_1 \mathop{\bigcup\!\!\!\!\bigcup}_{A}^{A'} A'_2$,

(ℓ) $A_1 \subseteq A_2, A \subseteq A_2$ and $p(\bar{x}) \in \mathbf{S}^m(A)$ is orthogonal to A_1.

(4) "T has DOP" is a Σ^1_1-relation (so NDOP is Π^1_2).

(5) "T has DIDIP" is Σ^1_1 (so NDIDIP is Π^1_1).

(6) "T has OTOP" in Σ^1_1.

Proof. Sometimes we give equivalent formulations to prove.

(1), (2) are obvious; for "does not fork" see clause (f) of part (3A).

(3) (a) $p \underset{\mathrm{wq}}{\perp} q$ just says: for some $A \subseteq M, p, q \in \mathbf{S}^{<\omega}(A, M)$ (or even $p, q \in \mathbf{S}^{\leq\omega}(A, M)$) satisfying: if $\varphi(\bar{x}, \bar{y}, \bar{z}) \in \mathbb{L}(\tau_T)$ and $\bar{a}$ from A we have $p(\bar{x}) \cup q(\bar{y}) \vdash \varphi(\bar{x}, \bar{y}, \bar{a})$ or $p(\bar{x}) \cup p(\bar{y}) \vdash \neg\varphi(\bar{x}, \bar{y}, \bar{a})$ and remember compactness,

(b) $p \perp q$, see clause (j) of part (3A),

(c) $p \perp M_0$ by clause (ℓ) of part (3A).

(3A) E.g.,

Clause (e): Because Δ is finite, the value is a natural number and for $\theta \leq \aleph_0, k < \omega$ we have $R^m(p(\bar{x}), \Delta, \theta) \geq k$ iff some Borel set of formulas, see [Sh:c, II, §2] is consistent. Simiarly for $(R^m(p(\bar{x}), \Delta, \theta) > k_1) \vee (R^m(p(\bar{x}), \Delta, \theta) = k_1) \wedge \mathrm{Mlt}^m(p(\bar{x}), \Delta, \theta) \geq k_2)$.

Clause (f): This is equivalent to "if $A_3 = acl(A_2), \Delta \subseteq \mathbb{L}(\tau_{T^{\mathrm{eq}}})$ is finite then there is $q \in \mathbf{S}^m_\Delta(A_2)$, i.e., a definition of such type which extend $p \restriction \Delta$ and is definable over $\mathrm{acl}(A_1)$.

Clause (g): We can use the definition: for every finite Δ there is $q \in \mathbf{S}^m_\Delta(A_3)$ definable over $acl(A_1, A_3)$ such that $R^m(p(\bar{x}), \Delta, 2) = R^m(p(\bar{x}) \cup q(\bar{x}), \Delta, 2)$.

Clause (j): This is equivalent to: $p_\ell \in \mathbf{S}^{<\omega}(A_\ell)$ for $\ell = 1,2, A_\ell \subseteq M$, and for every $n < \omega$ and finite $\Delta_1 \subseteq \mathbb{L}(\tau_T)$ for some finite $\Delta_2 \subseteq \mathbb{L}(\tau_T)$, if $\langle a^\ell_0, \dots, a^\ell_{n-1}\rangle$ is as in clause (h) with $(A_\ell, A_1 \cup A_2, p_\ell)$ here standing for (A_1, A_2, p) but we have finitely many possibilities for each, then $\mathrm{tp}_{\Delta_2}(\bar{a}^1_0 {}^\frown \dots {}^\frown a^1_{n-1}, A_1 \cup A_2, M)$, $\mathrm{tp}_{\Delta_1}(\bar{a}^2_0 {}^\frown \dots {}^\frown \bar{a}^2_{n-1}, A_1 \cup A_2, M)$ determine the A_1-type of $\langle \bar{a}^1_0 {}^\frown \dots {}^\frown \bar{a}^1_{n-1} {}^\frown \bar{a}^2_0 {}^\frown \dots {}^\frown \bar{a}^2_{n-1}\rangle$ over $A_1 \cup A_2$ in M.

Clause (ℓ): First assume A_1, A_2, A are algebraically closed. We know that $p \perp A_1$ iff there are f, M such that $A_2 \subseteq M \models T^{\mathrm{eq}}$, $f \supseteq \mathrm{id}_A(M, M)$-elementary mapping (i.e., an automorphism of M) and mapping A_2 to $A'_2, A_1 \mathop{\bigcup\!\!\!\!\bigcup}_{A}^{M} A'_2$ such that $p \perp f(p)$. In the general case as in clause (j) work with "for every finite $\Delta_1 \dots$ ".

(4) Obvious by (3) and by the definition (there are countable models of $T, M_\ell (\ell \leq 3)$ such that $M_1 \mathop{\bigcup\!\!\!\!\bigcup}_{M_1}^{M_3} M_2, M_3$ is $\mathbf{F}^\ell_{\aleph_0}$-constructible over $M_1 \cup M_2$ and $p \in \mathbf{S}^{<\omega}(M_3)$ non-algebraic such that $p \perp M_1, p \perp M_2$).

(5) Obvious by (3) and the definition (equivalent to: there are countable models M_n of $T, M_n \prec M_{n+1}$, and countable N which is $\mathbf{F}^\ell_{\aleph_0}$-atomic over $\cup\{M_n : n < \omega\}$ and non-algebraic $p \in \mathbf{S}^{<\omega}(N)$ such that $n < \omega \Rightarrow p_n \perp M_n$). $\square_{6.1}$

6.2. Claim.

(0) Convention.

(a) *T's vocabulary, $\tau = \tau_T$ is well orderable and for simplicity $\subseteq \mathbf{L}$,*

(b) *M, N denote models of T with universe a set of ordinals,*

(c) *T a theory in $\mathbb{L}(\tau)$ so $|T|$ is a cardinal; without loss of generality $\tau \subseteq \mathbf{L}_\lambda$, $\lambda = |T| + \aleph_0$,*

(d) *"a model of T" means one with well ordered universe so without loss of generality a set of ordinals.*

(1) *DLST and ULST holds (for models as in clause* (b)*), short for the downward Löwenheim–Skolem–Tarski and the upward Löwenheim–Skolem–Tarski theorems respectively. If T is categorical in $\lambda \geq |T|$ then $T \cup \{\exists^{\geq n} x(x = x) \colon n < \omega\}$ is complete, etc., all that takes place in some $\mathbf{L}[Y]$ is fine.*

(2) [T complete] *T has an $\aleph_0$-saturated model iff every model of T has an $\aleph_0$-saturated elementary extension iff $D(T)$ can be well ordered.*

(3) *Define $\kappa(T) =: \sup\{\kappa(T)^{\mathbf{L}[T,Y]} \colon Y$ a set of ordinals$\}$ but probably better to use $\kappa^+(T) = \cup\{(\kappa(T)^+)^{\mathbf{L}(T,Y)} \colon Y$ a set of ordinals$\}$.*

(4) *Assume T is complete. Every model M of T of cardinality $\leq \lambda$ has a κ-saturated elementary extension of cardinality $\leq \lambda$ iff $|D(T)| \leq \lambda$ and* (a) $\vee$ (b) *where*

(a) *$|{}^{\kappa >}\lambda| = \lambda$, i.e., $[\lambda]^{<\kappa}$ is well ordered*

(b) *$T = \mathrm{Th}(M)$ is stable, $|\lambda^{<\kappa(T)}| = \lambda$ and $|\mathscr{P}(\omega)|$ is a cardinal $\leq \lambda$ if some $p \in \mathbf{S}(B)$, $B \subseteq M \models T$, M well orderable, $|B| < \kappa(T)$ has a perfect set of stationarization and $\lambda > \aleph_0$.*

[*Why? As in* [Sh:c, III], *particularly section* 5, *hopefully see the proof of* [Sh:F701, 1.1].]

(5) [T complete]

(a) *if $\neg(|D(T)| \leq |T|)$ <u>then</u> there is a family $\mathscr{P}$ of subsets of $D(T)$, each of cardinality $\leq |T|$ and $\cup\{\mathbf{P} \colon \mathbf{P} \in \mathscr{P}\} = D(T)$ and for each $\mathbf{P} \in \mathscr{P}$ there is a Φ proper for linear orders, with $\tau(T), \tau(\Phi) \subseteq \mathbf{L}$ such that every model $EM_{\tau(T)}(I, \Phi)$ satisfies: the model realizes $p \in D(T)$ iff $p \in \mathbf{P}$.*

(6) *Assume there is no set of $\aleph_1$ reals. If T is complete countable, $D(T)$ uncountable, $M \models T$ and $\mathbf{P}_M = \{p \in D(T) \colon M$ realized $p\}$ <u>then</u> $\mathbf{P}_M$ is countable.*

(6A) *Of course, it is possible that $|D(T)|^{\mathbf{L}[T,Y]}$ is large in $\mathbf{L}[T, Y]$! (e.g., there is a set of $|T|$ independent formulas see* 12)(c)).

(7) *If T is complete not superstable, $T_1 \supseteq T$ complete, $\lambda = \mathrm{cf}(\lambda) > |T_1|, \lambda \geq \theta(\mathscr{P}(\omega))$ and axiom Ax^3_λ (see* [Sh:835], *i.e., $|[\lambda]^{\aleph_0}|$ a cardinal) <u>then</u> there is $\langle M_u \colon u \subseteq \lambda\rangle$ such that*

(a) *$M_u \in \mathrm{PC}(T_1, T)$,*

(b) *$\|M_u\| = \lambda$,*

(c) *$u \neq v \subseteq \lambda \Rightarrow M_u \not\cong M_v$.*

[*Why? There is a sequence $\langle C_\delta \colon \delta \in S\rangle, S \subseteq S^\lambda_{\aleph_0}$ stationary $C_\delta \subseteq \delta = \sup(C_\delta)$, $otp(C_\delta) = 0$ (hence we can partition S to λ stationary sets)*].

(8) *Define $\beth'_\alpha(\lambda)$ by $\beth'_0(\lambda) = \lambda, \beth'_{\alpha+1}(\lambda) = \theta(\mathscr{P}(\beth'_\alpha(\lambda)), \beth'_\delta(\lambda) = \cup\{\beth'_\alpha(\lambda) \colon \alpha < \delta\}$, it is a cardinal. If T is countable, Γ is a countable set of $\mathbf{L}(\tau_T)$-types and for every $\alpha < \omega_1$ there is $M \in \mathrm{EC}_{\beth'_\alpha}(T, \Gamma)$ so $|M| \subseteq \mathbf{L}$ of power $\beth'_\alpha$ or just of power $\geq \beth^{\mathbf{L}[M]}_\alpha$ (but we do not say that an ω_1-sequence of such models exists!), <u>then</u> there is an $\Phi \in \Upsilon^{or}_{\aleph_0}[T]$ so $|\tau_\Phi| = \aleph_0$ such that $\mathrm{EM}_{\tau(T)}(I, \Phi) \in \mathrm{EC}(T, \Gamma)$ for every linear order I.*

[*Why*? *See proof of* (9), *but here the members of the tree are finite set of formulas hence the tree is* $\subseteq \mathbf{L}[T]$ *and we can define the rank in* $\mathbf{L}[T]$ *but*: *we let* $\langle \Delta_n : n < \omega \rangle$ *be an increasing sequence of finite sets of formulas, each* $\varphi \in \Delta_n$ *has a set of free variables* $\subseteq \{x_0, \dots, x_{n-1}\}$ *and* $\bigcup_n \Delta_n = \mathbb{L}(\tau_T)$, Δ_n *is closed under change of free variables* (*modulo the restriction above*). *We define* $\mathscr{T}_n$ *as in the proof of part* (9) *by* $p \in \mathscr{T}_n$ *is a complete* (Δ_n, n)-*type. The tree is really* $\subseteq {}^{\omega >}\omega$.]

(9) [DC] *Assume T is an* (*infinite*) *theory with Skolem functions*, Γ *a set of* $\mathbb{L}(\tau_T)$-*types and for every* $\alpha < \theta(\mathscr{P}(|T|))$ *there is* $M \in \mathrm{EC}(T, \Gamma)$ *of power* $\geq \beth_\alpha$ *in* $\mathbf{L}[T, M]$, *then there is* Φ *such that* $EM_{\tau(T)}(I, \Phi) \in EC(T, \Gamma)$ *for every linear order* I.
[*Why*? *A wrong way is to assume* $\theta(\mathscr{P}(|T|))$ *is regular and in stage n we have an n-indiscernible sequence* $\mathbf{I}^n_\alpha \subseteq M$ *of cardinality* $\beth'_\alpha$ *for* $\alpha < \theta(\mathscr{P}(|T|))$ *with n-tuple from* $\mathbf{I}^n_\alpha$ *realizing* p_n, *as in the ZFC proof. The problem is that there may be no regular cardinal* $\geq \theta(\mathscr{P}(|T|))$. *But more carefully let* $\mathscr{T}_n$ *be the set of complete types* $p_n(x_0, \dots, x_{n-1})$ *consistent with T, such that it is the type of a sequence of length n which is m-indiscernible for each* $m \leq n$. *The order on* $\mathscr{T} = \cup\{\mathscr{T}_n : n < \omega\}$ *is inclusion, so really* $\mathscr{T}_n$ *is the n-th level. We need* $\mathrm{DC}_{\aleph_0}$ *to have a rank function on this set which has power* $\leq \mathscr{P}(|T|)$. *We prove by induction on the ordinal* γ *for each n, that if* $p \in \mathscr{T}_n$ *has rank* γ *that no indiscernible* $\mathbf{I} \subseteq M$, $M \in EC(T, \Gamma)$ *of cardinality* $\geq \beth^{\mathbf{L}[T]}_{\omega\gamma}$ *exists*.]

(9A) *Of course, if* $EC(T, \Gamma)$ *has a model* M, $|M| \subseteq \mathbf{L}$ *of cardinality* $\geq \beth'_\delta$ *where* $\delta := \theta(\mathscr{P}(|T|))$ *then we do not need* DC.

(9B) *We can avoid "T has Skolem functions", see* [*Sh*:*F*701], *in both parts* (9) *and* (9A). *The point is that T needs not be complete, without loss of generality T has elimination of quantifiers and we can define* T^{SK} *which is* $T +$ *the axioms of Skolem functions*; *now for every* $\alpha < \theta(\mathscr{P}(|T|)$, *there is a model M of T of cardinality* $\geq \beth'_\alpha$ *or just* $\geq \beth^{\mathbf{L}[M]}_\alpha$, *it can be expanded to* M^+, *a model of* T^{SK} *and we can continue* (*with new function symbols*).

(10) *Assume T is complete uncountable. Then all the proofs in* §2 + §3 *holds except that we do not have the dichotomy OTOP/existence of primes over stable amalgamation. We intend to return to it in* [*Sh*:*F*701].

(11) *If* $p(x_0, \dots, x_{n-1})$ *is a set of* $\mathbb{L}(\tau_T)$-*formulas consistent with T then it is realized in some model M in some universe* $\mathbf{L}[T, Y]$ *hence can be extended to a complete type realized in such M, p hence* $\in D_n(T)$ *when T is complete.*
[*Why*? *Work in* $\mathbf{L}[T, p]$, *O.K. as* $p \subseteq \mathbf{L}$ *as* $T \subseteq L$.]

6.3. Lemma. [Sh:c] *can be done in* ZF + $(\forall\alpha)([\alpha]^{\aleph_0}$ *is well ordered*), *see* [Sh:835] *as long as*

(a) *the theory T is in a vocabulary which can be well ordered,*
(b) *we deal only with models whose power is a cardinal,*
(c) *all notions are in* $\mathbf{L}[T, Y]$, $Y \subseteq$ Ord *large enough* (*so* $\mathfrak{C}$ *is not constant it depends on the universe*),
(d) *in* [Sh:c, VIII], *the case* $\lambda > |T_1|$ *regular* ($\tau(T_1)$ *well orderable, too*) *is clear as using the well ordering* $[\lambda]^{\aleph_0}$ *we can find* $\langle C_\delta : \delta \in S^\lambda_{\aleph_0} \rangle \in \mathbf{L}[T, Y]$ *hence define a*

partition $\langle S_\alpha : \alpha < \lambda \rangle$ of $S^\lambda_{\aleph_0}$ such that $(\exists^\lambda \alpha)$ $(S_\alpha$ stationary (in $\mathbf{V}$), so increasing Y we are there but

(e) *Ch VI on ultrapower should be considered separately.*

§7. Powers which are not cardinals. We suggest to look at categoricity of countable theories in so-called reasonable cardinals. For them we have the completeness theorem in 7.7. We then uncharacteristically examine a classical example: Ehrenfeucht example (with 3 models in $\aleph_0$, see 7.10).

We naturally ask

QUESTION. Can an expansion of the theory of linear orders be categorical in some uncountable power?

We then deal with a criterion, i.e., sufficient conditions for categoricity. We intend to continue this in [Sh:F701].

7.1. CONVENTION. T not necessarily $\subseteq \mathbf{L}$.

We may consider

7.2. DEFINITION. (1) For a class $\mathbf{C}$ of powers we say $T_1 \leq^{\mathrm{ex}}_{\mathbf{C}} T_2$ when: for every set X of power $\in \mathbf{C}$ if T_2 has a model with universe X then T_1 has a model with universe X.

(2) For a class $\mathbf{C}$ of powers we say $T_1 \leq^{\mathrm{cat}}_{\mathbf{C}} T_2$ when: for every set X of power $\in \mathbf{C}$ if T_2 is categorical in $|X|$, (i.e., has one and only one model with universe X up to isomorphism) then T_1 is categorical in $|X|$.

(3) In both cases, if $\mathbf{C}$ is the class of all powers $\geq |T_2|$ we may omit it.

7.3. *Observation.* $\leq^{\mathrm{ex}}_{\mathbf{C}}, \leq^{\mathrm{cat}}_{\mathbf{C}}$ are partial orders.

We may also consider

7.4. QUESTION. (1) For which countable theories T is there a forcing extension $\mathbf{V}^{\mathbb{P}}$ of $\mathbf{V}$, model of ZF such that in $\mathbf{V}^{\mathbb{P}}$ the theory T is categorical in some uncountable power?

(2) As in (1) for reasonable powers, see below.

7.5. DEFINITION. We say that X is a set of reasonable power (or $|X|$ is a reasonable power) when:

(a) there is a linear order of X,

(b) $|X| = |X \times X|$.

7.6. CLAIM. *If T is countable theory and X a set of reasonable power then T has a model with universe X.*

PROOF. By 7.7. $\square_{7.6}$

7.7. CLAIM. [ZF]

(1) *For some first order sentence ψ we have: for a set X the following are equivalent:*

(a) *X is a set of reasonable power,*

(b) *if T is a countable theory then T has a model with universe X,*

(c) *ψ has a model with universe ψ.*

(2) *If T is categorical in $|X|$, a reasonable power then $T \cup \{(\exists^{\geq n} x)(x = x) : n < \omega\}$ is a complete theory.*

PROOF. (1) (b) $\Rightarrow$ (a).

First apply clause (b) to T_1 = (the theory of dense linear order with neither first nor last elements), or just $T_1' = \{\psi_1\} \subseteq T_1$, where $\psi_1 \vdash T_1$ so it has a model $M = (X, <^M)$, so $<^M$ linearly ordered X.

Second, apply clause (b) to $T_2 = \mathrm{Th}(\omega, F)$, F a one-to-one function from $\omega \times \omega$ onto ω, or just $T_2' = \{\psi_2\} \in T_2$ expresses this so there is a model $M = (X, F^M)$ of T, so F^M exemplifies $|X| = |X \times X|$.

Note that we have used (b) only for theories consisting of one sentence.

(a) $\rightarrow$ (b).

Use Ehrenfeucht–Mostowski models.

That is it is enough to prove: using $I = (X, <)$ a linear order

$\boxplus$ if T' is a countable complete theory with Skolem functions, every term $\sigma(x_0, \dots, x_{n-1})$ is (by T') equal to a function symbol, $M' \models T$ and $\langle a_n : n < \omega\rangle$ is an indiscernible sequence in M', $p_n = \mathrm{tp}_{\mathrm{qf}}(\langle a_0, \dots, a_{n+1}\rangle, \emptyset, M)$ for $n < \omega$ <u>then</u> we can find $M, \langle a_t : t \in I\rangle$ such that

$\circledast$ (a) M is a model of T',

(b) M has universe X,

(c) $\langle a_t : t \in I\rangle$ is an indiscernible sequence in M,

(d) $\langle a_{t_0}, \dots, a_{t_{n-1}}\rangle$ realizes p_n in M when $t_0 <_I \cdots <_I t_{n-1}$.

Let $<^*$ be a well order $\tau(T)$.

Let $\langle (k_n, F_n) : n < \alpha \le \omega\rangle$ list with no repetition the pairs (k, F) satisfying $(*)_{k,F}$ such that $k_0 = 1, M \models \forall x[F_0(x) = x]$ where

$(*)_{k,F}$ (a) $F \in \tau(T)$ is a k-place function symbol,

(b) there is no $u \subset \{0, \dots, k-1\}$ such that $F^{M'}(a_0, a_1, \dots, a_{k-1}) \in \mathrm{Sk}_M(\{a_\ell : \ell \in u\})$,

(c) there is no k-place function symbol $F_1 \in \tau(T')$ such that $F_1 <^* F$ and $F_1^{M'}(a_0, \dots, a_{k_1}) = F^{M'}(a_0, \dots, a_{k-1})$.

Let $Y = \bigcup_{n<\omega} Y_n$ where $Y_n = \{(n, t_0, \dots, t_{k_n - 1}) : n < \alpha \text{ and } t_0 <_I \cdots < t_{k_{n-1}}\}$.

Let $g : X \times X \rightarrow X$ be one to one onto.

Clearly there is a model as required with universe Y, hence it is enough to prove $|Y| = |X|$. Clearly $|X| \le |Y|$ as $\{(0, t) : t \in I\} \subseteq Y$. Also $|Y_n| = |X|^{k_n}$ which is 1 if $k_n = 0$ and is $|X|$ if $k_n \ge 1$ as we can prove by induction on n. Moreover, we can choose $\langle f_n : n < \alpha, k_n \ge 1\rangle$ such that f_n is one-to-one from Y_n onto X as f_n is gotten by composition $k_n - 1$ times of g. This leads to $|Y| \le |X \times \omega| + |\omega|$. But trivially $\aleph_0 \le |X|$ by g hence $|X| \le |Y| \le |X| \times |X| + \aleph_0 = |X|$ hence we are done proving (b) $\Rightarrow$ (a).

Let ψ say "$<$ is a linear order and $F(x, y)$ is a one-to-one function onto".

Now

(c) $\Rightarrow$ (a): as in the proof of (b) $\Rightarrow$ (a)

and also

(b) $\Rightarrow$ (c): should be clear.

(2) Easy, too. $\square_{7.7}$

7.8. DISCUSSION. We can use an $\aleph_0$-saturated model M of T as a set of urelements, i.e., we use a Fraenkel–Mostowski model for the triple (M', a copy of M; finite support; finite partial automorphism of M). Is T categorical in $|M'|$? The problem is that maybe some $\psi \in \mathbb{L}_{(2^{\aleph_0})^+,\omega}$ define in M' with finitely many parameters, a model M'' of T with universe $|M'|$ such that there is no permutation f of $|M'|$ definable similarly such that f is an isomorphism from M' onto M''. But we may consider $(D, \aleph_0)$-homogeneous models of some extension of T (in bigger vocabulary). This seems related to [Sh:199], [Sh:750].

7.9. Definition. T_1 is the theory of dense linear order with neither first nor last element and $c_n < c_{n+1}$ for $n < \omega$ (so $\tau(T_1) = \{<\} \cup \{c_n : n < \omega\}$).

Remark. (1) This is the Ehrenfeucht example for $\dot{I}(\aleph_0, T) = 3$.
(2) We can replace T_2 by $T_{i,n}$ with 3 below replaced by $3 + n$.

7.10. Claim. [ZF]

(1) *T_1 is a complete countable first order which is not categorical in any infinite power.*
(2) *In fact if T_1 has a model with universe X <u>then</u> T_1 has at least three non-isomorphic models with this universe.*
(3) *If in (2) the set X is uncountable (i.e., $|X| \neq |\omega|$) and moreover not the countable union of countable sets <u>then</u> T_1 has at least $\aleph_0$ non-isomorphic models with this universe.*

7.11. Question. (1) Consistently (with ZF), in some uncountable power, does Th$(\mathbb{Q}, <)$ has exactly 3 models.

Proof. (1) Follows by (2).
(2) Let X be a set. For $\ell = 0, 1, 2, 3$ let

$\ell = \{N :$ (a) N is a model of T_1 with universe X:
(b) if $\ell = 1$ then N omit $p(x) = \{c_n < x : n < \omega\}$;
(c) if $\ell = 2$ some $a \in N$ realizes $p(x)$ but no $a \in N$ is the first such element;
(d) if $\ell = 3$ some element $a \in N$ realizes $p(x)$ and is the first such element$\}$.

Clearly

⊞ (a) K_0 is the class of models of T_1 with universe X,
(b) K_0 is the disjoint union of K_1, K_2, K_3.

By $(*)_1, (*)_2, (*)_3$ below the result follows:

$(*)_1$ if $K_2 \neq \emptyset$ then $K_3 \neq \emptyset$.
[Why? Let $M \in K_3$ and we define a $\tau(T_1)$-model N as follows:
(i) the universe of N is $X = |M|$,
(ii) $c_n^N = c_{n+1}^M$ for $n < \omega$,
(iii) $N \models a < b$ <u>iff</u> $M \models$ "$a < b \wedge a \neq c_0 \wedge b \neq c_0$ or $a = c_0^M \wedge$ (b realizes $p(x)$ in M) or $b = c_0^N \wedge a \neq c_0 \wedge \bigvee_{m<\omega} b < c_m$".
Now check that $N \in K_3$ with c_0^M being the $<^N$-first member of X realizing $p(x)$.]

$(*)_2$ if $K_3 \neq \emptyset$ then $K_1 \neq \emptyset$,
[Why? Let $M \in K_3$ and $c \in M$ realizes $p(x)$ be the first such element. We define a $\tau(T_1)$-model N by
(i) the universe of N is $X = |M|$,
(ii) $c_n^N = c_n^M$,
(iii) $N \models a < b$ iff $M \models$ "$a < b < c$" or $M \models$ "$c < b < a$" or $M \models$ "$c \leq a \wedge b < c$" or $M \models$ "$b = c \wedge c < a$".
Now check that $N \in K_1$.]

$(*)_3$ if $K_1 \neq \emptyset$ then $K_2 \neq \emptyset$,
[Why? Let $M \in K_1$, let $Y = \{a \in X : M \models$ "$c_{2n+1} \leq a < c_{2n+2}$" for some $n < \omega\}$ and we define a $\tau(T_1)$-model N
(i) the universe of N is $X = |M|$,
(ii) $c_n^N \equiv c_{2n}^M$ for $n < \omega$,
(iii) $N \models a < b$ iff $M \models$ "$a < b \wedge (a \notin Y) \wedge (b \notin Y)$" or $M \models$ "$a < b \wedge a \in Y \wedge b \in Y$" or $(b \in Y) \wedge (a \notin Y)$.
Now check that $N \in K_2$.]

(3) Let $M \in K_1$ have universe X and stipulate $c_{-1} = -\infty$, $X_n = \{a : M \models c_{n-1} < a \leq c_n\}$ for $n < \omega$ so $\langle X_n : n < \omega \rangle$ is a partition of X.
Let $S_* = \{n < \omega : X_n \text{ is uncountable}\}$.
Case 1: S_* is infinite.
For any partition $\langle S_n : n < \omega \rangle$ of ω to infinite sets we can define $N \in K_1$ with universe X such that $\{c_n^N : n < \omega\} = \{c_n^M : n \in S_0\}$, on this set $<^M, <^N$ agree, and the set $\{n < \omega : (c_n, c_{n+1})_N \text{ is uncountable}\}$ is any infinite co-infinite set.
Case 2: S_* is finite.
We can find $N \in K_0$ with universe X such that $\max\{n : (c_n, c_{n+1})_N \text{ is uncountable}\}$ is any natural number. $\square_{7.10}$

7.12. Definition. (1) Let N be $(\mathbb{L}_{\infty,\kappa}, \lambda)$-interpretable in M means (without loss of generality τ_N consist of predicates only): there is $\bar{d} \in {}^{\lambda>}M$ and sequence $\langle \varphi_R(\bar{x}_R, \bar{d}) : R \in \tau_N \rangle$, including R being equality such that

$$\varphi_R(\bar{x}_R, \bar{y}) \in \mathbb{L}_{\infty,\kappa},$$
$$\ell g(\bar{x}_R) = \text{arity}(R),$$
$$|N| = \{a \in M : M \models \varphi_{=}(a, a, \bar{d})\},$$
$$R^N = \{\bar{a} \in {}^{\ell g(x_R)}|M| : M \models \varphi_R(\bar{a}, \bar{d})\}.$$

(2) We add "fully" if $\varphi_{=}(\bar{x}_R) = (x_0 = x_1)$ for R being the equality.

7.13. Claim. (1) *To prove the consistency of "a first order complete T is categorical in some power $\neq \aleph_0$" it is enough*
$(*)$ *find a model N of T and $\kappa > \aleph_0$ satisfying: if M is a model of T fully $\mathbb{L}_{\infty,\kappa}(\tau_M)$-interpretable in N then $M \cong N$; moreover there is a function which is $\mathbb{L}_{\infty,\kappa}(\tau_M)$-definable in N (with $< \kappa$ parameters) and is an isomorphism from N onto M.*

(2) *We can replace $\mathbb{L}_{\infty,\kappa}(\tau_N)$ by: there is a set $\mathscr{F}$ such that*
(a) $\mathscr{F} \subseteq \{f : f$ *a partial automorphism of N with domain of cardinality* $< \kappa\}$,
(b) $(\forall A \subseteq N)(|A| < \kappa \Rightarrow (\exists f \in \mathscr{F})(A \subseteq \text{Dom}(f))$,

(c) $\mathscr{F}$ *closed under inverse and composition,*
(d) *if* $f \in \mathscr{F}, A \in N$ *then* $(\exists g \in \mathscr{F})(f \subseteq g \cap a \in \mathrm{Dom}(g))$.

PROOF. Straight.

Remark. So this categoricity does not imply "not complicated". $\square_{7.13}$

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