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Random graphs in the monadic theory of order

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Abstract. We continue the works of Gurevich-Shelah and Lifsches-Shelah by showing that it is consistent with ZFC that the first-order theory of random graphs is not interpretable in the monadic theory of all chains. It is provable from ZFC that the theory of random graphs is not interpretable in the monadic second order theory of short chains (hence, in the monadic theory of the real line).

0. Introduction

We are interested in the monadic theory of order – the collection of monadic sentences that are satisfied by every chain (= linearly ordered set). The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language L can be described as the augmentation of L by a list of quantifiable set variables and by new atomic formulas $t \in X$ where t is a first order term and X is a set variable.

It is known that the monadic theory of order and the monadic theory of the real line are at least as complicated as second order logic ([GuSh2], [Sh1]). The question that we are dealing with in this paper is related to the expressive power of this theory: what can be interpreted in it?

In our notion of (semantic) interpretation, interpreting a theory T in the monadic theory of order is defining models of T in chains. Some problems about the interpretability power of the monadic theory of order, which is

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a stronger criterion for complicatedness, have been raised and answered. For example, second order logic was shown to be even interpretable in the monadic theory of order ([GuSh3]) but this was done by using a weaker, non-standard form of interpretation: into a Boolean valued model.

Using standard interpretation ([GMS]) it was shown that it is consistent that the second–order theory of ω_2 is interpretable in the monadic theory of ω_2 (hence in the monadic theory of well orders). On the other hand, by [GuSh], Peano arithmetic is not interpretable in the monadic theory of short chains, (chains that do not embed $(\omega_1, <)$ and $(\omega_1, >)$) and in particular in the monadic theory of the real line. In [LiSh] we filled the gap left by the previous results and showed that it is not provable from ZFC that Peano arithmetic is interpretable in the monadic theory of order.

Here we replace Peano arithmetic by a much simpler theory – the theory of random graphs, and obtain the same results by proving:

Theorem. *There is a forcing notion P such that in V^P, the theory of random graphs is not interpretable in the monadic second-order theory of chains.*

In fact we show that the model V^P in which Peano arithmetic is not interpreted is a model in which the theory of random graphs is not interpreted (an exact formulation of the non-interpretability theorem is given in section 2).

The proof is similar in its structure to the proof in [LiSh]: we start by defining, following [Sh], our basic objects of manipulation - partial theories. Next, we present the notion of interpretation and the main theorem. We show in $Sect.$ 3 that an interpretation in a chain C 'concentrates' on an initial segment $D \subseteq C$ called a major segment. One of the main differences from [GuSh] and [LiSh] is that the notion of a major segment is not as sharp as there; this results in the need to apply more complicated combinatorial arguments.

The most widely used idea in the proof is applying the operation of shuffling subsets $X, Y \subseteq C$: given a partition of C, $\langle S_j : j \in J \rangle$ and a subset $a \subseteq J$, the shuffling of X and Y with respect to J and a is the set: $\bigcup_{j\in a}(X \cap S_j) \cup \bigcup_{j \notin a}(Y \cap S_j)$. One of the main results in [LiSh] was to show that this operation preserves partial theories; this is stated and used here as well.

To prove the main theorem we try to derive a contradiction from the existence of an interpretation in a chain $(C, <) \in V^P$. We start by making two special assumptions: that C itself is the minimal major initial segment, and that C is an uncountable regular cardinal. The spirit of the proof and main tools are similar to [LiSh], but some of the techniques have to be more tortuous. The proof in this case contains all the main ingredients and disposing of the special assumptions is essentially a formality.

Although we use many definitions and techniques from [Sh], [GuSh] and [LiSh] we have tried to make this paper as self contained as possible. The only main proof we have omitted is that of the theorem on preservation of partial theories under shufflings, as its proof is quite long and involves ideas that are not directly related to this paper.

1. Composition and preservation of partial theories

In this section we define formally the monadic theory of a chain and our main objects of interest: its finite approximations (partial theories). We state the useful properties of partial theories, namely the composition theorem and the theorem about preservation under shuffling.

The monadic theory of a chain is defined to be the first order theory of its power set.

Definition 1.1. Let $(C, <)$ be a chain. The *monadic second-order theory of* C is the first-order theory of the model

$$C^{\text{mon}} = (\mathcal{P}(C)\,;\ \subseteq,\ <^*,\ \text{EM},\ \text{SING})$$

where $\mathcal{P}(C)$ is the power set of C, $<$ and $\subseteq$ are binary relations, SING and EM are unary relations and:

(i) $C^{\text{mon}} \models \text{SING}(X)$ iff X is a singleton,

(ii) $C^{\text{mon}} \models X <^* Y$ iff $X = \{x\}$, $Y = \{y\}$ (where $x, y \in C$) and $C \models x < y$,

(iii) $C^{\text{mon}} \models \text{EM}(X)$ iff $X = \emptyset$,

(iv) $\subseteq$ is interpreted as the usual inclusion relation between subsets of C.

Remark. We denote the first order language above by $L(mon)$. However we will be slightly informal about that and identify it with the monadic version of the first-order language of order, L. Now each $\varphi \in$ L can be translated to a first-order formula $\varphi' \in L(mon)$ by the rules: $(\exists x)\psi(x)$ (individual quantification) will be translated to $(\exists X)[\text{SING}(X) \,\&\, \psi'(X)]$ and $x \in Y$ to $\text{SING}(X) \,\&\, (X \subseteq Y)$. So when we write $C \models \varphi$ (for $\varphi \in$ L) we mean $C^{\text{mon}} \models \varphi'$ and $x < y$ is translated as $X <^* Y$.

Notations 1.2. We denote individual variables by x, y, z and set variables by X, Y, Z. a, b, c are elements and A, B, C are sets. $\bar{a}$ and $\bar{A}$ denote finite sequences having lengths $\lg(\bar{a})$ and $\lg(\bar{A})$. We will write $\bar{a} \in C$ and $\bar{A} \subseteq C$ instead of $\bar{a} \in {}^{\lg(\bar{a})}C$ or $\bar{A} \in {}^{\lg(\bar{A})}\mathcal{P}(C)$, we may also write $a_0 \in \bar{a}$ or $A_0 \in \bar{A}$.

Next is the definition of the partial n-theory of $\bar{A}$ in C

Definition 1.3. Let $(C, <)$ be a chain and $\bar{A} \subseteq C$. We define

$$t = \mathrm{Th}^n(C; \bar{A})$$

by induction on n:
for $n = 0$: $t = \{\varphi(\bar{X}) : \varphi \in L(mon),\ \varphi \text{quantifier free},\ C^{\mathrm{mon}} \models \varphi(\bar{A})\}$
for $n = m + 1$: $t = \{\mathrm{Th}^m(C; \bar{A}{}^\wedge B) : B \subseteq C\}$.

Lemma 1.4. *(A) For every formula $\psi(\bar{X}) \in L$ there is an n such that from $\mathrm{Th}^n(C; \bar{A})$ we can decide effectively whether $C \models \psi(\bar{A})$. We call the minimal such n the depth of ψ and write $\mathrm{dp}(\psi) = n$.*
(B) For every n and l there is a finite set of monadic formulas (effectively computable from n and l) $\Psi(n, l) = \{\psi_m(\bar{X}) : m < m^*,\ \lg(\bar{X}) = l\} \subseteq L$ such that for any chains C, D and $\bar{A} \subseteq C$, $\bar{B} \subseteq D$ of length l the following hold:

(1) $\mathrm{dp}(\psi_m(\bar{X})) \leq n$ for $m < m^*$,
(2) $\mathrm{Th}^n(C; \bar{A})$ can be computed from $\{m < m^* : C \models \psi_m[\bar{A}]\}$,
(3) $\mathrm{Th}^n(C; \bar{A}) = \mathrm{Th}^n(D; \bar{B})$ iff for every $m < m^*$, $C \models \psi_m[\bar{A}] \iff D \models \psi_m[\bar{B}]$.

Proof. In [Sh], Lemma 2.1. □

Definition 1.5. When $\Psi(n, l)$ is as in 1.4(B), for each chain C and $\bar{A} \subseteq C$ of length l we can identify $\mathrm{Th}^n(C; \bar{A})$ with a subset of $\Psi(n, l)$. Denote by $T_{n,l}$ the collection of subsets of $\Psi(n, l)$ that arise as some $\mathrm{Th}^n(C; \bar{A})$ and call it *the set of formally possible (n, l)-theories*.

Remark. For given $n, l \in \mathbb{N}$, each $\mathrm{Th}^n(C; \bar{A})$ is hereditarily finite, (where $\lg(\bar{A}) = l$, C is a chain), and we can effectively compute the set of formally possible theories $T_{n,l}$. (See [Sh], Lemma 2.2).

Definition 1.6. If $(C, <_C)$ and $(D, <_D)$ are chains then $(C + D, <)$ is the chain that is obtained by adding a copy of D after C (where $<$ is naturally defined).

If $(I, <)$ is a chain and $\langle (C_i, <_i) : i \in I\rangle$ is a sequence of chains then $\sum_{i \in I}(C_i, <_i)$ is the chain that is the concatenation of the C_i's along I equipped with the obvious order.

Given $\bar{A} = \langle A_0, \ldots, A_{l-1}\rangle$ and $\bar{B} = \langle B_0, \ldots, B_{l-1}\rangle$ we denote by $\bar{A} \cup \bar{B}$ the sequence $\langle A_0 \cup B_0, \ldots, A_{l-1} \cup B_{l-1}\rangle$. The heavily used composition theorem for chains states that the partial theory of a chain is determined by the partial theories of its convex parts.

Theorem 1.7. *(Composition theorem for chains).*
(1) If C, C', D and D' are chains, $\bar{A} \subseteq C$, $\bar{A}' \subseteq C'$, $\bar{B} \subseteq D$ and $\bar{B}' \subseteq D'$ are of the same length and if

$$\mathrm{Th}^m(C;\bar{A}) = \mathrm{Th}^m(C';\bar{A}')$$

and

$$\mathrm{Th}^m(D;\bar{B}) = \mathrm{Th}^m(D';\bar{B}')$$

then

$$\mathrm{Th}^m(C+D;\bar{A}\cup\bar{B}) = \mathrm{Th}^m(C'+D';\bar{A}'\cup\bar{B}').$$

(2) If I is a chain and $\mathrm{Th}^m(C_i;\bar{A}^i) = \mathrm{Th}^m(D_i;\bar{B}^i)$ for each $i \in I$ (with all sequences of subsets having the same length) then

$$\mathrm{Th}^m\Big(\sum_{i\in I} C_i;\cup_i \bar{A}^i\Big) = \mathrm{Th}^m\Big(\sum_{i\in I} D_i;\cup_i \bar{B}^i\Big).$$

Proof. By [Sh] Theorem 2.4 (where a more general theorem is proved), or directly by induction on m. See also theorem 1.9 below. □

Using the composition theorem we can define a formal operation of addition of partial theories.

Notation 1.8. (1) When $t_1, t_2, t_3 \in T_{m,l}$ for some $m, l \in \mathbb{N}$, then $t_1 + t_2 = t_3$ means: there are chains C and D, and $\bar{A} \subseteq C$, $\bar{B} \subseteq D$ such that

$$t_1 = \mathrm{Th}^m(C;A_0,\dots,A_{l-1}) \text{ \& } t_2 = \mathrm{Th}^m(D;B_0,\dots,B_{l-1}) \text{ \& }$$
$$t_3 = \mathrm{Th}^m(C+D;\bar{A}\cup\bar{B}).$$

(By the composition theorem, the choice of C and D is immaterial).
(2) $\sum_{i\in I}\mathrm{Th}^m(C_i;\bar{A}^i)$ is $\mathrm{Th}^m(\sum_{i\in I} C_i;\cup_{i\in I}\bar{A}^i)$, (assuming $\lg(\bar{A}^i) = \lg(\bar{A}^j)$ for $i, j \in I$).
(3) If D is a sub-chain of C and $\bar{A} \subseteq C$ then $\mathrm{Th}^m(D;\langle A_0\cap D, A_1\cap D,\dots\rangle)$ is abbreviated by $\mathrm{Th}^m(D;\bar{A})$.
(4) For $a < b \in C$ and $\bar{P} \subseteq C$ we denote by $\mathrm{Th}^n(C;\bar{P})\restriction_{[a,b)}$ the theory $\mathrm{Th}^n([a,b);\bar{P}\cap[a,b))$.

We conclude this part by giving a monadic version of the Feferman-Vaught theorem. Note that the composition theorem is a consequence.

Theorem 1.9. *For every $n, l < \omega$ there is $m = m(n,l) < \omega$, effectively computable from n and l, such that if*
(i) *I is a chain,*
(ii) *$\langle C_i : i \in I\rangle$ is a sequence of chains,*
(iii) *for $i \in I$, $\bar{Q}_i \subseteq C_i$ is of length l,*
(iv) *for $t \in T_{n,l}$, $P_t := \{i \in I : \mathrm{Th}^n(C_i;\bar{Q}_i) = t\}$,*
(v) *$\bar{P} := \langle P_t : t \in T_{n,l}\rangle$,*
then $\mathrm{Th}^n(\sum_{i\in I} C_i;\cup\bar{Q}_i)$ is computable from $\mathrm{Th}^m(I;\bar{P})$.

Proof. This is theorem 2.4. in [Sh]. □

Next we define semi–clubs and shufflings and we quote the important preservation theorem.

Definition 1.10. Let $\lambda > \aleph_0$ be a regular cardinal
1) We say that $a \subseteq \lambda$ is a *semi–club subset of* λ if for every $\alpha < \lambda$ with $\mathrm{cf}(\alpha) > \aleph_0$:
if $\alpha \in a$ then there is a club subset of $\alpha,\ C_\alpha$ such that $C_\alpha \subseteq a$ and
if $\alpha \notin a$ then there is a club subset of $\alpha,\ C_\alpha$ such that $C_\alpha \cap a = \emptyset$.
(Note that λ and $\emptyset$ are semi–clubs and that a club $J \subseteq \lambda$ is a semi–club provided that the first and the successor points of J are of cofinality $\leq \aleph_0$. Also, if $a \subseteq \lambda$ is a semi–club then $\lambda \setminus a$ is one as well.)
2) Let $X, Y \subseteq \lambda,\ J = \{\alpha_i : i < \lambda\}$ a club subset of λ, and let $a \subseteq \lambda$ be a semi–club of λ. We will define the *shuffling of* X *and* Y *with respect to* a *and* J, denoted by $[X, Y]^J_a$, as:

$$[X, Y]^J_a = \bigcup_{i \in a} (X \cap [\alpha_i, \alpha_{i+1})) \cup \bigcup_{i \notin a} (Y \cap [\alpha_i, \alpha_{i+1}))$$

3) When $\bar{X}, \bar{Y} \subseteq \lambda$ are of the same length, we define $[\bar{X}, \bar{Y}]^J_a$ naturally.
4) We can naturally define shufflings of subsets of an ordinal δ with respect to a club $J \subseteq \delta$ and a semi–club $a \subseteq \mathrm{otp}(J)$.
6) a-Th$^n(\lambda; \bar{P})$ is Th$^n(\lambda; \bar{P}, a)$ where $a \subseteq \lambda$ is a semi–club.

The next theorem, which will play a crucial role in contradicting the existence of interpretations, states that the result of the shuffling of subsets of the same type is an element with the same partial theory. The proof of the preservation theorem in §4 of [LiSh] requires some amount of computations and uses some auxiliary definitions that are not material in the other parts of the paper. For example, the partial theories WTh$^n(C; \bar{P})$, ATh$^n(\beta,\ (C; \bar{P}))$ and a-WA$^n(C; \bar{P})$ are used in the proof and even the formulation of the theorem but we can avoid defining them by noticing that (for a large enough m) a-Th$^m(C; \bar{P})$ computes all these partial theories. We also avoid the definition of an n-suitable club (which relies on ATh^n). All the details can be found of course in [LiSh].

Theorem 1.11 (preservation theorem). *Let* $\bar{P}_0, \bar{P}_1 \subseteq \lambda$ *be of length* l, $n < \omega$ *and* $a \subseteq \lambda$ *be a semi–club.*
Then there are an $m = m(n, l) < \omega$ *and a club* $J = J(n, \bar{P}_0, \bar{P}_1) \subseteq \lambda$ *such that if* $\bar{X} := [\bar{P}_0, \bar{P}_1]^J_a$ *then*

$$(*)\quad \Big[a\text{-Th}^m(\lambda; \bar{P}_0) = a\text{-Th}^m(\lambda, \bar{P}_1)\Big]$$
$$\Rightarrow \Big[\text{Th}^n(\lambda; \bar{P}_0) = \text{Th}^n(\lambda; \bar{P}_1) = \text{Th}^n(\lambda; \bar{X})\Big].$$

Moreover, there is $t^* = t^*(\bar{P}_0, \bar{P}_1) \in T_{n,l}$ *such that, for every* $\gamma \in J$ *with* $\mathrm{cf}(\gamma) = \aleph_0$,

$$(**) \qquad \left[a\text{-Th}^m(\lambda; \bar{P}_0) = a\text{-Th}^m(\lambda; \bar{P}_1)\right] \Rightarrow$$

$$\left[\text{Th}^n(\lambda; \bar{P}_0) \restriction_{[0,\gamma)} = \text{Th}^n(\lambda; \bar{P}_1) \restriction_{[0,\gamma)} = \text{Th}^n(\lambda; \bar{X}) \restriction_{[0,\gamma)} = t^*\right].$$

Proof. By [LiSh] 4.5, 4.12. □

Definition 1.12. Let $\bar{P}_0, \bar{P}_1 \subseteq \lambda$ be as above. Call a club $J \subseteq \lambda$ *an* n*-suitable club for* $\bar{P}_0$ *and* $\bar{P}_1$ if for every semi–club $a \subseteq \lambda$, $(*)$ and $(**)$ of 1.11 hold.

Fact 1.13. *For every finite sequence* **P**$=\langle \bar{P}_i : \bar{P}_i \subseteq \lambda,\ \lg(\bar{P}_i) = l,\ i < k\rangle$ *and for every* $n < \omega$ *there is a club* $J \subseteq \lambda$ *that is* n*-suitable for every pair from* **P**.

Proof. By [LiSh] 4.3, 4.4. □

2. Random graphs and uniform interpretations

The notion of semantic interpretation of a theory T in a theory T' is not uniform. Usually it means that models of T are defined inside models of T' but the definitions vary with context. In [LiSh] we gave the general definition of the notion of interpretation of one first order theory in another. In our case, in which we deal with interpreting a class of theories, another notion emerges, that of a *uniform interpretation*.

First we define the theory of K-random graphs:

Definition 2.1. Let $1 < K \leq \omega$. An undirected graph $\mathcal{G} = (G, R)$ is a K*-random graph* if

$$\left[A_0, A_1 \subseteq G \;\&\; |A_0|, |A_1| < K \;\&\; A_0 \cap A_1 = \emptyset\right]$$
$$\Rightarrow \left[(\exists x \in G)(\forall a \in A_0)(\forall b \in A_1)[xRa \;\&\; \neg xRb]\right].$$

(When this holds we will say that x *separates* A_0 from A_1).

Definition 2.2. (1) RG_K is the theory of all K-random graphs (that is all the sentences, in the first-order language of graphs, that are satisfied by every K-random graph). RG^i_K is theory of all the infinite graphs that are K-random.
(2) Γ_K is the class of all the K-random graphs, (clearly $K < L \leq \omega \Rightarrow \Gamma_L \subseteq \Gamma_K$). Γ^i_K is the class of infinite K-random graphs.
(3) Γ_{fin} is the class $\{\Gamma_K\}_{1,<K<\omega}$, Γ^i_{fin} is $\{\Gamma^i_K\}_{1,<K<\omega}$.

The next definition is the one used in [LiSh]. It is applicable in dealing with RG_ω, but will have to be modified for dealing with finitely-random graphs.

Definition 2.3. An *interpretation* of a model $\mathcal{G}$ of RG_K in the monadic theory of a chain C is a sequence of formulas in the language L of the monadic theory of order

$$\mathcal{I} = \langle U(\bar{X}, \bar{W}),\ E(\bar{X}, \bar{Y}, \bar{W}),\ R(\bar{X}, \bar{Y}, \bar{W})\rangle$$

where:

1) $U(\bar{X}, \bar{W})$ is the *universe formula* that says which sequences of subsets of C represent elements of $\mathcal{G}$. We denote by C^U the set $\{\bar{X} \subseteq C : C \models U(\bar{X}, \bar{W})\}$.

2) $E(\bar{X}, \bar{Y}, \bar{W})$ is the *equality formula*, an equivalence relation on C^U. We write $\bar{A} \sim \bar{B}$ when $C \models E(\bar{A}, \bar{B}, \bar{W})$.

3) $R(\bar{X}, \bar{Y}, \bar{W})$ is the *interpretation of the graph relation*, a binary relation on C^U which respects $\sim$ i.e. "$C \models R(\bar{A}, \bar{B}, \bar{W})$" depends only on the E-equivalence classes of $\bar{A}$ and $\bar{B}$.

4) $\bar{W} \subseteq C$ is a finite set of parameters allowed in the interpreting formulas.

5) $\langle C^U/\sim,\ R\rangle \cong \mathcal{G}$.

Definition 2.4. Let $\mathcal{I}$ be an interpretation of $\mathcal{G}$ in the monadic theory of a chain C.

The *dimension* of the interpretation, denoted by $d(\mathcal{I})$, is $\lg(\bar{X})$. We will usually assume without loss of generality that $\lg(\bar{W}) = d(\mathcal{I})$ as well.

The *depth* of the interpretation, denoted by $n(\mathcal{I})$, is $\max\{\mathrm{dp}(U), \mathrm{dp}(E), \mathrm{dp}(R)\}$.

Definition 2.5. Let RG^* be one of the theories defined in 2.2(1) and Γ^* be the respective class. We say that *the monadic theory of order interprets RG^* (or Γ^*)* if there is a chain C, a random graph $\mathcal{G} \in \Gamma^*$ and an interpretation $\mathcal{I} = \langle U(\bar{X}, \bar{W}),\ E(\bar{X}, \bar{Y}, \bar{W}),\ R(\bar{X}, \bar{Y}, \bar{W})\rangle$ with $\langle C^U/\sim,\ R\rangle \cong \mathcal{G}$.

Common notions of an interpretation of a theory T_1 in a theory T_2 demand that every model of T_1 is interpretable in a model of T_2 (as in [BaSh]) or that inside every model of T_2 there is a definable model of T_1 (see [TMR]). Here we seem to require the minimum: a single model is interpreted in a single chain. This is often useful, but not always:

Fact 2.6. *For every $1 < K < \omega$ there is a chain C and a sequence $\mathcal{I}$ such that $\mathcal{I}$ is an interpretation of a model of RG^i_K (hence of RG_K) in C. (That is, RG_K, RG^i_K, RG_{fin}, RG^i_{fin} are interpretable in the monadic theory of order).*

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Proof. We shall demonstrate the construction for $K = 2$; the other cases are similar.

Let $C = (\omega, <)$, we will show that there is a one-dimensional interpretation of an infinite model of RG_2 in C without parameters. For that we have to define $U(X)$, $E(X, Y)$ and $R(X, Y)$. Let:

$U(X) := [X = \{a\} \;\&\; a > 1] \vee [X = \{x, a, b\} \;\&\; x \in \{0, 1\} \;\&\; a, b > 1]$;

$E(X, Y) := U(X) \;\&\; U(Y) \;\&\; X = Y$;

$R(X, Y) := U(X) \;\&\; U(Y)$ and either:

$[X = \{a\} \;\&\; Y = \{0, a, b\} \;\&\; a < b]$ or

$[Y = \{a\} \;\&\; X = \{0, a, b\} \;\&\; a < b]$ or

$[X = \{b\} \;\&\; Y = \{1, a, b\} \;\&\; a > b]$ or

$[Y = \{b\} \;\&\; X = \{1, a, b\} \;\&\; a > b]$ or

$[X = \{a\} \;\&\; Y = \{x, c, d\} \;\&\; x \in \{0, 1\} \;\&\; a \notin \{c, d\}]$ or

$[Y = \{a\} \;\&\; X = \{x, c, d\} \;\&\; x \in \{0, 1\} \;\&\; a \notin \{c, d\}]$.

Clearly everything is expressible in L and $R(X, Y)$ defines on $\{X \subseteq \omega : (\omega, <) \models U(X)\}$ a graph relation that is 2-random. □

Motivated by the previous fact we will define now the suitable modification of the previous definitions. The idea is to interpret, in a uniform way, an infinite set of random graphs.

Definition 2.7. A *uniform interpretation* of Γ_{fin} in the monadic theory of order is a sequence

$$\{\langle C_K, \; I, \; \bar{W}_K \rangle : K \in A\}$$

where

1) C_K is a chain,

2) A is an infinite subset of ω,

3) $\bar{W}_K \subseteq C_K$ for $K \in A$,

4) $I = \langle U(\bar{X}, \bar{Z}), E(\bar{X}, \bar{Y}, \bar{Z}), R(\bar{X}, \bar{Y}, \bar{Z}) \rangle$ is a sequence of formulas in L,

5) $\mathcal{I}_K := \langle U(\bar{X}, \bar{W}_K), E(\bar{X}, \bar{Y}, \bar{W}_K), R(\bar{X}, \bar{Y}, \bar{W}_K) \rangle$ is an interpretation of a model of RG_K in C_K for $K \in A$.

Given d and n in $\mathbb{N}$ there is only a finite number of possible interpretations $\mathcal{I}$ having dimension d and depth n. The following is therefore clear:

Proposition 2.8. *The following are equivalent:*

(A) *There is no uniform interpretation of Γ_{fin} in the monadic theory of order.*

(B) *For every $n, d \in \mathbb{N}$ there is $K^* = K^*(n, d) \in \mathbb{N}$ such that if $K \geq K^*$ and $\mathcal{I}$ is an interpretation of some $\mathcal{G} \models RG_K$ in a chain C, then either $d(\mathcal{I}) > d$ or $n(\mathcal{I}) > n$.*

(C) *For every sequence $\mathcal{I} = \langle U(\bar{X}, \bar{Z}), \; E(\bar{X}, \bar{Y}, \bar{Z}), \; R(\bar{X}, \bar{Y}, \bar{Z}) \rangle$ there is $K^* = K^*(\mathcal{I}) < \omega$ such that there are no chain C, $\bar{W} \subseteq C$, $K \geq$*

K^* and $\mathcal{G} \in \Gamma_K$ such that $\langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W}\rangle$ is an interpretation of $\mathcal{G}$ in C. □

Our main theorem has therefore the following form:

Theorem 2.9. *(Non-Interpretability Theorem). There is a forcing notion P such that in V^P the following hold:*
(1) RG_ω is not interpretable in the monadic theory of order.
(2) For every sequence of formulas $\mathcal{I} = \langle U(\bar{X}, \bar{Z}),\ E(\bar{X}, \bar{Y}, \bar{Z}),\ R(\bar{X}, \bar{Y}, \bar{Z})\rangle$ there is $K^* < \omega$, (effectively computable from $\mathcal{I}$), such that for no chain C, $\bar{W} \subseteq C$, and $K \geq K^*$ does $\langle U(\bar{X}, \bar{W}),\ E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})\rangle$ interpret RG_K in C.
(3) The above propositions are provable in ZFC if we restrict ourselves to the class of short chains.

Remark. As an ω-random graph is K-random for every $K < \omega$, an interpretation of RG_ω is a uniform interpretation of Γ_{fin}. Therefore clause (1) in the non-interpretability theorem follows from clause (2).

3. Major and minor segments

From now on we will assume that there exists (in the generic model V^P that is defined later) a uniform interpretation $\mathcal{I}$ of Γ_{fin} in the monadic theory of order. For reaching a contradiction we have to find a large enough $K = K(\mathcal{I}) < \omega$ (a function of the depth and dimension of $\mathcal{I}$) and show that no chain interprets a K-random graph by $\mathcal{I}$. The aim of this (and the next) section is to gather facts that will enable us to compute an appropriate K. The main observation is that an interpretation in a chain C "concentrates" on a segment (called a *major segment*). One of the factors in determining the size of K will be the relation between the major segment and the other, *minor*, segment.

Context 3.1. $\mathcal{I} = \langle U(\bar{X}, \bar{W}),\ E(\bar{X}, \bar{Y}, \bar{W}),\ R(\bar{X}, \bar{Y}, \bar{W})\rangle$ is an interpretation of a K-random graph $\mathcal{G} = (G, R)$ on a chain C. $\bar{W} \subseteq C$ are the parameters, $d = d(\mathcal{I}) = \lg(\bar{X}) = \lg(\bar{W})$ is the dimension of $\mathcal{I}$ and $n = n(\mathcal{I})$ is its depth.

Definition 3.2. $A \subseteq G$ is *big for* (K_1, K_2) if there is $B \subseteq G$ with $|B| \leq K_1$ such that :
$(*)$ for every disjoint pair $A_1, A_2 \subseteq G \setminus B$ with $|A_1 \cup A_2| \leq K_2$ there is some $x \in A \setminus (A_1 \cup A_2)$ that separates A_1 from A_2 i.e. $(\bigwedge_{y \in A_1} xRy) \wedge (\bigwedge_{y \in A_2} \neg xRy)$.
When $(*)$ holds we say that B witnesses the (K_1, K_2)-bigness of A.

Non-bigness is an additive property:

Proposition 3.3. *Let $A \subseteq G$ be big for (K_1, K_2) and suppose that $A = \bigcup_{i<m} A_i$. Then there is an $i < m$ such that A_i is big for $(K_1 + K_2, K_2/m)$.*

Proof. Let $B \subseteq G$ ($|B| \leq K_1$) witness the bigness of A. For $i < m$ we will try to define by induction counter-examples for bigness, that is a set $B_i \subseteq G$ and a function h_i so that:

(1) $|B_i| \leq K_2/m$,

(2) $B_i \subseteq G \setminus (B \cup \bigcup_{j<i} B_j)$,

(3) h_i: $B_i \to \{t, f\}$,

(4) for no $x \in A_i \setminus B_i$ we have $(\forall y \in B_i)[xRy \leftrightarrow h(y) = t]$.

Suppose we succeed. Let $C_1 := \{x : \bigvee_i (x \in B_i \;\&\; h_i(x) = t)\}$ and $C_2 := \{x : \bigvee_i (x \in B_i \;\&\; h_i(x) = f)\}$. But $|C_1 \cup C_2| \leq K_2$, $C_1 \cup C_2 \subseteq G \setminus B$ and of course $C_1 \cap C_2 = \emptyset$ so by the assumption on A there is some $x \in A$ that separates C_1 from C_2. Such an x belongs to some A_i and it separates $C_1 \cap B_i$ from $C_2 \cap B_i$. This contradicts clause (4).

Therefore at some stage $i < m$ we can't define B_i and look at $B^* := B \cup \bigcup_{j<i} B_j$. Now $|B^*| \leq K_1 + i \cdot K_2/m \leq K_1 + K_2$ and "being unable to continue" means: if $B_i \subseteq G \setminus B^*$ and $|B_i| \leq K_2/m$ then for every partition of B_i to B_i^1 and B_i^2 there is some $x \in A_i \setminus B_i$ such that $(\bigwedge_{y \in B_i^1} xRy) \wedge (\bigwedge_{y \in B_i^2} \neg xRy)$. In other words, A_i is big for $(K_1 + K_2, K_2/m)$ (witnessed by B^*) as required. $\square$

Notation 3.4. $\bar{A} \subseteq C$ is called *a representative* if it represents an element of $\mathcal{G}$ i.e. if $C \models U(\bar{A}, \bar{W})$ (of course $\lg(\bar{A}) = d$). The representatives $\bar{A}, \bar{B} \subseteq C$ are called *equivalent* and we write $\bar{A} \sim \bar{B}$ if they represent the same element in $\mathcal{G}$ i.e. if $C \models E(\bar{A}, \bar{B}, \bar{W})$. We use upper case letters such as $\bar{X}, \bar{A}, \bar{U}_i$ to denote representatives. The corresponding lower case letters (x, a, u_i) will denote the elements of $\mathcal{G}$ that are represented by the former. So e.g. $\bar{A} \sim \bar{B} \iff a = b$.

Definition 3.5. 1) A sub-chain $D \subseteq C$ is a *segment* if it is convex (i.e. $x < y < z \;\&\; x,z \in D \Rightarrow y \in D$).

2) A *Dedekind cut* of C is a pair (L, R) where L is an initial segment of C, R is a final segment of C, $L \cap R = \emptyset$ and $L \cup R = C$.

3) Let $\bar{A}, \bar{B} \subseteq C$. We will say that $\bar{A}, \bar{B}$ *coincide on (*resp. *outside)* a segment $D \subseteq C$, if $\bar{A} \cap D = \bar{B} \cap D$ (resp. $\bar{A} \cap (C \setminus D) = \bar{B} \cap (C \setminus D)$).

4) The *bouquet size* of a segment $D \subseteq C$ denoted by $\#(D)$ is the supremum of cardinals $|S|$ where S ranges over collections of nonequivalent representatives coinciding outside D. Thus $\#(D) \geq n$ iff there are nonequivalent representatives $A_1, A_2, \ldots, A_n$ coinciding outside D.

Definition 3.6. Let $D \subseteq C$ be a segment

1) D is i^**-fat* if $\#(D) \geq i^*$

2) D is (K_1, K_2)*-major* if there is a set $\{\bar{U}_i : i < i^*\}$ of representatives coinciding outside D and representing a subset of $\mathcal{G}$ that is big for (K_1, K_2).

3) D is called (K_1, K_2)-*minor* if it not (K_1, K_2)-major.

We denote by M_1 the number $|T_{n,3d}|$ (i.e. the number of possibilities for $\mathrm{Th}^n(C; \bar{X}, \bar{Y}, \bar{Z})$).

Proposition 3.7. *Let (L, R) be a Dedekind cut of C. If L [R] is (K_1, K_2)-major then R [L] is not K_3-fat where $K_3 = M_1(K_1 + K_2) + 1$.*

Proof. Suppose $\langle \bar{A}_i : i < i^L \rangle$ demonstrate that L is (K_1, K_2)-major, i.e. they represent a (K_1, K_2)-big set $\langle a_i : i < i^L \rangle$ in $\mathcal{G}$ and $\bar{A}_i \restriction_R = \bar{A}^*$. Assume towards a contradiction that $\langle \bar{B}_i : i < K_3 \rangle$ demonstrate that R is K_3-fat (i.e. $i < j < K_3 \Rightarrow b_i \neq b_j$ and $\bar{B}_i \restriction_L = \bar{B}^*$). Define an equivalence relation E_L on $\{0, 1, \ldots, i^L - 1\}$ by:

$$iE_Lj \iff \mathrm{Th}^n(L; \bar{A}_i, \bar{B}^*, \bar{W}) = \mathrm{Th}^n(L; \bar{A}_j, \bar{B}^*, \bar{W}).$$

By the definition of M_1, E_L has at most M_1 equivalence classes. By proposition 3.3 there is $a^L \subseteq \{0, 1, \ldots, i^L - 1\}$, an E_L equivalence class, such that $\{\bar{A}_i : i \in a^L\}$ represents a $(K_1 + K_2, K_2/M_1)$-big subset of $\mathcal{G}$. Let $B \subseteq \mathcal{G}$ witness the $(K_1 + K_2, K_2/M_1)$-bigness of $\{a_i : i \in a^L\}$. Since $|B| \leq (K_1 + K_2)$ and $K_3 = M_1(K_1 + K_2 + 1)$ we can choose some $j_1, j_2 < K_3$ with $b_{j_1}, b_{j_2} \notin B$ and with $\mathrm{Th}^n(R; \bar{A}^*, \bar{B}_{j_1}, \bar{W}) = \mathrm{Th}^n(R; \bar{A}^*, \bar{B}_{j_2}, \bar{W})$. Now by the composition theorem 1.7, and the choice of a^L and j_1, j_2 we have for every $i \in a^L$:

$$\begin{aligned}
\mathrm{Th}^n(C; \bar{A}_i, \bar{B}_{j_1}, \bar{W}) &= \mathrm{Th}^n(L; \bar{A}_i, \bar{B}_{j_1}, \bar{W}) + \mathrm{Th}^n(R; \bar{A}_i, \bar{B}_{j_1}, \bar{W}) \\
&= \mathrm{Th}^n(L; \bar{A}_i, \bar{B}^*, \bar{W}) + \mathrm{Th}^n(R; \bar{A}^*, \bar{B}_{j_1}, \bar{W}) \\
&= \mathrm{Th}^n(L; \bar{A}_i, \bar{B}^*, \bar{W}) + \mathrm{Th}^n(R; \bar{A}^*, \bar{B}_{j_2}, \bar{W})\ .
\end{aligned}$$

Therefore for every $i \in a^L$

$$C \models R(\bar{A}_i, \bar{B}_{j_1}, \bar{W}) \iff R(\bar{A}_i, \bar{B}_{j_2}, \bar{W}).$$

Since $b_{j_1}, b_{j_2} \notin B$ we get a contradiction to "A is $(K_1 + K_2, K_2/M_1)$-big as witnessed by B". □

Notation 3.8. Let M_2 be $|T_{n,2d}|$, M_3 be $M_1 + 1$ $(= |T_{n,3d}| + 1)$ and M_4 be such that for every colouring $f\colon [M_4]^3 \to \{0, 1, \ldots, 6\}$ there is a homogeneous subset of $\{0, 1, \ldots, M_4 - 1\}$ of size M_3, where $[M_4]^3$ is $\{\langle i, j, k \rangle :\ i < j < k < M_4\}$. ($M_4$ exists by Ramsey theorem).

The main lemma states that in every Dedekind cut one segment is major. Now we have to make an assumption on the degree of randomness of $\mathcal{G}$.

Lemma 3.9. *Assume $K > (M_3)^2$ (K is from "K-random"). Let (L, R) be a Dedekind cut of C. Then either L or R is (K_1, K_2)-major where $K_1 = K + \frac{K}{(M_2)^2}$ and $K_2 = \frac{K}{(M_2)^2 \cdot M_4}$.*

Proof. $\mathcal{G}$ is $(0, K)$-big and let $\{\bar{U}_i : i < i^*\}$ be a list of representatives for the elements of $\mathcal{G}$. Define a pair of equivalence relations E^0_L and E^0_R on $i^* = |\mathcal{G}|$ by:

$$\Big[iE^0_L j \iff \mathrm{Th}^n(L; \bar{U}_i, \bar{W}) = \mathrm{Th}^n(L; \bar{U}_j, \bar{W})\Big]$$
$$\Big[iE^0_R j \iff \mathrm{Th}^n(R; \bar{U}_i, \bar{W}) = \mathrm{Th}^n(R; \bar{U}_j, \bar{W})\Big].$$

By the definition of M_2 each relation has $\leq M_2$ equivalence classes; therefore by 3.3 there is a subset $A_1 \subseteq i^*$ and pair of theories (t_1, t_2) such that $\{u_i : i \in A_1\}$ is $(K, \frac{K}{(M_2)^2})$-big and

$$i \in A_1 \Rightarrow [\mathrm{Th}^n(L; \bar{U}_i, \bar{W}) = t_1 \;\&\; \mathrm{Th}^n(R; \bar{U}_i, \bar{W}) = t_2].$$

Denote by $\bar{X}\,{}^\wedge\bar{Y}$ the tuple $(\bar{X} \restriction_L) \cup (\bar{Y} \restriction_R)$.

(α) For $i, j \in A_1$ we have $C \models U(\bar{U}_i{}^\wedge\bar{U}_j, \bar{W})$ (hence $\bar{U}_i{}^\wedge\bar{U}_j$ is a representative).

Why? Because $C \models U(\bar{U}_i, \bar{W})$ and by the composition theorem
$\mathrm{Th}^n(C; \bar{U}_i, \bar{W}) = \mathrm{Th}^n(L; \bar{U}_i, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_i, \bar{W}) = t_1 + t_2 =$
$\mathrm{Th}^n(L; \bar{U}_i, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_j, \bar{W}) = \mathrm{Th}^n(C; \bar{U}_i{}^\wedge\bar{U}_j, \bar{W})$.
Define a pair of relations E_L and E_R on $\{\bar{U}_i : i \in A_1\}$ by:

$$\bar{U}_i E_L \bar{U}_j \iff (\exists r \in A_1)[\bar{U}_i{}^\wedge\bar{U}_r \sim \bar{U}_j{}^\wedge\bar{U}_r]$$

$$\bar{U}_i E_R \bar{U}_j \iff (\exists l \in A_1)[\bar{U}_l{}^\wedge\bar{U}_i \sim \bar{U}_l{}^\wedge\bar{U}_j]$$

(β) $\bar{U}_i E_L \bar{U}_j \Rightarrow (\forall r \in A_1)(\bar{U}_i{}^\wedge\bar{U}_r \sim \bar{U}_j{}^\wedge\bar{U}_r)$ $\bar{U}_i E_R \bar{U}_j \Rightarrow (\forall l \in A_1)(\bar{U}_l{}^\wedge\bar{U}_i \sim \bar{U}_l{}^\wedge\bar{U}_j)$.

Why? Suppose $\bar{U}_i E_L \bar{U}_j$, $\bar{U}_i{}^\wedge\bar{U}_r \sim \bar{U}_j{}^\wedge\bar{U}_r$ and let $r_1 \in A_1$.
Now $\mathrm{Th}^n(R; \bar{U}_r, \bar{W})$ = t_2 = $\mathrm{Th}^n(R; \bar{U}_{r_1}, \bar{W})$ hence $\mathrm{Th}^n(R; \bar{U}_r, \bar{U}_r, \bar{W})$ = $\mathrm{Th}^n(R; \bar{U}_{r_1}, \bar{U}_{r_1}, \bar{W})$. By the composition theorem

$$\begin{aligned}
&\mathrm{Th}^n(C; \bar{U}_i{}^\wedge\bar{U}_{r_1}, \bar{U}_j{}^\wedge\bar{U}_{r_1}, \bar{W})\\
&\quad = \mathrm{Th}^n(L; \bar{U}_i, \bar{U}_j, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_{r_1}, \bar{U}_{r_1}, \bar{W})\\
&\quad = \mathrm{Th}^n(L; \bar{U}_i, \bar{U}_j, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_r, \bar{U}_r, \bar{W})\\
&\quad = \mathrm{Th}^n(C; \bar{U}_i{}^\wedge\bar{U}_r, \bar{U}_j{}^\wedge\bar{U}_r, \bar{W}).
\end{aligned}$$

Therefore $\mathrm{Th}^n(C; \bar{U}_i{}^\wedge\bar{U}_{r_1}, \bar{U}_j{}^\wedge\bar{U}_{r_1}, \bar{W}) = \mathrm{Th}^n(C; \bar{U}_i{}^\wedge\bar{U}_r, \bar{U}_j{}^\wedge\bar{U}_r, \bar{W})$ and hence
$\bar{U}_i{}^\wedge\bar{U}_r \sim \bar{U}_j{}^\wedge\bar{U}_r \iff \bar{U}_i{}^\wedge\bar{U}_{r_1} \sim \bar{U}_j{}^\wedge\bar{U}_{r_1}$.

(γ) $|A_1/E_L| < M_4$ or $|A_1/E_R| < M_4$.

Otherwise, suppose $\langle \bar{X}_1, \bar{X}_2, \ldots, \bar{X}_{M_4-1}\rangle \subseteq \{\bar{U}_i : i \in A_1\}$ is a sequence of pairwise E_L-nonequivalent representatives and that $\langle \bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_{M_4-1}\rangle \subseteq \{\bar{U}_i : i \in A_1\}$ are pairwise E_R-nonequivalent. By (α) we know that for every $i, j < M_4$ there is some $h(i,j) < i^*$ with $\bar{X}_i{}^\wedge\bar{Y}_j \sim \bar{U}_{h(i,j)}$. Define a colouring $f\colon [M_4]^3 \to \{0, 1, \dots, 6\}$ by:

$$f(i,j,k) = \begin{cases} 0 & \text{if } h(i,i) = h(j,k) \\ 1 & \text{if } h(i,i) = h(k,j) \\ 2 & \text{if } h(j,j) = h(i,k) \\ 3 & \text{if } h(j,j) = h(k,i) \\ 4 & \text{if } h(k,k) = h(i,j) \\ 5 & \text{if } h(k,k) = h(j,i) \\ 6 & \text{otherwise.} \end{cases}$$

(If more then one of these cases occurs, f takes the minimal value.)
By the definition of M_4 there is $B \subseteq M_4$ with $|B| = M_3$ such that B is homogeneous with respect to f and we let $f \restriction_B \equiv m$. Is it possible that $m < 6$? Suppose for example that $m = 0$, and choose $i < j < j_1 < k$ from B. If $f(i,j,k) = 0 = f(i,j_1,k)$ we have $h(i,i) = h(j,k)$ and $h(i,i) = h(j_1,k)$. Hence $\bar{X}_i{}^\wedge\bar{Y}_i \sim \bar{X}_j{}^\wedge\bar{Y}_k$ and $\bar{X}_i{}^\wedge\bar{Y}_i \sim \bar{X}_{j_1}{}^\wedge\bar{Y}_k$. It follows that $\bar{X}_j{}^\wedge\bar{Y}_k \sim \bar{X}_{j_1}{}^\wedge\bar{Y}_k$ and hence $\bar{X}_j E_L \bar{X}_{j_1}$ and this is impossible. The other five possibilities are eliminated similarly and we conclude that

$$f \restriction_B \equiv 6.$$

Let $A_2 := \{l < i^* : (\exists i \in B)(h(i,i) = l)\}$ and $A_3 := \{l < i^* : (\exists i \neq j \in B)(h(i,j) = l)\}$. By the choice of B and the above we have $A_2 \cap A_3 = \emptyset$. Note that $|A_2| \le |B| = M_3 < K$ and $|A_3| \le |B|^2 = (M_3)^2 < K$. Hence by the K-randomness of $\mathcal{G}$ there is some $k < i^*$ such that

$$[l \in A_2 \;\Rightarrow\; C \models R(\bar{U}_k, \bar{U}_l, \bar{W})] \;\&\; [l \in A_3 \;\Rightarrow\; C \models \neg R(\bar{U}_k, \bar{U}_l, \bar{W})]$$

that is (as R respects $\sim$)

$$(\dagger) \quad i \neq j \in B \quad\Rightarrow\quad C \models [R(\bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_i, \bar{W}) \;\&\; \neg R(\bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_j, \bar{W})].$$

By the definition of $M_3 = |B|$ we have $i \neq j \in B$ with

$$(*) \quad \mathrm{Th}^n(R; \bar{U}_k, \bar{Y}_i, \bar{W}) = \mathrm{Th}^n(R; \bar{U}_k, \bar{Y}_j, \bar{W}).$$

But

$$\mathrm{Th}^n(C; \bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_i, \bar{W}) = \mathrm{Th}^n(L; \bar{U}_k, \bar{X}_i, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_k, \bar{Y}_i, \bar{W}) =_{\text{by } (*)}$$

$\mathrm{Th}^n(L; \bar{U}_k, \bar{X}_i, \bar{W}) + \mathrm{Th}^n(R; \bar{U}_k, \bar{Y}_j, \bar{W}) = \mathrm{Th}^n(C; \bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_j, \bar{W})$.
Therefore:

$$C \models R(\bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_i, \bar{W}) \iff C \models R(\bar{U}_k, \bar{X}_i{}^\wedge\bar{Y}_j, \bar{W})$$

and this is a contradiction to $(\dagger)$, so (γ) is proved.

To conclude, assume $|A_1/E_L| < M_4$. Then, by 3.3 and as $\{u_i : i \in A_1\}$ is $(K, \frac{K}{(M_2)^2})$-big, there is $A \subseteq A_1$ such that $\langle u_i : i \in A\rangle$ is $(K + \frac{K}{(M_2)^2}, \frac{K}{(M_2)^2 \cdot M_4})$-big and such that for every $i, j \in A$, $\bar{U}_i E_L \bar{U}_j$. Fix $k^* \in A$, and define a sequence $\langle \bar{V}_i : i \in A\rangle$ by:

$$\bar{V}_i \restriction_L = \bar{U}^*_k \restriction_L \text{ and } \bar{V}_i \restriction_R = \bar{U}_i \restriction_R .$$

We want to show that for every $i \in A$ we have $\bar{V}_i \sim \bar{U}_i$. Indeed, as $\bar{U}^*_k E_L \bar{U}_i$ and by (β) we know that for every $r \in A_1$, $\bar{U}_{k^*} {}^\wedge \bar{U}_r \sim \bar{U}_i {}^\wedge \bar{U}_r$ and choosing $r = i$ we get $\bar{U}_{k^*} {}^\wedge \bar{U}_i \sim \bar{U}_i {}^\wedge \bar{U}_i$ i.e. $\bar{V}_i \sim \bar{U}_i$. Hence $\langle v_i : i \in A\rangle$ is (K_1, K_2)-big and all the $\bar{V}_i$'s coincide outside R. Hence R is (K_1, K_2)-major.

By a similar argument we get: $|A_1/E_R| < M_4$ implies L is (K_1, K_2)-major. □

Notation. K_1 and K_2 will be from now on the numbers from lemma 3.9 above.

The computations below will be useful in the following stages. For the moment assume that $\mathcal{G}$ is finite.

Let (L, R) be a Dedekind cut of C and $K > (M_3)^2$ as before. First note that as $\mathcal{G}$ is K-random we have

$$|\mathcal{G}| = \#(C) \geq 2^{2(K-1)}$$

By 3.9 we may assume that L is (K_1, K_2)-major where $K_1 = K + \frac{K}{(M_2)^2}$ and $K_2 = \frac{K}{(M_2)^2 M_4}$. (The case R is (K_1, K_2)-major is symmetric.) If K is big enough we get

$$2^{2(K-1)} - K_1 = 2^{2(K-1)} - (K + \frac{K}{(M_2)^2}) > K > K_2 = \frac{K}{(M_2)^2 M_4}$$

and by the definition of (K_1, K_2)-major

$$\#(L) \geq 2^{K_2} = 2^{\frac{K}{(M_2)^2 M_4}}.$$

By 3.7 R is not $M_1(K_1 + K_2) + 1$-fat, i.e.,

$$\#(R) \leq M_1(K_1 + K_2) = M_1(K + \frac{K}{(M_2)^2} + \frac{K}{(M_2)^2 M_4}) \leq 2M_1 K$$

it follows that

$$(*)\quad \#(L)/(\#(R)+1)^2 \geq 2^{K_2}/(4(M_1)^2 K^2 + 4(M_1)K + 1)$$
$$= 2^{\frac{K}{(M_2)^2 M_4}}/(4(M_1)^2 K^2 + 4(M_1)K + 1).$$

Conclusion 3.10. For every $l < \omega$ there is $K^* = K^*(l, n, d) < \omega$ such that under the context in 3.1, if

$K \geq K^*$,

(L, R) is a Dedekind cut of C,

$M = M(K, n, d)$ denotes the bouquet size of the major segment,

$m = m(K, n, d)$ denotes the bouquet size of other segment,

then $M/(m+1)^2 > l \cdot K^2$.

Proof. By the inequality $(*)$ above and noting that M_1, M_2, M_3 and M_4 do not depend on K. □

Remark. By 3.7, if K is big enough then the segment that is not (K_1, K_2)-major is minor. We will always assume that.

If we assume that the interpreted graph $\mathcal{G}$ is infinite then we can say that, if K is big enough, one segment will have an infinite bouquet size while the other will have an a priori bounded bouquet size.

Lemma 3.11. *For every $n, d < \omega$ there is $K^* = K^*(n, d) < \omega$ such that if $\mathcal{I}$, of dimension d and depth n, is an interpretation of an infinite K-random graph $\mathcal{G}$ on C and $K > K^*$,*

then there is $m < \omega$, that depends only on K, n and d, such that if (L, R) is a Dedekind cut of C:

L (or R) has an infinite bouquet size,

R (or L) has bouquet size that is at most m.

Proof. By lemma 3.9 (letting $K^* = (M_3)^2$) we get that one of the segments is (K_1, K_2)-major and hence has infinite bouquet size. From 3.7 we get that the other segment is not K_3-fat where K_3 depends only on K and the interpretation $\mathcal{I}$ (i.e. n and d). The required m is that K_3, which is good for every Dedekind cut. □

4. Semi-homogeneous subsets

Our next step towards reaching a contradiction is of a combinatorial nature. In this section we introduce the notion of a semi-homogeneous subset and show that the gap between the size of a set and the size of a semi-homogeneous subset is reasonable.

Definition 4.1. Let $K, c < \omega$, let I be an ordered set and $f \colon [I]^2 \to \{0, \ldots, c\}$.

1) We call $T \subseteq I$ *right semi-homogeneous in I (for f and K)* if for every $i < i^*$ from T we have $|\{j \in I : j > i,\ f(i, j) = f(i, i^*)\}| \geq K$.

2) We call $T \subseteq I$ *left semi-homogeneous in I (for f and K)* if for every $i < i^*$ from T we have $|\{j \in I : j < i^*,\ f(j, i^*) = f(i, i^*)\}| \geq K$.

3) We call $T \subseteq I$ *semi-homogeneous in I (for f and K)* if T is both right semi-homogeneous and left semi-homogeneous.

4) We call $T \subseteq I$ *right–nice [left–nice]* for $S \subseteq I$ (and for f and K) if Max(S)<Min(T) [Min(S)>Max(T)] and for every $j \in T$, $S \cup \{j\}$ is right semi-homogeneous [left semi-homogeneous] in $T \cup S$.

Lemma 4.2. *Let K, c, I, f be as above. Suppose $|I| > c \cdot N \cdot K$. Then, there is a right semi-homogeneous subset $S \subseteq I$ of cardinality N.*

Proof. Let i_0 be Min(I) and $T_0 \subseteq I$ be of cardinality $\geq |I| - c \cdot (K-1)$ such that T_0 is right–nice for $\{i_0\}$, (just throw out every $j \in I$ such that $f(i_0, j)$ occurs less than K times, there being at most $c \cdot (K-1)$ such j's). Let i_1 be Min(T_0) and $T_1 \subseteq T_0$ be right–nice for i_1 of cardinality $\geq |I| - 2c \cdot (K-1)$, (use the same argument). Define $S_1 := \{i_0, i_1\}$. Clearly, for every $j \in T_1$, $S_1 \cup \{j\}$ is right semi-homogeneous in I.

Proceed to define i_2 (=Min(T_1)), T_2, S_2 and so on. After defining S_{N-2} and T_{N-2} we have thrown out $(N-1) \cdot c \cdot (K-1)$ elements and as $|I| > c \cdot N \cdot K$ we can define T_{N-1}, i_{N-1} and S_{N-1} which is the required right semi-homogeneous subset. □

Lemma 4.3. *Let K, c, I, f be as above. Suppose $|I| > c^2 \cdot N \cdot K^2 = c \cdot (cNK) \cdot K$. Then, there is a semi-homogeneous subset $T \subseteq I$ of cardinality N.*

Proof. Repeat the construction in the previous lemma to get $T^* \subseteq I$, right semi-homogeneous in I of cardinality $\geq c \cdot N \cdot K$ and now take $T \subseteq T^*$ left semi-homogeneous in T of cardinality $\geq N$.
T is semi-homogeneous in I. □

We return now to the previous section, and its context. Recall that, given a cut (L, R) we denoted by $M = M(K, n, d)$ the bouquet size of the major segment and by $m = m(K, n, d)$ the bouquet size of the minor segment.

Conclusion 4.4. In the context 3.1, for every $c, N < \omega$ there is $K < \omega$ such that if $|I| \geq M(K, n, d)$ then for every $f \colon [I]^2 \to \{0, \ldots, c\}$ there is a semi-homogeneous subset of $T \subseteq I$, for f and $m(K, n, d) + 1$, with $|T| \geq N$.

Proof. By lemma 4.3. we just need to ensure that $c^2 \cdot N \cdot (m(K, n, d)+1)^2 < M(K, n, d)$. i.e. $M/(m+1)^2 > c^2 \cdot N$. This holds by conclusion 3.10. □

5. The forcing

The universe V^P where no uniform interpretation exists is the same as in [LiSh]. The forcing P adds generic semi–clubs to each regular cardinal $> \aleph_0$.

Context. $V \models$ GCH

Definition 5.1. Let $\lambda > \aleph_0$ be a regular cardinal
1) $SC_\lambda := \{f : \quad f{:}\,\alpha \to \{0,1\},\ \alpha < \lambda,\ \mathrm{cf}(\alpha) \le \aleph_0\ \}$ where each f, considered to be a subset of α (or λ), is a semi–club. The order is inclusion. (So SC_λ adds a generic semi–club to λ).
2) Q_λ will be an iteration of the forcing SC_λ with length λ^+ and with support $< \lambda$.
3) $P := \langle P_\mu, \underset{\sim}{Q}_\mu{:}\ \mu \text{ a cardinal} > \aleph_0\ \rangle$ where $\underset{\sim}{Q}_\mu$ is forced to be Q_μ if μ is regular, otherwise it is $\emptyset$. The support of P is Easton's: each condition $p \in P$ is a function from the class of cardinals to names of conditions where the class S of cardinals that are matched to non-trivial names is a set. Moreover, when κ is an inaccessible cardinal, $S \cap \kappa$ has cardinality $< \kappa$.
4) $P_{<\lambda}, P_{>\lambda}, P_{\le\lambda}$ are defined naturally. For example $P_{<\lambda}$ is $\langle P_\mu, \underset{\sim}{Q}_\mu$: $\aleph_0 < \mu < \lambda\rangle$.

Discussion 5.2. Assuming GCH it is standard to see that Q_λ satisfies the λ^+ chain condition and that Q_λ and $P_{\ge\lambda}$ do not add subsets of λ with cardinality $< \lambda$. Hence, P does not collapse cardinals and does not change cofinalities, so V and V^P have the same regular cardinals.

Moreover, for a regular $\lambda > \aleph_0$ we can split the forcing into 3 parts, $P = P_0 * P_1 * P_2$ where P_0 is $P_{<\lambda}$, P_1 is a P_0-name of the forcing Q_λ and P_2 is a $P_0 * P_1$-name of the forcing $P_{>\lambda}$ such that V^P and $V^{P_0 * P_1}$ have the same $H(\lambda^+)$.

In the next sections, when we restrict ourselves to $H(\lambda^+)$ it will suffice to look only in $V^{P_0 * P_1}$.

6. The contradiction (reduced case)

Collecting the results from the previous sections we will reach a contradiction from the assumption that (for a sufficiently large K), the monadic theory of some chain C in V^P, interprets a radom graph $\mathcal{G} \in \Gamma_K$.

As we saw in Sect. 3, an interpretation has a major segment. We will show below that there is a minimal one (and without loss of generality the segment is an initial segment). In this section we restrict ourselves to a special case: we assume that the minimal major initial segment is the whole chain C. Moreover, the chain C is assumed to be regular cardinal $> \aleph_0$.

In the next section we will dispose of these special assumptions. However, the skeleton of those proofs will be the same as in this reduced case.

Definition 6.1. Assume that $(C, <)$ interprets $\mathcal{G} \in \Gamma_K$ by $\mathcal{I}$. $D \subseteq C$ is a *minimal* (K_1, K_2)*-major initial segment for* $\mathcal{I}$ if D is an initial segment of C which is a (K_1, K_2)-major segment and no proper initial segment $D' \subset D$ is (K_1, K_2)-major.

Fact 6.2. *Suppose that* $(C, <)$ *interprets* $\mathcal{G} \in \Gamma_K$ *by* $\mathcal{I}$*, where* K *and* $\mathcal{I}$ *satisfy the assumption of lemma 3.9. Then there is a chain* $(C^*, <^*)$ *that interprets* $\mathcal{G}$ *by some* $\mathcal{I}^*$ *having the same dimension and depth as* $\mathcal{I}$*, such that there is* $D^* \subseteq C^*$ *which is a minimal* (K_1, K_2)*-major initial segment for* $\mathcal{I}^*$*. (*K_1 *and* K_2 *are as in lemma 3.9).*

Proof. (By [Gu] lemma 8.2). Let L be the union of all the initial segments of C that are (K_1, K_2)-minor (note that if L is minor and $L' \subseteq L$ then L' is minor as well). If L is (K_1, K_2)-major then set $D = L, C^* = C, \mathcal{I}^* = \mathcal{I}$ and we are done.

Otherwise, let $D = C \setminus L$, by lemma 3.9 D is major. Now if there is a proper final segment $D' \subset D$ which is (K_1, K_2)-major then $C \setminus D'$ is minor. But $(C \setminus D') \supset L$, so that is impossible by maximality of L. Therefore D is a minimal (K_1, K_2)-major (final) segment. Now take C^* to be the inverse chain of C. Clearly D is a minimal (K_1, K_2)-major initial segment for an interpretation $\mathcal{I}^*$ of $\mathcal{G}$ (that is obtained be replacing '$<$' by '$>$' in $\mathcal{I}$) having the same depth and dimension. □

Sketch of the proof. Fixing an interpretation $\mathcal{I}$ (rather its depth and dimension) we are trying to show that if K is large enough then in V^P no chain C interprets some $\mathcal{G} \in \Gamma_K$ by $\mathcal{I}$. Towards a contradiction we choose K such that

$$\sqrt{K} > N_0 > N_1 > N_2 > N_3 > N_4 > N_5 > N_6$$

with:
(1) $N_6 = \max\{2, n_1, n_2, n_3\} + 1$ (n_1, n_2, n_3 are defined in assumption 5 below).
(2) $N_5 \to (N_6)^3_{32}$ i.e. a set of size N_5 has a homogeneous subset of size N_6 for colouring triplets into 32 colours (exists by Ramsey theorem).
(3) $N_4 = n_1 \cdot N_5$.
(4) $N_3 = 2 \cdot N_4$.
(5) $N_2 \to (N_3)^2_{n_3}$ (exists by Ramsey theorem).
(6) $N_1 \to (N_2)^3_{32}$ (exists by Ramsey theorem).
(7) $N_0 = n_1 \cdot N_1$.

We start with a sequence $\langle \bar{U}_i : i < M \rangle$ of representatives for the elements of $\mathcal{G}$ ($M = \#(D)$ i.e. the bouquet size of a minimal major segment, in our case it is $\#(C) = |\mathcal{G}|$, possibly infinite), and gradually reduce their number until we get pairs that will satisfy (for a suitable semi–club a and a club J):

$$i < j \;\Rightarrow\; [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_i.$$

These will be achieved at steps $1, 2, 3$. In steps $4, 5$ we will get also:

$$[\bar{U}_i, \bar{U}_j]^J_a \sim [\bar{U}_j, \bar{U}_i]^J_a.$$

Contradiction will be achieved when we show that some $[\bar{U}_i, \bar{U}_j]_a^J$ represents two different elements.

Assumptions. Our assumptions towards a contradiction are as follows:
1. $(C, <) \in V^P$ interprets $\mathcal{G} \in \Gamma_K$ by $\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})\rangle$,
$\lg(\bar{X}), \lg(\bar{Y})$ and w.l.o.g $\lg(\bar{W}) = d$, $n(\mathcal{I}) = n$.
2. C itself is the minimal (K_1, K_2)-major initial segment for $\mathcal{I}$. Moreover, $C = \lambda$, a regular cardinal $> \aleph_0$. For every proper initial segment $D \subset C$ we have $\#(D) < K_3$. (K_1, K_2 and K_3 are from $Sect.$ 3, they depend only on K, n and d).
3. $m(*) = m(*)(n + d, 4d)$ is as in the preservation theorem 1.11.
4. $J = \langle \alpha_i : i < \lambda\rangle \subseteq \lambda$ is an $m(*)$–suitable club for all the representatives that will be shuffled (there are only finitely many). $a \subseteq \lambda$ is a semi–club, generic with respect to every relevant element including J (again, finitely many), and see a remark later on.
5. n_1, n_2 and n_3 are defined as the number of possibilities for the following theories ($m(*)$ is as above):

$$n_1 := |\{a\text{-Th}^{m(*)}(C; \bar{X}, \bar{Y}) : \ \bar{X}, \bar{Y} \subseteq C, \ \lg(\bar{X}), \lg(\bar{Y}) = d\}|$$
$$n_2 := |\{a\text{-Th}^{m(*)}(C; \bar{X}, \bar{Y}, \bar{Z}) : \ \bar{X}, \bar{Y}, \bar{Z} \subseteq C, \ \lg(\bar{X}), \lg(\bar{Y}), \lg(\bar{Z}) = d\}|$$
$$n_3 := |\{a\text{-Th}^{m(*)}(C; \bar{X}, \bar{Y}, \bar{Z}, \bar{U}) : \ \bar{X}, \bar{Y}, \bar{Z}, \bar{U} \subseteq C, \ \lg(\bar{X}), \lg(\bar{Y}), \lg(\bar{Z}), \lg(\bar{U}) = d\}|$$

6. $\sqrt{K} > N_0$. In addition, K is large for $l := (n_2)^2 \cdot N_0 \cdot (2|T_{n,3d}| + 1)^2$ as in conclusion 3.10 i.e. $M/(m + 1)^2 > l \cdot K^2$ (this is possible as l depends only on $n(\mathcal{I})$ and $d(\mathcal{I})$).

To get started we need another observation that does not depend on the special assumption on the minimal major segment.

Definition 6.3. Suppose D is the minimal (K_1, K_2)-major initial segment for the interpretation. The *vicinity* of a representative $\bar{X}$ denoted by $[\bar{X}]$ is the collection of representatives
$\{\bar{Y}$: some $\bar{Z} \sim \bar{Y}$ coincides with $\bar{X}$ outside some proper (hence minor) initial segment of $D\}$.

Lemma 6.4. *(1) Every vicinity $[\bar{X}]$ is the union of at most $m = m(K, n, d)$ (the bouquet size of a minor segment) different equivalence classes.*
(2) From $\text{Th}^{n+d}(D; \bar{U}_1, \bar{U}_2, \bar{W})$ we can compute the truth value of: "$\bar{U}_1$ is in the vicinity of $\bar{U}_2$".

Proof. If (1) does not hold then there is a proper initial segment D' of the minimal major initial segment D with $\#(D') > m$ which is impossible. (2) is clear. □

We are ready now for a contradiction:

STEP 1: Let $N_0, \dots, N_6$ be as above and let K be as in assumption 6. Let $\langle \bar{U}_i : i < M \rangle$, be a list of representatives for the elements of $\mathcal{G} \in \Gamma_K$ that is interpreted by $\mathcal{I}$ on C. Let f be a colouring of $[M]^2$ into n_2 colours defined by

$$f(i,j) := a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{U}_j, \bar{W}).$$

We would like to get a semi-homogeneous subset of M for f and $m+1$ of size N_0. If $\mathcal{G}$ is finite then this is possible by assumption 6 and conclusion 4.4. Of course if $\mathcal{G}$ is infinite (i.e. $M \geq \aleph_0$) we can even get a homogeneous one.

Let then $S' \subseteq \{0, \dots, M-1\}$ be semi-homogeneous and look at $B' := \langle \bar{U}_i : i \in S' \rangle$. As $N_0 = n_1 \cdot N_1$ we can choose

$$B := \langle \bar{U}_i : i \in S \rangle$$

such that $S \subseteq S'$ is of size $|N_1|$ and such that $a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{W})$ is constant for every $i \in S$.

STEP 2: We start shuffling the members of B along a and J. Note that by the choice of B and $m(*)$ and by the preservation theorem

$$i, j \in S \Rightarrow \text{Th}^n(C; \bar{U}_i, \bar{W}) = \text{Th}^n(C; [\bar{U}_i, \bar{U}_j]_a^J, \bar{W}).$$

It follows that the results of the shufflings are representatives as well, that is

$$i, j \in S \Rightarrow C \models U([\bar{U}_i, \bar{U}_j]_a^J, \bar{W}).$$

Define for $i < j \in S$

$$k(i,j) := \min\Big\{k : (k \in S \;\&\; [\bar{U}_i, \bar{U}_j]_a^J \sim \bar{U}_k) \vee (k = M)\Big\}$$

By the choice of N_1 there is a subset $A \subseteq S$, of size N_2, such that for every $\bar{U}_i, \bar{U}_j, \bar{U}_l$ with $i < j < l$ and $i, j, l \in A$, the following five statements have a constant truth value:

$k(j,l) = i$,
$k(i,l) = j$,
$k(i,j) = i$,
$k(i,j) = j$,
$k(i,j) = l$.

Moreover, if there is a pair $i < j$ in A such that $k(i,j) \in A$ then:

$$\text{either for every } i < j \text{ from } A,\ k(i,j) = i$$
$$\text{or for every } i < j \text{ from } A,\ k(i,j) = j.$$

The reason is the following: suppose that $k(\alpha, \beta) = \gamma \in A$ for some $\alpha < \beta$ from A. If $\gamma < \alpha$ then $k(j,l) = i$ for all $i < j < l$ from A but k is one valued. Similarly, the possibilities $\alpha < \gamma < \beta$ and $\beta < \gamma$ are ruled out. We are left with $\gamma = \alpha$ or $\gamma = \beta$ and apply homogeneity.

STEP 3: The aim now is to find a pair $i < j$ from A with $k(i, j) \in A$. Define A^* to be the results of the shufflings:

$$A^* := \{k : (\exists i < j \in A)([\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k)\}$$

and it is enough to show that $A^* \cap A \neq \emptyset$.

If not, as $|A| = N_2 < K$ and $|A^*| \leq |A|^2 < K$ (we chose $\sqrt{K} > N_0$), there is a representative $\bar{V}_A$ such that

$$\bigwedge_{i \in A} [C \models R(\bar{U}_i, \bar{V}_A, \bar{W})] \wedge \bigwedge_{i \in A^* \setminus A} [C \models \neg R(\bar{U}_i, \bar{V}_A, \bar{W})].$$

As $N_2 > n_2$ there is $i < j \in A$ with:

$$a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{V}_A, \bar{W}) = a\text{-Th}^{m(*)}(C; \bar{U}_j, \bar{V}_A, \bar{W})$$

and by the preservation theorem

$$(*)\ \text{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{V}_A, \bar{W}) = \text{Th}^n(C; \bar{U}_i, \bar{V}_A, \bar{W}).$$

Now, $[\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k$ for some $k \in A^*$ but by $(*)$

$$C \models R([\bar{U}_k, \bar{V}_A, \bar{W}).$$

Therefore, by the choice of $\bar{V}_A$, we have $k \in A$. It follows that $k \in A \cap A^*$ so $A^* \cap A \neq \emptyset$ after all.
The aim is fulfilled and we may assume w.l.o.g that

$$i < j \in A \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_i.$$

STEP 4: The aim now is to show that

$$\otimes\ \ i < j \in A\ \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim [\bar{U}_j, \bar{U}_i]^J_a\ (= [\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a}).$$

Returning to the discussion in $Sect.\ 5$, we have mentioned so far only a finite number of elements from $H(\lambda^+)^{V^P}$, (including J). Everything already belongs to $H(\lambda^+)^{V^{P_0 * P_1}}$ where P_0 is $P_{<\lambda}$ and P_1 is a P_0-name for Q_λ which is an iteration of length λ^+ with support $< \lambda$ (we assume that the ground universe V satisfies GCH). Moreover, an initial segment of $P_0 * P_1$, denoted by $P_0 * P_1 \restriction_\beta$ adds all the relevant elements and we can choose the semi–club a as the one that is generated in the β'th stage of P_1.
Let $p \in P_0 * P_1$ be a condition that forces all the statements about the representatives we mentioned so far (e.g. $i < j \in A \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_i$). We think about p as a function with domain $\{-1\} \cup \lambda^+$ such that $p(-1) \in P_0$ and for $\alpha \in \lambda^+$, $p(\alpha) \in SC_\lambda$. under this notation $p(\beta)$ is an initial segment of a

and w.l.o.g a member of V^{P_0} (and not a name for one). Let $\gamma^* = \mathrm{Dom}(p(\beta))$. We may assume that $\mathrm{cf}(\gamma^*) = \aleph_0$. Let $\gamma := \alpha_{\gamma^*} \in J$ (so $\mathrm{cf}(\gamma) = \aleph_0$ as well).

By homogeneity of the forcing, $b := (a \cap \gamma) \cup [(\lambda \setminus a) \cap [\gamma, \lambda)]$ is a semi–club of λ that is also generic with respect to the relevant elements. We denote from now on, for $\bar{U}, \bar{V} \subseteq \lambda$,

$$\bar{U} \,{}^\wedge \bar{V} := (\bar{U} \cap \gamma) \cup (\bar{V} \cap [\gamma, \lambda)).$$

For proving $\otimes$ we will show that:

(α) $[\bar{U}_i, \bar{U}_j]^J_b \sim \bar{U}_i$ for all $i < j$ from A,

(β) $[\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a} {}^\wedge \bar{U}_k \sim \bar{U}_k$ for all i, j, k from A,

(γ) $[\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a} {}^\wedge \bar{U}_i \sim [\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}$ for all $i < j$ from A.

STEP 5: Let's prove the claims:

(α): By homogeneity of the forcing everything that p forces for a it forces for b.

(β): Recall that for every $i, j, k \in A$ we have

$$a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{W}) = a\text{-Th}^{m(*)}(C; \bar{U}_j, \bar{W}) = a\text{-Th}^{m(*)}(C; \bar{U}_k, \bar{W}).$$

As $m(*) = m(*)(n + d, 4d)$ and $\gamma \in J$ satisfies $\mathrm{cf}(\gamma) = \aleph_0$ we have by the second part of preservation Theorem 1.11

$$\begin{aligned}\text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} &= \text{Th}^{n+d}(C; \bar{U}_i, \bar{W}) \restriction_{[0,\gamma)} \\ &= \text{Th}^{n+d}(C; \bar{U}_j, \bar{W}) \restriction_{[0,\gamma)} .\end{aligned}$$

Similarly

$$\text{Th}^{n+d}(C; \bar{U}_i, \bar{W}) \restriction_{[0,\gamma)} = \text{Th}^{n+d}(C; \bar{U}_k, \bar{W}) \restriction_{[0,\gamma)}$$

and it follows that for every $i, j, k \in A$:

$$(\dagger)\ \ \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} = \text{Th}^{n+d}(C; \bar{U}_k, \bar{W}) \restriction_{[0,\gamma)} .$$

Now by the composition theorem

$$\begin{aligned}\text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a {}^\wedge \bar{U}_k, \bar{W}) &= \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} \\ &\quad + \text{Th}^{n+d}(C; \bar{U}_k, \bar{W}) \restriction_{[\gamma,\lambda)}\end{aligned}$$

and this equals by $(\dagger)$

$$\text{Th}^{n+d}(C; \bar{U}_k, \bar{W}) \restriction_{[0,\gamma)} + \text{Th}^{n+d}(C; \bar{U}_k, \bar{W}) \restriction_{[\gamma,\lambda)} = \text{Th}^{n+d}(C; \bar{U}_k, \bar{W}).$$

As the theories are equal and as $\bar{U}_k$ is a representative, there is some $l < M$ (not necessarily in A) such that

$$[\bar{U}_i, \bar{U}_j]^J_a {}^\wedge \bar{U}_k \sim \bar{U}_l.$$

If $l = k$ everything is fine. Otherwise assume $l > k$ (symmetrically for $l < k$) for a contradiction.

By the definition of vicinity we see that $\bar{U}_l \in [\bar{U}_k]$ and this is reflected in $\text{Th}^{n+d}(C; \bar{U}_k, \bar{U}_l, \bar{W})$. Now $k \in A \subseteq S$ and S was chosen to be semi-homogeneous in M. Therefore there are $l_0 < l_1 < \ldots < l_m < M$ with

$$\bigwedge_{i<(m+1)} \text{Th}^{n+d}(C; \bar{U}_k, \bar{U}_{l_i}, \bar{W}) = \text{Th}^{n+d}(C; \bar{U}_k, \bar{U}_l, \bar{W}).$$

Hence

$$\bigwedge_{i<(m+1)} (U_{l_i} \in [U_k])$$

but by 6.4 a vicinity contains at most m pairwise nonequivalent representatives, a contradiction. We conclude that $l = k$.

Therefore, for every i, j, k from A we have $[\bar{U}_i, \bar{U}_j]^J_a {}^\wedge \bar{U}_k \sim \bar{U}_k$. Substituting i and j we get: for every i, j, k from A, $[\bar{U}_j, \bar{U}_i]^J_a {}^\wedge \bar{U}_k \sim \bar{U}_k$ or:

$$[\bar{U}_i, \bar{U}_j]_{\lambda \setminus a} {}^\wedge \bar{U}_k \ \sim \ \bar{U}_k.$$

This is claim (β).

(γ): Now suppose $i < j$ are from A. By definition, for every $\bar{P} \subseteq C$ the theory $(\lambda \setminus a)$-$\text{Th}^{m(*)}(C; \bar{P})$ determines (and is determined by) a-$\text{Th}^{m(*)}(C; \bar{P})$. Therefore,

$$\begin{aligned} &a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{W}) = a\text{-Th}^{m(*)}(C; \bar{U}_j, \bar{W}) \\ &\quad \& \ (\lambda \setminus a)\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{W}) = (\lambda \setminus a)\text{-Th}^{m(*)}(C; \bar{U}_j, \bar{W}). \end{aligned}$$

Applying the preservation theorem for a and $\lambda \setminus a$ we get

$$\begin{aligned} \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} &= \text{Th}^{n+d}(C; \bar{U}_i, \bar{W}) \restriction_{[0,\gamma)} \\ &= \text{Th}^{n+d}(C; \bar{U}_j, \bar{W}) \restriction_{[0,\gamma)} \end{aligned}$$

and

$$\begin{aligned} \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a}, \bar{W}) \restriction_{[0,\gamma)} &= \text{Th}^{n+d}(C; \bar{U}_i, \bar{W}) \restriction_{[0,\gamma)} \\ &= \text{Th}^{n+d}(C; \bar{U}_j, \bar{W}) \restriction_{[0,\gamma)} \end{aligned}$$

so

$$\text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} = \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a}, \bar{W}) \restriction_{[0,\gamma)} .$$

Therefore, as $\text{Th}^{n+d}(C; \bar{P})$ determines $\text{Th}^{n+d}(C; \bar{P}, \bar{P})$:

$$\begin{aligned} (\ddagger) \quad &\text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_a, [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)} \\ &\quad = \text{Th}^{n+d}(C; [\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a}, [\bar{U}_i, \bar{U}_j]^J_{\lambda \setminus a}, \bar{W}) \restriction_{[0,\gamma)} . \end{aligned}$$

By (α) and (β) we know that

$$[\bar{U}_i, \bar{U}_j]_b \ \sim\ \bar{U}_i \ \sim\ [\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}{}^\wedge \bar{U}_i$$

and the equivalence is reflected by $\mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}{}^\wedge \bar{U}_i, [\bar{U}_i, \bar{U}_j]_b, \bar{W})$. Clearly Th^n is determined by Th^{n+d}. Hence:

$$\begin{aligned}
&\mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a{}^\wedge \bar{U}_i, [\bar{U}_i, \bar{U}_j]^J_b) = \text{ (by (‡))}\\
&\mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a, [\bar{U}_i, \bar{U}_j]^J_a, \bar{W}) \restriction_{[0,\gamma)}\\
&\quad + \mathrm{Th}^n(C; \bar{U}_i, [\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}, \bar{W}) \restriction_{[\gamma,\lambda)}\\
&= \mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]_{\lambda\setminus a}, [\bar{U}_i, \bar{U}_j]_{\lambda\setminus a}, \bar{W}) \restriction_{[0,\gamma)}\\
&\quad + \mathrm{Th}^n(C; \bar{U}_i, [\bar{U}_i, \bar{U}_j]_{\lambda\setminus a}, \bar{W}) \restriction_{[\gamma,\lambda)}\\
&= \mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]_{\lambda\setminus a}{}^\wedge \bar{U}_i, [\bar{U}_i, \bar{U}_j]_{\lambda\setminus a}).
\end{aligned}$$

Therefore

$$[\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}{}^\wedge \bar{U}_i \ \sim\ [\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a}$$

and (γ) is proved.

From (β) and (γ) we conclude

$$[\bar{U}_i, \bar{U}_j]^J_{\lambda\setminus a} \ \sim\ \bar{U}_i \ \sim\ [\bar{U}_i, \bar{U}_j]^J_a$$

and $\otimes$ is proved.

STEP 6: By definition of $N_2 = |A|$, N_3 and N_4 there is a sub-sequence of $\langle \bar{U}_i : i \in A\rangle$ that will be denoted for convenience (while preserving the order between the indices) by $\langle \bar{P}_i : i < N_3 = 2N_4\rangle$ such that for every $i < j < 2N_4$ and $r < l < 2N_4$:

(i) a-$\mathrm{Th}^{m(*)}(C; \bar{P}_i, \bar{P}_j, \bar{W}) = a$-$\mathrm{Th}^{m(*)}(C; \bar{P}_r, \bar{P}_l, \bar{W})$ (by defining a colouring of pairs from N_2).

(ii) $[\bar{P}_i, \bar{P}_j]^J_a \sim [\bar{P}_i, \bar{P}_j]_{\lambda\setminus a} \sim \bar{P}_i$ (by steps 3 and 5).

For $i < N_4$ let $\bar{Q}_i$ a representative that satisfies

$$\bigwedge_{\alpha\in[i,2N_4-i)} \Big(C \models R(\bar{P}_\alpha, \bar{Q}_i, \bar{W})\Big) \wedge$$
$$\bigwedge_{\alpha\in[0,i)\cup[2N_4-i,2N_4)} \Big(C \models \neg R(\bar{P}_\alpha, \bar{Q}_i, \bar{W})\Big).$$

As N_4 is big enough there is $T \subseteq \{0, \ldots, N_4 - 1\}$ with $|T| = N_6$ such that if $i, j \in T$ then either $[\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_i$ or $[\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_j$. To get T repeat steps 1, 2 and 3 while substituting $\langle \bar{U}_i : i < S'\rangle$ by $\langle \bar{Q}_i : i < N_4\rangle$, and N_0, N_1, N_2 by N_4, N_5 and N_6 respectively. Note that we lose generality by chosing one of the possibilities.

Now choose $i, j \in T$ (by $N_6 > n_3$) such that

$$a\text{-Th}^{m(*)}(C; \bar{P}_i, \bar{P}_{2N_4-i}, \bar{Q}_i, \bar{W}) = a\text{-Th}^{m(*)}(C; \bar{P}_j, \bar{P}_{2N_4-j}, \bar{Q}_j, \bar{W})$$

and shuffle along a and J:
$\text{Th}^n(C; \bar{P}_i, \bar{P}_{2N_4-i}, \bar{Q}_i, \bar{W}) =$
$\text{Th}^n(C; [\bar{P}_i, \bar{P}_j]^J_a, [\bar{P}_{2N_4-i}, \bar{P}_{2N_4-j}]^J_a, [\bar{Q}_i, \bar{Q}_j]^J_a, \bar{W}) =$
$\text{Th}^n(C; [\bar{P}_i, \bar{P}_j]^J_a, [\bar{P}_{2N_4-j}, \bar{P}_{2N_4-i}]^J_{\lambda\setminus a}, [\bar{Q}_i, \bar{Q}_j]^J_a, \bar{W})$
but $[\bar{P}_i, \bar{P}_j]^J_a \sim \bar{P}_i$, and by step 5,

$$[\bar{P}_{2N_4-j}, \bar{P}_{2N_4-i}]^J_{\lambda\setminus a} \sim \bar{P}_{2N_4-j}.$$

Now from "$[\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_i$ or $[\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_j$" and the equality of the theories Th^n:

$$C \models (R(\bar{P}_i, \bar{Q}_i, \bar{W}) \ \& \ R(\bar{P}_{2N_4-j}, \bar{Q}_i))$$

or

$$C \models (\neg R(\bar{P}_i, \bar{Q}_j, \bar{W}) \ \& \ \neg R(\bar{P}_{2N_4-j}, \bar{Q}_j).$$

Both possibilities contradict the choice of the $\bar{Q}_i$'s !

First remark. So J and a are chosen as follows: getting $\langle \bar{U}_i : i \in S\rangle$ at step 1 ($|S| = N_1$) choose for every subset $A \subseteq S$ a representative $\bar{V}_A$ that separates $\langle \bar{U}_i : i \in A\rangle$ from $\langle \bar{U}_i : i \in S \setminus A\rangle$ (some of these will be the $\bar{Q}_i$'s from step 6). J is an $m(*)$-suitable for all these elements and a is a semi–club that is generic with respect to all of these. Clearly, only finitely many elements are involved.

Second remark. Note that genericity was used only at stages 4 and 5 (i.e. to prove $[\bar{U}_i, \bar{U}_j]^J_a \sim [\bar{U}_j, \bar{U}_i]^J_a$).

We proved the following:

Theorem 6.5. *Let $\langle U(\bar{X}, \bar{Z}), E(\bar{X}, \bar{Y}, \bar{Z}), R(\bar{X}, \bar{Y}, \bar{Z})\rangle$ be a sequence of formulas of dimension d and depth n.*
Then there is $K < \omega$, that depends only on d and n such that, in V^P, for no chain C and parameters $\bar{W} \subseteq C$:

(i) *C is isomorphic to a regular cardinal $\lambda > \aleph_0$,*

(ii) *$\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})\rangle$ is an interpretation for some $\mathcal{G} \in \Gamma_k$ in C,*

(iii) *C is the minimal (K_1, K_2)-major initial (or final) segment for $\mathcal{I}$.*

□

7. Generality

Our aim in this section is to achieve full generality of the interpreting chain C and its minimal initial major segment D. There are three stages:

(I) $D \subseteq C$, $D \neq C$ but D is (isomorphic to) a regular cardinal $\lambda > \aleph_0$.
(II) $D = C$, D general.
(III) C and D are general.

Let us just remark that always $\mathrm{cf}(D) > \aleph_0$, otherwise we can prove the non existence of interpretations even from ZFC.

We will elaborate on stages (I) and (II), stage (III) is a simple combination of the techniques.

Chopping off the final segment. We are trying now to get a contradiction from the same assumptions as in the previous section except for the following: the minimal (K_1, K_2)-major initial segment D that is a regular cardinal is not necessarily equal to the interpreting chain C. a and J are therefore subsets of D.

The basic idea of the proof is that if t^* is fixed and known in advance then to know $t_i + t^*$ all we need to know is t_i. Here t_i are the restrictions of the information (partial theories) to D and t^* is the restriction to $C \setminus D$ which can be assumd to be fixed, as many representatives coincide outside D.

We do not specify the exact size of K (which should be slightly bigger than in the previous case). It should be apparent however that K depends on n and d only and is obtained by repeated applications of the Ramsey functions.

Preliminary step: Let $\langle \bar{U}_i : i < |\mathcal{G}|\rangle$ a list of representatives for the elements of $\mathcal{G}$. By definition of D, we may assume that $\langle \bar{U}_i : i < M = \#(D)\rangle$ is a list of representatives for a (K_1, K_2)-major subset of $\mathcal{G}$ and all of them coincide outside D. Denote $D^{@} := C \setminus D$ and for $i < M$:

$\bar{U}_i^* := \bar{U}_i \cap D$,
$\bar{U}^{@} := \bar{U}_i \cap D^{@}$,
$\bar{W}^* := \bar{W} \cap D$,
$\bar{W}^{@} := \bar{W} \cap D^{@}$.

Definition 7.1. (1) Define on $\mathcal{P}(D)$ a unary relation $U^*(\bar{X})$ and binary relations $\bar{X} \sim^* \bar{Y}$ and $R^*(\bar{X}, \bar{Y})$, with arity d by:

$$U^*(\bar{X}) \iff C \models U(\bar{X} \cup \bar{U}^{@}, \bar{W}),$$
$$\bar{X} \sim^* \bar{Y} \iff C \models E(\bar{X} \cup \bar{U}^{@}, \bar{Y} \cup \bar{U}^{@}, \bar{W}),$$
$$R^*(\bar{X}, \bar{Y}) \iff C \models R(\bar{X} \cup \bar{U}^{@}, \bar{Y} \cup \bar{U}^{@}, \bar{W}).$$

(When $i, j < M$ for instance, then $R^*(\bar{U}_i^*, \bar{U}_j^*)$ holds if and only if $C \models R(\bar{U}_i, \bar{U}_j, \bar{W})$).

(2) If $i, j < |\mathcal{G}|$ and $\bar{U}_i \cap D^{@} = \bar{U}_j \cap D^{@}$ we denote

$$[\bar{U}_i, \bar{U}_j]^J_a := [\bar{U}_i \cap D, \bar{U}_j \cap D]^J_a \cup (\bar{U}_i \cap D^{@})$$

(If $i, j < M$ for instance then $[\bar{U}_i, \bar{U}_j]^J_a$ is $[\bar{U}^*_i, \bar{U}^*_j]^J_a \cup \bar{U}^{@}$).

Fact 7.2. *$H(\lambda^+)^{VP}$ computes correctly $\sim^*$, U^* and R^* from a-Th$^{m(*)}$*

Proof. Take for example $\sim^*$: $\bar{X} \sim^* \bar{Y}$ is determined by $\mathrm{Th}^n(C; \bar{X} \cup \bar{U}^{@}, \bar{Y} \cup \bar{U}^{@}, \bar{W}) =$

$$\mathrm{Th}^n(D; \bar{X}, \bar{Y}, \bar{W}^*) + \mathrm{Th}^n(D^{@}; \bar{U}^{@}, \bar{U}^{@}, \bar{W}^{@}).$$

The second theory is fixed for every $\bar{X}, \bar{Y} \subseteq D$. Hence (e.g. by the finite number of possibilities) all we need to know is the first theory, which is computed correctly in $H(\lambda^+)^{VP}$ from a-Th$^{m(*)}$. □

We proceed by immitating the previous proof:

STEP 1: Define $B' := \langle \bar{U}_i : i \in S' \rangle$ where $S' \subseteq \{0, \dots, M-1\}$ is semi-homogeneous , and

$$B := \langle \bar{U}_i : i \in S \rangle$$

such that $S \subseteq S'$, $|S|$ finite and big enough, with a-Th$^{m(*)}(D; \bar{U}^*_i, \bar{W}^*)$ constant for $i \in S$.

STEP 2: Shuffle the members of B along a and J as in definition 7.1. Note that by the choice of B and the preservation theorem

$$i, j \in S \Rightarrow \mathrm{Th}^n(C; \bar{U}_i, \bar{W}) = \mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{W})$$

and therefore the resuts are representatives as well i.e.

$$i, j \in S \Rightarrow C \models U([\bar{U}_i, \bar{U}_j]^J_a, \bar{W}).$$

Define for $i < j \in S$

$$k(i,j) := \min\Big\{k : (k \in S \;\&\; [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k) \vee (k = M)\Big\}$$

equivalently

$$k(i,j) := \min\Big\{k : (k \in S \;\&\; [\bar{U}^*_i, \bar{U}^*_j]^J_a \sim^* \bar{U}^*_k) \vee (k = M)\Big\}$$

Let $A \subseteq S$ be large enough, homogeneous with the colouring into 32 colours we used before.

STEP 3: The aim is to find $i < j$ from A with $k(i,j) \in A$. Let

$$A^* := \{k < |\mathcal{G}| : (\exists i < j \in A)([\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k)\}$$

and let's show that $A^* \cap A \neq \emptyset$.

Othewise, there is some $\bar{V}_A$ (not necessarilly from $\langle \bar{U}_i : i < M\rangle$) that separates these two disjoint collections of representatives:

$$\bigwedge_{i \in A} [C \models R(\bar{U}_i, \bar{V}_A, \bar{W})] \wedge \bigwedge_{i \in A^* \setminus A} [C \models \neg R(\bar{U}_i, \bar{V}_A, \bar{W})].$$

We may assume that there are $i < j$ from A with

$$a\text{-Th}^{m(*)}(C; \bar{U}_i, \bar{V}_A, \bar{W}) \restriction_D = a\text{-Th}^{m(*)}(C; \bar{U}_j, \bar{V}_A, \bar{W}) \restriction_D .$$

By the preservation theorem

$$(*) \;\; \text{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{V}_A, \bar{W}) \restriction_D = \text{Th}^n(C; \bar{U}_i, \bar{V}_A, \bar{W}) \restriction_D$$

and in addition

$$\begin{aligned}(**)\text{Th}^n(C; [\bar{U}_i, \bar{U}_j]^J_a, \bar{V}_A, \bar{W}) \restriction_{D^@} &= \text{Th}^n(D^@; \bar{U}^@, \bar{V}_A \cap D^@, \bar{W}^@) \\ &= \text{Th}^n(C; \bar{U}_i, \bar{V}_A, \bar{W}) \restriction_{D^@} .\end{aligned}$$

Now $[\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_k$ for some $k \in A^*$ but by $(*)$ and $(**)$ and the composition theorem:

$$C \models R([\bar{U}_i, \bar{U}_j]^J_a, \bar{V}_A, \bar{W}).$$

Therefore, by the choice of $\bar{V}_A$, $k \in A$. As $k \in A^*$ it follows that $A^* \cap A \neq \emptyset$ after all.

As before we may conclude that, without loss of generality:

$$i < j \in A \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim \bar{U}_i$$

equivalently (and this is known even by $H(\lambda^+)^{V^P}$)

$$i < j \in A \Rightarrow [\bar{U}^*_i, \bar{U}^*_j]^J_a \sim^* \bar{U}^*_i$$

STEPS 4,5: We work inside $H(\lambda^+)^{V^P}$ and concentrate on $\langle \bar{U}^*_i : i \in A\rangle$. The aim is to show that

$$\otimes \;\; i < j \in A \Rightarrow [\bar{U}^*_i, \bar{U}^*_j]^J_a \sim^* [\bar{U}^*_j, \bar{U}^*_i]^J_a.$$

Let $p \in P_0 * P_1$ be a condition that forces all the facts we showed so far about $\sim^*$, U^* and R^* and the $\bar{U}^*_i$'s such as $i < j \in A \Rightarrow [\bar{U}^*_i, \bar{U}^*_j]^J_a \sim \bar{U}^*_i$. As before we define a generic semi–club $b \subseteq \lambda$ and show that:

(α) $[\bar{U}^*_i, \bar{U}^*_j]^J_b \sim^* \bar{U}^*_i$ for every $i < j$ from A,

(β) $[\bar{U}^*_i, \bar{U}^*_j]^J_{\lambda \setminus a} {}^\wedge \bar{U}^*_k \;\sim^*\; \bar{U}^*_k$ for every $i, j, k \in A$,

(γ) $[\bar{U}^*_i, \bar{U}^*_j]^J_{\lambda \setminus a} {}^\wedge \bar{U}^*_i \;\sim^*\; [\bar{U}^*_i, \bar{U}^*_j]^J_{\lambda \setminus a}$ for every $i < j$ from A.

The proofs are exactly the same as in the previous section (substituting C, $\bar{W}$ and $\sim$ by D, $\bar{W}^*$ and $\sim^*$), and from these facts we can deduce $\otimes$. Leaving $H(\lambda^+)^{V^P}$ we find that what we proved in V^P is:

$$\odot \ i < j \in A \Rightarrow [\bar{U}_i, \bar{U}_j]^J_a \sim [\bar{U}_j, \bar{U}_i]^J_a$$

STEP 6: As $|A|$ is big enough we have a sub-sequence of $\langle \bar{U}_i : i \in A \rangle$ that will be denoted for convenience (while preserving the order between the indices) by $\langle \bar{P}_i : i < N^*_3 = 2N^*_4 \rangle$ such that for every $i < j < 2N^*_4$ and $r < l < 2N^*_4$ (N^*_4 is a sufficiently big number as usual):

(i) a-Th$^{m(*)}(D; \bar{P}^*_i, \bar{P}^*_j, \bar{W}^*) = a$-Th$^{m(*)}(D; \bar{P}^*_r, \bar{P}^*_l, \bar{W}^*)$

(ii) Th$^n(D^@; \bar{P}^@_i, \bar{P}^@_j, \bar{W}^@) = $ Th$^n(D^@; \bar{P}^@_r, \bar{P}^@_l, \bar{W}^@)$ ($\bar{P}_i \cap D^@$ is constant),

(iii) $[\bar{P}_i, \bar{P}_j]^J_a \sim [\bar{P}_i, \bar{P}_j]_{\lambda \setminus a} \sim \bar{P}_i$.

For $i < N^*_4$ let $\bar{Q}_i$ a representative (not necessarily from $\langle \bar{U}_i : i < M \rangle$) that satisfies:

$$\bigwedge_{\alpha \in [i, 2K_4 - i)} \Big(C \models R(\bar{P}_\alpha, \bar{Q}_i, \bar{W}) \Big) \wedge$$
$$\bigwedge_{\alpha \in [0,i) \cup [2N^*_4 - i, 2N^*_4)} \Big(C \models \neg R(\bar{P}_\alpha, \bar{Q}_i, \bar{W}) \Big)$$

From the $\bar{Q}_i$'s extract $\langle \bar{Q}_i : i \in T^1 \rangle$, with $|T^1| = N^*_5$ large enough such that for every $i < j$ from T^1:

$$\Big(a\text{-Th}^{m(*)}(D, \bar{Q}^*_i, \bar{W}^*) = a\text{-Th}^{m(*)}(D, \bar{Q}^*_j, \bar{W}^*) \Big)$$
$$\& \ \Big(\text{Th}^n(D^@, \bar{Q}^@_i, \bar{W}^@) = \text{Th}^n(D^@, \bar{Q}^@_j, \bar{W}^@) \Big)$$

where $\bar{Q}^*_i := \bar{Q}_i \cap D$ and $\bar{Q}^@_i := \bar{Q}_i \cap D^@$.

For $i < j$ in T^1 denote

$$[\bar{Q}_i, \bar{Q}_j]^J_a := [\bar{Q}^*_i, \bar{Q}^*_j]^J_a \cup \bar{Q}^@_i$$

and note that by the preservation theorem the results of the shufflings are representatives for elements of $\mathcal{G}$ i.e.

$$i, j \in T^1 \Rightarrow C \models U([\bar{Q}_i, \bar{Q}_j]^J_a, \bar{W}).$$

Define $k^*(i,j)$ for $i < j$ from T^1 by

$$k^*(i,j) := \min \Big\{ k : (k \in T^1 \ \& \ [\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_k) \vee (k = N^*_5) \Big\}.$$

There is a subset $T \subseteq T^1$ of size N^*_6, large enough, such that for all $\bar{Q}_i, \bar{Q}_j, \bar{Q}_l$ with $i < j < l$ from T the following five statements have a constant truth value: $k^*(j,l) = i$, $k^*(i,l) = j$, $k^*(i,j) = i$, $k^*(i,j) = j$, $k^*(i,j) = l$. Moreover, as usual if there are $i < j$ in T with $k^*(i,j) \in T$ then either for every $i < j$ in T, $k^*(i,j) = i$ or for every $i < j$ in T, $k^*(i,j) = j$.

If there isn't such a pair choose $\bar{V}_T$ such that

$$\bigwedge_{i \in T} [C \models R(\bar{Q}_i, \bar{V}_T, \bar{W})] \wedge \bigwedge_{i<j\in T} [C \models \neg R([\bar{Q}_i, \bar{Q}_j]^J_a, \bar{V}_T, \bar{W})]$$

as N^*_6 is big enough there are $i < j$ from T with:

$$a\text{-Th}^{m(*)}(D; \bar{Q}^*_i, \bar{V}_T \cap D, \bar{W}^*) = a\text{-Th}^{m(*)}(D; \bar{Q}^*_j, \bar{V}_T \cap D, \bar{W}^*)$$
$$\text{Th}^n(D^@; \bar{Q}^@_i, \bar{V}_T \cap D^@, \bar{W}^@) = \text{Th}^n(D^@; \bar{Q}^@_j, \bar{V}_T \cap D^@, \bar{W}^@)$$

By the preservation theorem and the composition theorem we get

$$(*)\ \ \text{Th}^n(C; [\bar{Q}_i, \bar{Q}_j]^J_a, \bar{V}_T, \bar{W}) = \text{Th}^n(C; \bar{Q}_i, \bar{V}_T, \bar{W}).$$

Therefore $C \models R([\bar{Q}_i, \bar{Q}_j]^J_a, \bar{V}_T, \bar{W})$ which is a contradition.

It follows: either $i, j \in T \Rightarrow [\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_i$ or $i < j \in T \Rightarrow [\bar{Q}_i, \bar{Q}_j]^J_a \sim \bar{Q}_j$.

Now choose $i, j \in T$ such that

$$a\text{-Th}^{m(*)}(D; \bar{P}^*_i, \bar{P}^*_{2N^*_4-i}, \bar{Q}^*_i, \bar{W}^*) = a\text{-Th}^{m(*)}(D; \bar{P}^*_j, \bar{P}^*_{2N^*_4-j}, \bar{Q}^*_j, \bar{W}^*)$$
$$\text{Th}^n(D^@; \bar{P}^@_i, \bar{P}^@_{2N^*_4-i}, \bar{Q}^@_i, \bar{W}^@) = \text{Th}^n(D^@; \bar{P}^@_j, \bar{P}^@_{2N^*_4-j}, \bar{Q}^@_j, \bar{W}^@)$$

Shuffle along a and J and get a contradiction as before to the definition of the $\bar{Q}_i$'s.

We have proved the following:

Theorem 7.3. *Let $\langle U(\bar{X}, \bar{Z}), E(\bar{X}, \bar{Y}, \bar{Z}), R(\bar{X}, \bar{Y}, \bar{Z})\rangle$ be a sequence of formulas of dimension d and depth n.*
Then there is $K < \omega$, that depends only on d and n such that, in V^P, for no chain C and parameters $\bar{W} \subseteq C$:

(i) *$\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})\rangle$ is an interpretation for some $\mathcal{G} \in \Gamma_k$ in C,*

(ii) *D, the minimal (K_1, K_2)-major initial (or final) segment for $\mathcal{I}$, is isomorphic to a regular cardinal $\lambda > \aleph_0$.* □

Reduced shufflings: There are two main difficulties that face us in the general context. The first one is that the preservation theorem is formulated only in the context of well ordered chains. We can try and solve this by choosing a cofinal sequence through the chain and shuffle along this sequence. However the second difficulty is that a semi–club that has the cardinality of $\text{cf}(D)$ (where D is the minimal major initial segment) can't be generic with respect to subsets of D when $|D| > \text{cf}(D)$. The solution for both this difficulties lies in the observation that what we really shuffle are not subsets of the chain but rather partial theories.

Suppose that we are given a chain C, with $\mathrm{cf}(C) = \lambda > \aleph_0$ and some $\bar{A} \subseteq C$ of length l. For simplicity we assume that the chains have a first element min(C). Choosing a cofinal sequence $E = \langle \alpha_i : i < \lambda \rangle$ in C such that $\alpha_0 = \min(C)$ and defining $s_i := \mathrm{Th}^n(C, \bar{A}) \restriction_{[\alpha_i, \alpha_{i+1})}$ we get by the composition theorem that

$$\mathrm{Th}^n(C, \bar{A}) = \sum_{i<\lambda} s_i.$$

Concentrating on the chain $(\lambda, <)$ we define a sequence $\bar{P} = \bar{P}_{\bar{A}} = \langle P_t : t \in T_{n,l} \rangle$ where for $t \in T_{n,l}$, $P_t := \{i < \lambda : s_i = t\}$. By the Feferman-Vaught theorem (1.9) we know that $\mathrm{Th}^n(C; \bar{A})$ is determined by $\mathrm{Th}^m(\lambda; \bar{P})$ where $m = m(n, l)$ depends only on n and l.

Lemma 7.4. *Let C be a chain with cofinality $\lambda > \aleph_0$ and let $n, l \in \mathbb{N}$. Then, there are $m(*), l(*), \beta(*) \in \mathbb{N}$, all depending only on n and l, such that*

(a) there is a 1-1 function $\bar{X} \mapsto \bar{P}_{\bar{X}}$ such that for every $\bar{A} \subseteq C$ of length l there is $\bar{P}_{\bar{A}} \subseteq \lambda$ of length $l(*)$ and $\mathrm{Th}^n(C; \bar{A})$ is determined by $\mathrm{Th}^{m(*)}(\lambda; \bar{P}_{\bar{A}})$,

(b) $\beta(*)$ codes a Turing machine that computes $\mathrm{Th}^n(C; \bar{A})$ from $\mathrm{Th}^{m(*)}$ $(\lambda; \bar{P}_{\bar{A}})$.

Proof. Choose a cofinal $E = \langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ ($\alpha_0 = \min(C)$). Let $\bar{P}_{\bar{A}}$ be as above and $\ell(*) = |T_{n,l}|$. Then (a) is clear from the previous discussion. The computability in clause (b) is clear from the fact that $T_{m(*),l(*)}$ and $T_{n,l}$ are both finite. □

Remark. Of course we don't really lose generality by assuming that C has a minimal element. If C interprets $\mathcal{G}$ by $\mathcal{I}$ and doesn't have one then we can always construct $C^* = C \cup \{-\infty\}$ and interpret $\mathcal{G}$ on C^* by some $\mathcal{I}^*$ having the same depth and dimension $d + 1$ (add $-\infty$ as a parameter). So instead of taking $K = K(n, d)$ we use $K = K(n, d + 1)$ for getting a contradiction.

The discussions above justify the following definition:

Definition 7.5. Let $n, d \in \mathbb{N}$, and $\langle t_k : k < |T_{n,d}| \rangle$ be the list of the possibilities $T_{n,d}$.

(1) $\mathcal{T} = (\lambda, \bar{P})$ is a *pre-chain* if $\lambda > \aleph_0$ is a regular cardinal and $\bar{P} = \langle P_k : k < |T_{n,d}| \rangle$ is a partition of λ.

(2) We identify $(\lambda, \bar{P})$ with $\langle s_i : i < \lambda \rangle$ when $i \in P_k \iff s_i = t_k$.

(3) $((\lambda, \bar{P}), E)$ is a *guess* for $(C, \bar{A})$ if:

(i) $E = \langle \alpha_i : i < \lambda \rangle \subseteq \lambda$ is cofinal in C and $\alpha_0 = \min(C)$,

(ii) $\bar{A} \subseteq C$ and $\lg(\bar{A}) = d$,

(iii) $\mathrm{Th}^n(C; \bar{A}) \restriction_{[\alpha_i, \alpha_{i+1})} = s_i$ when $(\lambda, \bar{P}) = \langle s_i : i < \lambda \rangle$.

Next we claim that the guesses (which are well ordered chains of the correct cardinality) represent faithfully the guessed chain.

Definition 7.6. Suppose that
(a) C is a chain with cofinality $\lambda > \aleph_0$,
(b) $\bar{A}, \bar{B} \subseteq C$ have length d,
(c) $E = \langle \alpha_i : i < \lambda \rangle$ is cofinal in C and $\alpha_0 = \min(C)$,
(d) $J = \langle \beta_j : j < \lambda \rangle \subseteq \lambda$ is a club and $a \subseteq \lambda$ a semi–club.

The *reduced shuffling* of $\bar{A}$ and $\bar{B}$ along E, J and a, denoted by $[\bar{A}, \bar{B}]_a^{J,E}$ is defined by:

$$[\bar{A}, \bar{B}]_a^{J,E} := \bigcup_{j \in a} (\bar{A} \cap [\alpha_{\beta_j}, \alpha_{\beta_{j+1}})) \cup \bigcup_{j \notin a} (\bar{B} \cap [\alpha_{\beta_j}, \alpha_{\beta_{j+1}}))$$

Fact 7.7. *If $C, \bar{A}, \bar{B}, J$ and a are as above, $((\lambda, \bar{P}_{\bar{A}}), E)$ a guess for $(C, \bar{A})$ and $((\lambda, \bar{P}_{\bar{B}}), E)$ a guess for $(C, \bar{B})$ then*

$$[\bar{P}_{\bar{A}}, \bar{P}_{\bar{B}}]_a^J = \bar{P}_{[\bar{A}, \bar{B}]_a^{J,E}}$$

Proof. Straightforward. □

Definition 7.8. For C, $\bar{A} \subseteq C$, $E \subseteq C$ as above and $a \subseteq \lambda$ a semi–club, define

$$a\text{-Th}_E^n(C; \bar{A}) := a\text{-Th}^n(\lambda, \bar{P}_{\bar{A}})$$

Lemma 7.9. *For every $n, d \in \mathbb{N}$ there is $k(*) = k(n, d) \in \mathbb{N}$ such that if*
1. *C is a chain and $\mathrm{cf}(C) = \lambda > \aleph_0$,*
2. *$\bar{A}, \bar{B} \subseteq C$ are of length d,*
3. *E is cofinal in C,*
4. *$a \subseteq \lambda$ is a semi–club,*

then

$$a\text{-Th}_E^{k(*)}(C; \bar{A}) = a\text{-Th}_E^{k(*)}(C; \bar{B}) \;\Rightarrow\; \mathrm{Th}^n(C; \bar{A}) = \mathrm{Th}^n(C; \bar{B}) = \mathrm{Th}^n(C; [\bar{A}, \bar{B}]_a^{J,E}).$$

Proof. Let $k(*)$ be $k(m(*), |Tn, d|)$ where $m(*)$ is $m(*)(n, d)$ from the preservation theorem, and $k(\alpha, \beta)$ is the "Feferman-Vaught" number as in theorem 1.9. □

Lemma 7.10. *Let $\langle U(\bar{X}, \bar{Z}), E(\bar{X}, \bar{Y}, \bar{Z}), R(\bar{X}, \bar{Y}, \bar{Z}) \rangle$ be a sequence of formulas of dimension d and depth n.*

Then there is $K < \omega$, that depends only on d and n such that in V^P, for no chain C and parameters $\bar{W} \subseteq C$:

(i) *$\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W}) \rangle$ is an interpretation for some $\mathcal{G} \in \Gamma_k$ in C,*

(ii) *D, the minimal (K_1, K_2)-major initial (or final) segment for $\mathcal{I}$, satisfies $\mathrm{cf}(D) = \lambda > \aleph_0$.*

Proof. We will follow the previous procedures, this time choosing K big enough with respect to $k(*)$ as above and not $m(*)$ as usual. Assume that $\mathcal{I}$ is an interpretation of $\mathcal{G} \in \Gamma_K$ and suppose first that $C = D$. Fix $E \subseteq C$ cofinal, of order type λ. As $|E| = \lambda$, it belongs to the intermediate $V^{P_{\leq\lambda}}$.

Let $\langle \bar{U}_i : i < |\mathcal{G}| \rangle$ a list of the representatives and after the preliminary colouring we remain with a semi-homogeneous list $B := \langle \bar{U}_i : i \in S \rangle$, ($|S| = N_1$ big enough) having now the same a-$\mathrm{Th}_E^{k(*)}(C; \bar{U}_i)$. Let $B_1 = \langle \bar{V}_j : j < |S|^{N_2} \rangle$ a list of the representatives for elements separating subsets of B of size N_2 from their complements.

Let $\langle ((\lambda, \bar{P}_{\alpha,\beta,\gamma}, E) : \alpha, \beta, \gamma < N_1^{N_2} \rangle$ be a list of all the guesses for chains of the form
$(C; \bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{W})$ with $\bar{A}_i \in B \cup B_1$ for $i < 3$.

Choose $J \subseteq \lambda$, a $k(*)$ suitable club for all the guesses, and a generic semi–club $a \subseteq \lambda$. Start shuffling $\langle \bar{U}_i : i \in S \rangle$ (i.e. the respective guesses). A statement of the form $\bar{U}_\alpha \sim \bar{U}_\beta$ is translated to "$\mathrm{Th}^{m(*)}(\lambda; \bar{P}_{\bar{U}_\alpha}, \bar{P}_{\bar{U}_\beta})$ is such that $C \models E(\bar{U}_\alpha, \bar{U}_\beta, \bar{W})$".

Repeating the usual steps we get $\langle \bar{U}_i : i \in A \rangle$ such that w.l.o.g $[\bar{U}_i, \bar{U}_j]_a^{J,E} \sim \bar{U}_i$ for every $i < j$ from A. Using genericity we can show also that $[\bar{U}_j, \bar{U}_i]_a^{J,E} \sim \bar{U}_i$ as well.

Now choose a sequence of separating representatives $\langle \bar{Q}_i : i < |A|/2 \rangle$ from B_1 above (so J is suitable for them as well) and get a contradiction as usual.

In the case $D \neq C$ we combine the above with the previous proof: the result of the shuffling of a pair of representatives $\bar{U}_\alpha$ and $\bar{U}_\beta$ (coinciding outside D) is:

$\{$the result of the reduced shuffling of $\bar{U}_\alpha \cap D$ and $\bar{U}_\beta \cap D$ $\} \bigcup \{\bar{U}_\alpha \cap (C \setminus D)\}$.
And we work in D. □

As an ω-random graph is K-random for each $K < \omega$ we proved:

Theorem 7.11. *In V^P:*

A. If $\langle C_K,\ I,\ \{\bar{W}_K : K \in A\}\ \rangle$ is a uniform interpretation of Γ_{fin} in the monadic theory of order and $D_K \subseteq C_K$ are the minimal major initial (or final) segments of the interpretations, then $\mathrm{cf}(D_K) \leq \aleph_0$ for every large enough K.

B. If $\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W}) \rangle$ is an interpretation for RG_ω in a chain C and $D \subseteq C$ is the minimal major initial (or final) segment then $\mathrm{cf}(D) \leq \aleph_0$. □

In the next section we will show that "$\mathrm{cf}(D) \leq \aleph_0$" is impossible even from ZFC.

8. Short chains

Recall that a short chain is a chain that does not embed $(\omega_1, <)$ and the inverse chain $(\omega_1, >)$. Our aim in this section is to prove, from ZFC, the nonexistence of interpretations in short chains. In fact we show (and this is the only possibility when C is short) the nonexistence of interpretations with $\mathrm{cf}(D) \leq \aleph_0$.

Definition 8.1. An interpretation $\mathcal{I}$ of $\mathcal{G} \in \Gamma_K$ in a chain C is a *short interpretation* if the minimal (K_1, K_2)-major initial (or final) segment for $\mathcal{I}$, has cofinality $\aleph_0$.

The case $\mathrm{cf}(D) < \aleph_0$ is impossible:

Fact 8.2. *Let $\mathcal{I}$ be an interpretation of some $\mathcal{G} \in \Gamma_K$ in C. Let D be the (K_1, K_2)-major initial segment. Then (if K is sufficiently big with respect to $d(\mathcal{I})$ and $n(\mathcal{I})$), D does not have a last element.*

Proof. When K is big enough we have $M(K, n, d)/m(K, n, d) > 2$ (by 3.10) and this is what we need. Now if $D = D' \cup \{x\}$ where x is the last element of D then, from the definitions, easily $\#(D)/\#(D') \leq 2$. But D' is minor and this is a contradiction. □

Assumptions. From now on we are assuming towards a contradiction:

1. $\mathcal{I} = \langle U(\bar{X}, \bar{W}), E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})$ is an interpretation for some $\mathcal{G} \in \Gamma_K$ in a chain C. $n(\mathcal{I}) = n$ and $d(\mathcal{I}) = d$,
2. $K = K(n, d)$ is big enough (we will elaborate later),
3. C has a minimal element (almost w.l.o.g by a previous remark),
4. C is the minimal major initial segment for $\mathcal{I}$,
5. $\mathrm{cf}(C) = \aleph_0$.

The next definition is the current replacement of $m(*)$-suitable club:

Definition 8.3. Let $\langle \bar{U}_i : i < i^* \rangle$ be with $\bar{U}_i \subseteq C$, $\lg(\bar{U}_i) = d$. Let $E = \langle \alpha_k : k < \omega \rangle \subseteq C$ be increasing in C. E is an r*-suitable sequence for* $\langle \bar{U}_i : i < i^* \rangle$ if

1. E is cofinal in C and $\alpha_0 = \min(C)$,
2. For every $i < j < i^*$ there is $t_{i,j} \in T_{r,3d}$ such that for every $0 < k < \omega$:
$$\mathrm{Th}^r(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_0, \alpha_k)} = t_{i,j},$$
3. For every $i < j < i^*$ there is $s_{i,j} \in T_{r,3d}$ such that for every $0 < k < l < \omega$:
$$\mathrm{Th}^r(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_k, \alpha_l)} = s_{i,j}.$$

r-suitable sequences exist:

Claim 8.4. 1. Suppose that $\bar{U}, \bar{V} \subseteq C$ are of length d and $E = \langle \alpha_k : k < \omega \rangle$ is r-suitable for $\bar{U}, \bar{V}$. Let $E_1 \subseteq E$ be infinite with $\alpha_0 \in E_1$. Then E_1 is r-suitable for $\bar{U}, \bar{V}$.

2. Let $\bar{U}, \bar{V} \subseteq C$ be as above and let $E = \langle \alpha_k : k < \omega \rangle$ be cofinal with $\alpha_0 = \min(C)$. Then there is $E_1 \subseteq E$ that is r-suitable for $\bar{U}, \bar{V}$.

3. For every finite family $\langle \bar{U}_i : i < i^* \rangle$ with $\bar{U}_i \subseteq C$, $\lg(\bar{U}_i) = d$ there is an r-suitable $E \subseteq C$.

Proof. The first part is immediate. For proving 2. let $\bar{U}$, $\bar{V}$, E be given. Let $f\colon [\omega \setminus \{0\}]^2 \to |T_{r,3d}| \times |T_{r,3d}|$ be a colouring defined (for $0 < k < l < \omega$) by

$$f(k,l) = \langle \mathrm{Th}^r(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_0,\alpha_k)}, \mathrm{Th}^r(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_k,\alpha_l)} \rangle$$

Let $u \subseteq \omega$ be infinite, homogeneous with respect to f ($T_{r,3d}$ is finite). Define $E_1 := \{\alpha_0\} \cup \{\alpha_k : k \in u\}$.

The third part is immediate by 1. and 2. □

We will assume $\sqrt{K} \gg N_0 \gg N_1 \gg N_2 \gg 0$, all depending only on n and d.

Let $M = |\mathcal{G}|$ and let $\langle \bar{U}_i : i < M \rangle$ be a list of representatives for the elements of $\mathcal{G}$. Let $f\colon [M]^2 \to |T_{n+d,3d}|$ be defined by

$$f(i,j) = \mathrm{Th}^{n+d}(C; \bar{U}_i, \bar{U}_j, \bar{W}).$$

We may assume that there is S of size N_0, semi-homogeneous with respect to f and $(m+1)$, where m is the bouquet size of minor segments.

Let $E = \langle \alpha_k : k < \omega \rangle \subseteq C$ be $(n+d)$-suitable for $\langle \bar{U}_i {}^\wedge \bar{W} : i \in S' \rangle$, (by 8.4). Let $S \subseteq S'$ be of size N_1 such that for every $i < j$ and $r < s$ from S, and for every $0 < k < l < \omega$:

$$\mathrm{Th}^{n+d}(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_0,\alpha_k)} = \mathrm{Th}^{n+d}(C; \bar{U}_r, \bar{U}_s, \bar{W}) \restriction_{[\alpha_0,\alpha_k)} := t$$

and

$$\mathrm{Th}^{n+d}(C; \bar{U}_i, \bar{U}_j, \bar{W}) \restriction_{[\alpha_k,\alpha_l)} = \mathrm{Th}^{n+d}(C; \bar{U}_r, \bar{U}_s, \bar{W}) \restriction_{[\alpha_k,\alpha_l)} := s.$$

This is possible by the definition of $(n+d)$-suitability (and as N_0 is big enough). By the composition theorem for every $i < j$ in S:

$$\mathrm{Th}^{n+d}(C; \bar{U}_i, \bar{U}_j, \bar{W}) = t + \sum_{k<\omega} s\,.$$

Definition 8.5. For $u \subseteq \omega$ define the *shuffling of* $\bar{U}_i$ *and* $\bar{U}_j$ *along* u by

$$[\bar{U}_i, \bar{U}_j]_u := \bigcup_{k \in u} (\bar{U}_i \cap [\alpha_k, \alpha_{k+1}) \cup \bigcup_{k \notin u} (\bar{U}_j \cap [\alpha_k, \alpha_{k+1})$$

Claim 8.6. For every $i < j$ in S, for every $u \subseteq \omega$, $C \models U([\bar{U}_i, \bar{U}_j]_u, \bar{W})$.

Proof. By suitability of E and definition of S there are t_0 and s_0 such that for every $i \in S$:

(i) $\mathrm{Th}^n(C; \bar{U}_i, \bar{W}) \restriction_{[\alpha_0, \alpha_k)} = t_0$ for every $0 < k < \omega$,

(ii) $\mathrm{Th}^n(C; \bar{U}_i, \bar{W}) \restriction_{[\alpha_k, \alpha_l)} = s_0$ for every $0 < k < l < \omega$,

(iii) $\mathrm{Th}^n(C; \bar{U}_i, \bar{W}) = t_0 + \sum_{k<\omega} s_0$.

By the definition of shuffling, for every $u \subseteq \omega$ and $i < j$ in S,

$$\mathrm{Th}^n(C; [\bar{U}_i, \bar{U}_j]_u, \bar{W}) = t_0 + \sum_{k<\omega} s_0 = \mathrm{Th}^n(C; \bar{U}_i, \bar{W}).$$

Therefore $C \models U([\bar{U}_i, \bar{U}_j]_u, \bar{W})$. $\square$

Define now:

$e := \{2k : k < \omega\}$,

$o := \{2k + 1 : k < \omega\}$,

$p := \omega \setminus \{0\}$,

$q := \{0\}$.

Let

$$k(i,j) := \min\Big\{k : (k \in S \;\&\; [\bar{U}_i, \bar{U}_j]_e \sim \bar{U}_k) \vee (k = |\mathcal{G}|)\Big\}.$$

By bigness of N_1 there is $A \subseteq S$ of size N_2 such that for every $\bar{U}_i$, $\bar{U}_j$, $\bar{U}_l$ with $i < j < l$ from A, the following, usual, five statements have the same truth value:

$k(j,l) = i$,

$k(i,l) = j$,

$k(i,j) = i$,

$k(i,j) = j$,

$k(i,j) = l$.

Moreover, (the usual proof) if for some $i < j$ in A, $k(i,j) \in A$ then: either for every $i < j$ in A, $k(i,j) = i$ or for every $i < j$ in A, $k(i,j) = j$.

Let's find $i < j$ in A with $k(i,j) \in A$: if we can't then there is some $\bar{V}_A$ that separates between $\mathcal{A}_1 := \{\bar{U}_i : i \in A\}$ and $\mathcal{A}_2 := \{\bar{U}_l : (\exists i < j \in A)([\bar{U}_i, \bar{U}_j]_e \sim \bar{U}_l\}$. i.e.

$$\bigwedge_{\bar{U}_i \in \mathcal{A}_1} \Big(C \models R(\bar{U}_i, \bar{V}_A, \bar{W})\Big) \wedge \bigwedge_{\bar{U}_i \in \mathcal{A}_2} \Big(C \models \neg R(\bar{U}_i, \bar{V}_A, \bar{W})\Big).$$

We may assume that E is suitable also for $\bar{V}_A$ (there are finitely many possibilities for $\bar{V}_A$ after choosing $\langle \bar{U}_i : i \in S' \rangle$). As N_2 is big enough there are $i < j$ in A such that for every $0 < k < l < \omega$

$$\mathrm{Th}^n(C; \bar{U}_i, \bar{V}_A, \bar{W}) \restriction_{[\alpha_0, \alpha_k)} = \mathrm{Th}^n(C; \bar{U}_j, \bar{V}_A, \bar{W}) \restriction_{[\alpha_0, \alpha_k)}$$

and

$$\text{Th}^n(C;\bar{U}_i,\bar{V}_A,\bar{W})\restriction_{[\alpha_k,\alpha_l)} = \text{Th}^n(C;\bar{U}_j,\bar{V}_A,\bar{W})\restriction_{[\alpha_k,\alpha_l)} .$$

It follows that

$$\text{Th}^n(C;\bar{U}_i,\bar{V}_A,\bar{W}) = \text{Th}^n(C;[\bar{U}_i,\bar{U}_j]_e,\bar{V}_A,\bar{W})$$

and $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$, a contradiction. We conclude,

$$(*)\quad (\exists i < j \text{ in } A) \text{ such that } [\bar{U}_i,\bar{U}_j]_e \sim \bar{U}_i \textit{ or } [\bar{U}_i,\bar{U}_j]_e \sim \bar{U}_j.$$

Fact 8.7. *For every i, j in A (in fact in S'): $[\bar{U}_i,\bar{U}_j]_q \sim \bar{U}_j$ and $[\bar{U}_i,\bar{U}_j]_p \sim \bar{U}_i$.*

Proof. Let's prove the first statement (the second is proved similarly). By claim 8.6, $[\bar{U}_i,\bar{U}_j]_q$ is a representative hence is equivalent to $\bar{U}_l$ for some $l < |\mathcal{G}|$. Suppose that $l > j$. By semi-homogeneity of S' (therefore of A) there are $j < l_0 < l_1 \ldots < l_{m+1}$ such that

$$\bigwedge_{r<m+1} \text{Th}^{n+d}(C;\bar{U}_j,\bar{U}_{l_r}) = \text{Th}^{n+d}(C;\bar{U}_j,\bar{U}_l).$$

By definition of q, $\bar{U}_l$ belongs to the vicinity of $\bar{U}_j$. As "belonging to the vicinity" is determined by Th^{n+d} we get $m+1$ pairwise nonequivalent representatives in $[\bar{U}_j]$. This is impossible by lemma 6.4. The same holds if we assume $l < j$. Therefore we must conclude $l = j$ i.e. $[\bar{U}_i,\bar{U}_j]_q \sim \bar{U}_j$. □

Returning to the representatives $\bar{U}_i$ and $\bar{U}_j$ we got in $(*)$ above, suppose first that $[\bar{U}_i,\bar{U}_j]_e \sim \bar{U}_i$. We will show that

(1) $[\bar{U}_i,\bar{U}_j]_e \sim \bar{U}_i \Rightarrow [\bar{U}_i,\bar{U}_j]_o \sim [\bar{U}_i,\bar{U}_j]_q$,

(2) $[\bar{U}_i,\bar{U}_j]_e \sim [\bar{U}_i,\bar{U}_j]_o$.

It will follow that $[\bar{U}_i,\bar{U}_j]_e \sim [\bar{U}_i,\bar{U}_j]_q$ and by the previous fact $\bar{U}_i \sim \bar{U}_j$ which is a contradiction.

For showing (1) it is enough to show that

$$\text{Th}^n(C;\bar{U}_i,[\bar{U}_i,\bar{U}_j]_e,\bar{W}) = \text{Th}^n(C;[\bar{U}_i,\bar{U}_j]_o,[\bar{U}_i,\bar{U}_j]_q,\bar{W}).$$

Remembering how S was chosen we get

$$\begin{aligned}\text{Th}^n(C;\bar{U}_i,[\bar{U}_i,\bar{U}_j]_e,\bar{W})\restriction_{[\alpha_0,\alpha_1)} &= \text{Th}^n(C;\bar{U}_i,\bar{U}_i,\bar{W})\restriction_{[\alpha_0,\alpha_1)}\\ &= \text{Th}^n(C;[\bar{U}_i,\bar{U}_j]_o,[\bar{U}_i,\bar{U}_j]_q,\bar{W})\restriction_{[\alpha_0,\alpha_1)} .\end{aligned}$$

$$\begin{aligned}\text{Th}^n(C;\bar{U}_i,[\bar{U}_i,\bar{U}_j]_e,\bar{W})\restriction_{[\alpha_{2k},\alpha_{2k+1})} &= \text{Th}^n(C;\bar{U}_i,\bar{U}_i,\bar{W})\restriction_{[\alpha_{2k},\alpha_{2k+1})}\\ &= \text{Th}^n(C;\bar{U}_j,\bar{U}_j,\bar{W})\restriction_{[\alpha_{2k},\alpha_{2k+1})}\\ &= \text{Th}^n(C;[\bar{U}_i,\bar{U}_j]_o,[\bar{U}_i,\bar{U}_j]_q,\bar{W})\restriction_{[\alpha_{2k},\alpha_{2k+1})}\end{aligned}$$

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$$\begin{aligned}
&\mathrm{Th}^n(C;\bar U_i,[\bar U_i,\bar U_j]_e,\bar W)\restriction_{[\alpha_{2k+1},\alpha_{2k+2})}\\
&\quad=\mathrm{Th}^n(C;\bar U_i,\bar U_j,\bar W)\restriction_{[\alpha_{2k+1},\alpha_{2k+2})}\\
&\quad=\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,[\bar U_i,\bar U_j]_q,\bar W)\restriction_{[\alpha_{2k+1},\alpha_{2k+2})}
\end{aligned}$$

and $\mathrm{Th}^n(C;\bar U_i,[\bar U_i,\bar U_j]_e,\bar W) = \mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,[\bar U_i,\bar U_j]_q,\bar W)$ follows from the composition theorem.

For (2) note that

$$\begin{aligned}
\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_e,\bar U_i,\bar W) &= \mathrm{Th}^n(C;[\bar U_i,\bar U_j]_e,\bar U_i,\bar W)\restriction_{[\alpha_0,\alpha_1)}\\
&+\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_e,\bar U_i,\bar W)\restriction_{[\alpha_1,\alpha_2)}\\
&+\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_e,\bar U_i,\bar W)\restriction_{[\alpha_2,\alpha_3)}\\
&+\ldots=\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_0,\alpha_1)}\\
&+\mathrm{Th}^n(C;\bar U_j,\bar U_i,\bar W)\restriction_{[\alpha_1,\alpha_2)}\\
&+\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_2,\alpha_3)}\\
&+\mathrm{Th}^n(C;\bar U_j,\bar U_i,\bar W)\restriction_{[\alpha_3,\alpha_4)}\\
&+\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_4,\alpha_5)}+\ldots
\end{aligned}$$

and that

$$\begin{aligned}
\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,\bar U_i,\bar W) &= \mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,\bar U_i,\bar W)\restriction_{[\alpha_0,\alpha_2)}\\
&+\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,\bar U_i,\bar W)\restriction_{[\alpha_2,\alpha_3)}\\
&+\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,\bar U_i,\bar W)\restriction_{[\alpha_3,\alpha_4)}\\
&+\ldots=\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_0,\alpha_2)}\\
&+\mathrm{Th}^n(C;\bar U_j,\bar U_i,\bar W)\restriction_{[\alpha_2,\alpha_3)}\\
&+\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_3,\alpha_4)}\\
&+\mathrm{Th}^n(C;\bar U_j,\bar U_i,\bar W)\restriction_{[\alpha_4,\alpha_5)}\\
&+\mathrm{Th}^n(C;\bar U_i,\bar U_i,\bar W)\restriction_{[\alpha_5,\alpha_6)}+\ldots
\end{aligned}$$

By the composition theorem:

$$\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_e,\bar U_i,\bar W)=\mathrm{Th}^n(C;[\bar U_i,\bar U_j]_o,\bar U_i,\bar W).$$

That is:

$$[\bar U_i,\bar U_j]_e\sim\bar U_i\sim[\bar U_i,\bar U_j]_o.$$

Collecting the results we get:

$[\bar U_i,\bar U_j]_e\sim\bar U_i$ (this is the assumption),
$\bar U_i\sim[\bar U_i,\bar U_j]_o$ (by (2) above),
$[\bar U_i,\bar U_j]_o\sim[\bar U_i,\bar U_j]_q$ (by (1) above),
$[\bar U_i,\bar U_j]_q\sim\bar U_j$ (by fact 8.7).

Therefore, $\bar{U}_i \sim \bar{U}_j$ a contradiction.
We are therefore forced to assume that $[\bar{U}_i, \bar{U}_j]_e \sim \bar{U}_j$ but then we get the same way $\bar{U}_i \sim [\bar{U}_i, \bar{U}_j]_o$ (like (2) above), $[\bar{U}_i, \bar{U}_j]_o \sim [\bar{U}_i, \bar{U}_j]_p$ (like (1) above), $[\bar{U}_i, \bar{U}_j]_p \sim \bar{U}_j$ (by 8.7), and again $\bar{U}_i \sim \bar{U}_j$.

We assumed that C is equal to D, the minimal major initial segment for simplicity. However, if $D \neq C$ then following previous procedures we can easily chop off $C \setminus D$ and basically work inside D, getting a contradiction.

So we have eliminated the possibilities that were left by theorem 7.11 and proved:

Theorem 8.8. *(Non-Interpretability Theorem). There is a forcing notion P such that in V^P the following hold:*
(1) RG_ω is not interpretable in the monadic theory of order.
(2) For every sequence of formulas $\mathcal{I} = \langle U(\bar{X}, \bar{Z}),\ E(\bar{X}, \bar{Y}, \bar{Z}),\ R(\bar{X}, \bar{Y}, \bar{Z})\rangle$ there is $K^* < \omega$, (effectively computable from $\mathcal{I}$), such that for no chain C, $\bar{W} \subseteq C$, and $K \geq K^*$ does $\langle U(\bar{X}, \bar{W}),\ E(\bar{X}, \bar{Y}, \bar{W}), R(\bar{X}, \bar{Y}, \bar{W})\rangle$ interpret RG_K in C.
(3) The above propositions are provable in ZFC. if we restrict ourselves to the class of short chains. □

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