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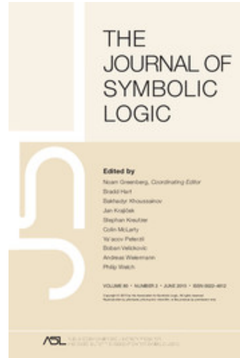

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STRONG NEGATIVE PARTITION ABOVE THE CONTINUUM

SAHARON SHELAH

Introduction. For e.g. $\lambda = \mu^+$, μ regular, λ larger than the continuum, we prove a strong nonpartition result (stronger than $\lambda \to [\lambda; \lambda]^2$). As a consequence, the product of two topological spaces of cellularity $< \lambda$ may have cellularity λ, or, in equivalent formulation, the product of two λ-c.c. Boolean algebras may lack the λ-c.c. Also λ-S-spaces and λ-L-spaces exist. In fact we deal not with successors of regular λ but with regular λ above the continuum which has a nonreflecting stationary subset of ordinals with uncountable cofinalities; sometimes we require λ to be not strong limit.

The paper is self-contained. On the nonpartition results see the closely related papers of Todorčević [T1], Shelah [Sh276] and [Sh261], and Shelah and Steprans [ShSt1].

On the cellularity of products see Todorčević [T2] and [T3], where such results were obtained for (e.g.) cf $[2^{\aleph_0}]$ and $[2^{\aleph_0}]^{+(\omega+1)}$; the class of cardinals he gets is quite disjoint from ours. In [Sh282] such results were obtained for more successors of singulars (mainly λ^+, $\lambda > 2^{\mathrm{cf}\,\lambda}$). Also, concerning S and L spaces, Todorčević gets existence.

Todorčević's work on cardinals like $[2^{\aleph_0}]^{+(\omega+1)}$ relies on [Sh68] (see more in [ShA2, Chapter XIII]) (the scales appearing in the proof of $\aleph_\delta^{\mathrm{cf}\,\delta} < \aleph_{(|\delta|^{\mathrm{cf}\,\delta})^+}$). The problem was stressed in a preliminary version of the surveys of Juhasz and Monk. We give a detailed proof for one strong nonpartition theorem (1.1) and then give various strengthenings. We then use 1.10 to get the consequences (in 1.11 and 1.12).

I thank Todorčević for asking me questions which led me to this paper. More specifically, he suggests that "productivity of λ-c.c. of Boolean algebras" is the real hard-core generalization of MA which we should consider. As generalizations of MA for $\lambda = \aleph_2$ and $2^{\aleph_0} = \aleph_1$ exist, I look at such cardinals. However, as he pointed out, we still have no example of an "absolutely defined cardinal" as $\aleph_2$ or $\aleph_{\omega+1}$.

The problem for λ^+ ($\lambda > \aleph_1$, λ regular) has meanwhile been solved; see [Sh327], which is close in spirit to the present article. (It improves on 1.1 (replacing $\lambda > 2^{\aleph_0}$ by $\lambda > \aleph_1$), but the proof does not give 1.7, 1.8, nor the case $\lambda = \aleph_1$, so the proofs here

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are not obsolete.) More recently the non-λ^+-productivity of Boolean algebras was proved for all singular λ (in [Sh355]).[1]

Notation. η, ν, and ρ are finite sequences of ordinals (e.g. natural numbers). We define

$$\eta = \langle \eta(l): l < \lg(\eta) \rangle, \qquad \mathrm{Rang}(\eta) = \{\eta(l): l < \lg(\eta)\},$$

$$\mathrm{Max}(\eta) = \mathrm{Max}[\mathrm{Rang}(\eta)].$$

For a sequence ρ, let us define $\rho^{[m]}(1 \le m < \omega)$ by induction on m: $\rho^{[1]} = \rho$, $\rho^{[m+1]} = (\rho^{[m]})^\wedge \rho$ (and $\rho^{[0]}$ the empty sequence).

1.1. THEOREM. *Suppose*

I) *λ is regular,*

II) $\lambda > 2^{\aleph_0}$,

III) *$S \subseteq \lambda$ is stationary and nonreflecting, $\delta \in S \Rightarrow \mathrm{cf}(\delta) > \aleph_0$, and*

IV) *$\mu < \lambda \le 2^\mu$ (i.e. λ is not a strong limit).*

Then $\mathrm{Pr}_0(\lambda, \lambda)$; *where:*

1.2. DEFINITION. $\mathrm{Pr}_0(\lambda, \mu)$ means that there is a function c witnessing it, which means c is a 2-place function from λ to μ such that ***if*** $n < \omega$, $\zeta^1_\alpha < \zeta^2_\alpha < \cdots < \zeta^n_\alpha < \lambda$ for $\alpha < \lambda$, and the sets $\{\zeta^1_\alpha, \ldots, \zeta^n_\alpha\}$ are pairwise disjoint for $\alpha < \lambda$, *then*, for any two-place function h from $\{1, \ldots, n\}$ to μ for some $\alpha < \beta < \lambda$, ($\zeta^n_\alpha < \zeta^1_\beta$ and)

$$[1 \le l \le n \wedge 1 \le k \le n \rightarrow c(\zeta^l_\beta, \zeta^k_\alpha) = h(l, k)].$$

Explanation of the proof of 1.1. We use Todorčević's walks $\langle \gamma_l(\beta, \alpha): l < n(\beta, \alpha) \rangle$ (for specifically chosen club C_α to avoid S (as in [Sh276, §3])) to determine the coloring. Now if we partition S to stationary $\langle S_\alpha: \alpha < \alpha(*) \rangle$, it is not hard to get $\delta \in S_i$ and $\alpha < \beta$ such that $\langle \gamma_n(\zeta^l_\beta, \zeta^k_\alpha): n < \mathbf{n}(\zeta^l_\beta, \zeta^k_\alpha) \rangle$ "pass through" δ. The problem is that when we define the coloring we "do not know" which one to choose. For this we use a function $H: \lambda \rightarrow \omega$. Now $\langle H(\gamma_l(\beta, \alpha)): l < \mathbf{n}(\beta, \alpha) \rangle$ is a finite sequence of natural numbers, and we devise a function d from the family of such finite sequences to the set of natural numbers. If in 1.2 h is constant, this is okay. As this is not generally the case, we use subsets $\langle A_\alpha: \alpha < \lambda \rangle$ of μ and the family G and function Γ defined below to translate for us. In the case when h is constantly l^* we could have let $G: \lambda \rightarrow \lambda$ and $c(\beta, \alpha) = G(\gamma_e(\beta, \alpha))$, where $e = d[\langle H(\gamma_m(\beta, \alpha)): m < \mathbf{n}(\beta, \alpha) \rangle]$.

The heart of the matter is to find, for given ζ^l_α ($\alpha < \lambda$, $l = 1, \ldots, n$), sequences ν^l, ρ and ν_l ($l = 1, \ldots, n$) in ${}^{\omega >}\omega$ that satisfy the following two conditions:

(i) For any natural number m, for some $\alpha < \beta$, and for $l, k \in \{1, \ldots, n\}$ (for the definition of $\rho^{[m]}$, see the notation section)

$$\langle H(\gamma_p(\zeta^l_\beta, \zeta^k_\alpha)): p < \mathbf{n}(\zeta^l_\beta, \zeta^k_\alpha) \rangle = \nu^{l\wedge} \rho^{[m]\wedge} \nu_k.$$

(ii) $\mathrm{Max}\,\rho > \mathrm{Max}\{\mathrm{Max}\,\nu^1, \ldots, \mathrm{Max}\,\nu^n, \mathrm{Max}\,\nu_1, \ldots, \mathrm{Max}\,\nu_n\}$.

Hence we can decipher m from $\nu^{l\wedge} \rho^{[m]\wedge} \nu_k$.

PROOF. Without loss of generality we suppose that S consists of limit ordinals only.

For every nonzero $\alpha < \lambda$, let C_α be $\{0, \alpha - 1\}$ if α is a successor ordinal, and a closed unbounded subset of α of order type $\mathrm{cf}(\alpha)$ disjoint to S if α is a limit ordinal.

[1] Since this was written, nonproductivity of λ-c.c. has been proved also for $\lambda = \aleph_2$, λ inaccessible not ω-Mahlo, and more; see [Sh 365].

For $0 < \alpha < \beta < \lambda$, let $\gamma(\beta, \alpha) \stackrel{\text{def}}{=} \text{Min}\{\gamma \in C_\beta : \gamma \geq \alpha\}$. Let us define $\gamma_l(\beta, \alpha)$ by putting $\gamma_0(\beta, \alpha) = \beta$ and $\gamma_{l+1}(\beta, \alpha) = \gamma(\gamma_l(\beta, \alpha), \alpha)$ (if defined).

If $0 < \alpha < \beta < \lambda$, let $\mathbf{n}(\beta, \alpha)$ be the largest $n < \omega$ such that $\gamma_n(\beta, \alpha)$ is defined ((c), below, justifies this definition). Now clearly

(a) if $0 < \alpha < \beta < \lambda$, then $\alpha \leq \gamma(\beta, \alpha) < \beta$;

hence

(b) if $0 < \alpha < \beta < \lambda$, $0 < l < \omega$, and $\gamma_l(\beta, \alpha)$ is defined, then $\alpha \leq \gamma_l(\beta, \alpha) < \beta$

and

(c) if $0 < \alpha \leq \beta < \lambda$, then $\mathbf{n}(\beta, \alpha)$ is well defined and, letting $\gamma_l \stackrel{\text{def}}{=} \gamma_l(\beta, \alpha)$, we have

$$\alpha = \gamma_{\mathbf{n}(\beta,\alpha)} < \gamma_{\mathbf{n}(\beta,\alpha)-1} < \cdots < \gamma_1 < \gamma_0 = \beta.$$

Also

(d) if $\delta \in S$ (so δ is a limit ordinal), and $\delta < \beta < \lambda$, then, for some $\alpha_0 < \delta$,

$$\left[\alpha_0 \leq \alpha \leq \delta \Rightarrow \bigwedge_{l=0}^{\mathbf{n}(\beta,\delta)} \gamma_l(\beta, \delta) = \gamma_l(\beta, \alpha)\right].$$

[Just let $\alpha_0 = \text{Max}\{\text{Max}\, C_{\gamma_l(\beta,\delta)} \cap \delta) + 1 : l < \mathbf{n}(\beta, \delta)\}$. The outer maximum is well defined as it is over a finite set of ordinals. The inner is also well defined, as $C_{\gamma_l(\beta,\delta)}$ is a closed subset of $\gamma_l(\beta, \delta)$, $\delta < \gamma_l(\beta, \delta)$ and $\delta \notin C_{\gamma_l(\beta,\delta)}$ or $C_{\gamma_l(\beta,\delta)} = \{\delta, 0\}$— because $\delta \in S$.]

Let A_α ($\alpha < \lambda$) be distinct subsets of μ, no one of which is a subset of another. Let G be the family of two-place functions g, whose domain is the finite family of all subsets of some finite subset of μ, called $w(g)$, and whose range is $\subseteq \lambda$. Clearly $|G| = \lambda$. So we can find a function Γ from λ onto G such that, for every $g \in G$, $S \cap \Gamma^{-1}(g)$ is a stationary subset of λ. For $g \in G$, $A \subseteq \mu$ and $B \subseteq \mu$, let $g(A, B)$ be

$$g(A \cap w(g), B \cap w(g)).$$

Now let H be a function from λ onto ω such that, for every n, $H^{-1}(\{n\}) \cap S$ is stationary.

Lastly, let us define the coloring c which, as we shall prove, witnesses $\text{Pr}_0(\lambda, \lambda)$ (thus finishing the proof of 1.1). Let $0 < \alpha < \beta$, so $\mathbf{n}(\beta, \alpha) > 0$ and

$$\alpha = \gamma_{\mathbf{n}(\beta,\alpha)}(\beta, \alpha) < \gamma_{\mathbf{n}(\beta,\alpha)-1}(\beta, \alpha) < \cdots < \gamma_1(\beta, \alpha) < \gamma_0(\beta, \alpha) = \beta.$$

Then let

$$e = e(\beta, \alpha) = d[\langle H(\gamma_m(\beta, \alpha)) : m < \mathbf{n}(\beta, \alpha)\rangle],$$

where d is a function as constructed below in Fact 1.3, and let

$$g(\beta, \alpha) = g_{\beta,\alpha} \stackrel{\text{def}}{=} \Gamma(\gamma_e(\beta, \alpha))$$

(so $g_{\beta,\alpha} \in G$) and, lastly, $c(\beta, \alpha) = g_{\beta,\alpha}(A_\alpha, A_\beta)$.

1.3. *Fact. There is a function d from finite sequences of ordinals to* ω *such that* ***if*** ρ *is a nonempty sequence of ordinals, and* $k < \omega$, ***then*** *for some m,* $1 < m < \omega$, *and for all finite sequences* η *and* ν *of ordinals,*

$$[\mathrm{Max}(\rho) > \mathrm{Max}(\eta \char`^ \nu) \Rightarrow d(\eta \char`^ \rho^{[m]} \char`^ \nu) = \lg(\eta) + \lg(\rho) + k].$$

1.3A. *Notation.* 1) Let $d_{\mathrm{mx}}(\rho) = \mathrm{Min}\{i < \lg(\rho): \rho(i) = \mathrm{Max}(\rho)\}$ for ρ a nonempty sequence of ordinals.

2) If we demand just

(*) for any $n < \omega$, ν_1^l, ν_2^l $(l < n)$, and k there is a natural number m such that for $l < n$

$$d(\nu_1^l \char`^ \rho^{[m]} \char`^ \nu_2^l) = \lg(\nu_1^l) + \lg(\rho) + k$$

(which is sufficient for our present application but seemingly not for later ones), then we can use

$$d(\tau) = d_{\mathrm{mx}}(\tau) + [\log_2(\lg(\tau))] - [\sqrt{\log_2(\lg(\tau))}]^2;$$

this works as, for any ρ, ν_1^l, ν_2^l $(l < n)$ and $k(1) < \omega$, for infinitely many $m < \omega$, $[\log_2(\lg(\nu_1^l \char`^ \rho^{[m]} \char`^ \nu_2^l))]$ has the form $i^2 + k(1)$ for some natural number $i > k(1)$. (The length depends on l, of course, but the "rounding" of $\log_2$ makes it disappear often enough.)

1.3B. *Claim. There is a function* d_f *from the class of finite sequences of ordinals to the set of natural numbers such that* ***if*** η, ρ *and* ν *are finite sequences of ordinals, and for no* $k > 1$ *and* ρ_1 *is it true that* $\rho = (\rho_1)^{[k]}$ *and* $\mathrm{Max}(\rho) > \mathrm{Max}(\eta)$ *and* $\mathrm{Max}(\rho) > \mathrm{Max}(\nu)$, ***then*** for $m > 4$

$$d_f(\eta \char`^ \rho^{[m]} \char`^ \nu) = m.$$

Proof of 1.3B. It is clearly enough to show that for every finite sequence τ of ordinals, if, for some η, ρ and ν as above, $\tau = \eta \char`^ \rho^{[m]} \char`^ \nu$, $m > 4$, then m is uniquely reconstructed from τ. So suppose:

$$(*) \qquad \text{for } e = 1, 2, \qquad \tau = \eta_e \char`^ \rho_l^{[m(e)]} \char`^ \nu_e, \qquad \zeta_e = \mathrm{Max}(\rho_e) > \mathrm{Max}(\eta_e \char`^ \nu_e), \qquad m_e > 4.$$

As above, we are assuming

$$(1) \qquad \text{for no } \rho'_e \text{ and } k > 1 \text{ is } \rho_e = (\rho'_e)^{[k]}.$$

Clearly,

$$\mathrm{Max}(\tau) = \mathrm{Max}[\eta_e \char`^ \rho_e^{[m(e)]} \char`^ \nu_e] = \mathrm{Max}\{\mathrm{Max}(\eta_e), \mathrm{Max}(\rho_e), \mathrm{Max}\, \nu_e\} = \mathrm{Max}(\rho_e) = \zeta_e.$$

Hence

$$(2) \qquad \zeta_1 = \zeta_2 \quad (\text{call it } \zeta).$$

Let $n_a(e) = \mathrm{Min}\{i: \rho_e(i) = \zeta\}$ and $n_b(e) = \mathrm{Max}\{i: \rho_e(i) = \zeta\}$. Clearly $0 \le n_a(e) \le n_b(e) < \lg(\rho_e)$.

Let $n_a = \mathrm{Min}\{i: \tau(i) = \zeta\}$ and $n_b = \mathrm{Max}\{i: \tau(i) = \zeta\}$. As $\tau = \eta_e \char`^ (\rho_e)^{[m(e)]} \char`^ \nu_e$, clearly

$$n_a = \lg(\eta_e) + n_a(e), \qquad n_b = \lg(\tau) - \lg(\nu_e) - (\lg(\rho_e) - n_b(e)).$$

Now

$$n_b - n_a = m(e) \times \lg(\rho_e) - n_a(e) - (\lg(\rho_e) - n_b(e));$$

hence (as $0 \leq n_a(e) \leq n_b(e) < \lg(\rho_e)$ by their definitions)

$$(m(e) - 1)\lg(\rho_e) \leq n_b - n_a < m(e)\lg(\rho_e).$$

If $\lg(\rho_1) = \lg(\rho_2)$, necessarily $m(1) = m(2)$, as desired. So without loss of generality we assume that

(3) $$\lg(\rho_1) > \lg(\rho_2).$$

For any integer i, define $\rho_e(i) = \rho_e(i + k \times \lg(\rho_e))$, where k is the unique integer such that $0 \leq i + k \times \lg(\rho_e) < \lg(\rho_e)$ (i.e. we consider the integers modulo $\lg(\rho_e)$).

If $e = 1, 2$ and $0 \leq i < m(e) \times \lg(\rho_e)$, then $\tau(\lg(\eta_e) + i) = \rho_e(i)$. Hence if

$$\lg(\eta_e) \leq i < i + \lg(\rho_e) < \lg(\eta_e{}^\wedge \rho_e^{[m(e)]}),$$

then $\tau(i) = \tau(i + \lg(\rho_e))$. So this holds if $n_a \leq i < i + \lg(\rho_e) \leq n_b$,

$$\cdots|\underset{\rho}{\overset{n_a}{\overline{\qquad\qquad}}}|\underset{\rho}{\overset{j\quad i}{\overline{\qquad\qquad}}}|\underset{\rho}{\overset{n_b}{\overline{\qquad\qquad}}}|\cdots$$

But as $m(e) \geq 4$, if $n_a \leq i \leq n_b$, then $n_a \leq i - \lg(\rho_e)$ or $i + \lg(\rho_e) \leq n_b$ [really $m(e) \geq 3$ suffices for this]. So if i is a natural number, we can find i_1 with $\lg(\eta_1) + i \equiv i_1 \bmod \lg(\rho_1)$ and $n_a \leq i_1 < n_a + \lg(\rho_1)$. So i_1 and $i_1 + \lg(\rho_1)$ are in the interval $[n_a, n_b]$, but $i_1 < i_1 + \lg(\rho_2) < i_1 + \lg(\rho_1)$; hence $\tau(i_1) = \tau(i_1 + \lg(\rho_2))$; hence

$$\rho_1(i) = \tau(i_1) = \tau(i_1 + \lg(\rho_2)) = \rho_1(i + \lg(\rho_2)).$$

Let $I = \{j: j$ a natural number and, for every natural number i, $\rho_1(i) = \rho_1(i + j)\}$. This is an ideal of $\mathbf{Z}$, with $\lg(\rho_1) \in I$ (by definition of $\rho_1(i)$) and $\lg(\rho_2) \in I$ (as proved above). So their greatest common divisor is in I. As ρ_1 is not of the form $\rho^{[m]}$, $m > 1$, necessarily $\lg(\rho_1)$ divides $\lg(\rho_2)$; but it is bigger, a contradiction.

Proof of 1.3. Trivial. For example, look at all possible triples $(\lg(\rho), d_{\mathrm{mx}}(\rho), k)$ or, to be specific, recalling that $[x]$ is the integral part of a real x, let

$$d_1(\tau) = [\log_2(\log_2(d_f(\tau)))],$$
$$d(\tau) \stackrel{\text{def}}{=} d_{\mathrm{mx}}(\tau) + d_1(\tau) - (\sqrt{d_1(\tau)})^2.$$

For every ρ and k let ρ_1 be of minimal length such that, for some $n \geq 1$, $\rho = (\rho_1)^{[n]}$. We choose m: let

$$m(0) = k + (\lg(\rho) - d_{\mathrm{mx}}(\rho)), \qquad m(2) = (m(0) + n + 8)^2 + m(0),^{2} \qquad m = 2^{2^{m(2)}}.$$

Let us compute $d(\tau)$ when $\tau = \eta^\wedge \rho^{[m]\wedge} \nu$ and $\mathrm{Max}(\rho) > \mathrm{Max}(\eta^\wedge \nu)$. So $d_{\mathrm{mx}}(\tau) = \lg(\eta) + d_{\mathrm{mx}}(\rho) = \lg(\eta) + d_{\mathrm{mx}}(\rho_1)$ and $\tau = \eta^\wedge \rho_1^{[nm]\wedge} \nu$. Also $d_f(\tau) = nm$ (by Fact 1.3B). So $\log_2(d_f(\tau)) = \log_2(m) + \log_2(n)$ and

$$0 \leq \log_2(n) < m(2) = \log_2 \log_2(m) \leq \log_2(\log_2(d_f(\tau)))$$
$$\leq \log_2 \log_2(m) + \log_2\left(1 + \frac{\log_2(n)}{\log_2(m)}\right) \leq m(2) + \log_2 2 < m(2) + 1.$$

[2] The eight is to help with the ≥ 4 demanded above.

So $d_1(\tau) = [\log_2 \log_2(d_f(\tau))] = m(2)$; hence $d_1(\tau) - (\sqrt{d_1(\tau)})^2 = m(0)$ and, lastly,

$$d(\tau) = d_{\mathrm{mx}}(\tau) + m(0) = \lg(\eta \frown \rho) + k.$$

Continuation of the Proof of 1.1. Now we want to prove that c witnesses $\Pr_0(\lambda, \lambda)$. So let $n(*) < \omega$, for $\alpha < \lambda$ and $\zeta_\alpha^1 < \cdots < \zeta_\alpha^{n(*)}$ let $\{\zeta_\alpha^1, \ldots, \zeta_\alpha^{n(*)}\}$ be pairwise disjoint for $\alpha < \lambda$, and let h^* be a two-place function from from $\{1, \ldots, n(*)\}$ to λ. We must find $\alpha < \beta$ as in Definition 1.2. For each α, the sets $\{A_{\zeta_\alpha^e}: e = 1, \ldots, n(*)\}$ are pairwise incomparable subsets of μ; hence for some finite $u_\alpha \subseteq \mu$ the sets $\{A_{\zeta_\alpha^e} \cap u_\alpha: e = 1, \ldots, n(*)\}$ are pairwise distinct. So, for some stationary $S^* \subseteq S$ and for every $\alpha \in S^*$, $u_\alpha = u^*$ and $A_{\zeta_\alpha^e} \cap u^* = a_e^*$ (for some fixed u^* and $a_1^*, \ldots, a_{n(*)}^*$). Without loss of generality, for $\alpha < \beta$ from S^* we assume that $\zeta_\alpha^{n(*)} < \zeta_\beta^1$. Let us define a $g \in G$ as follows: $w(g) = u^*$, and $g(a_k^*, a_l^*) = h^*(k, l)$; g is zero on other pairs. So $S_g = S \cap \Gamma^{-1}(g)$ is a stationary subset of S. We now define, by induction on $\varepsilon \in S_g$, an ordinal α_ε as follows: α_ε is the first ordinal $\alpha < \lambda$ such that $\alpha \in S^*$, $(\forall \xi < \varepsilon)[\alpha_\xi < \alpha]$ and $\zeta_\alpha^1 > \varepsilon$.

Now α_ε is well defined, as each demand holds for every large enough $\alpha \in S^*$. By renaming the ζ_α^k's, without loss of generality for every $\alpha \in S_g$ we have $u_\alpha = u^*$, $A_{\zeta_\alpha^e} \cap u^* = a_e^*$ for $e = 1, \ldots, n$ and $\zeta_\zeta^1 > \alpha$; and rename $S^* = S_g$. Let $S_n \stackrel{\mathrm{def}}{=} S \cap H^{-1}(\{n\})$.

Now define, for any set $T \subseteq \lambda$ and limit ordinal $\delta < \lambda$,

$$F_T(\delta) = \bigcap_{\alpha < \delta} \{\langle H(\gamma_0(\delta, \beta)), H(\gamma_1(\delta, \beta)), \ldots, H(\gamma_{\mathbf{n}(\delta,\beta)-1}(\delta, \beta))\rangle: \beta \in T \cap (\alpha, \delta)\}.$$

Now Range $F_T \subseteq \mathscr{P}({}^{\omega>}\omega)$.

Clearly the following statements hold:

(α) If cf $\delta > \aleph_0$ and T is unbounded below δ, then $F_T(\delta)$ is nonempty.

(β) If $T_1 \subseteq T_2$, then $F_{T_1}(\delta) \subseteq F_{T_2}(\delta)$.

(γ) If T is an unbounded subset of λ, then, for a closed unbounded set of $\delta < \lambda$ of uncountable cofinality,

$$\omega = \bigcup \{\mathrm{Rang}(\rho): \rho \in F_T(\delta)\}.$$

[*Proof of* (γ). Recall that $S_n = H^{-1}(\{n\}) \cap S$. Let $C = \{\delta < \lambda: \delta$ limit, and $\delta = \sup(T \cap \delta)\}$, so C is a club of λ. Let C^1 be the set of $\delta < \lambda$ such that $S_n \cap C$ is unbounded below δ for each n (so C^1 too is a club of λ). Suppose $\delta \in C^1$ and $\mathrm{cf}(\delta) > \aleph_0$. Let $n < \omega$. For $\alpha < \delta$ there is $\beta_{\alpha,n} \in (S_n \cap C) \cap (\alpha, \delta)$. Now $T \cap \beta_{\alpha,n}$ is unbounded below $\beta_{\alpha,n}$ (which is a limit ordinal as it belongs to C), and so by (d) above (in the beginning of the proof), for some $\xi = \xi_{\alpha,n} \in T \cap \beta_{\alpha,n}$ we have $\xi > \alpha$, and $\gamma_e(\delta, \beta_{\alpha,n}) = \gamma_e(\delta, \xi)$ for $e \leq \mathbf{n}(\delta, \beta_{\alpha,n})$. So

$$H(\gamma_{\mathbf{n}(\delta,\beta_{\alpha,n})}(\delta, \beta_{\alpha,n}))$$

appears in

$$\langle H_e(\gamma_e(\delta, \xi_{\alpha,n})): e < \mathbf{n}(\delta, \xi_{\alpha,n})\rangle.$$

But the former is n (as $\gamma_{\mathbf{n}(\delta,\beta_{\alpha,n})}(\delta, \beta_{\alpha,n}) = \beta_{\alpha,n}$, $\beta_{\alpha,n} \in S_n = S \cap H^{-1}(\{n\})$), and the latter is a sequence of natural numbers depending on α. As $\mathrm{cf}(\delta) > \aleph_0$, $\alpha < \delta$ arbitrary, some such sequence appears for arbitrarily large $\alpha < \delta$, so n appears in some sequence from $F_T(\delta)$. This holds for every $\delta \in C^1$, so we finish our proof of (γ).]

Let us define for $\delta < \lambda$

$$f^*(\delta) = \bigcap_{\alpha<\delta} \{\langle\langle H(\gamma_m(\delta,\zeta^e_\beta)): m < \mathbf{n}(\delta,\zeta^e_\beta)\rangle: e = 1,\dots,n\rangle: \text{ for some } \beta \in S^* \cap [\alpha,\delta)\}.$$

Clearly, we have

(δ) If $\delta < \lambda$, $\mathrm{cf}(\delta) > \aleph_0$, and $\delta \cap S^*$ is unbounded below δ, then $f^*(\delta)$ is nonempty.

By induction on $n < \omega$ we now define W_n, C^*_n and T_η ($\eta \in W_n$) such that:

(A) W_n is a subset of ${}^n(2^{\aleph_0})$, T_η a subset of λ.

(B) $W_0 = \{\langle\ \rangle\}$, and, for $n \geq 0$, if $\eta \in W_n$ then $[n > 0 \Rightarrow \eta \restriction (n-1) \in W_{n-1}]$ and, for at least one $i < 2^{\aleph_0}$, $\eta^\wedge\langle i\rangle \in W_{n+1}$.

(C) C^*_n is a club of λ, $C^*_{n+1} \subseteq C^*_n$, and every T_η ($\eta \in W_n$) is unbounded below any $\delta \in C^*_n$.

(D) $T_{\langle\ \rangle} = S^*$ (and in fact $S^* \cap C^*_n = \bigcup\{T_\eta \cap C^*_n: \eta \in W_n\}$).

(E) For $\eta \in W_n$, $T_\eta \cap C^*_{n+1}$ is the disjoint union of

$$\{T_{\eta^\wedge\langle i\rangle} \cap C^*_{n+1}: \eta^\wedge\langle i\rangle \in W_{n+1}\}.$$

(F) Each $T_\eta(\eta \in W_n)$ is a stationary subset of λ.

(G) For $\alpha, \beta \in T_\eta \cap C^*_{\lg(\eta)+1}$,

$$(\exists i)[\alpha, \beta \in T_{\eta^\wedge\langle i\rangle}] \quad \textit{iff} \quad F_{T_\eta}(\alpha) = F_{T_\eta}(\beta) \text{ and } f^*(\alpha) = f^*(\beta).$$

There is no difficulty in carrying out the induction (using $\lambda > 2^{\aleph_0}$). For $n = 0$, (B) + (D) determines W_0 and T_η ($\eta \in W_0$). For $n + 1$ define, for each $\eta \in W_n$, an equivalence relation $\sim_n$ on T_η:

$$\alpha \sim_n \beta \quad \text{iff} \quad F_{T_\eta}(\alpha) = F_{T_\eta}(\beta) \ \& \ f^*(\alpha) = f^*(\beta);$$

the number of equivalence classes is $\leq 2^{\aleph_0}$ (as F_{T_η} and f^* have ranges of power $\leq 2^{\aleph_0}$), and let them be $\langle T_{\eta^\wedge\langle i\rangle}: i < i_\eta \leq 2^{\aleph_0}\rangle$; for each $i < i_\eta$, if $T_{\eta^\wedge\langle i\rangle}$ is not stationary, choose a closed unbounded $C_{\eta,i}$ disjoint from it. We let

$$W_{n+1} = \{\eta^\wedge\langle i\rangle: T_{\eta^\wedge\langle i\rangle} \text{ stationary}\}$$

and

$$C^*_{n+1} = \bigcap\{C_{\eta,i}: \eta^\wedge\langle i\rangle \in W_{n+1}\} \cap C^*_n.$$

Check. Let $C^*_\omega = \bigcap_{m<\omega} C^*_n$, which is a club of λ, and define $W_\omega = \{\eta: \eta$ an ω-sequence of ordinals, and, for every $n < \omega$, $\eta \restriction n \in W_n\}$, and also $T_\eta = \bigcap_{n<\omega} T_{\eta\restriction n}$ for $\eta \in W_\omega$.

Clearly the T_η ($\eta \in W_n$) form a partition of $S^* \cap C^*_\omega$; hence, for some $\eta \in W_\omega$, T_η is a stationary subset of λ. (Here we use $\lambda > 2^{\aleph_0}$.)

Now we can find $\delta(*) \in T_\eta$ such that $F_{T_\eta}(\delta(*))$ is nonempty and moreover

$$\omega = \bigcup\{\mathrm{Rang}\ \rho: \rho \in F_{T_\eta}(\delta(*))\}.$$

(This is possible by (γ) above.) For $k = 1,\dots,n(*)$, let

$$\nu^k = \langle H(\gamma_e(\zeta^k_{\delta(*)}, \delta(*))): e < \mathbf{n}(\zeta^k_{\delta(*)}, \delta(*))\rangle$$

and let $\langle \nu_1, \ldots, \nu_{n(*)} \rangle$ be any member of $f^*(\delta)$ for some (all) $\delta \in T_{\eta \restriction 1}$. There is $\rho \in F_{T_\eta}(\delta(*))$ with

$$\mathrm{Max}(\rho) > \mathrm{Max}\{\mathrm{Max}(\nu^1), \ldots, \mathrm{Max}\, \nu^{n(*)}, \mathrm{Max}\, \nu_1, \ldots, \mathrm{Max}\, \nu_{n(*)}\}$$

(this is possible by the choice of $\delta(*)$).

Note that $\rho \in F_{T_{\eta \restriction n}}(\delta)$ if $\delta \in T_{\eta \restriction n}$, $n < \omega$ (by (β) above and (G) above). Now let $\alpha(0) < \delta(*)$ be such that

$(*)_0$ if $1 \le k \le n(*)$, $0 \le e \le \mathbf{n}(\zeta^k_{\delta(*)}, \delta(*))$, and $\alpha(0) \le \alpha < \delta(*)$, then $\gamma_e(\zeta^k_{\delta(*)}, \delta(*)) = \gamma_e(\zeta^k_{\delta(*)}, \alpha)$

(exists by (d) above).

Let $m(*) < \omega$ be arbitrary but fixed for a while. We define by induction on $m \le m(*)$, $\alpha(m)$ and $\delta(m)$ such that:

(i) $\alpha(m) < \alpha(m+1) < \delta(m+1) < \delta(m)$;

(ii) $\delta(0) = \delta(*)$ ($\alpha(0)$ defined above);

(iii) $\rho = \langle H(\gamma_e(\delta(m), \delta(m+1))) \colon e < \mathbf{n}(\delta(m), \delta(m+1)) \rangle$;

(iv) $\delta(m) \in T_{\eta \restriction (m(*)+8-m)}$;

(v) if $1 \le k \le n(*)$, $e \le \mathbf{n}(\zeta^k_{\delta(*)}, \delta(m))$ and $\alpha(m) \le \alpha \le \delta(m)$, then

$$\gamma_e(\zeta^k_{\delta(*)}, \delta(m)) = \gamma_e(\zeta^k_{\delta(*)}, \alpha);$$

(vi) $\gamma_e(\delta(m), \delta(m+1)) = \gamma_{e+n}(\zeta^k_{\delta(*)}, \delta(m+1))$ if $1 \le k \le n(*)$, $e \le \mathbf{n}(\delta(m), \delta(m+1))$ and $n = \mathbf{n}(\zeta^k_{\delta(*)}, \delta(m))$.

There is no problem in doing this. Now we can find $\delta(**) \in S^*$, $\alpha(m(*)) < \delta(**) < \delta(m(*))$, and

$$\nu_k = \langle H(\gamma_e(\delta(m(*)), \zeta^k_{\delta(**)})) \colon e < \mathbf{n}(\delta(m(*)), \zeta^k_{\delta(**)}) \rangle.$$

So for $1 \le k(1) \le n(*)$ and $1 \le k(2) \le n(*)$, the sequence

$$\langle H(\gamma_e(\zeta^{k(1)}_{\delta(*)}, \zeta^{k(2)}_{\delta(**)})) \colon e < \mathbf{n}(\zeta^{k(1)}_{\delta(*)}, \zeta^{k(2)}_{\delta(**)}) \rangle$$

is equal to $(\nu^{k(1)})^\wedge \rho^{[m(*)]\wedge} \nu_{k(2)}$.

By the demands on d (see 1.3), for some $m(*)$ and for every $k(1), k(2) \in [1, n(*)]$

$$d(\nu^{k(1)\wedge} \rho^{[m(*)]\wedge} \nu_{k(2)}) = \lg(\nu^{k(1)}) + \lg(\rho).$$

So

$$e(\zeta^{k(1)}_{\zeta(*)}, \zeta^{k(2)}_{\delta(**)}) = \lg(\nu^{k(1)}) + \lg(\rho)$$

(see the definition of $e(\beta, \alpha)$ just before 1.3), and so

$$\gamma_{e(\zeta^{k(1)}_{\delta(*)}, \zeta^{k(2)}_{\delta(**)})}(\zeta^{k(1)}_{\zeta(*)}, \zeta^{k(2)}_{\delta(**)}) = \delta(1).$$

As $\delta(1) \in T_{\eta \restriction (m(*)+8-1)}$, we have $\delta(1) \in S^* \subseteq S_g$, and also

$$g(\zeta^{k(1)}_{\delta(*)}, \zeta^{k(2)}_{\delta(**)}) = g(\delta(1)) = g$$

and

$$c(\zeta^{k(1)}_{\zeta(*)}, \zeta^{k(2)}_{\delta(**)}) = g(A_{\zeta^{k(1)}_{\delta(*)}} \cap w(g), A_{\zeta^{k(2)}_{\delta(**)}} \cap w(g));$$

but by the choice of g it is $h^*(k(1), k(2))$, as required.

1.4. THEOREM. *Suppose λ is regular $> 2^{\aleph_0}$, and $S \subseteq \{\delta < \lambda: \operatorname{cf} \delta > \aleph_0\}$ is stationary and does not reflect. Then* $\Pr_1(\lambda, \lambda)$, *where*:

1.5. DEFINITION. $\Pr_1(\lambda, \mu)$ holds if some $c: \lambda \to \mu$ witnesses it, which means that c is a two place function from λ to μ, and ***if*** $n(*) < \omega$, $\gamma < \mu$ for $\alpha < \lambda$, $\zeta^1_\alpha < \cdots < \zeta^{n(*)}_\alpha < \lambda$, and the sets $\{\zeta^1_\alpha, \ldots, \zeta^{n(*)}_\alpha\}$ are pairwise disjoint, ***then***, for some $\alpha < \beta$,

$$1 \leq l \leq n(*) \wedge 1 \leq k \leq n(*) \Rightarrow c(\zeta^k_\beta, \zeta^l_\alpha) = \gamma.$$

1.5A. REMARKS. 1) Thus in 1.4 we have omitted the assumption $(\exists \mu)(\mu < \lambda \leq 2^\mu)$.

2) In the case not covered by 1.1, i.e. λ is strongly inaccessible, for every 2-place function c from λ to λ, we can find $\bar{\zeta}_\alpha = \langle \zeta^i_\alpha: i < \alpha \rangle$, ζ^i_α increasing in i, and $\zeta^i_\alpha < \zeta^j_\beta$ for $\alpha < \beta$, such that

$$c(\zeta^i_\alpha, \zeta^{j(1)}_\beta) = c(\zeta^i_\alpha, \zeta^{j(2)}_\beta)$$

when $\alpha < \beta$, $i < \alpha$, and $j(1) < j(2) < \beta$, so the weakening of the conclusion is reasonable.

PROOF. We indicate the changes needed in the proof of 1.1. We omit G and A_α; we let g be a function from λ to λ such that, for every $\alpha < \lambda$, $S^\alpha = g^{-1}(\{\alpha\}) \cap S$ is stationary and defining the coloring c; we omit $g(\beta, \alpha)$; and we let

$$c(\beta, \alpha) = g(\gamma_e(\beta, \alpha)).$$

Also, given $\{\zeta^1_\alpha, \ldots, \zeta^{n(*)}_\alpha\}$ and $\gamma < \lambda$, we can arrange without loss of generality that $S^* = \Gamma^{-1}(\{\gamma\}) \cap S$ (instead of $S^* = S_g$).

1.6. DEFINITION. We define $\Pr_0(\lambda, \mu, \theta)$ and $\Pr_1(\lambda, \mu, \theta)$ as in Definitions 1.2 and 1.5 respectively but with "$n(*) < \omega$" replaced by "$n(*) < \theta$" (so $n(*)$ may be infinite, and θ may be finite or infinite). Now we strengthen 1.4.

1.7. THEOREM. *If λ is regular, $\theta > \aleph_0$ is regular, $2^\theta < \lambda$, S is a stationary subset of λ which does not reflect, and $(\forall \delta \in S)[\operatorname{cf} \delta > \theta]$, then* $\Pr_1(\lambda, \lambda, \theta)$.

PROOF. The proof is similar to that of 1.4, but the range of H is θ, and ${}^{\omega>}\theta$ replaces ${}^{\omega>}\omega$.

1.8. THEOREM. *Suppose λ is regular, $\lambda > \theta > \aleph_0$, θ regular, $\lambda > 2^\theta$, $S \subseteq \{\delta < \lambda: \operatorname{cf} \delta > \theta\}$ is stationary nonreflecting, and $\mu = \mu^{<\theta} < \lambda \leq 2^\mu$. Then* $\Pr_0(\lambda, \lambda, \theta)$.

PROOF. Change the proof of 1.1 as in 1.7. Note that now the family of functions G is:

$\{g$: for some $w(G) \subseteq \mu$ of power $< \theta$, g is a
two-place function from a family of $< \theta$ subsets of $w(G)$ to $\lambda\}$,

and we complete $g \in G$ to a function from the family of subsets of $w(G)$ to λ by letting $g(a, b) = 0$ when $a, b \subseteq w(G)$, $g(a, b)$ not defined.

1.9. *Conclusion. Suppose one of the following three conditions holds*:

(i) *λ is a successor of regular, $\lambda > 2^{\aleph_0}$, or*

(ii) *λ is regular $> 2^{\aleph_0}$ and $S \subseteq \{\delta < \lambda: \operatorname{cf} \delta > \aleph_0\}$ is stationary nonreflecting, or*

(iii) $\Pr_1(\lambda, 2)$.

Then there is a Boolean algebra B satisfying the λ-c.c. but for which $B \times B$ does not satisfy the λ-c.c.

Proof. By 1.4, (ii) $\to$ (iii), and trivially (i) $\to$ (ii). So assume $\Pr_1(\lambda, 2)$, and let c be a two-place function from λ to $\{0, 1\}$ exemplifying this. We define the two Boolean algebras, B_0 and B_1. For $e = 0$, 1, B_e is the Boolean algebra generated freely by

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$\{x_\alpha^e : \alpha < \lambda\}$ except for the relations $x_\alpha^e \cap x_\beta^e = 0$ when $c(\beta,\alpha) = e$ $(\alpha < \beta < \lambda)$. Clearly in B_e the x_α^e's are distinct, and $B_e \models x_\alpha^e \cap x_\beta^e = 0$ iff $c(\beta,\alpha) = e$. Now $B_0 \times B_1$ does not satisfy the λ-c.c., as $\{[x_\alpha^0, x_\alpha^1] : \alpha < \lambda\}$ exemplifies. Why does B_e satisfy the λ-c.c.? Suppose $a_\zeta \in B_e$ for $\zeta < \lambda$, $a_\zeta \neq 0$, and there are $n(\zeta)$, $\alpha(\zeta,0) < \cdots < \alpha(\zeta,n(\zeta)) < \lambda$, $\sigma(\zeta,0),\ldots,\sigma(\zeta,n(\zeta)) \in \{0,1\}$ and

$$y_{\zeta,m} = \begin{cases} x^e_{\alpha(\zeta,m)} & \text{if } \sigma(\zeta,m) = 0, \\ 1 - x^e_{\alpha(\zeta,m)} & \text{if } \sigma(\zeta,m) = 1, \end{cases}$$

such that

$$0 \neq \bigcap_{m=0}^{n(\zeta)} y_{\zeta,m} \subseteq a_\zeta.$$

Without loss of generality, for some m_0, n, $\sigma(0),\ldots,\sigma(n)$ and for every ζ we take $n(\zeta) = n$, $\sigma(\zeta,m) = \sigma(m)$ for $m \leq n$, $\alpha(\zeta,m) = \alpha(0,m)$ for $m < m_0$, and $\{\alpha(\zeta,m_0),\ldots,\alpha(\zeta,n)\}$ are pairwise disjoint for $\zeta < \lambda$. As c exemplifies $\mathrm{Pr}_1(\lambda,2)$, for some $\zeta < \xi < \lambda$

$$\bigwedge_{m=m_0}^{n} \bigwedge_{k=m_0}^{n} c(\alpha(\zeta,m), \alpha(\xi,k)) = 1 - e.$$

Clearly $B_e \models a_\zeta \cap a_\xi \neq 0$, and we are finished.

1.10. *Conclusion. Suppose one of the following three conditions holds*:

(i) $\lambda > 2^{\aleph_0}$ *is a successor of regular, or*

(ii) $\lambda > 2^{\aleph_0}$ *is not strong limited and* $S \subseteq \{\delta < \lambda : \mathrm{cf}\,\delta > \aleph_0\}$ *is stationary but does not reflect, or*

(iii) $\mathrm{Pr}_0(\lambda,2)$.

Then there are Hausdorff topological spaces X_0 *and* X_1 *of power* λ *with a basis of clopen sets, such that* X_0 *is a* λ-*S-space and* X_1 *a* λ-*L-space (i.e.* X_0 *and every subspace of it of cardinality* λ *has a density character* $<\lambda$, *but* X_0 *is not* λ-*Lindelof, whereas* X_1 *and every subspace of it of cardinality* λ *is* λ-*Lindelof but* X_1 *has density character* λ).

Proof. Let c exemplify $\mathrm{Pr}_0(\lambda,2)$, and for $e = 0, 1$ let X_e be the Boolean algebra of clopen sets which is generated by $\{u_\alpha^e : \alpha < \lambda\}$, where

$$u_\alpha^0 = \{\beta : \alpha < \beta < \lambda \text{ and } c(\beta,\alpha) = 0\} \cup \{\alpha\},$$

$$u_\alpha^1 = \{\beta : \beta < \alpha \text{ and } c(\beta,\alpha) = 1\} \cup \{\alpha\}.$$

The conclusion is then immediate by 1.1; see e.g. Roitman [R], using 1.1.

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