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Monotone hulls for $\mathcal{N} \cap \mathcal{M}$

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Abstract Using the method of decisive creatures [see Kellner and Shelah (J Symb Log 74:73–104, 2009)] we show the consistency of "there is no increasing ω_2–chain of Borel sets and $\mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N} \cap \mathcal{M}) = \omega_2 = 2^\omega$". Hence, consistently, there are no monotone Borel hulls for the ideal $\mathcal{M} \cap \mathcal{N}$. This answers Balcerzak and Filipczak (Math Log Q 57:186–193, 2011 [Questions 23, 24]). Next we use finite support iteration of ccc forcing notions to show that there may be monotone Borel hulls for the ideals $\mathcal{M}, \mathcal{N}$ even if they are not generated by towers.

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1 Introduction

Brendle and Fuchino [4, Sect. 3] considered the following spectrum of cardinal numbers

$$\begin{aligned}\mathfrak{D}\mathfrak{O} = \{\mathrm{cf}\ \left(\mathrm{otp}\left(\langle X, R{\restriction}X\rangle\right)\right) :\ & R \subseteq {}^\omega 2 \times {}^\omega 2 \text{ is a projective binary relation,}\\ & X \subseteq {}^\omega 2 \text{ and } R \cap X^2 \text{ is a well ordering of } X\}\end{aligned}$$

Dedicated to László Fuchs for his ninetieth birthday.

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and they introduced a cardinal invariant $\mathfrak{d}\mathfrak{o} = \sup \mathfrak{D}\mathfrak{O}$. The invariant $\mathfrak{d}\mathfrak{o}$ satisfies $\min\{\mathrm{non}(\mathcal{I}), \mathrm{cov}(\mathcal{I})\} \leq \mathfrak{d}\mathfrak{o}$ for every ideal $\mathcal{I}$ on $\mathbb{R}$ with Borel basis (see [4, Lemma 3.6]). The proof of Kunen [9, Theorem 12.7] essentially shows that adding any number of Cohen (or random) reals to a model of CH results in a model in which $\mathfrak{d}\mathfrak{o} = \aleph_1$. Thus both

$$\mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_2 + \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{d}\mathfrak{o} = \aleph_1, \text{ and}$$
$$\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2 + \mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{d}\mathfrak{o} = \aleph_1$$

are consistent (where $\mathcal{M}, \mathcal{N}$ stand for the ideals of meager and null sets, respectively). This naturally leads to the question if

$$(\circledast) \quad \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{N} \cap \mathcal{M}) = \aleph_2 + \mathfrak{d}\mathfrak{o} = \aleph_1 = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M})$$

is consistent. In this note we show the consistency of $(\circledast)$ using the method of *decisive creatures* developed in Kellner and Shelah [8], and this method is in turn a special case of the method of *norms on possibilities* of Rosłanowski and Shelah [11].

Note that if there is a $\subset$–increasing κ–chain of Borel subsets of ${}^{\omega}2$, then $\mathrm{cf}(\kappa) \in \mathfrak{D}\mathfrak{O}$. (Just consider a relation R on ${}^{\omega}2 \simeq {}^{\omega}2 \times {}^{\omega}2$ given by: $(x, y)\ R\ (x', y')$ if and only if "y, y' are Borel codes and x belongs to the set coded by y'"; cf. Elekes and Kunen [6, Lemma 2.4].) Thus if we set

$$\mathfrak{d}_{\mathcal{B}} = \sup\left\{\mathrm{cf}(\gamma) : \text{ there is a } \subset\text{–increasing chain of Borel subset of } \mathbb{R} \text{ of length } \gamma\right\}$$

then $\mathfrak{d}_{\mathcal{B}} \leq \mathfrak{d}\mathfrak{o}$. If $\mathfrak{d}_{\mathcal{B}}$ is smaller than the cofinality of the uniformity number $\mathrm{non}(\mathcal{I})$ of a Borel ideal $\mathcal{I}$, then there is no monotone Borel hull operation on $\mathcal{I}$ (see Elekes and Máthé [7, Theorem 2.1], Balcerzak and Filipczak [1, Theorem 5]). Thus

$(\otimes)$ if $\mathcal{I}$ is an ideal with Borel basis on $\mathbb{R}$, $\mathfrak{d}_{\mathcal{B}} < \mathrm{non}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ is a regular cardinal, then there is no $\subset$–monotone mapping $\psi : \mathcal{I} \longrightarrow \mathrm{Borel}(\mathbb{R}) \cap \mathcal{I}$.

Therefore in our model for $(\circledast)$ we will have (Corollary 4.2)

"there are no monotone Borel hull operations on the ideals $\mathcal{M}, \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N}$".

This answers Balcerzak and Filipczak [1, Question 23].

We also obtain a positive result providing a new situation in which monotone hulls exist. Consistently, the ideals $\mathcal{M}, \mathcal{N}$ do not possess tower–basis but they do admit monotone Borel hulls (Corollary 4.9). This model is obtained by finite support iterations of partial Amoeba for Category and Amoeba for Measure $\mathbb{A}$ forcing notions.

Notation *Most of our notation is standard and compatible with that of classical textbooks (like Bartoszyński and Judah* [2]*). However, in forcing we keep the older convention that* a stronger condition is the larger one.

- For two sequences η, ν we write $\nu \vartriangleleft \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \vartriangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\ell g(\eta)$. A *tree* is a family T of finite sequences closed under initial segments. For a tree T, the family of all ω–branches through T is denoted by $[T]$.
- The Cantor space ${}^{\omega}2$ is the space of all functions from ω to 2, equipped with the product topology generated by sets of the form $[\nu] = \{\eta \in {}^{\omega}2 : \nu \vartriangleleft \eta\}$ for $\nu \in {}^{\omega>}2$. This space is also equipped with the standard product measure μ.
- For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g. $\underset{\sim}{A}, \underset{\sim}{\eta}$). The canonical name for a $\mathbb{P}$–generic filter over $\mathbf{V}$ is denoted $\underset{\sim}{G}_{\mathbb{P}}$. Our notation and terminology concerning creatures and forcing with creatures will be compatible with that in [8] (except of the reversed orders). While this is a slight departure

from the original terminology established for creature forcing in [11], the reader may find it more convenient when verifying the results on decisive creatures that are quoted in the next section.

2 Background on decisive creatures

As declared in the introduction, we will follow the notation and the context of [8] (which slightly differs from that of [11]). For reader's convenience we will recall here all relevant definitions and results from that paper.

Let $\mathbf{H} : \omega \longrightarrow \mathcal{H}(\aleph_0)$ (where $\mathcal{H}(\aleph_0)$ is the family of all hereditarily finite sets). A *creating pair* for $\mathbf{H}$ is a pair $(\mathbf{K}, \Sigma)$, where

- $\mathbf{K} = \bigcup_{n<\omega} \mathbf{K}(n)$, where each $\mathbf{K}(n)$ is a finite set; elements of $\mathbf{K}$ are called *creatures*, each creature $\mathfrak{c} \in \mathbf{K}(n)$ has some norm $\mathrm{nor}(\mathfrak{c})$ (a non-negative real number) and a non-empty set of possible values $\mathrm{val}(\mathfrak{c}) \subseteq \mathbf{H}(n)$,
- if $\mathfrak{c} \in \mathbf{K}(n)$, $\mathrm{nor}(\mathfrak{c}) > 0$, then $|\mathrm{val}(\mathfrak{c})| > 1$
- $\Sigma : \mathbf{K} \longrightarrow \mathcal{P}(\mathbf{K})$ is such that if $\mathfrak{c} \in \mathbf{K}(n)$ and $\mathfrak{c}' \in \Sigma(\mathfrak{c})$, then $\mathfrak{c}' \in \mathbf{K}(n)$,
- $\mathfrak{c} \in \Sigma(\mathfrak{c})$ and $\mathfrak{c}' \in \Sigma(\mathfrak{c})$ implies $\Sigma(\mathfrak{c}') \subseteq \Sigma(\mathfrak{c})$,
- if $\mathfrak{c}' \in \Sigma(\mathfrak{c})$, then $\mathrm{nor}(\mathfrak{c}') \leq \mathrm{nor}(\mathfrak{c})$ and $\mathrm{val}(\mathfrak{c}') \subseteq \mathrm{val}(\mathfrak{c})$.

If $\mathfrak{c} \in \mathbf{K}$ and $x \in \mathbf{H}(n)$, then we write $x \in \Sigma(\mathfrak{c})$ if and only if $x \in \mathrm{val}(\mathfrak{c})$. For $x \in \mathbf{H}(n)$ we also set $\Sigma(x) = \mathrm{val}(x) = \{x\}$.

Definition 2.1 (See [8, Definitions 3.1, 4.1]) Let $0 < r \leq 1$, B, K, m be positive integers and $(\mathbf{K}, \Sigma)$ be a creating pair for $\mathbf{H}$.

(1) A creature $\mathfrak{c}$ is r–halving if there is a $\mathrm{half}(\mathfrak{c}) \in \Sigma(\mathfrak{c})$ such that
- $\mathrm{nor}(\mathrm{half}(\mathfrak{c})) \geq \mathrm{nor}(\mathfrak{c}) - r$, and
- if $\mathfrak{d} \in \Sigma(\mathrm{half}(\mathfrak{c}))$ and $\mathrm{nor}(\mathfrak{d}) > 0$, then there is a $\mathfrak{d}' \in \Sigma(\mathfrak{c})$ such that

$$\mathrm{nor}(\mathfrak{d}') \geq \mathrm{nor}(\mathfrak{c}) - r \quad \text{and} \quad \mathrm{val}(\mathfrak{d}') \subseteq \mathrm{val}(\mathfrak{d}).$$

$\mathbf{K}(n)$ is r–halving, if all $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ are r–halving.

(2) A creature $\mathfrak{c}$ is (B, r)–big if for every function $F : \mathrm{val}(\mathfrak{c}) \longrightarrow B$ there is a $\mathfrak{d} \in \Sigma(\mathfrak{c})$ such that $\mathrm{nor}(\mathfrak{d}) \geq \mathrm{nor}(\mathfrak{c}) - r$ and the restriction $F{\restriction}\mathrm{val}(\mathfrak{d})$ is constant. We say that $\mathfrak{c}$ is hereditarily (B, r)-big, if every $\mathfrak{d} \in \Sigma(\mathfrak{c})$ with $\mathrm{nor}(\mathfrak{d}) > 1$ is (B, r)-big. Also, $\mathbf{K}(n)$ is (B, r)–big if every $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ is (B, r)–big.

(3) We say that $\mathfrak{c}$ is (K, m, r)–decisive, if for some $\mathfrak{d}^-, \mathfrak{d}^+ \in \Sigma(\mathfrak{c})$ we have: $\mathfrak{d}^+$ is hereditarily $(2^{K^m}, r)$–big, and $|\mathrm{val}(\mathfrak{d}^-)| \leq K$ and $\mathrm{nor}(\mathfrak{d}^-), \mathrm{nor}(\mathfrak{d}^+) \geq \mathrm{nor}(\mathfrak{c}) - r$. The creature $\mathfrak{c}$ is (m, r)–decisive if $\mathfrak{c}$ is (K', m, r)–decisive for some K'.

(4) $\mathbf{K}(n)$ is (m, r)–decisive if every $\mathfrak{c} \in \mathbf{K}(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ is (m, r)–decisive.

Lemma 2.2 (See [8, Lemma 4.3]) *Assume that* $(\mathbf{K}, \Sigma)$ *is a creating pair for* $\mathbf{H}$, $k, m, t \geq 1$, $0 < r \leq 1$. *Suppose that* $\mathbf{K}(n)$ *is* (k, r)*–decisive and* $\mathfrak{c}_0, \ldots, \mathfrak{c}_{k-1} \in \mathbf{K}(n)$ *are hereditarily* $(2^{m^t}, r)$*–big with* $\mathrm{nor}(\mathfrak{c}_i) > 1 + r \cdot (k-1)$ *(for each* $i < k$*). Let* $F : \prod_{i<k} \mathrm{val}(\mathfrak{c}_i) \longrightarrow 2^{m^t}$. *Then there are* $\mathfrak{d}_0, \ldots, \mathfrak{d}_{k-1} \in \mathbf{K}(n)$ *such that:*

$$\mathfrak{d}_i \in \Sigma(\mathfrak{c}_i), \quad \mathrm{nor}(\mathfrak{d}_i) \geq \mathrm{nor}(\mathfrak{c}_i) - r \cdot k, \quad \textit{and } F{\restriction}\prod_{i \in k} \mathrm{val}(\mathfrak{d}_i) \textit{ is constant.}$$

A creating pair $(\mathbf{K}, \Sigma)$ determines the forcing notion $\mathbb{Q}^*_\infty(\mathbf{K}, \Sigma)$ and its special product $\mathbb{P}_I(\mathbf{K}, \Sigma)$ as described by the following definition. (The forcing notion $\mathbb{P}_I(\mathbf{K}, \Sigma)$ is a relative of the CS product of $\mathbb{Q}^*_\infty(\mathbf{K}, \Sigma)$ indexed by the set I.)

Definition 2.3 (See [8, Definitions 2.1, 5.2, 5.3])

(1) A condition in the forcing $\mathbb{Q}^*_\infty(\mathbf{K},\Sigma)$ is an ω–sequence $p=\langle p(i): i<\omega\rangle$ such that for some $n<\omega$ (called the trunk-length of p) we have $p(i)\in\mathbf{H}(i)$ if $i<n$, $p(i)\in\mathbf{K}(i)$ and $\mathrm{nor}(p(i))>0$ if $i\geq n$, and $\lim_{i\to\infty}(\mathrm{nor}(p(i)))=\infty$. The order on $\mathbb{Q}^*_\infty(\mathbf{K},\Sigma)$ is defined by $q\geq p$ if and only if (both belong to $\mathbb{Q}^*_\infty(\mathbf{K},\Sigma)$ and) $q(i)\in\Sigma(p(i))$ for all i.[1]

(2) Let I be a non-empty (index) set. A condition p in $\mathbb{P}_I(\mathbf{K},\Sigma)$ consists of a countable subset $\mathrm{dom}(p)$ of I, of objects $p(\alpha,n)$ for $\alpha\in\mathrm{dom}(p)$, $n\in\omega$, and of a function $\mathrm{trunklg}(p,\cdot):\mathrm{dom}(p)\longrightarrow\omega$ satisfying the following demands for all $\alpha\in\mathrm{dom}(p)$:
(α) If $n<\mathrm{trunklg}(p,\alpha)$, then $p(\alpha,n)\in\mathbf{H}(n)$.
(β) If $n\geq\mathrm{trunklg}(p,\alpha)$, then $p(\alpha,n)\in\mathbf{K}(n)$ and $\mathrm{nor}(p(\alpha,n))>0$.
(γ) Setting $\mathrm{supp}(p,n)=\{\alpha\in\mathrm{dom}(p):\ \mathrm{trunklg}(p,\alpha)\leq n\}$, we have $|\mathrm{supp}(p,n)|<n$ for all $n>0$ and $\lim_{n\to\infty}(|\mathrm{supp}(p,n)|/n)=0$.
(δ) $\lim_{n\to\infty}(\min(\{\mathrm{nor}(p(\alpha,n)):\ \alpha\in\mathrm{supp}(p,n)\}))=\infty$.

The order on $\mathbb{P}_I(\mathbf{K},\Sigma)$ is defined by $q\geq p$ if and only if (both belong to $\mathbb{P}_I(\mathbf{K},\Sigma)$ and) $\mathrm{dom}(q)\supseteq\mathrm{dom}(p)$ and
(ε) if $\alpha\in\mathrm{dom}(p)$ and $n\in\omega$, then $q(\alpha,n)\in\Sigma(p(\alpha,n))$,
(ζ) the set $\{\alpha\in\mathrm{dom}(p):\mathrm{trunklg}(q,\alpha)\neq\mathrm{trunklg}(p,\alpha)\}$ is finite.

Note that for $\alpha\in\mathrm{dom}(p)$ the sequence $\langle p(\alpha,n):n\in\omega\rangle$ is in $\mathbb{Q}^*_\infty(\mathbf{K},\Sigma)$. However, $\mathbb{P}_I(\mathbf{K},\Sigma)$ is not a subforcing of the CS product of I copies of $\mathbb{Q}^*_\infty(\mathbf{K},\Sigma)$ because of a slight difference in the definition of the order relation.

Proposition 2.4 (See [8, Lemmas 5.4, 5.5])

(1) *If $J\subseteq I$, then $\mathbb{P}_J(K,\Sigma)=\{p\in\mathbb{P}_I(\mathbf{K},\Sigma):\mathrm{dom}(p)\subseteq J\}$ is a complete subforcing of $\mathbb{P}_I(\mathbf{K},\Sigma)$.*
(2) *Assume* CH. *Then $\mathbb{P}_I(\mathbf{K},\Sigma)$ satisfies the $\aleph_2$–chain condition.*

Definition 2.5 (See [8, Definition 5.6])

(1) For a condition $p\in\mathbb{P}_I(\mathbf{K},\Sigma)$ we define[2]

$$\mathrm{val}^{\Pi}(p,{<}n)=\prod_{\alpha\in\mathrm{dom}(p)}\prod_{m<n}\mathrm{val}(p(\alpha,m)).$$

(2) If $w\subseteq\mathrm{dom}(p)$ and $t\in\prod\limits_{\alpha\in w}\prod\limits_{m<n}\mathbf{H}(m)$, then $p\wedge t$ is defined by

$$\mathrm{trunklg}(p\wedge t,\alpha)=\begin{cases}\max(\mathrm{trunklg}(p,\alpha),n) & \text{if } \alpha\in w,\\ \mathrm{trunklg}(p,\alpha) & \text{otherwise}\end{cases}$$

and

$$(p\wedge t)(\alpha,m)=\begin{cases}t(\alpha,m) & \text{if } m<n \text{ and } \alpha\in w,\\ p(\alpha,m) & \text{otherwise.}\end{cases}$$

(3) If $\underset{\sim}{\tau}$ is a name of an ordinal, then we say that p ${<}n$–decides $\underset{\sim}{\tau}$, if for every $t\in\mathrm{val}^{\Pi}(p,{<}n)$ the condition $p\wedge t$ forces a value to $\underset{\sim}{\tau}$. The condition p essentially decides $\underset{\sim}{\tau}$, if p ${<}n$-decides $\underset{\sim}{\tau}$ for some n.

1 Remember our convention that for $x,y\in\mathbf{H}(i)$ and $\mathfrak{c}\in\mathbf{K}(i)$ we write $x\in\Sigma(\mathfrak{c})$ iff $x\in\mathrm{val}(\mathfrak{c})$, and $x\in\Sigma(y)$ iff $x=y$.

2 Remember our convention that, for $x\in\mathbf{H}(i)$, $\mathrm{val}(x)=\{x\}$.

Proposition 2.6 (1) *$p \wedge t \in \mathbb{P}_I(\mathbf{K}, \Sigma)$, and if $t \in \mathrm{val}^{\Pi}(p, <n)$, then $p \wedge t \geq p$.*
(2) $\mathrm{val}^{\Pi}(p, <n) \leq \prod_{m<n} |\mathbf{H}(m)|^m$.
(3) *$\{p \wedge t : t \in \mathrm{val}^{\Pi}(p, <n)\}$ is predense above p*

Theorem 2.7 (See [8, Theorems 5.8, 5.9]) *Let $\varphi(<n) = \prod_{m<n} |\mathbf{H}(m)|^m$ and $0 < r(n) \leq 1/(n^2\varphi(<n))$. Assume that each $\mathbf{K}(n)$ is $(n, r(n))$-decisive and $r(n)$–halving (for $n \in \omega$).*

1. *The forcing notion $\mathbb{P}_I(\mathbf{K}, \Sigma)$ is proper and ${}^{\omega}\omega$-bounding. If $|I| \geq 2$ and $\lambda = |I|^{\aleph_0}$, then $\mathbb{P}_I(\mathbf{K}, \Sigma)$ forces $|I| \leq 2^{\aleph_0} \leq \lambda$.*
2. *Moreover, if $\underset{\sim}{\tau}(n)$ is a $\mathbb{P}_I(\mathbf{K}, \Sigma)$–name for an ordinal (for $n < \omega$) and $p \in \mathbb{P}_I(\mathbf{K}, \Sigma)$, then there is a condition $q \geq p$ which essentially decides all the names $\underset{\sim}{\tau}(n)$.*
3. *Assume, additionally, that each $\mathbf{K}(n)$ is $(g(n), r(n))$– big, where $g \in {}^{\omega}\omega$ is strictly increasing. Suppose that $\underset{\sim}{\nu}(n)$ is a $\mathbb{P}_I(\mathbf{K}, \Sigma)$–name and $p \in \mathbb{P}_I(\mathbf{K}, \Sigma)$ forces that $\underset{\sim}{\nu}(n) < 2^{g(n)}$ for all $n < \omega$. Then there is a $q \geq p$ which $<n$–decides $\underset{\sim}{\nu}(n)$ for all n.*

The next theorem is a consequence of (the proof of) [4, Corollaries 4.8(e), 3.9(b)]. However, the results in [4] are stated for products, while $\mathbb{P}_I(\mathbf{K}, \Sigma)$ is not exactly a product (though it does have all the required features). Therefore we will present the relatively simple proof of this result fully.

Theorem 2.8 *Assume* CH. *Let r, φ, $\mathbf{K}$ and Σ be as in the assumptions of Theorem 2.7. Then $\Vdash_{\mathbb{P}_I(\mathbf{K},\Sigma)} \mathfrak{d}_0 = \mathfrak{d}_{\mathcal{B}} = \aleph_1$.*

Proof If $|I| \leq \aleph_1$, then $\Vdash_{\mathbb{P}_I(\mathbf{K},\Sigma)}$ CH, so let us assume $|I| \geq \aleph_2$.

Every bijection $\pi : I \xrightarrow{\text{onto}} I$ determines an automorphism $\tilde{\pi}$ of the forcing $\mathbb{P}_I(\mathbf{K}, \Sigma)$ in a natural way. Then, for $J \subseteq I$, $\tilde{\pi} \restriction \mathbb{P}_J(\mathbf{K}, \Sigma)$ is an isomorphism from $\mathbb{P}_J(\mathbf{K}, \Sigma)$ onto $\mathbb{P}_{\pi[J]}(\mathbf{K}, \Sigma)$. Also, π gives rise to a natural bijection from $\mathrm{val}^{\Pi}(p, <n)$ onto $\mathrm{val}^{\Pi}(\tilde{\pi}(p), <n)$; we will denote this mapping by $\tilde{\pi}$ as well.

Suppose that $\varphi(x, y, \underset{\sim}{\tau})$ is a projective definition of a binary relation on ${}^{\omega}2$, where $\underset{\sim}{\tau}$ is a $\mathbb{P}_I(\mathbf{K}, \Sigma)$–name for a real parameter. Assume towards contradiction that there are $\mathbb{P}_I(\mathbf{K}, \Sigma)$–names $\underset{\sim}{\eta}_\alpha$ (for $\alpha < \omega_2$) and a condition $p \in \mathbb{P}_I(\mathbf{K}, \Sigma)$ such that

(i) $p \Vdash_{\mathbb{P}_I(\mathbf{K},\Sigma)}$ " $\big(\forall \alpha, \beta < \omega_2\big)\big(\varphi(\underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}_\beta, \underset{\sim}{\tau}) \Leftrightarrow \alpha < \beta\big)$ ".

For each $\alpha < \omega_2$ choose a condition $p_\alpha \geq p$ which essentially decides all $\underset{\sim}{\eta}_\alpha(n)$ (for $n < \omega$). Then we may also pick an increasing sequence $\bar{N}^\alpha = \langle N^\alpha_n : n < \omega \rangle \subseteq \omega$ and a mapping $f_\alpha : \bigcup_{n<\omega} \mathrm{val}^{\Pi}(p_\alpha, <N^\alpha_n) \longrightarrow 2$ such that for each $t \in \mathrm{val}^{\Pi}(p_\alpha, <N^\alpha_n)$ we have $(p_\alpha \wedge t) \Vdash \underset{\sim}{\eta}_\alpha(n) = f_\alpha(t)$.

By CH, we may use a standard Δ–system argument and the fact that $\mathbb{P}_I(\mathbf{K}, \Sigma)$ satisfies the $\aleph_2$–cc (see 2.4) to choose $J \in [I]^{\aleph_1}$, $X \in [\omega_2]^{\aleph_2}$ and bijections $\pi_{\alpha,\beta} : \mathrm{dom}(p_\alpha) \xrightarrow{\text{onto}} \mathrm{dom}(p_\beta)$ such that

(ii) $\mathrm{dom}(p) \subseteq J$ and $\underset{\sim}{\tau}$ is a $\mathbb{P}_J(\mathbf{K}, \Sigma)$–name, and for distinct $\alpha, \beta \in X$:
(iii) $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(q_\beta) = \mathrm{dom}(p_\alpha) \cap J$ and $\pi_{\alpha,\beta} \restriction (\mathrm{dom}(p_\alpha) \cap J)$ is the identity,
(iv) $\tilde{\pi}_{\alpha,\beta}(p_\alpha) = p_\beta$, $\bar{N}^\alpha = \bar{N}^\beta$, and $f_\alpha = f_\beta \circ \tilde{\pi}_{\alpha,\beta}$.

Pick $\alpha < \beta$ from X. Let π be a bijection from I onto I such that $\pi_{\alpha,\beta} \subseteq \pi$, $(\pi_{\alpha,\beta})^{-1} \subseteq \pi$ and $\pi \restriction J$ is the identity. Then

(v) $\tilde{\pi}(p_\alpha) = p_\beta$, $\tilde{\pi}(p_\beta) = p_\alpha$ and $\tilde{\pi}(\underset{\sim}{\tau}) = \underset{\sim}{\tau}$.

Note that $p_\alpha \cup p_\beta$ does not have to be a condition in $\mathbb{P}_I(\mathbf{K}, \Sigma)$ as the demand 2.3 (2) (γ) may fail. But extending finitely many trunks will easily resolve this problem and we get a condition q stronger than both p_α and p_β. We may even do this in such a manner that the condition q satisfies $\tilde{\pi}(q) = q$. Since $q \geq p_\alpha, p_\beta$, clause (iv) implies

(vi) $q \Vdash$ " $\tilde{\pi}(\underset{\sim}{\eta}_\alpha) = \underset{\sim}{\eta}_\beta \;\&\; \tilde{\pi}(\underset{\sim}{\eta}_\beta) = \underset{\sim}{\eta}_\alpha$ ".

Since $q \geq p$ and $\alpha < \beta$ we have $q \Vdash \varphi(\underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}_\beta, \underset{\sim}{\tau})$. Applying the automorphism $\tilde{\pi}$ and remembering (vi) we conclude that then also $\tilde{\pi}(q) = q \Vdash \varphi(\underset{\sim}{\eta}_\beta, \underset{\sim}{\eta}_\alpha, \underset{\sim}{\tau})$, contradicting (i). □

3 Consistency of $\mathfrak{d}_\mathfrak{o} < \mathrm{non}(\mathcal{M} \cap \mathcal{N})$

Definition 3.1 Let $n < \omega$.

(1) A *basic n–block* is a finite non-empty set B of functions from some non-empty $v \in [\omega]^{<\omega}$ to 2 (i.e., $B \subseteq {}^v 2$) such that $|B|/2^{|v|} < 2^{-n}$. If $\eta \in {}^{\omega>}2 \cup {}^\omega 2$ and $B \subseteq {}^v 2$ is a basic block, then we write $\eta \prec B$ whenever $\eta \restriction v \in B$. For an n–block $B \subseteq {}^v 2$ we set $v(B) = v$.
(2) Let H_n be the family of all pairs $(b, \mathcal{B})$ such that b is a positive integer and $\mathcal{B}$ is a non-empty finite set of basic n–blocks.
(3) We define a function $\mathrm{pnor} : H_n \longrightarrow \omega$ by declaring inductively when $\mathrm{pnor}(b, \mathcal{B}) \geq k$. We set $\mathrm{pnor}(b, \mathcal{B}) \geq 0$ always, and then
  - $\mathrm{pnor}(b, \mathcal{B}) \geq 1$ if and only if $(\forall F \in [{}^\omega 2]^b)(\exists B \in \mathcal{B})(\forall \eta \in F)(\eta \prec B)$,
  - $\mathrm{pnor}(b, \mathcal{B}) \geq k+1$ if and only if there are positive integers $b_0, \ldots, b_{M-1}$ and disjoint sets $\mathcal{B}_0, \ldots, \mathcal{B}_{M-1} \subseteq \mathcal{B}$ such that
    (α) $M > b^{k+1}$, $b_0 \geq b$ and
    (β) $\mathrm{pnor}(b_i, \mathcal{B}_i) \geq k$ and $(b_i)^2 \cdot 2^{|\mathcal{B}_i|^n} < b_{i+1}$ for all $i < M$.

Proposition 3.2 *Let $n < \omega$, $(b, \mathcal{B}), (b', \mathcal{B}') \in H_n$.*

(1) $\mathrm{pnor}(b, \mathcal{B}) \in \omega$ *is well defined and* $2^{\mathrm{pnor}(b,\mathcal{B})} \leq |\mathcal{B}|$.
(2) *If $\mathcal{B} \subseteq \mathcal{B}'$ and $b' \leq b$, then* $\mathrm{pnor}(b, \mathcal{B}) \leq \mathrm{pnor}(b', \mathcal{B}')$.
(3) *For each N there is $(b^*, \mathcal{B}^*) \in H_n$ such that*

$$b^* \geq N \text{ and } \mathrm{pnor}(b^*, \mathcal{B}^*) \geq N \text{ and } \min(v(B)) > N \text{ for all } B \in \mathcal{B}^*.$$

(4) If $\mathrm{pnor}(b, \mathcal{B}) \geq k+1 \geq 2$ *and* $c : \mathcal{B} \longrightarrow \{0, \ldots, b-1\}$, *then for some $\ell < b$ we have* $\mathrm{pnor}(b, c^{-1}[\{\ell\}]) \geq k$.

Proof (1,2) Easy induction on $\mathrm{pnor}(b, \mathcal{B})$.

(3) Note that if $w \in [\omega]^{<\omega}$, $2^n \cdot N < 2^{|w|}$ and $\mathcal{B}_w$ consists of all basic n–blocks B with $v(B) = w$, then $\mathrm{pnor}(N, \mathcal{B}_w) \geq 1$. Now proceed inductively.

(4) Induction on $k \geq 1$. Assume $\mathrm{pnor}(b, \mathcal{B}) \geq 2$ and $c : \mathcal{B} \longrightarrow b$. We claim that for some $\ell < b$ we have $\mathrm{pnor}(b, c^{-1}[\{\ell\}]) \geq 1$. If not, then for each $\ell < b$ we may choose $F_\ell \in [{}^\omega 2]^b$ such that

$$\big(\forall B \in \mathcal{B}\big)\big(\exists \eta \in F_\ell\big)\big(c(B) = \ell \;\Rightarrow\; \eta \not\prec B\big).$$

Set $F = \bigcup\limits_{\ell<b} F_\ell$. Let $b_0, \ldots, b_{M-1}, \mathcal{B}_0, \ldots, \mathcal{B}_{M-1}$ witness $\mathrm{pnor}(b, \mathcal{B}) \geq 2$, in particular, $b_1 > b^2$ and $\mathrm{pnor}(b_1, \mathcal{B}_1) \geq 1$. Since $|F| \leq b^2$ we conclude that there is $B \in \mathcal{B}_1$ such that $(\forall \eta \in F)(\eta \prec B)$. Then B contradicts the choice of $F_{c(B)}$.

Now, for the inductive step, assume our statement holds for k. Let $\mathrm{pnor}(b, \mathcal{B}) \geq k+2$ and $c : \mathcal{B} \longrightarrow \{0, \ldots, b-1\}$. Let $\{(b_i, \mathcal{B}_i) : i < M\}$ witness $\mathrm{pnor}(b, \mathcal{B}) \geq (k+1)+1$, so $M > b^{k+2}$ and $\mathrm{pnor}(b_i, \mathcal{B}_i) \geq k+1$ and $b_i \geq b$. For each $i < M$ apply the inductive hypothesis to choose $\ell_i < b$ such that $\mathrm{pnor}(b_i, \mathcal{B}_i \cap c^{-1}[\{\ell_i\}]) \geq k$. Choose $\ell^* < b$ such that $|\{i < M : \ell^* = \ell_i\}| > b^{k+1}$. Then $\{(b_i, \mathcal{B}_i \cap c^{-1}[\{\ell_i\}]) : \ell_i = \ell^*\}$ witnesses that $\mathrm{pnor}(b, c^{-1}[\{\ell^*\}]) \geq k+1$. □

Now, by induction on $n < \omega$ we define the following objects

$(\oplus)_n$ $\varphi_{\mathbf{H}^*}(<n), r_{\mathbf{H}^*}(n), a(n), N_n, g(n), \mathbf{H}^*(n), \mathbf{K}^*(n), \Sigma^*{\restriction}\mathbf{K}^*(n), \varphi_{\mathbf{H}^*}(=n)$.

We start with stipulating $N_0 = 0$, $\varphi_{\mathbf{H}^*}(<0) = 1$.

Assume we have defined objects listed in $(\oplus)_k$ for $k < n$, and that we also have defined integers $N_n, \varphi_{\mathbf{H}^*}(<n)$. We set

(i) $g(n) = 2^{N_n} + \varphi_{\mathbf{H}^*}(<n)$, $r_{\mathbf{H}^*}(n) = \frac{1}{(n+2)^2 \varphi_{\mathbf{H}^*}(<n)}$ and $a(n) = 2^{1/r_{\mathbf{H}^*}(n)}$.

Choose $(b^*, \mathcal{B}^*) \in H_n$ such that

(ii) $b^* > g(n)$, $\min(v(B)) > N_n$ for all $B \in \mathcal{B}^*$ and $\mathrm{pnor}(b^*, \mathcal{B}^*) > a(n)^{n+972}$

[possible by 3.2 (3)]. Set

(iii) $N_{n+1} = \max\left(\bigcup\{v(B) : B \in \mathcal{B}^*\}\right) + 1$.

We let $\mathbf{H}^*(n)$ be the set of all basic n–blocks B such that $v(B) \subseteq [N_n, N_{n+1})$, and $\mathbf{K}^*(n)$ consist of all triples $\mathfrak{c} = (k^{\mathfrak{c}}, b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}})$ such that

$$(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) \in H_n, \quad \mathcal{B}^{\mathfrak{c}} \subseteq \mathbf{H}^*(n), \quad b^{\mathfrak{c}} > g(n), \quad \text{and } k^{\mathfrak{c}} \in \omega, \quad k^{\mathfrak{c}} < \mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - 1.$$

For $\mathfrak{c} \in \mathbf{K}^*(n)$ we set

(iv) $\mathrm{nor}(\mathfrak{c}) = \log_{a(n)}\left(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}\right)$, $\mathrm{val}(\mathfrak{c}) = \mathcal{B}^{\mathfrak{c}}$ and
$\Sigma^*(\mathfrak{c}) = \{\mathfrak{d} \in \mathbf{K}^*(n) : k^{\mathfrak{c}} \leq k^{\mathfrak{d}},\ b^{\mathfrak{c}} \leq b^{\mathfrak{d}},\ \mathcal{B}^{\mathfrak{d}} \subseteq \mathcal{B}^{\mathfrak{c}}\}$.

Finally, we put $\varphi_{\mathbf{H}^*}(=n) = |\mathbf{H}^*(n)|^n$ and $\varphi_{\mathbf{H}^*}(<n+1) = \varphi_{\mathbf{H}^*}(<n) \cdot \varphi_{\mathbf{H}^*}(=n)$. This completes our inductive definition.

Proposition 3.3 $(\mathbf{K}^*, \Sigma^*)$ *is a creating pair for* $\mathbf{H}^*$ *such that, for each* $n < \omega$, $\mathbf{K}^*(n)$ *is* $(n, r_{\mathbf{H}^*}(n))$*–decisive,* $r_{\mathbf{H}^*}(n)$*–halving and* $(g(n), r_{\mathbf{H}^*}(n))$*–big.*

Proof To verify halving, for each $\mathfrak{c} \in \mathbf{K}^*(n)$ with $\mathrm{nor}(\mathfrak{c}) > 1$ set

$$\mathrm{half}(\mathfrak{c}) = (k^{\mathfrak{c}} + \lfloor \frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}) \rfloor, b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}).$$

Note that $\mathrm{nor}(\mathfrak{c}) > 1$ implies $\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}} > 2$ and hence

$$k^{\mathfrak{c}} + \lfloor \frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}) \rfloor < \mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - 1.$$

Therefore, $\mathrm{half}(\mathfrak{c}) \in \Sigma^*(\mathfrak{c})$ and $\mathrm{nor}(\mathrm{half}(\mathfrak{c})) \geq \mathrm{nor}(\mathfrak{c}) - r_{\mathbf{H}^*}(n)$. Now suppose $\mathfrak{d} \in \Sigma^*(\mathrm{half}(\mathfrak{c}))$, so $k^{\mathfrak{c}} + \lfloor \frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}) \rfloor \leq k^{\mathfrak{d}}$, $b^{\mathfrak{c}} \leq b^{\mathfrak{d}}$ and $\mathcal{B}^{\mathfrak{d}} \subseteq \mathcal{B}^{\mathfrak{c}}$. Also, $k^{\mathfrak{d}} < \mathrm{pnor}(b^{\mathfrak{d}}, \mathcal{B}^{\mathfrak{d}}) - 1$, so $\mathrm{pnor}(b^{\mathfrak{d}}, \mathcal{B}^{\mathfrak{d}}) > k^{\mathfrak{c}} + \lfloor \frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}) \rfloor + 1$. Consider $\mathfrak{d}' = (k^{\mathfrak{c}}, b^{\mathfrak{d}}, \mathcal{B}^{\mathfrak{d}})$. Plainly $\mathfrak{d}' \in \Sigma^*(\mathfrak{c})$, $\mathrm{val}(\mathfrak{d}') \subseteq \mathrm{val}(\mathfrak{d})$ and

$$\begin{aligned} \mathrm{nor}(\mathfrak{d}') \geq \log_{a(n)}\left(\lfloor \frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}}) \rfloor + 1\right) &\geq \log_{a(n)}\left(\frac{1}{2}(\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}})\right) \\ &= \mathrm{nor}(\mathfrak{c}) - r_{\mathbf{H}^*}(n). \end{aligned}$$

It follows from 3.2(4) that

(∗) if $\mathfrak{c} \in \mathbf{K}^*(n)$, $\mathrm{nor}(\mathfrak{c}) > r_{\mathbf{H}^*}(n)$, then $\mathfrak{c}$ is $(b^{\mathfrak{c}}, r_{\mathbf{H}^*}(n))$–big.

Hence $\mathbf{K}^*(n)$ is $(g(n), r_{\mathbf{H}^*}(n))$–big (remember the definition of $\mathbf{K}^*(n)$).

Now suppose $\mathfrak{c} \in \mathbf{K}^*(n)$, $\mathrm{nor}(\mathfrak{c}) > 1$. Then $\mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - k^{\mathfrak{c}} > 2$, so by the definition of pnor [see 3.1(3)] we may find $b^{\mathfrak{c}} \le b_0 < b_1 < \cdots < b_{M-1}$ and disjoint $\mathcal{B}_0, \ldots, \mathcal{B}_{M-1} \subseteq \mathcal{B}^{\mathfrak{c}}$ such that $\mathrm{pnor}(b_i, \mathcal{B}_i) \ge \mathrm{pnor}(b^{\mathfrak{c}}, \mathcal{B}^{\mathfrak{c}}) - 1$ and $(b_i)^2 \cdot 2^{|\mathcal{B}_i|^n} < b_{i+1}$. Set

$$\mathfrak{d}^- = (k^{\mathfrak{c}}, b_0, \mathcal{B}_0), \quad \mathfrak{d}^+ = (k^{\mathfrak{c}}, b_1, \mathcal{B}_1), \quad \text{and } K = |\mathcal{B}_0|.$$

Plainly, $\mathfrak{d}^-, \mathfrak{d}^+ \in \Sigma(\mathfrak{c})$, $\min\{\mathrm{nor}(\mathfrak{d}^-), \mathrm{nor}(\mathfrak{d}^+)\} \ge \mathrm{nor}(\mathfrak{c}) - r_{\mathbf{H}^*}(n) > r_{\mathbf{H}^*}(n)$ and $|\mathrm{val}(\mathfrak{d}^-)| = K$. Also $\mathfrak{d}^+$ is hereditarily $(2^{K^n}, r_{\mathbf{H}^*}(n))$–big (remember $b_1 > 2^{K^n}$, use (∗)). Thus $\mathfrak{d}^-, \mathfrak{d}^+$ witness that $\mathfrak{c}$ is $(K, n, r_{\mathbf{H}^*}(n))$–decisive. □

Definition 3.4 (1) For a cardinal λ we consider the forcing notion $\mathbb{P}_\lambda(\mathbf{K}^*, \Sigma^*)$ determined by the creating pair $(\mathbf{K}^*, \Sigma^*)$ as in 2.3(2). For $\alpha < \lambda$, a $\mathbb{P}_\lambda(\mathbf{K}^*, \Sigma^*)$–name $\underset{\sim}{\rho}_\alpha$ is defined by

$$\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)} \underset{\sim}{\rho}_\alpha = \bigcup \big\{p(\alpha, n) : \alpha \in \mathrm{dom}(p) \ \&\ n < \mathrm{trunklg}(p, \alpha) \ \&\ p \in \underset{\sim}{G}_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)}\big\}.$$

(2) For $\rho \in \prod\limits_{n<\omega} \mathbf{H}^*(n)$ we set $F(\rho) = \big\{\eta \in {}^\omega 2 : (\forall^\infty n < \omega)(\eta \prec \rho(n)\big\}$.

Plainly, for each $\alpha < \lambda$, $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)} \underset{\sim}{\rho}_\alpha \in \prod\limits_{n<\omega} \mathbf{H}^*(n)$. Also, for $\rho \in \prod\limits_{n<\omega} \mathbf{H}^*(n)$, the set $F(\rho)$ is a meager and null Σ^0_2–subset of ${}^\omega 2$.

Theorem 3.5 *Assume CH. Let λ be an uncountable cardinal, $\lambda = \lambda^{\aleph_0}$.*

(1) *Forcing with $\mathbb{P}_\lambda(\mathbf{K}^*, \Sigma^*)$ preserves cardinalities and cofinalities and $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)}$ " $2^{\aleph_0} = \lambda$ ".*
(2) *If $\beta < \lambda$ and $\underset{\sim}{\nu}$ is a $\mathbb{P}_{\lambda\setminus\{\beta\}}(\mathbf{K}^*, \Sigma^*)$–name for a member of ${}^\omega 2$, then $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)}$ " $\underset{\sim}{\nu} \in F(\underset{\sim}{\rho}_\beta)$ ".*
(3) *Consequently, $\Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)}$ " $\mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \lambda$ ".*

Proof (1) It follows from 3.3 + 2.4(2) + 2.7.

(2) The proof is parallel to that of [8, Lemma 9.1]. Assume $p \in \mathbb{P}_\lambda(\mathbf{K}^*, \Sigma^*)$. Remembering 2.4(1) we may use 2.7(3) to find a condition $q \ge p$ such that

$(*)_1$ the condition $q{\restriction}(\lambda \setminus \{\beta\})$ $<n$–decides the value of $\underset{\sim}{\nu}{\restriction}N_n$ (for each n), and
$(*)_2$ $\mathrm{trunklg}(q, \alpha) \ge 972$ for all $\alpha \in \mathrm{dom}(q)$ and $\mathrm{nor}(q(\alpha, m)) \ge 972$ whenever $\alpha \in \mathrm{supp}(q, m)$, and
$(*)_3$ $\beta \in \mathrm{dom}(q)$ and if $\mathrm{supp}(q, m) \ne \emptyset$, then $|\mathrm{supp}(q, m)| \ge 972$.

Thus, for each n, we have a mapping $E_n : \mathrm{val}^\Pi(q{\restriction}(\lambda \setminus \{\beta\}), <n) \longrightarrow {}^{N_n}2$ such that

$$\big(q{\restriction}(\lambda \setminus \{\beta\})\big) \wedge t \Vdash_{\mathbb{P}_{\lambda\setminus\{\beta\}}(\mathbf{K}^*,\Sigma^*)} \text{"}\underset{\sim}{\nu}{\restriction}N_n = E_n(t)\text{''}.$$

We will further strengthen q to a condition q^* such that $\mathrm{dom}(q^*) = \mathrm{dom}(q)$ and

$(*)^{\mathrm{goal}}$ for all $n \ge \mathrm{trunklg}(q^*, \beta)$ and $t \in \mathrm{val}^\Pi\big(q^*{\restriction}(\lambda \setminus \{\beta\}), <(n+1)\big)$ we have

$$\big(\forall B \in q^*(\beta, n)\big)\big(E_{n+1}(t) \prec B\big).$$

Then clearly we will have $q^* \Vdash_{\mathbb{P}_\lambda(\mathbf{K}^*,\Sigma^*)}$ "$\underset{\sim}{\nu} \in F(\underset{\sim}{\rho}_\beta)$'' and the proof of 3.5(2) will follow by the standard density argument.

To construct the condition q^* we set $\operatorname{dom}(q^*) = \operatorname{dom}(q)$, $\operatorname{trunklg}(q^*, \alpha) = \operatorname{trunklg}(q, \alpha)$, and we define $q^*(\alpha, m)$ by induction on m so that:

$q^*(\alpha, m) = q(\alpha, m)$ whenever $\alpha \notin \operatorname{supp}(q, m)$ or $\beta \notin \operatorname{supp}(q, m)$, and

$q^*(\alpha, m) \in \Sigma^*(q(\alpha, m))$, $\operatorname{nor}(q^*(\alpha, m)) \geq \operatorname{nor}(q(\alpha, m)) - 2$ for $\alpha \in \operatorname{supp}(q, m)$.

These demands guarantee that q^* is a condition in $\mathbb{P}_\lambda(\mathbf{K}^*, \Sigma^*)$ stronger than q.

Fix an $n \geq \operatorname{trunklg}(q, \beta)$. Put $A = \operatorname{supp}(q, n)$ and note that $\beta \in A$, A has at least 972 elements (remember $(*)_3$), and $|A| < n$ (by 2.3(2)(γ)).

Set $\mathfrak{c}^0_\alpha = q(\alpha, n)$ for $\alpha \in A$.

We choose inductively an enumeration $\{\alpha_0, \dots, \alpha_{|A|-1}\}$ of A and creatures $\mathfrak{c}^\ell_{\alpha_k}$ (for $\ell \leq k < |A|$) and $\mathfrak{d}_{\alpha_k}$ from $\Sigma^*(\mathfrak{c}^0_{\alpha_k})$. So assume that for some $\ell \geq 0$ we already have defined a list $\{\alpha_k : k < \ell\}$ of distinct elements of A and creatures $\mathfrak{c}^\ell_\alpha$ for $\alpha \in A \setminus \{\alpha_0, \dots, \alpha_{\ell-1}\}$. Each $\mathfrak{c}^\ell_\alpha$ is $(K^\ell_\alpha, n, r_{\mathbf{H}^*}(n))$–decisive for some K^ℓ_α. Put $K_\ell = \min(\{K^\ell_\alpha : \alpha \in A \setminus \{\alpha_0, \dots, \alpha_{\ell-1}\}\})$, and choose α_ℓ such that $K^\ell_{\alpha_\ell} = K_\ell$. Let $\mathfrak{d}_{\alpha_\ell} \in \Sigma^*(\mathfrak{c}^\ell_{\alpha_\ell})$ be such that $|\operatorname{val}(\mathfrak{d}_{\alpha_\ell})| \leq K_\ell$ and $\operatorname{nor}(\mathfrak{d}_{\alpha_\ell}) \geq \operatorname{nor}(\mathfrak{c}^\ell_{\alpha_\ell}) - r_{\mathbf{H}^*}(n)$. For $\alpha \in A \setminus \{\alpha_0, \dots, \alpha_\ell\}$, let $\mathfrak{c}^{\ell+1}_\alpha \in \Sigma^*(\mathfrak{c}^\ell_\alpha)$ be hereditarily $(2^{(K_\ell)^n}, r_{\mathbf{H}^*}(n))$–big and such that $\operatorname{nor}(\mathfrak{c}^{\ell+1}_\alpha) \geq \operatorname{nor}(\mathfrak{c}^\ell_{\alpha_\ell}) - r_{\mathbf{H}^*}(n)$. Iterate this procedure $|A| - 1$ times. At the end, there remains one α that has not been listed as an α_ℓ, so we set $\alpha_{|A|-1} = \alpha$ and $\mathfrak{d}_{\alpha_{|A|-1}} = \mathfrak{c}^{|A|-1}_\alpha$.

Since $\mathfrak{c}^{\ell+1}_{\alpha_{\ell+1}}$ is hereditarily $(2^{(K_\ell)^n}, r_{\mathbf{H}^*}(n))$–big, we see that $2^{(K_\ell)^n} < K_{\ell+1}$. Let m be such that $\beta = \alpha_m$, and put

$$K = K_m, \quad S = \{\alpha_\ell : \ell < m\}, \quad L = \{\alpha_\ell : \ell > m\}.$$

It is possible that (at most) one of the sets S, L is empty. By our choices,

$(*)_4$ (a) $\mathfrak{d}_\alpha \in \Sigma^*(q(\alpha, n))$, $\operatorname{nor}(\mathfrak{d}_\alpha) \geq \operatorname{nor}(q(\alpha, n)) - (n-1) \cdot r_{\mathbf{H}^*}(n) > 900$, and

(b) if $S \neq \emptyset$ then $\mathfrak{d}_\beta$ is $(2^{(K_{m-1})^n}, r_{\mathbf{H}^*}(n))$–big and hence in particular $(K_{m-1})^{n-2} < K$; if $S = \emptyset$ then $K = K_0$,

(c) $\prod_{\alpha \in S} |\operatorname{val}(\mathfrak{d}_\alpha)| \leq (K_{m-1})^{n-2} < K$ and $|\operatorname{val}(\mathfrak{d}_\beta)| \leq K$,

(d) $\varphi_{\mathbf{H}^*}(<n) < K_0 \leq K$ (remember that $\mathbf{K}(n)$ is $(g(n), r_{\mathbf{H}^*}(n))$–big and $g(n) > \varphi_{\mathbf{H}^*}(<n)$),

(e) if $\alpha \in L$, then $\mathfrak{d}_\alpha$ is $(2^{K^n}, r_{\mathbf{H}^*}(n))$–big.

Let $Z = \{t \in \operatorname{val}^\Pi(q \restriction (\lambda \setminus \{\beta\}), <(n+1)) : t(\alpha, n) \in \operatorname{val}(\mathfrak{d}_\alpha) \text{ for } \alpha \in A \setminus \{\beta\}\}$ and for $s \in \prod_{\alpha \in L} \operatorname{val}(\mathfrak{d}_\alpha)$ let $Z_s = \{t \in Z : t(\alpha, n) = s(\alpha) \text{ for } \alpha \in L\}$. Next, for $t \in Z$ put $\mathcal{C}_t = \{B \in \mathcal{B}^{\mathfrak{d}_\beta} : E_{n+1}(t) \not\prec B\}$.

If $S = \emptyset$, then in what follows ignore $\prod_{\alpha \in S} \operatorname{val}(\mathfrak{d}_\alpha)$ and set $K_{m-1} = 1$. Assume L is non-empty (otherwise move to $(*)_6$). For each $s \in \prod_{\alpha \in L} \operatorname{val}(\mathfrak{d}_\alpha)$ consider a function

$$c(s) : \operatorname{val}^\Pi(q \restriction (\lambda \setminus \{\beta\}), <n) \times \prod_{\alpha \in S} \operatorname{val}(\mathfrak{d}_\alpha) \longrightarrow \mathcal{P}(\operatorname{val}(\mathfrak{d}_\beta))$$

such that $c(s)(t_0, t_1) = \mathcal{C}_{t_0 \frown t_1 \frown s}$, where $t_0 \frown t_1 \frown s \in Z_s$ is obtained by natural concatenation. This determines a coloring c on $\prod_{\alpha \in L} \operatorname{val}(\mathfrak{d}_\alpha)$ with the range of size at most

$$\left(2^K\right)^{\varphi_{\mathbf{H}^*}(<n) \cdot (K_{m-1})^{n-2}} \leq \left(2^K\right)^{K \cdot K} = 2^{K^3} < 2^{K^n}.$$

Since $\mathbf{K}^*(n)$ is $(n, r_{\mathbf{H}^*}(n))$–decisive, and each $\mathfrak{d}_\alpha$ is hereditarily $(2^{K^n}, r_{\mathbf{H}^*}(n))$–big (for $\alpha \in L$), $\mathrm{nor}(\mathfrak{d}_\alpha) > 900$ and $|L| \le n-2$, therefore we may use Lemma 2.2 to find $q^*(\alpha, n) \in \Sigma^*(\mathfrak{d}_\alpha)$ for $\alpha \in L$ such that

$(*)_5$ (a) $\mathrm{nor}(q^*(\alpha,n)) \ge \mathrm{nor}(\mathfrak{d}_\alpha) - r_{\mathbf{H}^*}(n)\cdot n \ge \mathrm{nor}(q(\alpha,n)) - 2$, and
(b) $c{\restriction}\prod\limits_{\alpha\in L} \mathrm{val}(q^*(\alpha,n))$ is constant.

If $L = \emptyset$ then the procedure described above is not needed. In any case, letting

$$X = \mathrm{val}^{\Pi}(q{\restriction}(\lambda\setminus\{\beta\}), {<}n) \times \prod_{\alpha\in S}\mathrm{val}(\mathfrak{d}_\alpha),$$

we have a mapping $d : X \longrightarrow \mathcal{P}(\mathrm{val}(\mathfrak{d}_\beta))$ and $q^*(\alpha, n)$ for $\alpha \in L$ such that

$(*)_6$ if $t \in Z$ and $t(\alpha,n) \in \mathrm{val}(q^*(\alpha,n))$ for $\alpha \in L$, then $\mathcal{C}_t = d(t_0, t_1)$, where $t_0 = t{\restriction}\big((\mathrm{dom}(q)\setminus\{\beta\})\times n\big) \in \mathrm{val}^{\Pi}(q{\restriction}(\lambda\setminus\{\beta\}), {<}n)$ and $t_1 = t{\restriction}(S\times\{n\}) \in \prod\limits_{\alpha\in S}\mathrm{val}(\mathfrak{d}_\alpha)$.

For each $(t_0, t_1) \in X$ fix one $t = t[t_0, t_1] \in Z$ such that $t(\alpha, n) \in \mathrm{val}(q^*(\alpha, n))$ for $\alpha \in L$, $t_0 = t{\restriction}\big((\mathrm{dom}(q)\setminus\{\beta\})\times n\big)$ and $t_1 = t{\restriction}(S\times\{n\})$. Now, for $B \in \mathrm{val}(\mathfrak{d}_\beta)$ we (try to) choose $(t_0^B, t_1^B) \in X$ such that $B \in \mathcal{C}_{t[t_0^B, t_1^B]}$, if possible. Consider a coloring $e : \mathrm{val}(\mathfrak{d}_\beta) \longrightarrow {}^{N_{n+1}}2 \cup \{*\}$ defined by

$$e(B) = \begin{cases} E_{n+1}(t[t_0^B, t_1^B]) & \text{if } (t_0^B, t_1^B) \in X \text{ is defined,}\\ * & \text{otherwise.}\end{cases}$$

Since $|X| \le \varphi_{\mathbf{H}^*}({<}n)\cdot(K_{m-1})^{n-2} \le \max\{(K_{m-1})^{n-1}, \varphi_{\mathbf{H}^*}({<}n)\}$, we know that the range of the coloring e has at most $\max\{(K_{m-1})^{n-1}, \varphi_{\mathbf{H}^*}({<}n)\} + 1$ members. Thus $\mathfrak{d}_\beta$ is $(|\mathrm{rng}(e)|, r_{\mathbf{H}^*}(n))$–big and we may choose $q^*(\beta, n) \in \Sigma^*(\mathfrak{d}_\beta)$ such that $\mathrm{nor}(q^*(\beta,n)) \ge \mathrm{nor}(\mathfrak{d}_\beta) - r_{\mathbf{H}^*}(n) \ge \mathrm{nor}(q(\alpha,n)) - 2 > 900$ and $e{\restriction}\mathrm{val}(q^*(\alpha,n))$ is constant. If the constant value were $\eta \in {}^{N_{n+1}}2$, then we would have $\eta \not\prec B$ for all $B \in \mathrm{val}(q^*(\alpha,n))$, contradicting $\mathrm{nor}(q^*(\beta,n)) > 0$. Therefore,

$(*)_7$ (t_0^B, t_1^B) is defined for no $B \in \mathrm{val}(q^*(\beta,n))$ and hence

$$\mathrm{val}(q^*(\beta,n)) \cap \bigcup\{\mathcal{C}_{t[t_0,t_1]} : (t_0,t_1)\in X\} = \emptyset.$$

For $\alpha \in S$ we set $q^*(\alpha,n) = \mathfrak{d}_\alpha$. Now note that

$(*)_8$ if $t \in Z$ is such that $t(\alpha,n) \in q^*(\alpha,n)$ for $\alpha \in S\cup L$ and $B \in \mathrm{val}(q^*(\beta,n))$, then $E_{n+1}(t) \prec B$.

Why? Assume towards contradiction that $E_{n+1}(t) \not\prec B$, i.e., $B \in \mathcal{C}_t$. Represent t as $t = t_0{}^\frown t_1{}^\frown s$ where $(t_0,t_1) \in X$. Then $\mathcal{C}_t = \mathcal{C}_{t[t_0,t_1]}$ (by $(*)_6$) and therefore $B \in \mathcal{C}_{t[t_0,t_1]}$, contradicting $(*)_7$.

This completes the definition of q^*. It follows from $(*)_8$ (for $n \ge \mathrm{trunklg}(q^*,\beta)$) that $(*)^{\mathrm{goal}}$ is satisfied.

(3) Follows from (2) and the fact that $F(\rho) \in \mathcal{N}\cap\mathcal{M}$ for $\rho \in \prod\limits_{n<\omega}\mathbf{H}(m)$. □

Corollary 3.6 *It is consistent that*

$$\mathrm{non}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}\cap\mathcal{M}) = \aleph_2 = 2^{\aleph_0} \textit{ and } \mathfrak{d}_0 = \aleph_1.$$

Proof Start with a model of CH and force with $\mathbb{P}_{\aleph_2}(\mathbf{K}^*, \Sigma^*)$. It follows from 3.5 to 2.8 that the resulting model is as required. □

In models for the statement in Corollary 3.6 necessarily $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$. However, it is not clear if we could not get a parallel result for $\mathfrak{d}_{\mathcal{B}}$ and cov.

Problem 3.7 *Is it consistent that*

$$\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_2 = 2^{\aleph_0} \text{ and } \mathfrak{d}_{\mathcal{B}} = \aleph_1 \text{ ?}$$

In particular, is it consistent that $\mathfrak{d}_{\mathfrak{o}} > \mathfrak{d}_{\mathcal{B}}$ *?*

Directly from 3.6 we also obtain

Corollary 3.8 *It is consistent that* $\mathrm{non}(\mathcal{N}\cap\mathcal{M}) = \aleph_2$ *and there is no* $\subset$*–increasing chain of Borel subset of* ${}^{\omega}2$ *of length* ω_2.

4 Monotone hulls

The interest in Corollary 3.8 came from the questions concerning Borel hulls.

Definition 4.1 Let $\mathrm{Borel}({}^{\omega}2)$ be the family of all Borel subsets of ${}^{\omega}2$, $\mathcal{I}$ be a σ–ideal on ${}^{\omega}2$ with Borel basis and $\mathcal{S}_{\mathcal{I}}$ be the σ–algebra of subsets of ${}^{\omega}2$ generated by $\mathrm{Borel}({}^{\omega}2)\cup\mathcal{I}$. Let $\mathcal{F}\subseteq\mathcal{S}_{\mathcal{I}}$. A *monotone Borel hull* on $\mathcal{F}$ with respect to $\mathcal{I}$ is a mapping $\psi:\mathcal{F}\longrightarrow\mathrm{Borel}({}^{\omega}2)$ such that

- $A\subseteq\psi(A)$ and $\psi(A)\setminus A\in\mathcal{I}$ for all $A\in\mathcal{F}$, and
- if $A_1\subseteq A_2$ are from $\mathcal{F}$, then $\psi(A_1)\subseteq\psi(A_2)$.

If the range of ψ consists of sets of some Borel class $\mathcal{K}$, then we say that ψ is a monotone $\mathcal{K}$ hull operation.

As discussed in Balcerzak and Filipczak [1, Question 24], 3.8 implies the following.

Corollary 4.2 *It is consistent that*

- *there are no monotone Borel hulls on* $\mathcal{M}$ *with respect to* $\mathcal{M}$*, and*
- *there are no monotone Borel hulls on* $\mathcal{N}$ *with respect to* $\mathcal{N}$*, and*
- *there are no monotone Borel hulls on* $\mathcal{M}\cap\mathcal{N}$ *with respect to* $\mathcal{M}\cap\mathcal{N}$.

The non-existence of monotone Borel hulls on $\mathcal{I}$ implies non-existence of such hulls on $\mathcal{S}_{\mathcal{I}}$. While some partial results were presented in [7] and [1], not much is known about the converse implication.

Problem 4.3 (Cf. Balcerzak and Filipczak [1, Question 26]) *Let* $\mathcal{I}\in\{\mathcal{M},\mathcal{N}\}$. *Is it consistent that there exists a monotone Borel hull on* $\mathcal{I}$ *(with respect to* $\mathcal{I}$*) but there is no such hull on* $\mathcal{S}_{\mathcal{I}}$ *? In particular, is it consistent that* $\mathrm{add}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$ *but there is no monotone Borel hull on* $\mathcal{S}_{\mathcal{I}}$ *?*

It was noted in [1, Proposition 7] (see also Elekes and Máthé [7, Theorem 2.4]) that $\mathrm{add}(\mathcal{I}) = \mathrm{cof}(\mathcal{I})$ implies that there exists a monotone Borel hull on $\mathcal{I}$ (with respect to $\mathcal{I}$). It appears that was the only situation in which the positive result of this kind was known. Using a finite support iteration of ccc forcing notions we will show in this section that, consistently, we may have $\mathrm{add}(\mathcal{I}) < \mathrm{cof}(\mathcal{I})$ (for $\mathcal{I}\in\{\mathcal{N},\mathcal{M}\}$) and yet there are monotone hulls for $\mathcal{I}$.

Definition 4.4 Let $\mathcal{I}$ be an ideal of subsets of ${}^{\omega}2$.

(1) We say that a family $\mathcal{B} \subseteq \mathrm{Borel}({}^{\omega}2) \cap \mathcal{I}$ is *an mhg–base for $\mathcal{I}$* if[3]

  (a) $\mathcal{B}$ is a basis for $\mathcal{I}$, i.e., $(\forall A \in \mathcal{I})(\exists B \in \mathcal{B})(A \subseteq B)$, and
  (b) if $\langle B_i : i < \omega_1 \rangle$ is a sequence of elements of $\mathcal{B}$, then for some $i < j < \omega_1$ we have $B_i \subseteq B_j$.

(2) Let α^*, β^* be limit ordinals. *An $\alpha^* \times \beta^*$–base for $\mathcal{I}$* is a sequence $\langle B_{\alpha,\beta} : \alpha < \alpha^* \ \& \ \beta < \beta^* \rangle$ of Borel sets from $\mathcal{I}$ such that it forms a basis for $\mathcal{I}$ (i.e., (a) above holds) and

  (c) for each $\alpha_0, \alpha_1 < \alpha^*$, $\beta_0, \beta_1 < \beta^*$ we have

$$B_{\alpha_0,\beta_0} \subseteq B_{\alpha_1,\beta_1} \quad \Leftrightarrow \quad \alpha_0 \le \alpha_1 \ \& \ \beta_0 \le \beta_1.$$

Proposition 4.5 *Assume that $\langle B_{\alpha,\beta} : \alpha < \alpha^* \ \& \ \beta < \beta^* \rangle$ is an $\alpha^* \times \beta^*$–base for $\mathcal{I}$. Then:*

(i) $B_{\alpha,\beta} \neq B_{\alpha',\beta'}$ *whenever* $(\alpha, \beta) \neq (\alpha', \beta')$, $\alpha, \alpha' < \alpha^*$, $\beta, \beta' < \beta^*$.
(ii) $\{B_{\alpha,\beta} : \alpha < \alpha^* \ \& \ \beta < \beta^*\}$ *is an mhg–base for* $\mathcal{I}$.
(iii) $\mathrm{add}(\mathcal{I}) = \min\{\mathrm{cf}\,(\alpha^*), \mathrm{cf}\,(\beta^*)\}$ *and* $\mathrm{cof}(\mathcal{I}) = \max\{\mathrm{cf}\,(\alpha^*), \mathrm{cf}\,(\beta^*)\}$.

Proof Straightforward. □

Proposition 4.6 *Suppose that an ideal $\mathcal{I}$ on ${}^{\omega}2$ has an mhg–base $\mathcal{B} \subseteq \mathrm{Borel}({}^{\omega}2) \cap \mathcal{I}$. Then there exists a monotone hull operation $\psi : \mathcal{I} \longrightarrow \mathrm{Borel}({}^{\omega}2) \cap \mathcal{I}$ on $\mathcal{I}$. If, additionally, $\mathcal{B} \subseteq \Pi^0_\xi$, $\xi < \omega_1$, then ψ can be taken to have values in Π^0_ξ.*

Proof For a set $A \in \mathcal{I}$ let $\mathcal{S}_A$ be the family of all sequences $\bar{B} = \langle B_i : i < \gamma \rangle \subseteq \mathcal{B}$ satisfying

$(*)_1$ $(\forall i < \gamma)(A \subseteq B_i)$ and
$(*)_2$ $(\forall i < j < \gamma)(B_i \nsubseteq B_j)$.

Note that for each $\bar{B} \in \mathcal{S}_A$ we have $\ell g(\bar{B}) < \omega_1$ (by 4.4(1)(b) and $(*)_2$). Clearly, every $\trianglelefteq$–increasing chain of elements of $\mathcal{S}_A$ has a $\trianglelefteq$–upper bound in $\mathcal{S}_A$, so we may choose $\bar{B}_A = \langle B^A_i : i < \gamma_A \rangle \in \mathcal{S}_A$ which has no proper extension in $\mathcal{S}_A$. Put $\psi(A) = \bigcap\limits_{i<\gamma_A} B^A_i$. Plainly, $A \subseteq \psi(A) \in \mathcal{I}$ and $\psi(A)$ is a Borel set, and if $\mathcal{B} \subseteq \Pi^0_\xi$ then also $\psi(A) \in \Pi^0_\xi$. □

Claim 3.6.1 $\psi(A) = \bigcap\{B \in \mathcal{B} : A \subseteq B\}$

Proof of the Claim By $(*)_1$ we see that $\psi(A) \supseteq \bigcap\{B \in \mathcal{B} : A \subseteq B\}$. To show the converse inclusion suppose $B \in \mathcal{B}$, $A \subseteq B$. By the choice of $\bar{B}_A$ we know that $\bar{B}_A{}^\frown\langle B\rangle \notin \mathcal{S}_A$ and hence $B^A_i \subseteq B$ for some $i < \gamma_A$. Consequently $\psi(A) \subseteq B$.

It follows from the above claim that $A_1 \subseteq A_2 \in \mathcal{I}$ implies $\psi(A_1) \subseteq \psi(A_2)$. □

Bartoszyński and Kada [3] showed that for any σ–directed partial order Q there is a ccc forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \text{“}\mathcal{M} \text{ has a basis order isomorphic to } Q \text{ with respect to set–inclusion”}.$$

A parallel result for $\mathcal{N}$ was given by Burke and Kada [5]. These theorems imply that for uncountable cardinals κ and λ we may force that $\mathcal{M}$ has a $\kappa \times \lambda$–basis, and we may also force that $\mathcal{N}$ has a $\kappa \times \lambda$–basis. The corresponding forcing notions (for both cases) were essentially versions of “FS iterations with partial memories” used in Shelah [13–15], Mildenberger and Shelah [10] and Shelah and Thomas [16]. We will use explicitly the method of “FS iterations with partial memories” to construct a model in which *both* ideals have $\kappa \times \lambda$–bases.

[3] “mhg” stands for “monotone hull generating”.

Theorem 4.7 *Let κ, λ be cardinals of uncountable cofinality, $\kappa \leq \lambda$. There is a ccc forcing notion $\mathbb{Q}^{\kappa,\lambda}$ of size $\lambda^{\aleph_0}$ such that*

$$\Vdash_{\mathbb{Q}^{\kappa,\lambda}} \text{“the meager ideal } \mathcal{M} \text{ has a } \kappa \times \lambda - \text{basis consisting of } \Sigma^0_2 \text{ sets, and the null ideal } \mathcal{N} \text{ has a } \kappa \times \lambda - \text{ basis consisting of } \Pi^0_2 \text{ sets''.}$$

Proof The forcing notion $\mathbb{Q}^{\kappa,\lambda}$ will be obtained by means of finite support iteration of ccc forcing notions. The iterands will be products of the Amoeba for Category $\mathbb{B}$ and Amoeba for Measure $\mathbb{A}$ but *considered over partial sub-universes only*.

We will use the notation and some basic facts stated in the third section of [16].

Let us recall the forcings $\mathbb{A}$ and $\mathbb{B}$ used as iterands.

- A condition in $\mathbb{A}$ is a tree $T \subseteq {}^{\omega>}2$ such that $\mu([T]) > \frac{1}{2}$ and $\mu([t] \cap [T]) > 0$ for all $t \in T$. The order $\leq_{\mathbb{A}}$ of $\mathbb{A}$ is the reverse inclusion.
- A condition in $\mathbb{B}$ is a pair (n, T) such that $n \in \omega$, $T \subseteq {}^{\omega>}2$ is a tree with no maximal nodes and $[T]$ is a nowhere dense subset of ${}^{\omega}2$. The order $\leq_{\mathbb{B}}$ of $\mathbb{B}$ is given by: $(n, T) \leq_{\mathbb{B}} (n', T')$ if and only if $n \leq n'$, $T \subseteq T'$ and $T \cap {}^{n}2 = T' \cap {}^{n}2$.

Both $\mathbb{A}$ and $\mathbb{B}$ are (nice definitions of) ccc forcing notions, $\mathbb{B}$ is σ–centered and if $\mathbf{V}' \subseteq \mathbf{V}''$ are universes of set theory then $\mathbb{A}^{\mathbf{V}'}$ is still ccc in $\mathbf{V}''$. We will use the following immediate properties of these forcing notions.

$(\circledast)_1$ If $G \subseteq \mathbb{A}$ is generic over $\mathbf{V}$, $F = \bigcap\{[T] : T \in G\}$, then F is a closed subset of ${}^{\omega}2$, $\mu(F) = \frac{1}{2}$ and F is disjoint from every Borel null set coded in $\mathbf{V}$. Hence the set $F^* = \{x \in {}^{\omega}2 : (\forall y \in F)(\exists^{\infty} n)(x(n) \neq y(n))\}$ is a null Π^0_2 set and it includes all Borel null sets coded in $\mathbf{V}$.
Let $\underset{\sim}{F}_{\mathbb{A}}, \underset{\sim}{F}^*_{\mathbb{A}}$ be $\mathbb{A}$–names for the sets F, F^*, respectively.

$(\circledast)_2$ If $G \subseteq \mathbb{B}$ is generic over $\mathbf{V}$, $F = \bigcup\{[T] : (\exists n)((n, T) \in G)\}$, then F is a closed nowhere dense subset of ${}^{\omega}2$. Letting $F^* = \{x \in {}^{\omega}2 : (\exists y \in F)(\forall^{\infty} n)(x(n) = y(n))\}$ we get a meager Σ^0_2 set including all Borel meager sets coded in $\mathbf{V}$.
Let $\underset{\sim}{F}_{\mathbb{B}}, \underset{\sim}{F}^*_{\mathbb{B}}$ be $\mathbb{B}$–names for the sets F, F^*, respectively.

$(\circledast)^a_3$ If $T \in \mathbb{A}$, $t \in T$, then there is $T' \geq_{\mathbb{A}} T$ such that $T' \Vdash_{\mathbb{A}} [t] \cap \underset{\sim}{F}_{\mathbb{A}} \neq \emptyset$.

$(\circledast)^b_3$ If $T \in \mathbb{A}$, $n \in \omega$, then there is $N > n$ such that for each $\nu \in {}^{[n,N)}2$ there is $T' \geq_{\mathbb{A}} T$ with $T' \Vdash_{\mathbb{A}} (\forall y \in \underset{\sim}{F}_{\mathbb{A}})(y{\restriction}[n, N) \neq \nu)$.

$(\circledast)^a_4$ If $(n, T) \in \mathbb{B}$, $t \in T$, $\ell g(t) \leq n$, $m_1 > m_0 \geq n$ and $\nu \in {}^{[m_0,m_1)}2$, then there are $(n', T') \geq_{\mathbb{B}} (n, T)$ and $s \in T'$ such that $t \vartriangleleft s$ and $s{\restriction}[m_0, m_1) = \nu$ (and $(n', T') \Vdash_{\mathbb{B}} [s] \cap \underset{\sim}{F}_{\mathbb{B}} \neq \emptyset$).

$(\circledast)^b_4$ If $(n, T) \in \mathbb{B}$, $m_0 < \omega$, then there are $m_1 > m_0$ and $\nu \in {}^{[m_0,m_1)}2$ and $(n', T') \geq_{\mathbb{B}} (n, T)$ such that $(n', T') \Vdash_{\mathbb{B}} (\forall y \in \underset{\sim}{F}_{\mathbb{B}})(y{\restriction}[m_0, m_1) \neq \nu)$.

Fix an ordinal γ and a bijection $\pi : \kappa \times \lambda \xrightarrow{\text{onto}} \gamma$ such that

$$\alpha_0 \leq \alpha_1 < \kappa \ \&\ \beta_0 \leq \beta_1 < \lambda \quad \Rightarrow \quad \pi(\alpha_0, \beta_0) \leq \pi(\alpha_1, \beta_1).$$

For $i = \pi(\alpha_1, \beta_1)$ let $a_i = \{\pi(\alpha_0, \beta_0) : \alpha_0 \leq \alpha_1 \ \&\ \beta_0 \leq \beta_1\} \setminus \{i\}$. We say that a set $b \subseteq \gamma$ is *closed* if $a_i \subseteq b$ for all $i \in b$. It follows from our choice of π that for each $i < \gamma$ we have

$(\circledast)_5$ $a_i \subseteq i$ and the sets $a_i, i, a_i \cup \{i\}$ are closed.

Now, by induction we define $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{F}^0_i, \underset{\sim}{F}^1_i, \underset{\sim}{F}^{\mathbb{A}}_i, \underset{\sim}{F}^{\mathbb{B}}_i : i < \gamma\rangle$ and $\mathbb{P}^*_b$ for closed $b \subseteq \gamma$ simultaneously proving the correctness of the definition and the desired properties listed below.[4]

[4] See [16, 3.1–3.7] for the order in which these should be shown.

$(\circledast)_6$ $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i : j \leq \gamma, i < \gamma \rangle$ is a finite support iteration of ccc forcing notions.

$(\circledast)_7$ $\mathbb{P}^*_b = \{p \in \mathbb{P}_\gamma : \text{supp}(p) \subseteq b$ & $p(i)$ is a $\mathbb{P}^*_{a_i}$–name (for a member of $\underset{\sim}{\mathbb{Q}}_i$) for each $i \in \text{supp}(p)\}$.

$(\circledast)_8$ $\mathbb{P}^*_b$ is a complete suborder of $\mathbb{P}_\gamma$, $\mathbb{P}^*_{a_i \cup \{i\}}$ is isomorphic with the composition $\mathbb{P}^*_{a_i} * \underset{\sim}{\mathbb{Q}}_i$.

$(\circledast)_9$ $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}^*_{a_i}$–name for the product[5] $\mathbb{A} \times \mathbb{B}$.

$(\circledast)_{10}$ $\underset{\sim}{F}^0_i, \underset{\sim}{F}^1_i, \underset{\sim}{F}^{\mathbb{A}}_i, \underset{\sim}{F}^{\mathbb{B}}_i$ are $\mathbb{P}^*_{a_i \cup \{i\}}$–names for the sets $\underset{\sim}{F}_{\mathbb{A}}, \underset{\sim}{F}_{\mathbb{B}}, \underset{\sim}{F}^*_{\mathbb{A}}, \underset{\sim}{F}^*_{\mathbb{B}}$ added by the forcings at the last coordinate of $\mathbb{P}^*_{a_i \cup \{i\}} \simeq \mathbb{P}^*_{a_i} * (\mathbb{A} \times \mathbb{B})$.

$(\circledast)_{11}$ (a) $\mathbb{P}^*_i$ is a dense subset of $\mathbb{P}_i$ (for $i \leq \gamma$).

(b) If $q \in \mathbb{P}^*_\gamma$, then $q \restriction b \in \mathbb{P}^*_b$.

(c) If $p, q \in \mathbb{P}^*_\gamma$, $p \leq q$ and $i \in \text{supp}(q)$ then $p \restriction a_i \leq_{\mathbb{P}^*_{a_i}} q \restriction a_i$ and $q \restriction a_i \Vdash_{\mathbb{P}^*_{a_i}} p(i) \leq q(i)$.

(d) If $q \in \mathbb{P}^*_\gamma$, $p \in \mathbb{P}^*_b$ and $p \leq q$, then $p \leq_{\mathbb{P}^*_b} q \restriction b$.

(e) If $q \in \mathbb{P}^*_b$, $p \in \mathbb{P}^*_\gamma$, $p \restriction b \leq_{\mathbb{P}^*_b} q$ and r is defined by

$$r(\xi) = \begin{cases} q(\xi) & \text{if } \xi \in b, \\ p(\xi) & \text{otherwise} \end{cases} \qquad \text{for } \xi < \gamma$$

then $r \in \mathbb{P}^*_\gamma$ and $r \geq q, r \geq p$.

Also,

$(\circledast)_{12}$ if $\underset{\sim}{\tau}$ is a canonical[6] $\mathbb{P}^*_\gamma$–name for a member of ${}^\omega 2$, then $\underset{\sim}{\tau}$ is a $\mathbb{P}^*_{a_i}$–name for some $i < \gamma$.

[Why? Note that if $(\alpha_n, \beta_n) \in \kappa \times \lambda$, $n < \omega$, then there is $(\alpha^*, \beta^*) \in \kappa \times \lambda$ such that $\alpha_n \leq \alpha^*$, $\beta_n \leq \beta^*$ for all $n < \omega$.]

The main technical point of our argument is given in the following observation.

$(\circledast)_{13}$ Suppose $i, j < \gamma$, $i \notin a_j$, $j \notin a_i$, $i \neq j$, $\ell \in \{0, 1\}$. Assume that $p \in \mathbb{P}^*_\gamma$, $\eta \in {}^n 2$, $n < \omega$ and $p \Vdash_{\mathbb{P}^*_\gamma} [\eta] \cap \underset{\sim}{F}^\ell_i \neq \emptyset$. Then there are $\nu \in {}^{[n,N)} 2$, $n < N < \omega$ and $q \geq_{\mathbb{P}^*_\gamma} p$ such that

$$q \Vdash_{\mathbb{P}^*_\gamma} \text{“}[\eta^\frown \nu] \cap \underset{\sim}{F}^\ell_i \neq \emptyset \text{ and } \big(\forall y \in \underset{\sim}{F}^\ell_j\big)\big(y \restriction [n, N) \neq \nu\big)\text{''}.$$

[Why? Let us provide detailed arguments for $\ell = 0$. By $(\circledast)^b_3 + (\circledast)_9 + (\circledast)_{11}$ we may find $N > n$ and a condition $p'_0 \in \mathbb{P}^*_{a_j}$ such that $p'_0 \geq p \restriction a_j$ and

$$p'_0 \Vdash_{\mathbb{P}^*_{a_j}} \text{“ for each } \nu \in {}^{[n,N)} 2 \text{ there is } p_j \geq_{\underset{\sim}{\mathbb{Q}}_j} p(j) \text{ such that } p_j \Vdash_{\underset{\sim}{\mathbb{Q}}_j} (\forall y \in \underset{\sim}{F}_{\mathbb{A}})(y \restriction [n, N) \neq \nu)\text{''}.$$

Let $p_0 \in \mathbb{P}^*_\gamma$ be such that $p_0(\xi) = p'_0(\xi)$ for $\xi \in a_j$ and $p_0(\xi) = p(\xi)$ otherwise (see $(\circledast)_{11}(e)$; so p_0 is a common extension of p'_0 and p). Note that $p_0(j) = p(j)$. Use $(\circledast)^a_3$ to choose $\nu \in {}^{[n,N)} 2$ and a condition $p'_1 \in \mathbb{P}^*_{a_i \cup \{i\}}$ such that $p'_1 \geq p_0 \restriction (a_i \cup \{i\})$ and $p'_1 \Vdash_{\mathbb{P}^*_{a_i \cup \{i\}}} [\eta^\frown \nu] \cap \underset{\sim}{F}^0_i \neq \emptyset$. Let $p_1 \in \mathbb{P}^*_\gamma$ be such that $p_1(\xi) = p'_1(\xi)$ if $\xi \in a_i \cup \{i\}$ and $p_1(\xi) = p_0(\xi)$ otherwise. Then p_1 is stronger than both p'_1 and p_0, and $p_1(j) = p_0(j) = p(j)$. Hence

$$p_1 \restriction a_j \Vdash_{\mathbb{P}^*_{a_j}} \text{“ there is } p_j \geq_{\underset{\sim}{\mathbb{Q}}_j} p_1(j) \text{ such that } p_j \Vdash_{\underset{\sim}{\mathbb{Q}}_j} (\forall y \in \underset{\sim}{F}_{\mathbb{A}})(y \restriction [n, N) \neq \nu) \text{''}.$$

Let $q(j)$ be a $\mathbb{P}^*_{a_j}$–name for a p_j as above and let $q(\xi) = p_1(\xi)$ for $\xi \neq j$. Clearly $q \in \mathbb{P}^*_\gamma$ and $q \restriction (a_j \cup \{j\}) \Vdash_{\mathbb{P}^*_{a_j \cup \{j\}}} (\forall y \in \underset{\sim}{F}^0_j)(y \restriction [n, N) \neq \nu)$, and (as $q \restriction (a_i \cup \{i\}) = p_1 \restriction (a_i \cup \{i\}) = p'_1$)

[5] Since $\mathbb{B}^{\mathbf{V}^{\mathbb{P}^*_{a_i}}}$ is σ–centered we know that the product is ccc.

[6] i.e., determined in a standard way by a sequence of maximal antichains.

$q \restriction (a_i \cup \{i\}) \Vdash_{\mathbb{P}^*_{a_i \cup \{i\}}} [\eta^\frown \nu] \cap \underset{\sim}{F}^0_i \neq \emptyset$. Using $(\circledast)_8 + (\circledast)_{10} + (\circledast)_{11}$ we get that the condition q is as required. If $\ell = 1$ then the arguments are similar, but instead of $(\circledast)^a_3$, $(\circledast)^b_3$ we use $(\circledast)^a_4$, $(\circledast)^b_4$.]

For $\alpha < \kappa$, $\beta < \lambda$ let $\underset{\sim}{B}^{\mathbb{A}}_{\alpha,\beta} = \underset{\sim}{F}^{\mathbb{A}}_{\pi(\alpha,\beta)}$ and $\underset{\sim}{B}^{\mathbb{B}}_{\alpha,\beta} = \underset{\sim}{F}^{\mathbb{B}}_{\pi(\alpha,\beta)}$. Immediately from $(\circledast)_{12} + (\circledast)_1 + (\circledast)_2$ we conclude that

$(\circledast)_{14}$ $\Vdash_{\mathbb{P}^*_\gamma}$ "$\{\underset{\sim}{B}^{\mathbb{A}}_{\alpha,\beta} : \alpha < \kappa \ \&\ \beta < \lambda\}$ is a basis for $\mathcal{N}$ and
$\{\underset{\sim}{B}^{\mathbb{B}}_{\alpha,\beta} : \alpha < \kappa \ \&\ \beta < \lambda\}$ is a basis for $\mathcal{M}$"

and

$(\circledast)_{15}$ if $\alpha_0 \leq \alpha_1 < \kappa$, $\beta_0 \leq \beta_1 < \lambda$, $(\alpha_0, \beta_0) \neq (\alpha_1, \beta_1)$, then

$$\Vdash_{\mathbb{P}^*_\gamma} \text{"}\underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \subsetneq \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \ \&\ \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \subsetneq \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\text{"}.$$

Also

$(\circledast)_{16}$ if $\alpha_0, \alpha_1 < \kappa$, $\beta_0, \beta_1 < \lambda$ and $\neg\big(\alpha_0 \leq \alpha_1 \ \&\ \beta_0 \leq \beta_1\big)$ then

$$\Vdash_{\mathbb{P}^*_\gamma} \text{"}\underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \nsubseteq \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \ \&\ \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \nsubseteq \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\text{"}.$$

[Why? If $\alpha_1 \leq \alpha_0$ and $\beta_1 \leq \beta_0$, then $(\circledast)_{15}$ applies, so we may assume additionally $\neg\big(\alpha_1 \leq \alpha_0 \ \&\ \beta_1 \leq \beta_0\big)$. Then our assumptions on $\alpha_0, \alpha_1, \beta_0, \beta_1$ mean that, letting $j = \pi(\alpha_0, \beta_0)$ and $i = \pi(\alpha_1, \beta_1)$, we have $i \notin a_j$, $j \notin a_i$, $i \neq j$. So using $(\circledast)_{13}$ for $\ell = 0$ we easily build a $\mathbb{P}^*_\gamma$–name $\underset{\sim}{\eta}$ for a member of ${}^\omega 2$ such that

$$\Vdash_{\mathbb{P}^*_\gamma} \text{"}\underset{\sim}{\eta} \in [\underset{\sim}{F}^0_i] \subseteq {}^\omega 2 \setminus \underset{\sim}{F}^{\mathbb{A}}_i = {}^\omega 2 \setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \ \&\ \underset{\sim}{\eta} \in \underset{\sim}{F}^{\mathbb{A}}_j = \underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0}\text{"}.$$

Similarly, using $(\circledast)_{13}$ for $\ell = 1$ and interchanging the role of i and j we may construct a $\mathbb{P}^*_\gamma$–name $\underset{\sim}{\eta}'$ such that $\Vdash_{\mathbb{P}^*_\gamma}$ "$\underset{\sim}{\eta}' \in \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}$".]

Finally we note that $\mathbb{P}^*_\gamma$ has a dense subset of size $\lambda^{\aleph_0}$, so we may choose it as our desired forcing $\mathbb{Q}^{\kappa,\lambda}$. □

Remark 4.8 In a manner similar to our proof of $(\circledast)_{13}$ above one may argue for the following stronger property.

$(\circledast)^2_{13}$ Suppose $i, j < \gamma$, $i \notin a_j$, $j \notin a_i$, $i \neq j$, $\ell \in \{0, 1\}$. Assume that $p \in \mathbb{P}^*_\gamma$, $\eta \in {}^n 2$, $n < \omega$ and $p \Vdash_{\mathbb{P}^*_\gamma} [\eta] \cap \underset{\sim}{F}^\ell_i \neq \emptyset$. Then there are $\nu_0, \nu_1 \in {}^{[n,N)}2$, $n < N < \omega$ and $q \geq_{\mathbb{P}^*_\gamma} p$ such that $\nu_0 \neq \nu_1$ and

$$q \Vdash_{\mathbb{P}^*_\gamma} \text{"}[\eta^\frown \nu_0] \cap \underset{\sim}{F}^\ell_i \neq \emptyset \neq [\eta^\frown \nu_1] \cap \underset{\sim}{F}^\ell_i \text{ and } \big(\forall y \in \underset{\sim}{F}^\ell_j\big)\big(y \restriction [n, N) \notin \{\nu_0, \nu_1\}\big)\text{"}.$$

Then, if $i = \pi(\alpha_0, \beta_0)$, $j = \pi(\alpha_1, \beta_1)$, $i \notin a_j$, $j \notin a_i$ and $i \neq j$, we may use this property to construct $\mathbb{P}^*_\gamma$–names $\underset{\sim}{T}^{\mathbb{A}}$ and $\underset{\sim}{T}^{\mathbb{B}}$ for perfect subtrees of ${}^{\omega>}2$ such that

$$\Vdash_{\mathbb{P}^*_\gamma} \text{"}[\underset{\sim}{T}^{\mathbb{A}}] \subseteq \underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \text{ and } [\underset{\sim}{T}^{\mathbb{B}}] \subseteq \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\text{"}.$$

Also $(\circledast)_{15}$ can easily strengthen to

$(\circledast)^+_{15}$ if $\alpha_0 \leq \alpha_1 < \kappa$, $\beta_0 \leq \beta_1 < \lambda$, $(\alpha_0, \beta_0) \neq (\alpha_1, \beta_1)$, then

$$\Vdash_{\mathbb{P}^*_\gamma} \text{" both } \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1} \setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \text{ and } \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1} \setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \text{ are uncountable ".}$$

Consequently, in $\mathbf{V}^{\mathbb{P}^*_\gamma}$, the $\kappa \times \lambda$–bases $\{\underset{\sim}{B}^{\mathbb{A}}_{\alpha,\beta} : \alpha < \kappa,\ \beta < \lambda\}$ and $\{\underset{\sim}{B}^{\mathbb{B}}_{\alpha,\beta} : \alpha < \kappa,\ \beta < \lambda\}$ have the additional property that

$$\Vdash_{\mathbb{P}^*_\gamma} \text{“}\alpha_0 > \alpha_1 \ \vee\ \beta_0 > \beta_1 \quad \Rightarrow \quad \left|\underset{\sim}{B}^{\mathbb{A}}_{\alpha_0,\beta_0} \setminus \underset{\sim}{B}^{\mathbb{A}}_{\alpha_1,\beta_1}\right| = \left|\underset{\sim}{B}^{\mathbb{B}}_{\alpha_0,\beta_0} \setminus \underset{\sim}{B}^{\mathbb{B}}_{\alpha_1,\beta_1}\right| = 2^{\aleph_0}\text{''}.$$

This is used in Rosłanowski and Shelah [12].

Corollary 4.9 *It is consistent that*

- $\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) < \mathrm{cof}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}) = 2^\omega$ *(and hence the ideals $\mathcal{M}, \mathcal{N}$ do not poses tower bases) , and*
- *there is a monotone Π^0_3 hull operation on $\mathcal{M}$ with respect to $\mathcal{M}$, and*
- *there is a monotone Π^0_2 hull operation on $\mathcal{N}$ with respect to $\mathcal{N}$, and*
- *there is a monotone Π^0_3 hull operation on $\mathcal{M} \cap \mathcal{N}$ with respect to $\mathcal{M} \cap \mathcal{N}$.*

Proof Start with a universe satisfying CH and use the forcing given by Theorem 4.7 for $\kappa = \aleph_1$ and $\lambda = \aleph_2$. Propositions 4.6 and 4.5 imply that the resulting model is as required. □

Remark 4.10 In Theorem 4.7 we obtained a universe of set theory in which both $\mathcal{N}$ and $\mathcal{M}$ have bases that are (with respect to the inclusion) order isomorphic to $\kappa \times \lambda$. We may consider any partial order $(S, \sqsubseteq)$ such that

(a) $|S| = \lambda$ and $(S, \sqsubseteq)$ is well founded, and
(b) every countable subset of S has a common $\sqsubseteq$–upper bound.

Then by a very similar construction we get a forcing extension in which both $\mathcal{N}$ and $\mathcal{M}$ have bases order isomorphic to $(S, \sqsubseteq)$. If additionally

(c) for every sequence $\langle s_i : i < \omega_1\rangle \subseteq S$ there are $i < j < \omega_1$ such that $s_i \sqsubseteq s_j$,

then those bases will be mhg. (Note that forcings with the Knaster property preserve the demand described in (c).)

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