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JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 162 (2001) 367–379

www.elsevier.com/locate/jpaa

Spectra of the Γ-invariant of uniform modules

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Received 4 December 1998; received in revised form 25 April 2000
Communicated by A. Blass

Abstract

For a ring R, denote by $\mathrm{Spec}_\Gamma(\kappa, R)$ the κ-spectrum of the Γ-invariant of strongly uniform right R-modules. Recent realization techniques of Goodearl and Wehrung show that $\mathrm{Spec}_\Gamma(\aleph_1, R)$ is full for a suitable von Neumann regular algebra R, but the techniques do not extend to cardinals $\kappa > \aleph_1$. By a direct construction, we prove that for any field F and any regular uncountable cardinal κ there is an F-algebra R such that $\mathrm{Spec}_\Gamma(\kappa, R)$ is full. We also derive some consequences for the Γ-invariant of strongly dense lattices of two-sided ideals, and for the complexity of Ziegler spectra of infinite-dimensional algebras.

MSC: 16D70; 16D50; 06C05; 03C60

0. Introduction

The Γ-invariant method introduced by Eklof in [3,4] provides an efficient tool for classification of algebraic objects which are defined by existence of infinite filtrations of particular forms. The method has been used to develop a structure theory of almost free groups [6], uniserial modules [15], and bilinear spaces [1,2].

More recently, Γ-invariants were defined also in the dual setting, for objects possessing dual filtrations. This resulted in a classification of dense lattices [7], and of strongly uniform modules [17,18].

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[1] Research supported by a Fulbright Scholarship at UCI.

PII: S0022-4049(00)00118-3

For a regular uncountable cardinal κ, denote by $B(\kappa)$ the Boolean algebra consisting of all subsets of κ modulo the filter of subsets containing a closed unbounded set. The Γ-invariant of objects of dimension κ takes values in $B(\kappa)$. The value measures an obstruction for an object of dimension κ to have a certain algebraic property. For example, for almost free groups, the property is "to be a free group" [5]. For bilinear spaces, the property is "to decompose orthogonally into subspaces of dimension $< \kappa$" [2]. For dense lattices, it is "to be relatively complemented" [7], etc.

For each Γ-invariant, two natural problems arise:

(1) Given a regular uncountable cardinal κ and $i \in B(\kappa)$, is there an object of dimension κ whose Γ-invariant value equals i?

The set of all $i \in B(\kappa)$ for which the answer to (1) is positive is called the *κ-spectrum* of the Γ-invariant, and denoted by $\mathrm{Spec}_\Gamma(\kappa)$. The κ-spectrum is said to be *full* provided that $\mathrm{Spec}_\Gamma(\kappa) = B(\kappa)$, [2].

(2) For $i \in \mathrm{Spec}_\Gamma(\kappa)$, describe all the objects of dimension κ whose Γ-invariant value equals i.

Solutions to problems (1) and (2) depend substantially on the particular form of the Γ-invariant. For almost free groups, the κ-spectrum is full for each $\kappa = \aleph_n$, $n < \omega$, [11, Theorem 5.6], but the fullness for $\kappa = \aleph_{\omega^2+1}$ is independent of ZFC [6,9]. For bilinear spaces, the κ-spectrum is full for $\kappa = \aleph_1$ [1], but it is not full for any regular $\kappa \geq \aleph_2$ [16, Lemma 2]. For dense lattices, the κ-spectrum is full for all regular uncountable cardinals κ [7, Theorem 1.15].

Since isomorphic objects have the same value of the Γ-invariant, fullness of the κ-spectrum always implies that there exist many (at least 2^κ) non-isomorphic objects of dimension κ. In that case, (2) gives a strategy for a fine classification of all objects of dimension κ.

In the present paper, we provide a complete solution to problem (1) for the Γ-invariant of strongly uniform modules introduced in [17]. Answering the questions of [17, Section 3, Problem 3; 7, Section 2; 18, Section 2], we prove that the κ-spectrum is full for all regular uncountable cardinals κ. Our main result is as follows:

Theorem 2.7. *Let λ be an uncountable cardinal and F be a field. Then, there exists an F-algebra R such that for any regular uncountable cardinal $\kappa \leq \lambda$ and any $i \in B(\kappa)$ there is a strongly uniform module $L \in$ Mod-R such that $\mathrm{End}_R(L) = F$ and $\Gamma(L) = i$. In particular, $\mathrm{Spec}_\Gamma(\kappa, R)$ is full.*

Section 1 contains basic facts about strongly uniform modules. The proof of Theorem 2.7 is presented in Section 2. In Section 3, we deal with consequences for the Γ-invariant of two-sided ideal lattices. We also relate our construction to the Goodearl–Wehrung one (cf. [8, Theorem 4.4; 18, Theorem 2.4]). The latter works only for $\kappa = \aleph_1$, but provides for additional properties of the algebras and modules. In Section 4, we derive consequences for the structure of Ziegler spectra of infinite-dimensional algebras.

1. Strongly uniform modules

Let R be an associative ring with unit. Denote by $L_2(R)$ the lattice of all two-sided ideals of R, and by Mod-R the category of all (unitary right R-) modules. If $M \in$ Mod-R, then $\mathrm{End}_R(M)$ denotes the endomorphism ring of M. (Endomorphisms are always written as acting on the opposite side from scalars).

A non-zero module $U \in$ Mod-R is called *uniform* provided that $V \cap W \neq 0$ for all non-zero submodules V and W of U. So uniform modules coincide with non-zero submodules of indecomposable injective modules. Uniform modules play an important role in module theory: for example, they form building blocks for Goldie dimension theory of modules, [10]. (For the model-theoretic role of injective uniform modules, we refer to [13,14]; see also Section 4.)

A trivial sufficient condition for uniformity of a module over an arbitrary ring is the existence of a minimal non-zero submodule. Such uniform modules are called *cocyclic*. Cocyclic modules are exactly the strongly uniform modules of dimension 1 in the sense of the following:

Definition 1.1. Let R be a ring and $U \in$ Mod-R. A sequence of non-zero submodules of U, $\mathscr{U} = (U_\alpha \mid \alpha < \kappa)$, is called a *c.d.c.* in U provided that $\mathscr{U}$ is

- continuous ($U_0 = U$, and $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$ for all limit ordinals $\alpha < \kappa$),
- strictly decreasing ($U_{\alpha+1} \subset U_\alpha$ for all $\alpha < \kappa$), and
- cofinal (for each non-zero submodule $V \subseteq U$ there is $\alpha < \kappa$ such that $U_\alpha \subseteq V$).

U is *strongly uniform* provided that there is a c.d.c. in U. The ordinal κ is called the *length* of $\mathscr{U}$. The least ordinal κ such that there is a c.d.c. $\mathscr{U}$ of length κ in U is called the *dimension* of U.

It is easy to see that any strongly uniform module U is uniform, and either $d = 1$ or d is a regular infinite cardinal, where d is the dimension of U.

Clearly, $d = 1$ iff U is cocyclic. Moreover, any module with a countable submodule lattice is uniform iff it is strongly uniform. This is not true in general: if $R = k[x]$ is the polynomial ring of one variable x over a field k then $U = R$ is uniform, but U is strongly uniform iff k is countable, cf. [17, Section 2].

Definition 1.2. Let U be a strongly uniform module. Let $0 \neq V \subset W \subseteq U$. Then W is *complemented* over V (in U) provided that there is a submodule $X \subseteq U$ such that $W \cap X = V$ and $W + X = U$. For example, U is complemented over any $0 \neq V \subset U$.

The case of the least infinite dimension, $d = \aleph_0$, is quite easy. Let U be a strongly uniform module of dimension $\aleph_0$. It is easy to see that either

(i) there is a c.d.c. $\mathscr{U}$ of length ω in U such that U_α is complemented over U_β for all $\alpha < \beta < \omega$, or

(ii) there is a c.d.c. $\mathscr{U}$ of length ω in U such that U_α is not complemented over U_β for all $0 \neq \alpha < \beta < \omega$.

In the former case, U is called *complementing*; in the latter, U is *narrow*. We refer to [17, Section 2; 7, Section 2] for properties and constructions of complementing and narrow modules of dimension $\aleph_0$.

For the more complex case of dimension $d \geq \aleph_1$, we employ the method of Γ-invariants as in [17, Section 2]:

Definition 1.3. Let κ be a regular uncountable cardinal. For any $E \subseteq \kappa$, define

$$\bar{E} = \{D \subseteq \kappa \mid \exists C \subseteq \kappa\text{: } C \text{ closed and unbounded in } \kappa \,\&\, D \cap C = E \cap C\}.$$

So $\bar{E} \in B(\kappa)$.

Let U be a strongly uniform module of dimension κ. Let $\mathscr{U} = (U_\alpha \mid \alpha < \kappa)$ be a c.d.c. in U. Let

$$E_{\mathscr{U}} = \{\alpha < \kappa \mid \exists \beta\text{: } \alpha < \beta < \kappa \,\&\, U_\alpha \text{ is not complemented over } U_\beta\}.$$

Define $\Gamma(U) = \overline{E_{\mathscr{U}}}$. By [7, Lemma 1.8], $\Gamma(U)$ does not depend on the particular choice of the c.d.c. $\mathscr{U}$.

$\Gamma(U)$ is called the *Γ-invariant value* of U. We denote by $\mathrm{Spec}_\Gamma(\kappa, R)$ the *κ-spectrum* of Γ, i.e., the set of all $i \in B(\kappa)$ such that there is a strongly uniform module $U \in$ Mod-R with $\Gamma(U) = i$. If $\mathscr{R}$ is a class of rings we define $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R}) = \bigcup_{R \in \mathscr{R}} \mathrm{Spec}_\Gamma(\kappa, R)$, the *$\kappa$-spectrum* of Γ for $\mathscr{R}$. A κ-spectrum is said to be *full* provided that it is equal to the whole of $B(\kappa)$.

The size of $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R})$ depends substantially on the properties of $\mathscr{R}$:

Theorem 1.4. (i) $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R}) = \{\bar{\kappa}\}$ *for all* $\kappa > \aleph_0$ *provided that* $\mathscr{R}$ *is the class of all commutative rings or* $\mathscr{R}$ *is the class of all rings with right Krull dimension.*

(ii) *For any field F,* $\mathrm{Spec}_\Gamma(\aleph_1, \mathscr{R})$ *is full provided that* $\mathscr{R}$ *is the class of all locally matricial F-algebras.*

Proof. (i) is by [17, Theorems 2.10 and 2.12], and (ii) by [18, Theorem 2.4]. □

The proof of (ii) makes use of a much stronger result, namely of a realization theorem for ideal lattices of bounded distributive lattices of size $\leq \aleph_1$ by ideal lattices of von Neumann regular rings (cf. [8, Theorem 4.4; 18, Theorem 2.4]). In particular, the strongly uniform modules are constructed with the additional property that they are *distributive*, that is, their submodule lattices are distributive.

Nevertheless, by a result of Wehrung [19, Corollary 2.5] the proof of (ii) does not extend to any $\kappa > \aleph_1$ (see also [12, Corollary 4.4]). It remains open whether $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R})$ is full for some $\kappa > \aleph_1$ where $\mathscr{R}$ is the class of all von Neumann regular rings. [2]

[2] *Added in proof*: By a different approach, Pavel Růžička recently proved that the spectrum is full for any regular uncountable cardinal κ when $\mathscr{R}$ is the class of all locally matricial algebras (see also footnote 3).

2. Fullness of the κ-spectra

In this section, we will prove that the $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R})$ is full for each regular $\kappa \geq \aleph_1$ where $\mathscr{R}$ is the class of all rings:

Let F be a field and κ be a regular uncountable cardinal. Fix $S \subseteq \kappa$ with $0 \in S$.

For each $\alpha < \kappa$, put

$$Y_\alpha = \{\langle(\alpha_i, \beta_i); i \leq n\rangle \mid n < \omega; \alpha_n = \alpha; \alpha_i < \beta_i < \kappa \text{ for all } i \leq n;$$
$$\alpha_i \in S \text{ for all } 0 < i \leq n; \alpha_i < \alpha_{i+1} \text{ for all } i < n\}.$$

Observe that $Y_\alpha = \{\langle(\alpha, \beta)\rangle \mid \alpha < \beta < \kappa\}$ if $\alpha \notin S$.

For each sequence $y \in Y_\alpha$, $y = \langle(\alpha_i, \beta_i);\ i \leq n\rangle$, put $\mathrm{amax}(y) = \alpha_n$, and $\mathrm{bmax}(y) = \max_{i \leq n} \beta_i$ ($> \mathrm{amax}(y)$).

Let $Y_{<\alpha} = \bigcup_{\beta<\alpha} Y_\beta$ and $Y_{\geq\alpha} = \bigcup_{\alpha \leq \beta < \kappa} Y_\beta$. Put $Y = \bigcup_{\alpha<\kappa} Y_\alpha$. Note that $\mathrm{card}(Y) = \kappa$.

Denote by L the F-linear space with the F-basis $\{x_\eta \mid \eta \in Y\}$, so

$$L = \bigoplus_{\eta \in Y} F x_\eta$$

has dimension κ. For each $\alpha < \kappa$, denote by L_α the F-subspace of L generated by $\{x_\eta \mid \eta \in Y_{\geq\alpha}\}$. For $\alpha < \beta < \kappa$ and $\alpha \in S$, we define a subspace

$$L_{\alpha\beta} = \bigoplus_{\eta \in Y_{<\alpha}} F(x_\eta - x_{\eta\frown(\alpha,\beta)}) \oplus L_\beta.$$

Definition 2.1. Let $v, \rho \in Y$ be such that

$(*)$ $\quad \mathrm{amax}(\rho) \geq \mathrm{bmax}(v)$.

We will define $T = T_{v\rho} \in \mathrm{End}_F(L)$. For $\eta \in Y$, $x_\eta T$ will always be zero or x_θ, where ρ is an initial segment of θ which is defined by induction as follows:

- if v is not an initial segment of η then $x_\eta T = 0$;
- if $\eta = v$ then $x_\eta T = x_\rho$;
- if v is a proper initial segment of η, so $\eta = \eta' \frown (\alpha, \beta)$ and v is an initial segment of η', we have $x_{\eta'} T = x_{\rho'}$ for some $\rho' \in Y$. If $\rho' \in Y_{\geq\alpha}$, we define $x_\eta T = x_{\rho'}$. If $\rho' \in Y_{<\alpha}$, we define $x_\eta T = x_{\rho'\frown(\alpha,\beta)}$.

Denote by R, the unital F-subalgebra of $\mathrm{End}_F(L)$ generated by the set $\{T_{v\rho} \mid v, \rho \in Y,\ \mathrm{amax}(\rho) \geq \mathrm{bmax}(v)\}$. Then $L = L_0$ is canonically a (right R-) module.

Lemma 2.2. (i) *L_α is a submodule of L for each $\alpha < \kappa$. Moreover, we have $L_\alpha = x_{\langle(0,1),(\alpha,\alpha+1)\rangle} R$ for each $0 \neq \alpha \in S$.*

(ii) *$L_{\alpha\beta}$ is a submodule of L for all $\alpha < \beta < \kappa$ such that $\alpha \in S$.*

Proof. Let $T = T_{v\rho}$, where $v, \rho \in Y$ satisfy $(*)$.

(i) Let $\eta \in Y_{\geq\alpha}$. If $v \in Y$ is not an initial segment of η then $x_\eta T = 0$. If $\eta = v$ then $x_\eta T = x_\rho \in L_\alpha$ by the assumption $(*)$.

Let v be a proper initial segment of η, so $\eta=\eta' \frown (\alpha',\beta')$ for some $\alpha \leq \alpha' < \beta'$, v is an initial segment of η', and $x_{\eta'}T = x_{\rho'}$ for some $\rho' \in Y$.

If $\rho' \in Y_{<\alpha'}$ then $x_\eta T = x_{\rho' \frown(\alpha',\beta')} \in L_\alpha$. If $\rho' \in Y_{\geq\alpha'}$ then $x_\eta T = x_{\rho'} \in L_\alpha$.

For $0 \neq \alpha \in S$, let $\mu = \langle(0,1)\rangle$ and $\mu' = \langle(0,1),(\alpha,\alpha+1)\rangle$. Then for each $\eta \in Y_{\geq\alpha+1}$, we have $x_\eta = x_{\mu'}T_{\mu'\eta}$. Similarly, for each $\eta \in Y_\alpha$ we have $x_\eta = x_{\mu'}T_{\mu\eta}$.

(ii) In view of (i), it suffices to prove that $(x_\eta - x_{\eta\frown(\alpha,\beta)})T \in L_{\alpha\beta}$ for all $\eta \in Y_{<\alpha}$.

If v is not an initial segment of $\eta \frown (\alpha,\beta)$ then $(x_\eta - x_{\eta\frown(\alpha,\beta)})T = 0$.

If $v = \eta \frown (\alpha,\beta)$ then $(x_\eta - x_{\eta\frown(\alpha,\beta)})T = (-x_{\eta\frown(\alpha,\beta)})T = -x_\rho \in L_\beta$ by $(*)$.

If $v=\eta$ then $(x_\eta - x_{\eta\frown(\alpha,\beta)})T = x_\rho - (x_{\eta\frown(\alpha,\beta)})T$. If $\rho \in Y_{<\alpha}$, then $(x_{\eta\frown(\alpha,\beta)})T = x_{\rho\frown(\alpha,\beta)}$, so $(x_\eta - x_{\eta\frown(\alpha,\beta)})T \in L_{\alpha\beta}$. If $\rho \in Y_{\geq\alpha}$, then $(x_{\eta\frown(\alpha,\beta)})T = x_\rho$, so $(x_\eta - x_{\eta\frown(\alpha,\beta)})T = 0$.

Assume that v is a proper initial segment of η, so $\eta = \eta' \frown (\alpha',\beta')$ for some $\alpha' < \alpha$, and v is an initial segment of η'. We have $x_{\eta'}T = x_{\rho'}$ where $\rho' \in Y$.

If $\rho' \in Y_{<\alpha'}$ then $x_\eta T = x_{\rho'\frown(\alpha',\beta')}$ while $(x_{\eta\frown(\alpha,\beta)})T = x_{\rho'\frown(\alpha',\beta')\frown(\alpha,\beta)}$, because $\alpha' < \alpha$. So $(x_\eta - x_{\eta\frown(\alpha,\beta)})T \in L_{\alpha\beta}$.

Assume $\rho' \in Y_{\geq\alpha'}$, so $x_\eta T = x_{\rho'}$. If $\rho' \in Y_{<\alpha}$, then $(x_{\eta\frown(\alpha,\beta)})T = x_{\rho'\frown(\alpha,\beta)}$, so $(x_\eta - x_{\eta\frown(\alpha,\beta)})T \in L_{\alpha\beta}$. If $\rho' \in Y_{\geq\alpha}$, then $(x_{\eta\frown(\alpha,\beta)})T = x_{\rho'}$, so $(x_\eta - x_{\eta\frown(\alpha,\beta)})T = 0$. □

Lemma 2.3. $\mathscr{L} = (L_\alpha \mid \alpha < \kappa)$ is a c.d.c. in L.

Proof. Clearly, $\mathscr{L}$ is strictly decreasing and continuous. Let X be a non-zero submodule of L and take $0 \neq x \in X$. So $x = \sum_{\eta\in Y} f_\eta x_\eta$ and $f_\eta = 0$ for almost all, but not all, $\eta \in Y$. Take $v \in Y$ such that $f_v \neq 0$ and v is not a proper initial segment of any $\eta \in Y$ with $f_\eta \neq 0$. Let $\alpha = \mathrm{bmax}(v)$. Take any $\rho \in Y_{\geq\alpha}$ and let $T = T_{v\rho}$. Then $xT = (f_v x_v)T = f_v x_\rho$, so $x_\rho \in X$. This proves that $L_\alpha \subseteq X$, and $\mathscr{L}$ is cofinal. □

Proposition 2.4. *Let $\gamma < \kappa$. Then $\mathscr{L}_\gamma = (L_\alpha \mid \gamma \leq \alpha < \kappa)$ is a c.d.c. in L_γ such that $E_{\mathscr{L}_\gamma} = [\gamma,\kappa) \setminus S$. In particular, $\Gamma(L_\gamma) = \overline{\kappa \setminus S}$.*

Proof. By Lemma 2.3, $\mathscr{L}_\gamma$ is a c.d.c. in L_γ.

We prove that L_α is complemented over L_β in L_γ provided that $\gamma < \alpha < \beta < \kappa$ and $\alpha \in S$. By modularity, it is enough to prove this for $\gamma = 0$:

Clearly, $L = L_\alpha + L_{\alpha\beta}$. Take $x \in L_\alpha \cap L_{\alpha\beta}$. Then $x = y + z$, where $y \in \bigoplus_{\eta\in Y_{<\alpha}} F(x_\eta - x_{\eta\frown(\alpha,\beta)})$ and $z \in L_\beta$. Since $x \in L_\alpha$, we have $y = 0$, so $L_\beta = L_\alpha \cap L_{\alpha\beta}$.

It remains to prove that L_α is not complemented over L_β in L_γ provided that $\gamma < \alpha < \beta < \kappa$ and $\alpha \notin S$:

Assume there is a submodule X in L such that $L_\gamma = L_\alpha + X$ and $L_\beta = L_\alpha \cap X$. Let $v = \langle(\gamma,\gamma+1)\rangle \in Y_{\geq\gamma}$, $\rho = \langle(\alpha,\alpha+1)\rangle \in Y_{\geq\alpha}$ and take $T = T_{v\rho}$. By assumption, there are $x \in X$ and $y \in L_\alpha$ such that $x_v = x + y$. Since $\alpha \notin S$, we have $(L_\alpha)T \subseteq L_{\alpha+1}$. So $xT = x_\rho - yT \in L_\alpha \setminus L_{\alpha+1}$. On the other hand, $xT \in X$, so $xT \in L_\beta$, a contradiction. □

The following lemma says that each L_α, $\alpha < \kappa$, is a rigid module in the sense that $\mathrm{End}_R(L_\alpha)$ is minimal possible.

Lemma 2.5. $\mathrm{End}_R(L_\alpha) = F$ *for all* $\alpha < \kappa$.

Proof. Let $0 \neq e \in \mathrm{End}_R(L_\alpha)$.

First, we prove that $\mathrm{Ker}\, e = 0$. If not, by Lemma 2.3, there is $\beta < \kappa$ such that $L_\beta \subseteq \mathrm{Ker}\, e \cap \mathrm{Im} e$. Take $v \in Y_{\geq\beta}$. Let $x \in L_\alpha$ be such that $ex = x_v$. Then $x = \sum_{\eta \in Y_{\geq\alpha}} f_\eta x_\eta$, and the set $A = \{\eta \in Y_{\geq\alpha} \mid f_\eta \neq 0\}$ is finite. W.l.o.g., we may assume that $ex_\eta \neq 0$ for all $\eta \in A$. Then, for each $\eta \in A$, v is not an initial segment of η. Take $\rho \in Y_{\geq\beta}$ such that $(*)$ holds. Put $T = T_{v\rho}$. Then $0 = e(xT) = (ex)T = x_\rho$, a contradiction.

Next, we prove that for each $\eta \in Y_{\geq\alpha}$, there is $f_\eta \in F$ such that $ex_\eta = f_\eta x_\eta$. Clearly, $ex_\eta = \sum_{\tau \in Y_{\geq\alpha}} f_\tau x_\tau$, and the set $A = \{\tau \in Y_{\geq\alpha} \mid f_\tau \neq 0\}$ is finite. Since $\mathrm{Ker}\, e = 0$, at least one $\tau \in A$ must contain η as an initial segment. Let $\tau_0 \in A$ be maximal such. If $\tau_0 \neq \eta$, then taking $\rho \in Y_{\geq\alpha}$ such that $\mathrm{amax}(\rho) \geq \mathrm{bmax}(\tau_0)$, we see that $T_{\tau_0\rho}$ maps ex_η to $f_{\tau_0} x_\rho$, while $x_\eta T_{\tau_0\rho} = 0$, a contradiction. This shows that $\tau_0 = \eta$.

Let $\tau \in A \setminus \{\eta\}$ be maximal. If τ is not an initial segment of η, then taking $\rho \in Y_{\geq\alpha}$ such that $\mathrm{amax}(\rho) \geq \mathrm{bmax}(\tau)$, we see that $T_{\tau\rho}$ maps x_η to 0, but $(ex_\eta)T_{\tau\rho} = f_\tau x_\rho$, a contradiction. So τ is a proper initial segment of η, $\eta = \eta' \frown (\beta,\gamma)$, and τ is an initial segment of η'. Take $\rho \in Y$ such that $\mathrm{amax}(\rho) \geq \mathrm{bmax}(\eta)$ and let $T = T_{\tau\rho}$. Then $x_{\eta'} T = x_{\rho'}$ for some ρ' containing ρ as an initial segment. Then $x_\eta T = x_{\rho'}$, so $ex_{\rho'} = (ex_\eta)T = f_\tau x_\rho + f_\eta x_{\rho'}$. On the other hand, $ex_{\rho'} = (ex_\eta)T_{\eta\rho'} = f_\eta x_{\rho'}$. So $f_\tau = 0$, a contradiction.

Finally, we prove that $f_v = f_\rho$ for all $v, \rho \in Y_{\geq\alpha}$. This is clear when $(*)$ holds. But then $f_v = f_\rho = f'_v$, where $v, v' \in Y_{\geq\alpha}$ are arbitrary, and $\rho = \langle(\beta,\gamma)\rangle$ is such that $(*)$ holds and $\beta = \mathrm{amax}(\rho) \geq \mathrm{bmax}(v')$. □

Theorem 2.6. *Let* κ *be a regular uncountable cardinal and* $i \in B(\kappa)$. *Let* F *be a field and* L *be an* F*-linear space of dimension* κ.

Then there exists an F*-subalgebra*, R, *of* $\mathrm{End}_F(L)$ *such that* L, *viewed as a right* R*-module*, *is strongly uniform with* $\Gamma(L) = i$ *and* $\mathrm{End}_R(L) = F$.

Proof. By Proposition 2.4 and Lemma 2.5. □

In the construction of Theorem 2.6, different elements of $B(\kappa)$ occur as values of the Γ-invariant of modules over different algebras. This is easily improved in our main result:

Theorem 2.7. *Let* λ *be an uncountable cardinal and* F *be a field. Then there exists an* F*-algebra* R *such that for any regular uncountable cardinal* $\kappa \leq \lambda$ *and any* $i \in B(\kappa)$ *there is a strongly uniform module* $L \in \text{Mod-}R$ *such that* $\mathrm{End}_R(L) = F$ *and* $\Gamma(L) = i$. *In particular*, $\mathrm{Spec}_\Gamma(\kappa, \mathscr{R})$ *is full.*

Proof. For each regular uncountable cardinal $\kappa \leq \lambda$ and each $i \in B(\kappa)$, denote by $R_{\kappa i}$ the F-algebra, and by $L_{\kappa i}$ the right $R_{\kappa i}$-module, constructed in Theorem 2.6. Let $R = \prod_{\kappa,i} R_{\kappa i}$ (the ring direct product). Then each $L = L_{\kappa i}$ is canonically a right R-module, and the R- and $R_{\kappa i}$-submodule lattices of L coincide. It follows that $\Gamma(L)=i$. Moreover, $\mathrm{End}_R(L)=\mathrm{End}_{R_{\kappa i}}(L)=F$. □

3. The Γ-invariant of two-sided ideal lattices

The Γ-invariant of strongly uniform modules as defined in Section 1 is completely determined by properties of submodule lattices of the respective modules. In fact, this is a particular instance of a more general Γ-invariant, the Γ-invariant of strongly dense lattices [7, Section 1].

Recall that a bounded modular lattice $(A,\wedge,\vee,0,1)$ is *strongly dense* provided that it contains a continuous strictly decreasing cofinal chain (*c.d.c.*) consisting of non-zero elements of A. If $0\neq b<a\leq 1\in A$, then a is *complemented* over b provided that there exists $c\in A$ with $a\wedge c=b$ and $a\vee c=1$. As in Definition 1.3, we can define for each c.d.c. $\mathscr{U}$ of length κ in A the set $\overline{E_{\mathscr{U}}}\in B(\kappa)$. Then $\Gamma(A)=\overline{E_{\mathscr{U}}}$ does not depend on the choice of the c.d.c. $\mathscr{U}$, and it is called the *Γ-invariant value* of the lattice A, [7, Section 1].

This Γ-invariant is of particular interest in the case when $A=L_2(S)$, the two-sided ideal lattice of an algebra S. Indeed, the proof of Theorem 1.4(ii) makes essential use of this case: for $\kappa=\aleph_1$, applying a construction due to Goodearl and Wehrung [8, Theorem 4.4] together with [7, Theorem 1.15], one can realize each $i\in B(\aleph_1)$ as $\Gamma(L_2(S))$ for a locally matricial F-algebra S. In particular, $L_2(S)$ is a distributive lattice. Let $R=S\otimes_F S^{\mathrm{op}}$, where S^{op} is the opposite F-algebra of S. Then S is a (right R-) module whose submodule lattice is canonically isomorphic to $L_2(S)$. So S is a strongly uniform module of dimension κ. Moreover, $i=\Gamma(L_2(S))=\Gamma(S)$, so i is realized as the Γ-invariant value of a distributive strongly uniform module.

For $\kappa>\aleph_1$, the question of the possible values of the Γ-invariant of strongly dense two-sided ideal lattices remains open. [3] Nevertheless, Theorem 2.6 provides a realization of any $i\in B(\kappa)$ as $\Gamma(A)$ where A is a lower interval in $L_2(S)$ for an F-algebra S:

Corollary 3.1. *Let F be a field, κ be a regular uncountable cardinal, $i\in B(\kappa)$, R be the F-algebra and L be the module constructed in Theorem* 2.6. *Let*

$$S=\left\{\begin{pmatrix} f & l \\ 0 & r\end{pmatrix} \mid f\in F,\ l\in L,\ r\in R\right\}.$$

Let $I=\left\{\begin{pmatrix} 0 & l \\ 0 & 0\end{pmatrix} \mid l\in L\right\}$. Then S is an F-algebra and $I\in L_2(S)$. Denote by A the interval in $L_2(S)$ consisting of all two-sided ideals contained in I. Then A is a strongly dense lattice of dimension κ and $\Gamma(A)=i$.

[3] *Added in proof*: Recently, Pavel Růžička proved that the ideal lattice of any bounded distributive lattice is isomorphic to the lattice of two-sided ideals of a locally matricial algebra. From [7, Theorem 1.15], it easily follows that the spectrum of the Γ-invariant of strongly dense two-sided ideal lattices is full for any $\kappa>\aleph_1$. More details appear in Růžička's manuscript "Lattices of two-sided ideals of locally matricial algebras and the Γ-invariant problem".

Proof. Clearly, A is isomorphic to the (right R-) submodule lattice of L, so the assertion follows by Theorem 2.6. □

Though our construction in Section 2 applies to an arbitrary regular uncountable cardinal κ, it neither produces R which is von Neumann regular nor L which has a distributive lattice of submodules. So Theorem 1.4(ii) provides a stronger result in the particular case of $\kappa = \aleph_1$:

Lemma 3.2. *Neither of the algebras R appearing in Theorems* 2.6 *and* 2.7 *is von Neumann regular. Neither of the strongly uniform modules L from Theorems* 2.6 *and* 2.7 *is distributive.*

Proof. To see that R in Theorem 2.6 (and hence in 2.7) is not von Neumann regular take $\alpha+1 < \beta < \kappa$, $\mu = \langle(\alpha,\alpha+1)\rangle$ and $\phi = \langle(\alpha+1,\beta)\rangle$. Then $T_{\mu\phi}$ has no pseudo-inverse in R.

Indeed, if $T \in R$ is such that $T_{\mu\phi}TT_{\mu\phi} = T_{\mu\phi}$, then $x_\phi T \in T_{\mu\phi}^{-1}(x_\phi) \cap L_{\alpha+1}$ by Lemma 2.2(i). It follows that $x_\phi T = x_\tau$, where $\tau = \mu \frown (\alpha+1,\gamma)$ for some $\alpha+1 < \gamma < \kappa$. Now, any $T_{\nu\rho}$, with $\nu,\rho \in Y$ satisfying $(*)$, maps x_ϕ to zero or to $x_{\phi'} \in L_{\alpha+2}$. On the other hand, we have $T = f.1 + t \in R$, where $f \in F$ and t is an F-linear combination of finite products of elements of the form $T_{\nu\rho}$, with $\nu,\rho \in Y$ satisfying $(*)$. Then $x_\tau = x_\phi T = fx_\phi + x_\phi t$, where $x_\phi t \in L_{\alpha+2}$, a contradiction.

To see that the module L in Theorem 2.6 (and hence in 2.7) is not distributive, fix $\alpha < \kappa$, and for each $\alpha+1 < \beta < \kappa$ let $\phi_\beta = \langle(\alpha+1,\beta)\rangle$. Then $(x_{\phi_\beta} + L_{\alpha+2})r = fx_{\phi_\beta} + L_{\alpha+2}$ for any $r = f.1 + t \in R$, where $f \in F$ and t is an F-linear combination of finite products of elements of the form $T_{\nu\rho}$, with $\nu,\rho \in Y$ satisfying $(*)$. So the R-submodules, and the F-subspaces, of $N_\alpha = \bigoplus_{\alpha+1<\beta<\kappa}(x_{\phi_\beta} + L_{\alpha+2})R \subseteq L/L_{\alpha+2}$ coincide. Since $\dim_F(N_\alpha) = \kappa > 1$, the module N_α, and hence L, is not distributive. □

The results above suggest the question of the structure of $L_2(R)$ for the F-algebra R constructed in Theorem 2.6. We will prove that $L_2(R)$ is strongly dense, but in contrast with the Goodearl–Wehrung construction, $L_2(R)$ is always narrow. First, we need more information about the arithmetic of the algebra R:

Let $\nu,\nu',\rho,\rho' \in Y$ be such that $(*)$ holds and $\mathrm{amax}(\rho') \geq \mathrm{bmax}(\nu')$. We will compute $T_{\nu\rho}T_{\nu'\rho'}$:

(1) If ν' is not an initial segment of ρ and ρ is not an initial segment of ν', then $T_{\nu\rho}T_{\nu'\rho'} = 0$.

(2) If $\rho = \nu' \frown \tau$, then $T_{\nu\rho}T_{\nu'\rho'} = T_{\nu,\rho'\frown\tau'}$ where $\tau' = \emptyset$ provided that $\mathrm{amax}(\rho) \leq \mathrm{amax}(\rho')$, and τ' is the final segment of τ consisting of all pairs whose first component is $> \mathrm{amax}(\rho')$ provided that $\mathrm{amax}(\rho) > \mathrm{amax}(\rho')$.

(3) If $\nu' = \rho \frown \tau$ and $\tau \neq \emptyset$, then

$$T_{\nu\rho}T_{\nu'\rho'} = \bigoplus_{\sigma\in X} T_{\nu\frown\sigma\frown\tau,\rho'}$$

where X consists of the empty set and of all elements of $Y_{\leq \mathrm{amax}(\rho)}$ whose initial pair has first component $> \mathrm{amax}(\nu)$.

Further, let $t = \prod_{i \leq n} T_{\nu_i \rho_i}$ where $n < \omega$ and $\mathrm{amax}(\rho_i) \geq \mathrm{bmax}(\nu_i)$ for all $i \leq n$. If $n > 0$ and t is irredundant (in the sense that the product cannot be simplified using (II) for successive factors), then (III) shows that

$$t = \bigoplus_{\sigma_0 \in X_0, \ldots, \sigma_n \in X_n} T_{\nu_0 \frown \sigma_0 \frown \tau_0 \frown \cdots \frown \sigma_n \frown \tau_n, \rho_n}$$

where $\nu_{i+1} = \rho_i \frown \tau_i$ for all $i < n$, $\tau_i \neq \emptyset$ for all $i \leq n$, and for each $i \leq n$, X_i consists of the empty set and of all elements of $Y_{\leq \mathrm{amax}(\rho_i)}$ whose initial pair has first component $> \mathrm{amax}(\nu_i)$. Note that $\mathrm{amax}(\nu_0) < \mathrm{amax}(\rho_0) < \mathrm{amax}(\nu_1) < \cdots < \mathrm{amax}(\rho_{n-1}) < \mathrm{amax}(\nu_n) < \mathrm{amax}(\rho_n)$.

Let $r \in R$. Then r can be expressed as an F-linear combination

$$(**) \qquad r = f.1 + \sum_{j<m} f_j t_j,$$

where $m < \omega$, $f \in F$, $0 \neq f_j \in F$ and t_j is a finite irredundant product of elements of the form $T_{\nu\rho}$ with $\nu, \rho \in Y$ satisfying $(*)$ for each $j < m$.

So each t_j is of the form

$$t_j = \bigoplus_{\sigma_{j0} \in X_{j0}, \ldots, \sigma_{jn_j} \in X_{n_j}} T_{\nu_{j0} \frown \sigma_{j0} \frown \tau_{j0} \frown \cdots \frown \sigma_{jn_j} \frown \tau_{jn_j}, \rho_{jn_j}}$$

(in order to unify our notation, we set $n_j = 0$, $X_j = \{\emptyset\}$ and $\tau_j = \emptyset$ in the case when $t_j = T_{\nu_{j0}\rho_{j0}}$ has exactly one factor).

We will say that $(**)$ is a *canonical form* of r provided that each t_j is irredundant and $t_j \neq t_{j'}$ for all $j \neq j' < m$.

Theorem 3.3. *$L_2(R)$ is a strongly dense lattice of dimension κ and $\Gamma(L_2(R)) = \bar{\kappa}$.*

Proof. For each $\alpha < \kappa$, define

$$I_\alpha = \{r \in R \mid \mathrm{Im}\, r \subseteq L_\alpha\}.$$

The proof is divided into three lemmas.

Lemma 3.4. *Let $r \in R$ be in the canonical form $(**)$. Let $\alpha > 0$. Then $r \in I_\alpha$ iff $f = 0$ and $\mathrm{amax}(\rho_{jn_j}) \geq \alpha$ for all $j < m$. In particular, I_α coincides with the ideal of R generated by all $T_{\nu\rho}$ such that $\nu, \rho \in Y$ satisfy $(*)$ and $\mathrm{amax}(\rho) \geq \alpha$.*

Proof. The 'if' part is clear, since r then maps into L_α.

For the 'only if' part, assume that $r \in L_\alpha$. If $f \neq 0$ then we take $\eta \in Y_0$ such that ν_{j0} is not an initial segment of η for all $j < m$. Then $x_\eta r = f x_\eta \notin L_\alpha$, a contradiction.

Proving indirectly, we can w.l.o.g. assume that $f = 0$ and $\mathrm{amax}(\rho_{jn_j}) < \alpha$ for all $j < m$. Let $i < m$ be such that ρ_{in_i} is minimal. Since $r \in I_\alpha$, we have $\mathrm{card}(J) \geq 2$

where $J=\{j<m \mid \rho_{jn_j}=\rho_{in_i}\}$. Let $j\in J$ be such that v_{j0} is minimal. Since $r\in I_\alpha$, we have $\mathrm{card}(J_0)\geq 2$ where $J_0=\{j'\in J \mid v_{j'0}=v_{j0}\}$.

If there is $k\in J_0$ such that $n_{k0}=0$, then there exists $k'\in J_0$ such that $k'\neq k$ and $t_{k'}=t_k=T_{v_{k0}\rho_{k0}}$ which contradicts the assumption that $(**)$ is canonical.

Otherwise, let $k\in J_0$ be such that $\mathrm{amax}(\sigma_{k0})$ is maximal. Then $\mathrm{card}(J_1)\geq 2$ where $J_1=\{k'\in J_0 \mid \mathrm{amax}(\rho_{k'0})=\mathrm{amax}(\rho_{k0})\}$. Let $l\in J_1$ be such that $\mathrm{amax}(\tau_{l0})$ is minimal. Then $\mathrm{card}(J_2)\geq 2$ where $J_2=\{l'\in J_1 \mid \tau_{l'0}=\tau_{l0}\}$. Proceeding similarly, after finitely many steps we obtain a pair $j\neq j'<m$ such that $t_j=t_{j'}$ which contradicts the assumption that $(**)$ is canonical. $\square$

Note that Lemma 3.4 implies that the canonical form $(**)$ is unique for each $r\in R$. That is, all the irredundant products together with $1\in R$ form an F-basis of R.

Lemma 3.5. *$\mathscr{I}=(I_\alpha \mid \alpha<\kappa)$ is a c.d.c. in $L_2(R)$.*

Proof. Clearly, $I_\alpha\in L_2(R)$. Since $T_{\langle(0,1)\rangle,\langle(\alpha,\alpha+1)\rangle}\in I_\alpha\setminus I_{\alpha+1}$, $\mathscr{I}$ is strictly decreasing. By definition, $I_\alpha=\bigcap_{\beta<\alpha} I_\beta$ for all limit ordinals $\alpha<\kappa$, so $\mathscr{I}$ is continuous.

Let $0\neq r\in R$. We will prove that $T_{v\rho}\in RrR$ for some $v,\rho\in Y$ satisfying $(*)$. Consider the canonical form of r, $(**)$.

If $f\neq 0$ then there is $T_{v\rho}$ satisfying $(*)$ such that $v\in Y_0$ is not an initial segment of ρ_{jn_j} for any $j<m$. Then $rT_{v\rho}=fT_{v\rho}$, so $T_{v\rho}\in RrR$.

Assume $f=0$. Multiplying r by an appropriate $T_{v'\rho'}$ on the right and using (III), we can w.l.o.g. assume that $\rho'=\rho_{jn_j}$ for all $j<m$. Since $(**)$ is canonical, an argument similar to the one in the proof of Lemma 3.4 shows that there exist $\rho''\in Y$ and $\beta<\kappa$ such that there is $j<m$ with $T_{\langle(0,\beta)\rangle\rho''}r=T_{\langle(0,\beta)\rangle\rho''}t_j=T_{\langle(0,\beta)\rangle\rho}$ where ρ' is an initial segment of ρ. Then $T_{\langle(0,\beta)\rangle\rho}\in RrR$.

Let $s=T_{v\rho}\in R$ with $v,\rho\in Y$ satisfying $(*)$. Put $\alpha=\mathrm{bmax}(\rho)$. To finish the proof it suffices to show that $I_\alpha\subseteq RsR$. By Lemma 3.4, it is enough to show that $T_{v'\rho'}\in RsR$ whenever $v',\rho'\in Y$ satisfy $\mathrm{amax}(\rho')\geq\mathrm{bmax}(v')$ and $\mathrm{amax}(\rho')\geq\alpha$.

If $v\in Y_{\geq 1}$, then $T_{\langle(0,1)\rangle v}s=T_{\langle(0,1)\rangle\rho}$ and $T_{\langle(0,1)\rangle\rho}T_{\rho\rho'}=T_{\langle(0,1)\rangle\rho'}$, so $T_{v'\rho'}=T_{v'\rho''}T_{\langle(0,1)\rangle,\rho'}\in RsR$, where ρ'' is obtained from ρ' by adding (replacing by) the initial pair $(0,1)$.

Let $v\in Y_0$ so $v=\langle(0,\beta)\rangle$ where $0<\beta<\alpha$. As above, we get $T_{v\rho'}\in sR$, and $T_{v'\rho'}\in RsR$. $\square$

Lemma 3.6. $\Gamma(L_2(R))=\bar{\kappa}$.

Proof. Let $0<\alpha<\beta<\kappa$. Assume there exists $C\in L_2(R)$ such that $I_\alpha+C=R$ and $I_\alpha\cap C=I_\beta$. In particular, there exists $r\in I_\alpha$ such that $1-r\in C$ and $I_\alpha(1-r)\subseteq I_\alpha\cap C=I_\beta$. Consider the canonical form of r, $(**)$. By Lemma 3.4, $f=0$. Moreover, there exists $\alpha<\gamma<\kappa$ such that for each $j<m$, if a pair (α,δ_j) occurs in v_{j0}, then $\gamma\neq\delta_j$. Then $s=T_{\langle(0,1)\rangle,\langle(\alpha,\gamma)\rangle}\in I_\alpha\setminus I_\beta$ and $sr=0$, so $s(1-r)=s\in I_\beta$, a contradiction. This proves that I_α is not complemented over I_β. By Lemma 3.5, $\Gamma(L_2(R))=\bar{\kappa}$. $\square$

4. Complexity of Ziegler spectra of infinite-dimensional algebras

Theorem 2.7 cannot be improved to produce a proper class of strongly uniform modules with different values of the Γ-invariant over a fixed ring R:

Lemma 4.1. *Let R be a ring. For each right ideal I of R such that R/I is strongly uniform, denote by d_I the dimension of R/I (for example, $d_I = 1$ for any maximal right ideal I). Let $\kappa_R = \sup_I d_I$. Then each strongly uniform module has dimension $\leq \kappa_R$.*

Proof. Let U be a strongly uniform module of dimension λ. Let E be the injective hull of U. Then E is strongly uniform, and has dimension λ. On the other hand, E is the injective hull of some cyclic module R/I. Then also R/I is strongly uniform of dimension λ, so $\lambda = d_I$. □

We do not know whether we can improve Theorem 2.7 to produce injective uniform (= indecomposable injective) modules with prescribed values of the Γ-invariant. Nevertheless, slightly modifying the invariant, we can produce the relevant examples:

Definition 4.2. Let κ be a regular uncountable cardinal. Let U be a strongly uniform module of dimension κ. By modularity of submodule lattices, we have $\Gamma(V) \leq \Gamma(W)$ for any non-zero submodules $V \subseteq W \subseteq U$. So the set

$$\mathscr{G}(U) = \{\Gamma(V) \mid 0 \neq V \subseteq U\}$$

is a lower directed subset of $B(\kappa)$.

If U is such that $\mathscr{G}(U)$ has a least element, we define

$$\Gamma^*(U) = \min \mathscr{G}(U);$$

otherwise, $\Gamma^*(U)$ is not defined.

Recall that for a ring R, the *Ziegler spectrum* of R, Zg(R), is a topological space whose points are (isomorphism classes of) indecomposable pure-injective modules and the topology has the property that closed subsets correspond bijectively to complete theories of modules closed under products. (A closed subset C corresponds to the complete theory of the module $M = \bigoplus_{N \in C} N^{(\omega)}$.) Despite being a set, the Ziegler spectrum captures most model theoretic properties of the class Mod-R, cf. [13,14].

We finish by showing that the point structure of Zg(R) is very complex in case R is the infinite-dimensional algebra constructed in Theorem 2.7:

Theorem 4.3. *Let λ be an uncountable cardinal and F be a field. Then there exists an F-algebra R such that for any regular uncountable cardinal $\kappa \leq \lambda$ and any $i \in B(\kappa)$ there is a strongly uniform module $I \in \mathrm{Zg}(R)$ such that $\Gamma^*(I) = i$.*

Proof. Let R be as in Theorem 2.7, and $L = L_{\kappa i} \in$ Mod-R $\cap$ Mod-$R_{\kappa i}$ be the strongly uniform module constructed in Theorem 2.7, with $\Gamma(L) = i$. By Proposition 2.4, the right R-submodule L_α has Γ-invariant value equal to i for all $\alpha < \kappa$. Denote by E the injective hull of the right R-module L. From Lemma 2.3 we get that $\{L_\alpha \mid \alpha < \kappa\}$ is cofinal in E. So i is the least element of the lower directed set $\mathscr{G}(E)$. This proves that $\Gamma^*(E) = \Gamma(L) = i$. Finally, there is a (unique) element $I \in \mathrm{Zg}(R)$ which is isomorphic to E, so $\Gamma^*(I) = i$. $\square$

Acknowledgements

Jan Trlifaj thanks Professor Paul Eklof for many stimulating discussions on the subject, and for his constant help. Thanks are also due to Rutgers University for supporting his trip to Rutgers.

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