EVERY TWO ELEMENTARILY EQUIVALENT MODELS HAVE ISOMORPHIC ULTRAPOWERS*

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ABSTRACT
We prove (without G.C.H.) that every two elementarily equivalent models have isomorphic ultrapowers, and some related results.

We prove here

THEOREM. *Let λ be any cardinality, $\mu = \min\{\mu: \lambda^\mu > \lambda\}$. Then there is an ultrafilter D over λ such that:*

1) *If M, N are elementarily equivalent models of power $< \mu$,* then M^λ/D, N^λ/D *are isomorphic;*

2) *If M is a model of power* $< \mu$, $2^\kappa \leqq 2^\lambda$, then M^λ/D *is* κ^+*-saturated;*

3) *If* M_k, N_k *are models of cardinality* $\leqq \chi < \mu$, *of the same language, and* $\Pi_{k<\lambda} M_k/D$, $\Pi_{k<\lambda} N_k/D$ *are elementarily equivalent then they are isomorphic.*

This theorem generalizes Keisler [6] (which proved a stronger result using G.C.H.) and the proof generalizes the proof of Kunen [12]. Part (1) of the theorem affirms a well-known conjecture; it is not clear who proposed it. It occurs as open problem 5 in Chang and Keisler [1]. The problem was attacked by several people in several ways. Keisler [6] proves: if $\lambda^+ = 2^\lambda$, then there is an ultrafilter D over λ such that: if $M \equiv N$, $\|M\| \leqq \lambda^+$, $\|N\| \leqq \lambda^+$, and the language is of cardinality $\leqq \lambda$ *then* $M^\lambda/D \cong N^\lambda/D$. By Keisler [8] this can be broken into the following stages: if $\lambda^+ = 2^\lambda$, there is a λ^+-good ultrafilter over λ; if D is a λ^+-good ultrafilter over I and M a model with language of cardinality $\leqq \lambda$, then M^I/D is λ^+-saturated, and any two elementarily equivalent μ-saturated models of cardinality μ are isomorphic. (See Keisler [8], Keisler [7] and Morley and Vaught [15]). Another approach was that of Kochen [11] (or Keisler [10] §5). He gen-

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eralizes ultrapower to ultralimits, a generalization which preserves most of the interesting properties of ultrapower, and proves that any two elementarily equivalent models of cardinalities $\leqq \beth_\alpha$ have isomorphic ultralimits of cardinality $\beth_{\alpha+\omega}$ Lately, Mansfield has generalized ultrapower in another way, to boolean ultrapower, and proved for them a parallel isomo-theorem. See [13]. Recently Kunen [12] succeeded in eliminating G.C.H. from the theorem on the existence of good ultrafilter, (we generalize his proof.) Silver and, independently, Rucker proved: it is consistent with ZFC + ($\aleph_1 < 2^{\aleph_0}$) that there is an ultrafilter D over ω such that for every countable model M with a countable language, M^ω/D is saturated. (In fact, this follows easily enough from Martin's axiom). It is yet an open question whether for any M, N $M \equiv N$, $\| M \| \leqq \mu$, $\| N \| \leqq \mu$, $|L(N)| \leqq \mu$; there is an ultrafilter D over μ such that M^μ/D, N^μ/D are isomorphic. Maybe this is independent from ZFC.

By part (1) of our theorem we can eliminate G.C.H. from some theorems which were used by Keisler [6], especially those concerning the characterization of elementary classes (Keisler [6]). Also from the theorem "a sentence is preserved under reduced products iff it is equivalent to a Horn sentence" (Keisler [5]) G.C.H. can be eliminated, by the technique used here. G.C.H. was already eliminated by Galvin [3], using a set theoretic consideration, and by Mansfield [14] using Boolean ultrapowers.

About ultrapowers and ultraproducts see Łos [3], Frayne Morel and Scott [2], the survey Keisler [9] or Bell and Slomson [16]. We use only the definition.

NOTATION. Through all the paper, λ will be a fixed (infinite) cardinal, $\mu = \min\{\mu : \lambda^\mu > \lambda\}$. Notice that μ is a regular cardinal. We use χ, κ for cardinals; $i, j, k, l, \alpha, \beta, \gamma, \delta$ for ordinals, m, n for natural numbers; f, for functions from λ into μ, and g for functions from λ to some $\chi(g) < \mu$. We use F and G for families of such functions. Speaking of functions $f \in F$ with different indexes, we mean they are different functions. D will denote a proper filter over λ. The filter $[E]$ generated by the family E of subsets of λ is

$$\left\{A : A \subset \lambda, \text{ and for some } A_1, \cdots, A_n \in E, \left(\bigcap_{m=1}^{n} A_m\right) \subset A\right\}.$$

Let $A = \emptyset \pmod D$ mean that for some $X \in D$, $A \subset (\lambda - X)$. Models will be denoted by M, N. The universe of M is $|M|$, and the cardinality of a set A is $|A|$, so that the cardinality of (the universe of) M is $\| M \|$.

DEFINITION 1. We say that (F, G, D) is κ-consistent if

A) D is generated by a family of $\leqq \kappa$ subsets of λ.

B) If $f_i \in F$, $j_i < \mu$ for $i < \chi < \mu$ and $f^m \in F$, $g^m \in G$ for $m \leqq n$ then

$$\{k < \lambda : f_i(k) = j_i \text{ for } i < \chi \text{ and } f^m(k) = g^m(k) \text{ for } m \leqq n\} \neq \emptyset \ (\operatorname{mod} D)$$

LEMMA 1. *There is a family F of 2^λ functions (from λ to μ) such that $(F,\emptyset,\{\lambda\})$ is μ-consistent (this generalizes Ketonen's lemma which was used by Kunen [12] but both had already appeared in Engelking and Karlowicz [1a].*

PROOF. Let H be the set of all pairs (A, h) such that: A is a subset of λ of cardinality $< \mu$; h is a function, from a family S of $< \mu$ subsets of A into μ. The number of $A \subset \lambda$, $|A| < \mu$ is $\sum_{\chi<\mu} \lambda^\chi \leqq \mu \cdot \lambda = \lambda$. For each such A, the number of suitable S is

$$|\{S : S \subset \{B : B \subset A\}, |S| < \mu\}| = \sum_{\chi<\mu} |\{S : S \subset \{B : B \subset A\}, |S| = \chi\}|$$

$$= \sum_{\chi<\mu} |\{B : B \subset A\}|^\chi = \sum_{\chi<\mu} (2^{|A|})^\chi \leqq \sum_{\chi<\mu} (\lambda^{|A|})^\chi = \sum_{\chi<\mu} \lambda^\chi \leqq \mu\lambda = \lambda$$

and for each such S the number of functions from S into μ is $\leqq \mu^{|S|} \leqq \lambda^{|S|} = \lambda$. So $|H| \leqq \lambda$, and in fact $|H| = \lambda$. Let $H = \{(A_k, h_k) : k < \lambda\}$. For every set $B \subset \lambda$ define f_B as follows: $f_B(i) = h_i(B \cap A_i)$ if $h_i(B \cap A_i)$ is defined, and $f_B(i) = 0$ otherwise. Let $F = \{f_B : B \subset \lambda\}$, and we shall prove that F satisfies our demands.

Let $f^i \in F$ $j_i < \mu$ for $i < \chi < \mu$, and let $f^i = f_{B_i}$. Clearly $i_1 \neq i_2$ implies $B_{i_1} \neq B_{i_2}$. As we have $\chi < \mu$ sets $B_i \subset \lambda$, there is $A \subset \lambda$, $|A| = \chi$, such that $i_1 \neq i_2$ implies $A \cap B_{i_1} \neq A \cap B_{i_2}$. Define

$$S = \{A \cap B_i : i < \chi\}, \quad h(A \cap B_i) = j_i \text{ for every } i < \chi.$$

Clearly $(A, h) \in H$, so for some $k < \lambda$, $(A, h) = (A_k, h_k)$. Hence

$$f^i(k) = f_{B_i}(k) = h_k(A \cap B_i) = j_i.$$

So

$$\{k : f^i(k) = j_i \text{ for every } i < \chi\} \neq \emptyset$$

so

$$\{k : f^i(k) = j_i \text{ for every } i < \chi\} \neq \emptyset \ (\operatorname{mod} \{\lambda\})$$

and the lemma is proved.

LEMMA 2. (A) *If (F, G, D) is κ-consistent, $\kappa \leqq \kappa_1$, then (F, G, D) is κ_1-consistent.*

B) *If for every* $i < \delta$, (F_i, G_i, D_i) *is* κ_i*-consistent, for* $i < j < \delta$, $D_i \subset D_j$, $G_i \subset G_j$, $F_i \supset F_j$; $D = \bigcup_{i<\delta} D_i$, $G = \bigcup_{i<\delta} G_i$, $F = \bigcap_{i<\delta} F_i$ *and* $\kappa \geqq \kappa_i$ *for every* $i < \delta$ *and* $\kappa \geqq \mathrm{cf}(\delta)$ *(the cofinality of* δ*) then* (F, G, D) *is* κ*-consistent.*

C) *If* (F, G, D) *is* κ*-consistent,* $F' \subset F$, $G' \subset G$, *then* (F', G', D) *is* κ*-consistent.*

PROOF. Immediate.

LEMMA 3. *Suppose* $(F, \emptyset, D)$ *is* κ*-consistent,* $\mu + |G| \leqq \kappa$, *(*G *a set of functions from* λ *to cardinals* $< \mu$*). Then there is* $F' \subset F$, $|F - F'| \leqq \kappa$ *such that* (F', G, D) *is* κ*-consistent.*

PROOF. Let D be generated by $E = \{J_\alpha : \alpha < \kappa\}$ $(J_\alpha \subset \lambda)$ and without loss of generality assume that E is closed under finite intersection. Clearly it suffices to prove that for every finite subset G_1 of G there is $F(G_1)$, $|F(G_1)| \leqq \kappa$ such that $(F - F(G_1), G_1, D)$ is κ-consistent, because then
$F' = F - \bigcup\{F(G_1) : G_1 \subset G, |G_1| < \aleph_0\}$
will satisfy our conclusion.

So let $G_1 = \{g_0, \cdots, g_n\}$. Suppose there is no $F(G_1)$ as required. So there is a case of violation of part (2) of the definition of κ-consistency of (F, G, D). We can remove the involved functions from F, and again we do not get κ-consistency. So we can repeat it κ^+ times. So we can define by induction on $\beta < \kappa^+$, (distinct) functions $f_i^\beta, f_m^{*\beta} \in F$; $i < \chi_\beta < \mu$, $m \leqq n$ and ordinals j_i^β $i < \chi_\beta < \mu$ such that:

1) $f_i^\beta, f_m^{*\beta} \in F - \{f_i^\gamma, f_m^{*\gamma} : \gamma < \beta, m \leqq n, i < \chi_\gamma\}$

2) for every β

$$A_\beta = \{k < \lambda : \text{for every } i < \chi_\beta \ f_i^\beta(k) = j_i^\beta, \text{ for every } m \leqq n \ f_m^{*\beta}(k) = g_m(k)\} = \emptyset \pmod D.$$

By the definition of D, for every $\beta < \kappa^+$, as $A_\beta = \emptyset \pmod D$ there is $\alpha_\beta < \kappa$ such that $A_\beta \subset (\lambda - J_{\alpha\beta})$. As the number of α_β's is κ, and the number of χ_β is $\leqq \mu \leqq \kappa$, whereas the number of $\beta < \kappa^+$ is κ^+, clearly there are $\alpha^0 < \kappa$, $\chi^0 < \mu$ such that $|\{\beta < \kappa^+ : \chi_\beta = \chi^0, \alpha_\beta = \alpha^0\}| = \kappa^+$. Without loss of generality assume that $\chi_\beta = \chi^0$, $\alpha_\beta = \alpha^0$ for every $\beta < \mu$. Let $\{\langle j_0^{*\beta}, \cdots, j_n^{*\beta}\rangle : \beta < \chi^*\}$ be the set of all sequences of length $n + 1$ of ordinals smaller than $\chi^* = \sup\{|g_m(k)|^+ : m \leqq n, k < \lambda\}$. (The cardinal χ^* is $< \mu$, as each g_m is by definition a function from λ into some $\chi < \mu$). (Clearly the number of such sequences is χ^*.) Let

$$A = \{k < \lambda : \text{for every } \beta < \chi^*, i < \chi^0, m \leqq n, f_i^\beta(k) = j_i^\beta, f_m^{*\beta}(k) = j_m^{*\beta}\}.$$

As $\chi^* < \mu$, $\chi^0 < \mu$ also $\chi^*\chi^0 + \chi^*\ (n + 1) < \mu$, so as $(F, \emptyset, D)$ is κ-consistent, clearly $A \neq \emptyset \pmod D$. Hence it cannot hold that $A \subset (\lambda - J_{\alpha 0})$. So we can choose $k \in A$, $k \notin (\lambda - J_{\alpha 0})$. As $k \in A$, for every $\beta < \chi^*$, $i < \chi^0$, $m \leqq n$, $f_i^\beta(k) = j_i^\beta$, $f_m^{*\beta}(k) = j_m^{*\beta}$. By the definition of the sequences $\langle j_0^{*\beta}, \cdots, j_n^{*\beta} \rangle$, there is $\beta < \chi^*$ such that

$$g_0(k) = j_0^{*\beta}, \cdots, g_n(k) = j_n^{*\beta}.$$

So by the definition of A_β, $k \in A_\beta$, but

$$A_\beta \subset \lambda - J_{\alpha 0},\ k \notin (\lambda - J_{\alpha 0})$$

contradiction.

Lemma 4. A) *Suppose (F, G, D) is κ-consistent $A \subset \lambda$. Then there is $F' \subset F$, $|F - F'| < \mu$ such that $(F', G, [D \bigcup \{A\}])$ is κ-consistent or $(F', G, [D \bigcup \{\lambda - A\}])$ is κ-consistent.*

B) *If (F, G, D) is κ-consistent, $A_\alpha \subset \lambda$ for $\alpha < \kappa$, and $\mu \leqq \kappa$,* then *there are $F' \subset F$, $|F - F'| \leqq \kappa$, and a filter D', $D \subset D'$* such that *(F', G, D') is κ-consistent and for every $\alpha < \kappa$ either $A_\alpha \in D'$ or $(\lambda - A_\alpha) \in D'$.*

Proof. Clearly it suffices to prove A) as B) follows by repeating A) and using Lemma 2B. Let $D_1 = [D \bigcup \{A\}]$ $D_2 = [D \bigcup \{\lambda - A\}]$. D_1 and D_2 are generated by families of $\leqq \kappa$ subsets of λ. (As if $D = [E]$ $|E| \leqq \kappa$, then $D_1 = [E \bigcup \{A\}]$, $D_2 = [E \bigcup \{\lambda - A\}]$.)

If (F, G, D_1) is κ-consistent — our conclusion follows. So we can assume that (F, G, D_1) is not κ-consistent. So there are $f_i \in F$ $j_i < \mu$ for $i < \chi < \mu$ and $f^m \in F$ $g^m \in G$ for $m \leqq n$ such that

$$B = \{k < \lambda : \text{for every } i < \chi\ m \leqq n\ f_i(k) = j_i, f^m(k) = g^m(k)\} = \emptyset \pmod{D_1}.$$

This implies that for some $X \in D$, $B \subset (\lambda - A) \bigcup (\lambda - X)$. Let

$$F' = F - \{f_i, f^m : i < \chi, m \leqq n\}.$$

If (F', G, D_2) is κ-consistent, our conclusion follows. So assume (F', G, D_2) is not κ-consistent, and we shall get a contradiction. So there are $f_i^* \in F'$, $j_i^* < \mu$ for $i < \chi^* < \mu$ and $f^{*m} \in F'$, $g^{*m} \in G$ for $m \leqq n^*$ such that

$$B^* = \{k < \lambda : \text{ for every } i < \chi^*,\ m \leqq n^*, f_i^*(k) = j_i^*, f^{*m}(k) = g^{*m}(k)\} = \emptyset \pmod{D_2}.$$

So for some $X^* \in D$, $B^* \subset (\lambda - X^*) \bigcup (\lambda - (\lambda - A)) = (\lambda - X^*) \bigcup A$. So $B^* \bigcap B \subset (\lambda - X^*) \bigcup (\lambda - X) = (\lambda - (X^* \bigcap X))$ and as D is a filter $X^* \bigcap X \in D$.

So $B^* \bigcap B = \emptyset$ (mod D). Observing what are B and B^*, we see that we get a contradiction to the κ-consistency of (F, G, D). (If one D_i is not a filter the proof is the same.)

LEMMA 5. *Let M be a model of cardinality $\chi < \mu$, and $\bar{a}_{l,m} \in |M|^\lambda$ for $l < l_0$, $1 \leqq m \leqq n_l$, $\mu \leqq \kappa$, and $(F,\emptyset,D)$ is κ-consistent. Assume moreover that $p = \{\phi_l(x, y_{l,1}, \cdots, y_{l,n_l}) : l < l_0 < \kappa^+\}$, ($\phi_l$—formula in the language L of M) and p is closed under conjunctions and for every $l < l_0$, $A^l = \{k < \lambda : M \models (\exists x)\phi_l(x, \bar{a}_{l,1}[k], \cdots, \bar{a}_{l,n_l}[k])\} \in D$.*

Then there are $\bar{a} \in |M|^\lambda$, $F' \subset F$, $D' \supset D$, such that: $|F - F'| \leqq \kappa$, $(F', \emptyset, D')$ is κ-consistent and for every $l < l_0 \{k < \lambda : M \models \phi_l(\bar{a}[k], \bar{a}_{l,1}[k], \cdots, \bar{a}_{l,n_l}[k])\} \in D'$.

REMARK. $|M|^\lambda$ is the set of functions from λ into $|M|$.

PROOF. Let $|M| = \{c_i : i < \chi < \mu\}$. For every $l < l_0$ let us define a function g_l from λ into $\chi(< \mu)$ such that:

$$\text{if } M \models (\exists x)\phi_l(x, \bar{a}_{l,1}[k], \cdots, \bar{a}_{l,n_l}[k]) \text{ and } j = g_l(k)$$

then $M \models \phi_l[c_j, \bar{a}_{l,1}[k], \cdots, \bar{a}_{l,n_l}[k])$.

Let $G = \{g_l : l < l_0\}$. As $l_0 < \kappa^+$ and $(F,\emptyset,D)$ is κ-consistent, and $\mu \leqq \kappa$, there is, by Lemma 3, $F_1 \subset F$, $|F - F_1| \leqq \kappa$ such that (F_1, G, D) is κ-consistent. Choose $f \in F_1$ and let:

$$F' = F_1 - \{f\},\ \bar{a}[k] = \begin{cases} c_{f(k)} & \text{if } f(k) < \chi \\ c_0 & \text{otherwise} \end{cases}$$

and $D' = [D \bigcup E]$ where $E = \{A_l : l < l_0\}$, $A_l = \{k < \lambda : M \models \phi_l[\bar{a}[k], \bar{a}_{l,1}[k], \cdots]\}$. We shall show that $(F', \emptyset, D')$ is κ-consistent, and hence prove the lemma.

As D is generated by a family E_1 of $\leqq \kappa$ subsets of λ, clearly D' is generated by $E_1 \bigcup E$, $|E_1 \bigcup E| \leqq \kappa$.

Suppose $(F', \emptyset, D')$ is not κ-consistent. So there are $f_i \in F'$, $j_i < \mu$ for $i < \chi_1 < \mu$ and $X' \in D'$ such that

$$A = \{k < \lambda : \text{for every } i < \chi_1, f_i(k) = j_i\} \subset \lambda - X'.$$

As p is closed under conjunctions, E is closed under intersection. So there are $X \in D$, $l < l_0$ such that $X' \supset X \bigcap A_l$. So $A \bigcap A_l \subset (\lambda - X)$. That is

$\{k < \lambda$: for every $i < \chi, f_i(k) = j_i, M \models \phi_l[\bar{a}[k], \bar{a}_{l,1}[k], \cdots, \bar{a}_{l,n_l}[k]\} \subset (\lambda - X)$.

Hence

$$\{k < \mu : \text{for every } i < \chi, f_i(k) = j_i,\ f(k) = g_l(k)\} \subset (\lambda - X) \bigcup (\lambda - A^l)$$

(A^l — defined in the lemma.)

A contradiction to the κ-consistency of (F_1, G, D). Thus we have proved the lemma.

REMARK. We did not prove explicitly that D' is a proper filter, but this can be viewed as a special case of the κ-consistency of $(F', \emptyset, D')$ (with empty set of f_i's).

PROOF OF THE MAIN THEOREM.

REMARK. 1) The theorem is formulated at the beginning.

2) For simplicity we omit the proof of part (3).

3) From part (2) and Keisler [8] it is clear that D is λ^+-good. As in Kunen [12] we can also prove it directly.

We can assume, without loss of generality, that the language of any model M, $\|M\| < \mu$, is of cardinality $\leqq 2^{\|M\|} \leqq \lambda^{\|M\|} = \lambda$. So let L^0 be a (first-order) language of cardinality λ, which contains, for every $n < \omega$, $n > 0$, λ predicates with n places, and λ function symbols with n places. So we can restrict ourselves to models whose language is included in L^0, and whose universe is $\chi = \{i : i < \chi\}$ for some $\chi < \mu$. Now the number of sublanguages of L^0 is 2^λ, and for each such L, and $\chi < \mu$, there are $|L|^{2^\chi} \leqq \lambda^\lambda = 2^\lambda$ L-models with universe χ. Let

$$\{(M_i, N_i) : i < 2^\lambda\}$$

be a list of all the pairs of elementarily equivalent models, whose language is $L_i \subset L^0$, and whose universes are some cardinals $< \mu$. We shall find an ultrafilter D over λ such that: M_i^λ/D is isomorphic to N_i^λ/D, and M_i^λ/D is κ^+-saturated if $2^\kappa \leqq 2^\lambda$. As the ultrapowers of isomorphic models are isomorphic this is sufficient.

Let

$$|M_i|^\lambda = \{\bar{a}^i_\alpha : \alpha < 2^\lambda\}$$
$$|N_i|^\lambda = \{\bar{b}^i_\alpha : \alpha < 2^\lambda\}.$$

From considerations of cardinalities, it is clear that there is a function R, defined for every $\gamma < 2^\lambda$ such that

A) For every $i < 2^\lambda$, $\alpha < 2^\lambda$ there is $\gamma < 2^\lambda$ such that

$$R(\gamma) = \langle i, 1, \bar{a}^i_\alpha \rangle.$$

B) For every $i < 2^\lambda$, $\alpha < 2^\lambda$, there is $\gamma < 2^\lambda$ such that

$$R(\gamma) = \langle i, 2, \bar{b}^i_\alpha \rangle.$$

C) *For every* $i < 2^\lambda$ and set of formulas p, $2^{|p|} \leqq 2^\lambda$, p is closed under conjunctions and

$$p = \{\phi_l(x, y_{\alpha_{l,1}}, \cdots,) : l < |p|\} \quad (\alpha_{l,m} < 2^\lambda, \phi_l \in L_i)$$

there are $\gamma < 2^\lambda$, $\gamma_{\alpha_{l,m}} < \gamma$

$$R(\gamma) = \langle i, p \rangle, R(\gamma_{\alpha_{l,m}}) = \langle i, 1, \bar{a}^i_{\alpha_{l,m}} \rangle .$$

D) For every subset A of λ, there is $\gamma < 2^\lambda$ such that $R(\gamma) = A$.

E) For every γ exactly one of A), B), C), D) occurs.

We shall now define by induction on $\gamma \leqq 2^\lambda$ a set of functions F_γ, a filter D_γ and functions H^i_γ, $i < 2^\lambda$ such that:

1) For every γ, $(F_\gamma, \varnothing, D_\gamma)$ is $(\lambda + |\gamma|)$-consistent, $|F_0| = 2^\lambda$, $D_0 = \{\lambda\}$, $|F_0 - F_\gamma| \leqq \lambda + |\gamma|$ and for $\beta < \gamma$ $F_\beta \subset F_\gamma$, $D_\beta \supset D_\gamma$.

2) H^i_γ is a function from a subset of $|M_i|^\lambda$ into $|N_i|^\lambda$, for $\beta < \gamma$, H^i_γ extends H^i_β, and $|\bigcup_{i<2^\lambda} \operatorname{Dom} H^i_\gamma| \leqq |\gamma|$.

3) If $\bar{a}^i_{\alpha_1}, \cdots, \bar{a}^i_{\alpha_n} \in \operatorname{Dom} H^i_\gamma$, $\bar{b}^i_{\beta_m} = H^i_\gamma(\bar{a}^i_{\alpha_m})$ for $1 \leqq m \leqq n$ and $\phi(x_1, \cdots, x_n) \in L_i$ *then*

$$\{k < \lambda : M_i \vDash \phi[a_{\alpha_1}{}^i[k], \cdots \bar{a}^i_{\alpha_n}[k]] \Leftrightarrow N_i \vDash \phi[\bar{b}^i_{\beta_1}[k], \cdots, \bar{b}^i_{\beta_n}[k]]\} \in D_\gamma .$$

4) If $\bar{a}^i_{\alpha_1}, \cdots, \bar{a}^i_{\alpha_n} \in \operatorname{Dom} H^i_\gamma$, $\phi \in L_i$ then either $\{k < \lambda : M_i \vDash \phi[\bar{a}^i_{\alpha_1}[k], \cdots, \bar{a}^i_{\alpha_n}[k]]\} \in D_\gamma$ or $\{k < \lambda : M_i \vDash \neg\phi[\bar{a}^i_{\alpha_1}[k], \cdots, \bar{a}^i_{\alpha_n}[k]]\} \in D_\gamma$.

5) If $R(\gamma) = \langle i, 1, \bar{a}^i_\alpha \rangle$ then $\bar{a}^i_\alpha \in \operatorname{Dom} H^i_{\gamma+1}$.

6) If $R(\gamma) = \langle i, 2, \bar{b}^i_\alpha \rangle$ then $\bar{b}^i_\alpha \in \operatorname{Range} H^i_{\gamma+1}$.

7) If $R(\gamma) = \langle i, p \rangle$ and for every $\phi(x, y_{\alpha_1}, \cdots, y_{\alpha_n}) \in p$ $\{k < \lambda : M_i \vDash (\exists x)\phi(x, \bar{a}^i_{\alpha_1}[k], \cdots, \bar{a}^i_{\alpha_n}[k])\} \in D_\gamma$ *then* there is $\bar{a}^i_\alpha \in |M_i|^\lambda$ such that for every $\phi(x, y_{\alpha_1}, \cdots, y_{\alpha_n}) \in p$ $\{k < \lambda : M_i \vDash \phi[\bar{a}^i_\alpha[k], \bar{a}^i_{\alpha_1}[k], \cdots, \bar{a}^i_\alpha [k]]\} \in D_{\gamma+1}$.

8) If $R(\gamma) = A \subset \lambda$ then either $A \in D_{\gamma+1}$ or $(\lambda - A) \in D_{\gamma+1}$.

* * *

If we succeed in the induction $D = D_{(2^\lambda)}$ will be the required ultrafilter. By (8) and (D) it is an ultrafilter. For every $i < 2^\lambda$, it is clear that $H^i_{2^\lambda}$ induces an isomorphism from M_i^λ/D onto N_i^λ/D. [By 5) and (A) the domain of $H^i_{2^\lambda}$ is $|M_i|^\lambda$, by (6) and (B) its range is $|N_i|^\lambda$, and by (3) it preserves all the formulas, hence all the relations and, in particular, the equality.]. By (7) and (C) M_i^λ/D is κ^+-saturated whenever $2^k \leqq 2^\lambda$.

Let us return to the definition by induction, which is the only thing remaining to be proved.

Case I. $\gamma = 0$.

This follows from Lemma 1. [(3) follows from the elementary equivalence of M_i and N_i.]

Case II. γ a limit ordinal.

Define $F_\gamma = \bigcap_{\beta<\gamma} F_\beta$, $D_\gamma = \bigcup_{\beta<\gamma} D_\beta$ and $H^i_\gamma = \bigcup_{\beta<\gamma} H^i_\beta$. It is easy to see that all the conditions (1)–(8) still hold. In particular (1) follows from Lemma 2.B.

Case III. $\gamma = \beta + 1$, $R(\beta) = \langle i, 1, \bar{a}^i_\alpha \rangle$.

First we use Lemma 4.B so that the type realized by $\bar{a}^i_\alpha$ over $\mathrm{Dom}\, H^i_\beta$ will be decided. Then we use Lemma 5 to extend H^i_β to $\{\bar{a}^i_\alpha\} \bigcup \mathrm{Dom}\, H^i_\beta$. (We depend on $|L_i| \leqq \lambda$.)

Case IV. $\gamma = \beta + 1$, $R(\beta) = \langle i, 2, \bar{b}^i \rangle$.

The same as Case III.

Case V. $\gamma = \beta + 1$, $R(\beta) = \langle i, p \rangle$.

It follows from Lemma 5.

Case VI. $\gamma = \beta + 1$, $R(\beta) = A(\subset \lambda)$.

It follows from Lemma 4.A.

So we prove the theorem.

REMARK. We actually proved more than we needed to know about G in order to prove our main theorem. We could have proved even more: we could have generalized all our lemmas, except for 2B, to the case where $< \chi^0$ equations of the form $f(k) = g(k)$ are allowed in Definition 1, with some natural restrictions imposed on χ^0. Maybe there is a use for these stronger lemmas.

REFERENCES

1. C. C. Chang and H. J. Keisler, *Model theory*, to appear.

1a. Engelking and Karlowicz, *Some theorems of set theory and their topological consequences*, Fund. Math. **57** (1965), 275–285.

2. T. Frayne, A. Morel and D. Scott, *Reduced direct products*, Fund. Math. **51** (1962), 195–228.

3. F. Galvin, *Horn sentences*, Annals Math. Logic, **1** (1970), 389–422.

4. J. Los, *Quelque remarques, théorèmes et problèmes sur les classes définissable d'algèbres*, Mathematical interpretations of formal systems, 98–113, North Holland Publ. Co., Amsterdam, 1955.

5. H. J. Keisler, *Reduced products and Horn classes*, Trans. Amer. Math. Soc., **117** (1965), 307–328.

6. H. J. Keisler, *Ultraproducts and elementary classes*, Ph. D. thesis, Univ. of Calif. Berkeley California, 1961. See also Kon, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 477–495. [Indag. Math. 23].

7. H. J. Keisler, *Good ideals in field of sets*, Annals of Math. **79** (1964), 338–359.

8. H. J. Keisler, *Ultraproducts and saturated models*, Koninkl. Nederl. Akad. Wetensch., Proc. Ser. A **67** and Indag. Math. **26** (1964), 178–186.

9. H. J. Keisler, *A survey of ultraproducts*, Proceedings of the 1964 International Congress for logic, methodology and philosophy of science, held in Jerusalem 1964, Bar-Hillel (ed.), North-Holland Publ. Co. Amsterdam, pp. 112–126.

10. H. J. Keisler, *Limit ultrapowers*, Trans. Amer. Math. Soc. **107** (1963), 382–408.

11. S. Kochen, *Ultraproducts in the theory of models*, Annals of Math. (2) **74** (1961), 221–261.

12. K. Kunen, *Ultrafilters and independent sets*, to appear in Trans. Amer. Math. Soc.

13. R. Mansfield, *The theory of Boolean ultrapowers*, Annals Math. Logic **2** (1971), 297–323.

14. R. Mansfield, *Horn classes and reduced direct products*, a preprint.

15. M. D. Morley and R. L. Vaught, *Homogeneous universal models*, Math. Scand. **11** (1962), 37–57.

16. J. L. Bell and A. B. Slomson, *Models and ultraproducts*, North-Holland Publ. Co. Amsterdam, 1969.

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