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The tree property at successors of singular cardinals

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Abstract. Assuming some large cardinals, a model of ZFC is obtained in which $\aleph_{\omega+1}$ carries no Aronszajn trees. It is also shown that if λ is a singular limit of strongly compact cardinals, then λ^+ carries no Aronszajn trees.

1 Introduction

The main results of this paper are (1) that the consistency of "ZFC and $\aleph_{\omega+1}$ carries no Aronszajn trees" follows from the consistency of some large cardinals (roughly a huge cardinal with ω supercompact cardinals above it), and (2) that if a singular cardinal λ is a limit of strongly compact cardinals, then there are no Aronszajn trees of height λ^+. The proof of (2) is in Sect. 3, and the forcing constructions which prove (1) are given in Sects. 4, 5, and 6. The generalization to higher singular cardinals of both (1) and (2) poses no problem.

2 Preliminaries

Recall that a tree is a partial ordering in which the set of predecessors of any point is well ordered. Usually trees have a single minimal node—the root—and no two distinct points have the same set of predecessors. Following an established practice, if $(T, <_T)$ is a tree then T may denote both the set of points and the ordering. A point $a \in T$ is of height α (an ordinal) iff α is the order type of the set of predecessors of a in T. The set of all points in T of height α is denoted T_α and is called the αth level of T. The supremum of the heights of the non-empty levels of T is called the *height* of T. A *branch* of T is a downward closed linearly ordered subset of T.

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A κ tree is a tree of height κ (a cardinal) in which every level has size $< \kappa$; it is an Aronszajn tree iff it has no branch of length κ. In this paper we will be interested in Aronszajn trees of height λ^+ where λ is a singular cardinal. It is convenient to assume that the universe of any λ^+ tree is the set $\lambda \times \lambda^+$; moreover, we stipulate that the αth level of this tree has the form $\lambda \times \{\alpha\}$(except the root).

A tree of height λ^+ and in which every level has cardinality $\leq \lambda$ (λ any cardinal) is said to be *special* iff there is a map σ from T into λ such that $a <_T b \Rightarrow \sigma(a) \neq \sigma(b)$. Any such special tree is clearly an Aronszajn tree, because σ is one–to–one on any branch. The existence of a special tree of height λ^+ is known to be equivalent to a weak square property.

We use the convention concerning forcing by which $p < q$ means that q is more informative than p. A forcing poset here is a separative partial order P, with a least informative point (denoted $\emptyset_P$) and with no maximal point. (A poset is separative if whenever $p \not\leq q$ then some extension of q in P is incompatible with p. There is a canonical way of producing a separative poset from non-separative: Define an equivalence relation $p_1 \sim p_2$ if "any x is compatible with p_1 iff it is compatible with p_2". Then, on the equivalence classes, define $[p_1] \leq [p_2]$ if every x compatible with p_2 is also compatible with p_1.)

V^P denotes the class of all P–terms, but when an expression such as "$\mathbf{T} \in V^P$ is a tree of hight λ^+" is used, we mean that $\emptyset_P$ forces this statement.

A *projection* from a poset P into Q is an order preserving map $\Pi : P \longrightarrow Q$ such that $\Pi(\emptyset_P) = \emptyset_Q$, and if $\Pi(p) = q$ and $q' > q$ in Q then for some $p' > p$, $\Pi(p') \geq q'$. (Some authors use a different definition!)

If $\Pi : P \longrightarrow Q$ is a projection and $G \subset Q$ is a V–generic filter, then P/G is the separative poset defined by taking $\Pi^{-1}G$ and turning it into a separative poset. Then P is isomorphic to a dense subset of the iteration $Q * (P/G)$: the isomorphism is the map taking $p \in P$ to $(\Pi(p), [p])$.

2.1 The preservation theorem

Theorem 2.1. *Let λ be a singular cardinal (of cofinality ω, for notational simplicity), and suppose that P and R are two posets such that:*

1. *$\|P\| = \chi < \lambda$, and $\mathbf{T}$ is a λ^+-tree in V^P. $\|P\|$ is the cardinality of P.)*
2. *R is χ^+–closed.*

Then any λ^+-branch of $\mathbf{T}$ in $V^{P \times R}$ is already in V^P.

Proof. The preservation theorem makes sense even when P is the trivial forcing consisting of a single point (no forcing). In this case V^P is V and the theorem then says that:

> If λ is a singular cardinal of cofinality ω, T is a λ^+ tree, and R is a countably closed poset, then forcing with R adds no new λ^+-branches to T.

For illustration of an important idea, let's prove this special case first. We need the following lemma which will also be used in the full proof.

Lemma 2.2. *Let S be any forcing poset. Suppose that λ and λ^+ are a cardinal and its successor, T is a λ^+–tree, and* **B** *is a name of a λ^+–branch of T in V^S. If* **B** *is a new branch (***B** *is not in V), then for some α there is a set $X \subset T_\alpha$ of cardinality λ, in V, such that every $x \in X$ is forced by some condition in S to be in* **B**.

Proof. We will say that $s \in S$ is λ–wide at T_α if there are λ extensions of s that force pairwise distinct values for $\mathbf{B} \cap T_\alpha$. If we start with an arbitrary condition, our proof will give that every $s \in S$ is λ–wide at some T_α, $\alpha < \lambda^+$.

Define E, in V, to be the set of possible nodes of **B**:

$$E = \{a \mid \text{ some condition in } S \text{ forces } a \in \mathbf{B}\}.$$

We want some $\alpha < \lambda^+$ such that $|E \cap T_\alpha| = \lambda$. So assume, on the contrary, that $| E \cap T_\alpha | < \lambda$ for every α. Then $E \subseteq T$ satisfies the following properties:

1. Any node in E has extensions in E at arbitrarily higher level.
2. E is downward closed in T.
3. Any node in E has two incomparable extensions in E (for otherwise, a condition would force that **B** is in V).
4. For every $\alpha < \lambda^+$, $| E \cap T_\alpha | < \lambda$.

This is not possible: let $U \subseteq \lambda^+$ be a closed unbounded set such that if $\alpha \in U$ then whenever $\gamma < \alpha$, if $a \in E \cap T_\gamma$, then a has two incomparable extensions of height $< \alpha$ in E. Then pick any $\alpha \in U$ such that $\alpha \cap U$ has order–type $\geq \lambda$ and a point a $a \in T_\alpha$, and conclude that $E \cap T_\alpha$ has size λ by splitting the points at levels in U below a. QED

Now to prove the special case of the theorem, assume that

$$r \Vdash \mathbf{B} \text{ is a new } \lambda^+ \text{ branch in } T.$$

We are going to associate with each node $\sigma \in \lambda^{<\omega}$ (the tree of finite sequences from λ) a condition $r_\sigma \in R$ and a point $t_\sigma \in T$ such that:

1. If $\sigma_1 \subset \sigma_2$ in $\lambda^{<\omega}$, then $r_{\sigma_1} \leq_R r_{\sigma_2}$.
2. For any $\sigma \in \lambda^{<\omega}$ there is some level $\alpha_\sigma < \lambda^+$ such that the points $\{t_{\sigma'} \mid \sigma'$ is an immediate successor of $\sigma\}$ are distinct points of T_{α_σ}.

Using the lemma, the definition of r_σ is done by induction. Let $\alpha < \lambda^+$ be the supremum of the level ordinals α_σ for $\sigma \in \lambda^{<\omega}$.

Any branch $s \in \lambda^\omega$ defines an increasing ω-sequence $\langle r_{s\restriction n} \mid n \in \omega\}$ of conditions, and hence has a supremum in R, denoted r_s.

Now extend r_s to force the value of $B \cap T_\alpha$ and let b_s be that point of T_α. Then $s_1 \neq s_2 \Longrightarrow b_{s_1} \neq b_{s_2}$ because s_1 and s_2 split in $\lambda^{<\omega}$. But as $|\lambda^\omega| \geq \lambda^+$, this shows that T_α has cardinality at least λ^+, which is surely impossible. *This proves the special case, and now we return to the theorem as stated.*

So P is a poset of cardinality $\chi < \lambda$ and $\mathbf{T}$ is a name forced by every condition in P to be a λ^+ tree. R is a χ^+-closed poset. As before, $T_\alpha = \lambda \times \{\alpha\}$ is the αth level of $\mathbf{T}$, for $\alpha < \mu = \lambda^+$. Let $\mathbf{B} \in V^{P \times R}$ be a name of a cofinal branch of $\mathbf{T}$, supposedly not in V^P. We also view $\mathbf{B}$ as a name in $(V^P)^R$ (that is, a name in R–forcing, in V^P).

Say (in V^P) that two conditions $r_1, r_2 \in R$ *force distinct values* for $\mathbf{B} \cap T_\alpha$ iff for some $a_1 \neq a_2$ in T_α, $r_i \Vdash a_i \in \mathbf{B}$, for $i = 1, 2$. A weaker property, which even may hold when r_1 or r_2 do not determine $\mathbf{B} \cap T_\alpha$, is that whenever r_1' and r_2' are extensions of r_1 and r_2 that determine the value of $\mathbf{B} \cap T_\alpha$ then r_1' and r_2' force distinct values for $\mathbf{B} \cap T_\alpha$. In this case we say that r_1 and r_2 *force contradictory information* on $\mathbf{B} \cap T_\alpha$". Observe that if $\alpha < \beta < \lambda^+$ and r_1, r_2 force contradictory information on $\mathbf{B} \cap T_\alpha$, then they force contradictory information on $\mathbf{B} \cap T_\beta$ (the argument is carried in V^P where $\mathbf{T}$ is defined). Observe also that if r_1, r_2 force distinct values for $B \cap T_\alpha$ then they force contradictory information on $B \cap T_\beta$ for any $\beta \geq \alpha$.

Working in V, our aim is to tag the nodes of the tree $\lambda^{<\omega}$ with conditions in R and we will denote the tag of $\sigma \in \lambda^{<\omega}$ with $r_\sigma \in R$. The required properties of this tagging are the following.

1. If $\sigma_1 \subset \sigma_2$ in $\lambda^{<\omega}$, then $r_{\sigma_1} \leq r_{\sigma_2}$ in R.
2. For every node $\sigma \in \lambda^{<\omega}$ there is an ordinal $\alpha < \lambda^+$ such that, for any two immediate extensions σ_1, σ_2 of σ, there is a dense set $D \subseteq P$, such that for every $p \in D$

$$p \Vdash_P \quad r_{\sigma_1} \textit{and } r_{\sigma_2} \textit{ force contradictory information on } \mathbf{B} \cap T_\alpha.$$

Why this suffices? Because, assuming such a construction, let $\beta < \lambda^+$ be above all the ordinals α mentioned in item 2 and look at the set of all full branches λ^ω. For each $f \in \lambda^\omega \cap V$, let $r_f \in R$ be an upper bound of the conditions $r_{f \restriction n}$, tagged along the branch f. We claim that if $f \neq g$ are full branches, then there is a dense set $D \subseteq P$ such that for every $p \in D$,

$$p \Vdash_P \quad r_f \textit{ and } r_g \textit{ force contradictory information on } \mathbf{B} \cap T_\beta.$$

Indeed, let $\sigma \subset f \cap g$ be the splitting node, then item 2 gives the required dense set. To conclude the proof, we find that, in V^P, any two branches of λ^ω give distinct values for T_β, and since $\lambda^{\aleph_0} \geq \lambda^+$, this shows that $T_\beta = \lambda \times \{\beta\}$ contains λ^+ distinct nodes in V^P, which is not possible since λ^+ is not collapsed in V^P. The tagging R_σ and the dense sets are defined below.

Consider $S = P \times R$; we will say that $(p, r) \in P \times R$ is λ–wide at T_α if there are λ extensions of (p, r) that force pairwise distinct values for $\mathbf{B} \cap T_\alpha$. (It is true that $\mathbf{T}$ is not assumed to be in V; however, its level–sets are, and so this definition is meaningful.)

Lemma 2.3. *1. Any condition $(p, r) \in P \times R$ is λ–wide at some T_α.*
2. If (p, r) is λ-wide at T_α, then it is also λ–wide at any higher level T_β.

Proof. Indeed, given $(p,r) \in P \times R$, let G be a V–generic filter over P containing p. In $V[G]$, λ and λ^+ are not collapsed, and we can use Lemma 2.2 to find some $\alpha < \lambda^+$ such that there are λ possible values for $\mathbf{B} \cap T_\alpha$ (forced by some extensions of r). Any such value is also a possible value for some extension of (p,r) (in V), and hence when the set of possible values for $\mathbf{B} \cap T_\alpha$ is calculated in V it must have cardinality λ as well.

For the second part assume that (p,r) is λ wide at α. given any $\beta > \alpha$, find first (p_i, r_i) extending (p,r), for $i < \lambda$, that determine distinct values of $\mathbf{B} \cap T_\alpha$ and then extend each pair to a condition (p'_i, r'_i) that determines $\mathbf{B} \cap T_\beta$. Even though it may be possible for two such extensions to determine the same point in $T_\beta = \lambda \times \{\beta\}$, it is not possible for $\|P\|^+$ extensions to determine the same point (because in such a case we would have two extensions with the same P coordinate, and this is not possible as the tree $\mathbf{T}$ is in V^P). So that the λ conditions are partitioned into classes of $\leq \|P\|$ members in each class, and thus there are λ classes, which gives λ possible values for $\mathbf{B} \cap T_\beta$. Observe, however, that if (p,r) is λ–wide at T_α, then extensions of (p,r) need not be λ–wide at the same T_α, and it may be necessary to go to higher levels.

Lemma 2.4. *If $\{r_j \mid j < \lambda\} \subseteq R$, and $p_0 \in P$ are given, then, for some ordinal α, there are extensions $r'_j \geq r_j$ in R, for every $j < \lambda$, such that for every pair $i < j$ there is $p_1 \geq p_0$ in P such that $p_1 \Vdash_P r'_i$ and r'_j force distinct values for $\mathbf{B} \cap T_\alpha$.*

Proof. First, by our last lemma, find for every $i < \lambda$ an ordinal α_i such that (p_0, r_i) is λ–wide at T_{α_i}, and then let α be above all of these α_i's. By the second part of the lemma, each (p_0, r_i) is λ–wide at T_α. Now, by induction on $i < \lambda$, we will define an extension $r'_i \geq r_i$, and two functions, e_i and f_i, where $e_i : P \to P$, and $f_i : P \to T_\alpha$, such that:

1. For every $a \in P$, $e_i(a)$ extends a, and $(e_i(a), r'_i) \Vdash_{P \times R} \mathbf{B} \cap T_\alpha = \{f_i(a)\}$.
2. If $k < i < \lambda$ then
$$f_i(p_0) \notin \{f_k(a) \mid a \in P\}.$$
That is, the value of $\mathbf{B} \cap T_\alpha$ that $(e_i(a), r'_i)$ determines is different from all the values determined by previous conditions.

Suppose that it is the turn of r'_i, e_i, f_i to be defined. Let $P = \{p(\xi) \mid \xi < \chi\}$ be an enumeration of P, starting with the given condition $p(0) = p_0$. By induction on $\xi < \chi$, we shall define a condition $r_i^\xi \in R$, and the values $e_i(p(\xi)) > p(\xi)$, and $f_i(p(\xi)) \in T_\alpha$ such that:

1. $\langle r_i^\xi \in R \mid \xi < \chi \rangle$ form an increasing sequence of conditions extending r_i
2. $(e_i(p(\xi)), r_i^\xi) \Vdash \mathbf{B} \cap T_\alpha = \{f_i(p(\xi))\}$.
3. $f_i(p(0)) \notin \{f_k(a) \mid a \in P,\ k < i\}$.

First, use the fact that (p_0, r_i) is λ–wide at T_α to find an extension $(e_i(p_0), r_i^0) \geq (p_0, r_i)$ that forces $\mathbf{B} \cap T_\alpha = \{f_i(p_0)\}$ for a value $f_i(p_o)$ that satisfies (3) above. Then construct the increasing sequence r_i^ξ and the values of e_i and f_i (using the χ^+ completeness of R at limit stages), and finally define r'_i to be an upper bound in R of that sequence.

Let us check that the requirements of the lemma are satisfied for r'_i. If $k < i$ is any index, look at $p' = e_i(p_0)$, and let $p_1 = e_k(p')$. Then p_1 is as required, because the value of $\mathbf{B} \cap T_\alpha$ determined by (p_1, r'_i) (namely $f_i(p_0)$) is distinct from the one determined by (p_1, r'_k) (namely $f_k(p')$). This proves the lemma, and the following completes the proof of the theorem by showing how the tagging can be done.

Lemma 2.5. *If $r \in R$, then there are extensions $r'_i \geq r$ for $i < \lambda$ such that, for some α, if $i < j < \lambda$ then for some dense set $D = D_{i,j} \subseteq P$, for every $p \in D$,*

$$p \Vdash_P r'_i \text{ and } r'_j \text{ force contradictory information on } B \cap T_\alpha.$$

Proof. Enumerate $P = \{p(\xi) \mid \xi < \chi\}$. Essentially, the proof is obtained by repeatedly applying the previous lemma, varying p_0 so as to get the dense sets. By induction on $\xi \leq \chi$ we define:

1. A sequence of conditions in R, $\langle r_i^\xi \mid i < \lambda \rangle$.
2. A family $D^\xi(i, j) \subset P$, increasing with ξ, for every $i < j < \lambda$. Finally, we will set $D(i, j) = D^\chi(i, j)$, and to ensure that $D(i, j)$ is dense we demand that $p(\xi)$ has an extension in $D^{\xi+1}(i, j)$.
3. An ordinal $\alpha(\xi) < \mu$.

We require that for each i, $\langle r_i^\xi \mid \xi < \chi \rangle$ forms an increasing sequence, beginning with $r_i^0 = r$. (Finally, $r'_i = r_i^\chi$ will be the required extension.)

At limit stages δ, r_i^δ is an upper bound in R of the conditions r_i^ξ, $\xi < \delta$. $D^\delta(i, j)$ is the union of $D^\xi(i, j)$ for $\xi < \delta$.

At successor stages, $\xi + 1$, the extensions $\{r_i^{\xi+1} \mid i < \lambda\}$ are defined using Lemma 2.4 for the collection $\{r_i^\xi \mid i < \lambda\}$ and the condition $p_0 = p(\xi)$. That lemma gives an ordinal $\alpha = \alpha(\xi)$ and extensions $p_1(i, j) \geq p(\xi)$ for every pair $i < j < \lambda$, such that

$$p_1(i, j) \Vdash r_i^{\xi+1} \text{ and } r_j^{\xi+1} \text{ force distinct values for } \mathbf{B} \cap T_\alpha.$$

Then we define $D^{\xi+1}(i, j)$ by $D^{\xi+1}(i, j) = D^\xi(i, j) \cup \{p_1(i, j)\}$.

Finally, define $r'_i = r_i^\chi$, $\alpha = \sup\{\alpha(\xi) \mid \xi < \chi\}$, and $D_{i,j} = D^\chi(i, j)$. $D_{i,j}$ is dense in P, because every $p(\xi)$ has some extension in $D^{\xi+1}(i, j)$. This ends the proof of Theorem 2.1.

2.2 On systems

Let K be a forcing poset and $\mathbf{T} \in V^K$ a λ^+-tree (λ^+ is a cardinal in V and in V^K). By our convention the underlying universe of $\mathbf{T}$ (namely $\lambda \times \lambda^+$) is in V, but the ordering $<_{\mathbf{T}}$ is in V^K of course.

Definition 2.1. *Let $\mathbf{T}$ be a K-name of a λ^+-tree as above, where λ and λ^+ are cardinals both in V and V^K, then the* pre-tree *of $\mathbf{T}$ is the sequence of relations $\langle R_p \mid p \in K \rangle$ defined by*

$$a \; R_p \; b \text{ iff } p \Vdash_K a <_{\mathbf{T}} b.$$

So each R_p is a binary relation on the universe of T, namely on $\lambda \times \lambda^+$.

It turns out that the consistency proof for "no Aronszajn trees on λ^+" relies on an investigation of such pre-trees for posets K such that $|K| < \lambda$. An abstract definition which captures the essential properties of these pre-trees but does not refer to any tree or forcing notion is given next.

Definition 2.2. *Systems: Suppose that $\lambda_0 \leq \lambda$ are cardinals, $D \subseteq \lambda^+$ is unbounded, and $T = \langle T_\alpha \mid \alpha \in D\rangle$ is a sequence of sets such that $T_\alpha \subseteq \lambda_0 \times \{\alpha\}$, for $\alpha \in D$. Let I be an index set of cardinality $\leq \lambda$, and $R = \{R_i \mid i \in I\}$ a collection of binary relations such that for every $i \in I$ $R_i \subseteq \bigcup\{T_\alpha \times T_\beta \mid \alpha < \beta$ are both in $D\}$. Then the pair $\mathscr{S} = (T, R)$ is called* a system *over λ^+ (or a λ^+–system) if the following hold:*

1. *For every $\alpha < \beta$ in D, there are $a \in T_\alpha,\ b \in T_\beta$ and $i \in I$ such that $\langle a, b\rangle \in R_i$.*
2. *For every $i \in I$, and $\alpha < \beta < \gamma$ in D, if $a \in T_\alpha,\ b \in T_\beta,\ c \in T_\gamma$ are such that $\langle a, c\rangle \in R_i$ and $\langle b, c\rangle \in R_i$, then $\langle a, b\rangle \in R_i$.*

The set D is called the domain of the system and I its index set. The cardinal λ_0 is the width, *and λ^+ the* height *of the system.*

An example of a λ^+–system is any λ^+–tree; in this example R consists of a single relation–the tree ordering. The pre-tree as defined in 2.1 is more illustrative; the number of relations is the cardinality of the forcing poset.

Strong systems: If condition (1) above is replaced by: "For every $\alpha < \beta$ in D, for every $b \in T_\beta$, there are $a \in T_\alpha$, and $i \in I$ such that $\langle a, b\rangle \in R_i$" (but (2) remains unchanged) then the system $\mathscr{S}$ is called a strong system.

A pre-tree relative to some forcing poset K is in fact a strong system.

Subsystems: Let $\mathscr{S}$ be a system of width λ_0 over λ^+ as above. Suppose that $D_0 \subseteq D$ is unbounded in λ^+, $\lambda_0' \leq \lambda_0$ is a cardinal, and $I_0 \subseteq I$ is any subset of indices. Then the restriction of $\mathscr{S}$ is obtained by taking the sequence $\langle T_\alpha \cap \lambda_0' \times \{\alpha\} \mid \alpha \in D_0\rangle$, and taking the restrictions of the relations R_i's for $i \in I_0$.

This restriction is not necessarily a system: Though item 2 is inherited automatically, item 1 may not be. A restriction that happens to a system is called a subsystem. A subsystem of a strong system may no longer be strong.

*Narrow systems: A system $\mathscr{S}$ is said to be (ρ, ι)–*narrow *iff $\lambda_0 < \rho$, and $|I| < \iota$. That is, its width is less than ρ and its index set has size $< \iota$. A λ^+–system is said to be* narrow *iff it is (λ, λ)–narrow.*

Branches: A "branch" of the system is a set B such that for some $i \in I$ for all $a, b \in B$ $\langle a, b\rangle \in R_i$ (if the level of a is below the level of b).

Thus, returning to a concrete example, a branch of a pre-tree gives a set B forced by a single condition to be linearly ordered.

Derived-systems Suppose that Q is a forcing poset and $\underset{\sim}{\mathscr{S}} = \langle \mathbf{T}, \mathbf{R}\rangle$ is in V^Q a system with domain λ^+, width $\lambda_0 \leq \lambda$, and index set some cardinal τ. Then the derived-system, *$Derived_Q(\underset{\sim}{\mathscr{S}})$ is defined as the following λ^+-system in V.*

Derived$_Q(\underset{\sim}{\mathscr{S}})$ *has as index set the product* $Q \times \tau$*, its width remains* λ_0*, and its relations* $R_{q,i}$ *are defined for* $q \in Q$ *and* $i \in \tau$ *by:*

$$\langle a, b\rangle \in R_{q,i} \text{ iff } q \Vdash_Q \langle a, b\rangle \in \mathbf{R}_i.$$

A pre-tree is an example of a derived-system. A derived-system is a system, and it is strong if the system $\underset{\sim}{\mathscr{S}}$ *is strong in* V^Q *(that is, forced by every condition to be strong). If* $P = Q \times K$ *is a product of two forcing posets and* $\underset{\sim}{\mathscr{S}}$ *is a* P*-name of a system, then* *Derived*$_P(\underset{\sim}{\mathscr{S}}) \sim$ *Derived*$_Q($*Derived*$_K(\underset{\sim}{\mathscr{S}}))$*. (Formally,* $\underset{\sim}{\mathscr{S}}$ *is not a* K*-name, so that the reader must interpret it accordingly.)*

2.3 On collapsing

For a regular cardinal κ and an ordinal $\lambda > \kappa$, $Coll(\kappa, \lambda)$ is the poset that collapses λ to κ, using functions from ordinals below κ and into λ. The poset $Coll(\kappa, < \lambda)$ is the product with support of cardinality $< \kappa$ of all collapses of ordinals between κ and λ (this is the "Levy" collapse, see Jech, Sect. 20). So if $q \in Coll(\kappa, < \lambda)$ then q is a function of size $< \kappa$ defined on ordinals $\alpha \in (\kappa, \lambda)$ such that $q(\alpha) \in Coll(\kappa, \alpha)$.

Now if $L = \langle \lambda_i \mid i < \omega\rangle$ is an increasing sequence of cardinals, then $C = Coll(L)$ is the full support iteration of the collapsing posets $Coll(\lambda_i, < \lambda_{i+1})$. In detail, define by induction on $1 \le n < \omega$ posets P_n as follows. $P_1 = Coll(\lambda_0, < \lambda_1)$, and

$$P_{n+1} = P_n * Coll(\lambda_n, < \lambda_{n+1})^{V^{P_n}}.$$

Then $Coll(L)$ is the full support limit of the posets P_n.

Lemma 2.6. *Suppose that* Q *is a* λ*–closed forcing poset (any increasing sequence of length* $< \lambda$ *has a* least *upper bound). Let* μ *be the cardinality of* Q*. Then there is a projection* $\Pi : Coll(\lambda, \mu) \longrightarrow Q$ *such that whenever* $G \subset Q$ *is* V *generic, then the quotient poset* $Coll(\lambda, \mu)/G$ *is* λ*–closed. In fact, the projection is* λ*–continuous: If* $f = \bigcup_{i<\lambda_0} f_i$ *is the supremum of an increasing sequence of length* $\lambda_0 < \lambda$ *of conditions in* $Coll(\lambda, \mu)$ *then* $\Pi(f)$ *is the supremum of* $\{\Pi(f_i) \mid i < \lambda_0\}$ *in* Q*.*

Proof. Let $Q = \{q_i \mid i < \mu\}$ be an enumeration of Q. Any condition in $Coll(\lambda, \mu)$ is a function $f : \alpha \longrightarrow \mu$, where $\alpha < \lambda$, and we define $\Pi(f)$ as follows. Define by induction a Q-increasing sequence $\langle a(\xi) \mid \xi \le \alpha\rangle$ by requiring that (1) $a(0)$ is the minimum of Q (2) at limit stages $\delta \le \alpha$, $a(\delta)$ is the least upper bound of $\langle a(\xi) \mid \xi < \delta\rangle$, and (3) if $q_{f(\xi)}$ extends $a(\xi)$ in Q, then $a(\xi + 1) = q_{f(\xi)}$, and otherwise $a(\xi + 1) = a(\xi)$. Finally, $\Pi(f) = a(\alpha)$. It is easy to see that Π is a λ-continuous projection.

Remark first that if $G \subset Q$ is V-generic, then $\Pi^{-1}G$ is already separative. That is, if $f, g \in Coll(\lambda, \mu)$ are such that $f \not\le g$ and $\Pi(f), \Pi(g) \in G$, then there is $g' \in Coll(\lambda, \mu)$ extending g and incompatible with f such that $\Pi(g') \in G$.

To prove the λ–closure of the quotient, suppose that

$$q \Vdash_Q \langle \tau_i \mid i < \lambda_0 < \lambda \rangle \text{ is an increasing sequence in } Coll(\lambda, \mu)/G = \Pi^{-1}G.$$

We will find an extension of q that forces a least upper bound to this sequence. Define by induction on $i < \lambda_0$ a Q-increasing sequence $\langle q_i \mid i < \lambda_0 \rangle$ beginning with $q_0 = q$, such that for every i, for some $f_i \in Coll(\lambda, \mu)$, $q_{i+1} \Vdash_Q \tau_i = f_i$. Let q' be an upper bound in Q to this sequence. Now $f_i \subset f_j$ for $i < j$, and $f = \bigcup_{i<\lambda_0} f_i$ is a condition. Since Q is separative, $\Pi(f_i) \leq_Q q_{i+1}$ follows from the fact that $q_{i+1} \Vdash \Pi(f_i) \in G$. The continuity of Π implies that $\Pi(f)$ is the least upper bound of all the conditions $\Pi(f_i)$, and hence $\Pi(f) \leq q'$. That is q' forces that f is in $Coll(\lambda, \mu)/G$.

A similar lemma holds for $Coll(\lambda, < \kappa)$ if Q is λ–closed and with cardinality less than κ, but we need to apply such a lemma in a slightly more complex situation (in Sect. 6). Suppose that:

1. $\lambda < \lambda_1 \leq \mu < \kappa$ are regular cardinals. $Q = Coll(\lambda, < \lambda_1)$, and the projection $\Pi_1 : Coll(\lambda, < \kappa) \longrightarrow Q$ is the obvious restriction projection.
2. $P = Q * R$ is a two stage iteration where R is a name in V^Q such that R is forced to be λ–closed (by every condition). The projection of P on Q is denoted Γ (so $\Gamma(q, \tau) = q$).
   Suppose that the cardinality of P is μ.

Lemma 2.7. *Under the conditions set above on* P, Q, *and* R, *there is a projection* $\Pi : Coll(\lambda, < \kappa) \longrightarrow P$ *such that* $\Gamma \circ \Pi = \Pi_1$, *and such that whenever* $G \subseteq P$ *is* V*-generic, then the quotient poset* $Coll(\lambda, < \kappa)/G$ *is* λ*-closed.*

Proof. Set an enumeration $\{\tau_i \mid i \in \mu\}$ of all the terms in V^Q that are forced by every condition to be in R, and where two names are identified if every condition forces them to be equal. Given any condition $q \in Coll(\lambda, < \kappa)$, let $q_1 = \Pi_1(q)$, and $f = q \restriction \{\mu\}$ be the component of q that collapses μ. Then $\Pi(q) = (q_1, \tau) \in Q * R$, where τ is defined by the following procedure. Suppose that the domain of f is $\alpha < \lambda$, and define an increasing sequence $\langle \eta_i \mid i \leq \alpha \rangle$ of terms, by induction on i as follows:

1. η_0 is an assumed empty condition in R (least informative). η_{i+1} is $\tau_{f(i)}$ if every condition in Q forces that $\tau_{f(i)}$ extends η_i, and $\eta_{i+1} = \eta_i$ otherwise.
2. If $\delta \leq \alpha$ is a limit ordinal and all the terms η_i for $i < \delta$ have been defined such that for $i < j < \delta$, $\eta_i <_R \eta_j$ is forced by every condition in Q, then η_δ is defined as (the name of) the least upper bound of this increasing sequence.

Finally, the projection is defined by setting $\tau = \eta_\alpha$. We leave it to the reader to verify that Π is indeed a projection as required, and in particular that if G is generic over P then the quotient $Coll(\lambda, < \kappa)/G$ is λ–closed.

2.4 On embeddings and ultrapowers

The dual characterization of supercompact cardinals is probably known to the reader: If $\mu > \kappa$ then κ is μ-supercompact if κ is the critical point of an elementary embedding $j : V \longrightarrow M$ of the universe V into a transitive inner model M such that $\mu < j(\kappa)$ and $M^{\mu} \subset M$ (which means that every function from μ to M is in fact in M). An equivalent, more tangible, definition is that $P_{\kappa}(\mu) = \{X \mid X \subset \mu,\ |X| < \kappa\}$ carries a non-principal, fine, κ-complete, normal ultrafilter (see Jech [2] Chapter 6, or Solovay [7]).

We say that κ is *huge* if κ is the critical point of an elementary embedding $j : V \longrightarrow M$ into a transitive substructure M such that $M^{j(\kappa)} \subseteq M$. If the stronger demand, $M^{j(j(\kappa))} \subseteq M$, holds then we say that κ is 2-huge. For our consistency result we need a cardinal that is slightly stronger than huge, but not quite 2-huge. Its definition is given by the following lemma on the equivalence between two characterizations, which we quote without proof (the proof is quite standard; see for example Solovay, Reinhardt, and Kanamori [8]).

If $\mu > \tau$ are cardinals, define $P^{\tau}(\mu) = \{X \subseteq \mu \mid \textit{order-type of } X \textit{ is } \tau\}$. If $\mathscr{U} \subset \mathscr{P}(P^{\tau}(\mu))$ is a non-principal ultrafilter, then $\mathscr{U}$ is said to be:

1. κ*–complete* if the intersection of fewer than κ sets in $\mathscr{U}$ is again in $\mathscr{U}$.
2. *normal* if any choice function f is constant on a set in $\mathscr{U}$. (f is a choice function if $f(A) \in A$ for every $A \in P^{\tau}(\mu)$. f is constant on $X \in \mathscr{U}$ if for some $\gamma \in \mu$, $f(A) = \gamma$ for all $A \in X$.)
3. *fine* if $\forall \alpha < \mu\ \{A \in P^{\tau}(\mu) \mid \alpha \in A\} \in \mathscr{U}$.
4. κ*–small below* τ if $\{A \in P^{\tau}(\mu) \mid\ |A \cap \tau| < \kappa\} \in \mathscr{U}$.

Lemma 2.8. *1. If $j : V \longrightarrow M$ is an elementary embedding into a transitive structure M, with critical point κ, and τ is such that $\kappa < \tau < j(\kappa) < j(\tau) = \mu$ and $M^{\mu} \subseteq M$, if $\mathscr{U} \subset \mathscr{P}(P^{\tau}(\mu))$ is defined by*

$$A \in \mathscr{U} \ \textit{ iff } (j''\tau) \in j(A),$$

then $\mathscr{U}$ is a non-principal, κ–complete, normal, fine, and κ–small below τ ultrafilter on $P^{\tau}(\mu)$.

2. If, on the other hand, $\kappa < \tau < \mu$ and $\mathscr{U}$ are such that $\mathscr{U} \subset \mathscr{P}(P^{\tau}(\mu))$ is an ultrafilter that satisfy the five properties above, and if the ultrapower $V^{P^{\tau}(\mu)}/\mathscr{U}$ is computed, then it is well–founded and the resulting elementary embedding, $i : V \longrightarrow N$, is such that κ is the critical point of i, $\tau < i(\kappa)$, $\mu = i(\tau)$ and $N^{i(\tau)} \subset N$.

Definition 2.3. *κ is said to be τ–huge iff either of the conditions 1 and 2 of Lemma 2.8 hold.*

2.5 The exact assumptions

For the consistency proof of "no Aronszajn trees on $\aleph_{\omega+1}$" we need the following:

A cardinal κ and an increasing sequence of cardinals $L = \langle \lambda_i \mid i < \omega \rangle$ with $\lambda_0 > \kappa$ such that:

A_1: For $\lambda = \sup\{\lambda_i \mid i < \omega\}$ and $\mu = \lambda^{+}$, each λ_i, for $i > 0$, is μ-supercompact.

A_2: If $P = Coll(L)$, then in $V_1 = V^P$ κ is τ-huge for $\tau = \kappa^{+(\omega+1)}$ with witness an embedding $j : V_1 \longrightarrow M_1$ (as in Definition 2.3) such that $j(\tau) = \mu$.

(The final model, the one with no Aronszajn trees on $\aleph_{\omega+1}$, will be obtained from a universe that satisfies A_1, A_2 by first collapsing with $Coll(L)$, and then using the resulting embedding j to force with the product $Coll(\kappa^{+(\omega+1)}, < j(\kappa)) \times Coll(\omega, \kappa^{+\omega})$. We shall return to this, and in detail of course, but the reader may want to see at this early stage how the final model is obtained.)

This requirement (A_1, A_2) of a "potentially" huge with ω supercompacts above it, is somewhat technical, but it may be obtained with the following more familiar assumptions:

A cardinal κ and an increasing sequence $\langle \lambda_i \mid i < \omega \rangle$ such that:

B_1: For $\lambda = \sup\{\lambda_i \mid i < \omega\}$ and $\mu = \lambda^+$, each λ_i, $i > 0$, is μ-supercompact.

B_2: κ is the critical point of an embedding $j : V \longrightarrow M$ where $j(\kappa) = \lambda_0$ and $M^\mu \subseteq M$.

Our aim in this subsection is to prove that if cardinals κ and $\langle \lambda_i \mid i < \omega \rangle$ satisfy B_1 and B_2, then there is a generic extension in which cardinals that satisfy A_1 and A_2 can be found.

Let ρ be a cardinal, and $L = \langle \lambda_i \mid i < \omega \rangle$ with $\lambda_0 = \rho$ be any increasing sequence of cardinals with limit λ such that, for $i > 0$, λ_i is λ^+-supercompact. Than L is called t*he minimal supercompact sequence above* ρ if λ is the least cardinal above ρ such that the interval (ρ, λ) contains an ω-sequence of λ^+-supercompact cardinals with $\lambda_0 = \rho$ and such that each λ_{i+1} is the first λ^+-supercompact cardinal above λ_i.

Fix a function g such that, for every ρ, $g(\rho) = \langle \lambda_i \mid i < \omega \rangle$ is such that $\lambda_0 = \rho$ and $\langle \lambda_i \mid i < \omega \rangle$ is the minimal supercompact sequence above λ_0 (if it exists, and g is undefined otherwise).

Suppose $j : V \to M$ is an elementary embedding with critical point κ and such that $M^\mu \subset M$ where $\mu \geq 2^{j(\kappa)}$. It is not difficult to see that κ is $j(\kappa)$-supercompact, not only in V but in M as well (use the combinatorial characterization of supercompactness, and for any $X \subseteq P_\kappa(j(\kappa))$ ask whether $j''j(\kappa) \in j(X)$). Hence, as high as we wish below κ, there are cardinals that are κ-supercompact. So, for every $\rho < \kappa$, $g(\rho)$ is defined and its supremum is below κ, and thus, for every $\rho < j(\kappa)$, $\sup(g(\rho)) < j(\kappa)$ as well.

Let κ, μ, and $\langle \lambda_i \mid i < \omega \rangle$ be as in B_1 and B_2. So κ is the critical point of an embedding $j : V \longrightarrow M$, where $M^\mu \subseteq M$, and (by taking the minimal sequence above $j(\kappa)$) we may assume that

$$g(j(\kappa)) = \langle \lambda_i \mid i < \omega \rangle = L.$$

Thus, if L_0 is the minimal supercompact sequence above κ, and $\tau = (\sup(L_0))^+$, then $j(L_0) = L$, and $j(\tau) = \mu$. κ is thus τ-huge.

For every ordinal $\alpha \leq j(\kappa)$ a cardinal $\rho_0(\alpha)$ and an iteration P_α of length α with Easton support, is defined below by induction. Then, we shall define $P_{j(\kappa)}$ as the required poset which gives an extension where κ and L satisfy A_1 and A_2.

1. P_0 is the trivial poset and $\rho_0(0) = \aleph_1$.
2. For limit α, P_α consists of all partial functions f defined on α such that $f \restriction \gamma \in P_\gamma$ for all $\gamma < \alpha$, and f has the Easton support property: $dom(f) \cap \gamma$ is bounded below γ for every inaccessible γ. The cardinal $\rho_0(\alpha)$ is the first inaccessible cardinal above all the $\rho_0(\gamma)$'s, $\gamma < \alpha$.
3. If P_α and $\rho_0(\alpha)$ are defined, then $P_{\alpha+1} = P_\alpha * Coll(L^\alpha)$ for $L^\alpha = g(\rho_0(\alpha))$. The first inaccessible above the cardinals in the sequence L is $\rho_0(\alpha + 1)$.

Standard arguments prove that for Mahlo γ's that are closed under the function $\alpha \mapsto \sup(g(\alpha))$, P_γ satisfies the γ-c.c. Also, for any $\alpha < \beta$, P_β can be decomposed as $P_\alpha * R$, where R is defined in V^{P_α} as an Easton support iteration of collapses determined by the same function g, but beginning with $\rho_0(\alpha)$. (Supercompact cardinals remain supercompact in any generic extension done via a poset of smaller size; see Levy and Solovay [6].)

Set $P = P_{j(\kappa)}$. We will show in V^P that κ and $L = g(j(\kappa))$ satisfy the properties A_1 and A_2. Recall that $\mu = \lambda^+$ where $\lambda = \sup(L)$. In V, we have an elementary embedding $j : V \longrightarrow M$ into a transitive inner model M such that $M^\mu \subseteq M$. Again, the argument that small forcing will not destroy supercompacness can show that the sumpercompact cardinals in L remain μ-supercompact in V^P; that is, A_1 is easy. We promised to prove that in $(V^P)^{Coll(L)}$, κ is τ-huge for $\tau = \kappa^{+(\omega+1)}$, but in fact we will find a condition in $P * Coll(L)$ and show that extensions through this condition satisfy this requirement. The argument is fairly standard, but we repeat it for completeness' sake.

Observe that

$$j(P_{\kappa+1}) = [P_{j(\kappa+1)}]^M,$$

but the closure of M under μ sequences implies that interpreting this iteration in V or in M results in the same poset Q. So

$$Q = j(P_{\kappa+1}) = P_{j(\kappa)+1} = P_{j(\kappa)} * Coll(L).$$

We will find a condition $q \in Q$ that forces κ to be τ-huge (as in Definition 2.3). In fact, the following suffices:

Lemma 2.9. *There is a condition $q \in Q$ such that if K is a V-generic filter over Q containing q, then the collection $\{j(p) \mid p \in K\}$ has an upper bound in $j(Q)/K$.*

The meaning and proof of this lemma are clarified by the following: Decompose

$$Q = P_{\kappa+1} * R \tag{1}$$

where R, the remainder, is an Easton support iteration, starting above $Coll(g(\kappa))$, of collapses guided by g, going up to $j(\kappa) + 1$. Now apply j to get

$$j(Q) = [P_{j(\kappa)+1} * j(R)]^M = [Q * j(R)]^M .$$

So Q is a factor of $j(Q)$, and $j(Q)/K$ can be formed in $M[K]$. The lemma claims first that each $j(p)$, for $p \in K$, is in $j(Q)/K$, and then that this collection has an upper bound.

Since $2^\lambda = \mu$, the cardinality of Q is μ. It follows that in $V[K]$ $(M[K])^\mu \subseteq M[K]$. Certainly, $j(Q)/K$ is μ^+-closed in $M[K]$ (in fact it is τ_0-closed, where τ_0 is the first inaccessible in M above λ). Hence $j(Q)/K$ is μ^+-closed in $V[K]$. So, to prove the lemma, we only need to choose $q \in Q$ which forces that

$$j(p) \in j(Q)/K$$

for $p \in K$.

Analyzing (1), we write $p \in Q$ as $p = \langle p_0, t, r\rangle$ where $p_0 \in P_\kappa$, t is a name forced to be in $Coll(L^\kappa)$, and r is forced to be in R. Then $j(p) = \langle p_0, j(t), j(r)\rangle$ (because $j(p_0) = p_0$ by the Easton condition). Now $j(p) \in j(Q)/K$ iff the projection of $j(p)$ on Q, namely $\langle p_0, j(t)\rangle$, is in K.

The definition of q can now be given, Define $q \in P_{j(\kappa)+1}$ as $\langle \emptyset, \sigma\rangle$, where $\sigma \in V^{P_{j(\kappa)}}$ is forced to be in $Coll(L)$. It is easier to describe the interpretation of σ in $V[H]$, where $H \subset P_{j(\kappa)}$ is V-generic. Well, look at all conditions $p \in P_{\kappa+1} \cap H$ (there are $< j(\kappa)$ of them); write each such p as $p = \langle p_0, t\rangle$; interpret $j(t)$ as a condition in $Coll(L)$, and take the supremum in $Coll(L)$ of all of these conditions. This proves the lemma and we now see how the result follows.

Lemma 2.10. *Assuming q is as in Lemma 2.9, $q \Vdash_Q \kappa$ is τ-huge.*

Proof. Let $K \subseteq Q$ be a V-generic filter containing q. Work in $V[K]$ and let $s_0 \in j(Q)/K$ be an upper bound of $\{j(p) \mid p \in K\}$. We are going to define in $V[K]$ an ultrafilter $\mathscr{U}$ over $P^\tau(\mu)$ that satisfy the properties of Lemma 2.3. For this, we fix $\langle A_\xi \mid \xi \in \mu^+\rangle$, an enumeration of all subsets of $P^\tau(\mu)$, and plan to decide inductively whether $A_\xi \in \mathscr{U}$ or not. Construct by induction an increasing sequence $\langle s_\xi \mid \xi < \mu^+\rangle$ of conditions in $j(Q)/K$ as follows:

1. At limit stages, $\delta < \mu^+$, use the μ^+-completeness of $j(Q)/K$ to find an upper bound to $\langle s_\xi \mid \xi < \delta\rangle$.
2. If s_ξ is defined, pick for A_ξ a name a_ξ such that $a_\xi[K]$, the interpretation of a_ξ in $V[K]$, is A_ξ. Then $j(a_\xi) \in M^{j(Q)}$, and we find an extension $s_{\xi+1}$ of s_ξ that decides whether $(j''\mu) \in j(a_\xi)$ or not. If the decision is positive, then put $A_\xi \in \mathscr{U}$, and otherwise not. Two comments are in order for this definition to make sense:
   a) First, $j(a_\xi)$ is not a name in $j(Q)/K$-forcing, but in $j(Q)$. Yet, from $M[K]$ any generic extension via $j(Q)/K$ takes us into a universe that is also a $j(Q)$ generic extension of M, and it is as such that we ask about the interpretation of $j(a_\xi)$.
   b) Apparently, this definition depends on a particular choice of a name for A_ξ, but in fact if a'_ξ is another name, then the same answer is obtained. The point of the argument is that some condition p in K forces $a_\xi = a'_\xi$,

and hence $[j(p) \Vdash_{j(Q)} j(a_\xi) = j(a'_\xi)]^M$. But since $j(p) \in j(Q)/K$ is extended by s_0, it can be seen that the answer to the ξ's question does not depend on the particular choice of the name.

We leave it to the reader to prove that $\mathcal{U}$ thus defined satisfies the required properties of Lemma 2.3. For example, let us prove that $\mathcal{U}$ is κ-small below $\tau = \kappa^{+(\omega+1)}$ in $V[K]$. For some $\xi \in \mu^+$, $A_\xi = \{A \in P^\tau(\mu) \mid |A \cap \tau| < \kappa\}$, and a name a_ξ for A_ξ was chosen and an extension $s_{\xi+1}$ deciding whether $j''\mu \in j(a_\xi)$ was thought after. But some condition $p \in K$ forces $a_\xi = \{A \in P^\tau(\mu) \mid |A \cap \tau| < \kappa\}$, and hence $j(p)$, and decidedly s_0, forces $j(a_\xi) = \{A \in P^{j(\tau)}_{j(\mu)} \mid |A \cap j(\tau)| < j(\kappa)\}$. Now $j''\mu = A$ has order-type $\mu = j(\tau)$, and $A \cap j(\tau) = j''\tau$ has cardinality $\tau < j(\kappa)$.

3 There are no Aronszajn trees on successors of singular limits of compact cardinals

The paper really begins here with the following ZFC theorem.

Theorem 3.1. *If λ is singular and a limit of strongly compact cardinals, then there are no λ^+-Aronszajn trees.*

Proof. For notational simplicity, assume that $cf(\lambda) = \omega$. Let $\langle \lambda_i \mid i < \omega \rangle$ be an increasing ω-sequence of strongly compact cardinals with limit λ. (Recall that a cardinal κ is strongly compact if every κ-complete filter can be extended to a κ-complete ultrafilter.) Let T be a λ^+–tree (i.e., of height λ^+ and levels of size $\leq \lambda$) and we will find a λ^+ branch in T. We may assume that T_α, the αth level of T, is the set $\lambda \times \{\alpha\}$. Accordingly, we define $T_{\alpha,n} = \lambda_n \times \{\alpha\}$, so that $T_\alpha = \bigcup_{n<\omega} T_{\alpha,n}$. The proof for the existence of the branch is divided into two steps:

Step one: We claim that there is an unbounded $D \subseteq \lambda^+$ and a fixed $n \in \omega$ such that whenever $\alpha < \beta$ are both in D, then, for some $a \in T_{\alpha,n}$ and $b \in T_{\beta,n}$, $a <_T b$. We call an unbounded set D and a collection $\langle T_{\alpha,n} \mid \alpha \in D \rangle$ as above a *spine* of T. Thus the first part of the proof provides a spine for every λ^+ tree.

Indeed, using the fact that λ_0 is strongly compact, extend the filter of co–bounded subsets of T (that is, those subsets whose complement has cardinality $\leq \lambda$) to a countably complete uniform ultrafilter u over T. Given $\alpha \in \lambda^+$ (considered as a level of T) define $n_\alpha \in \omega$ by the following procedure: For every $x \in T$ of level $> \alpha$, let $r^\alpha_x \in T_\alpha$ be such that $r^\alpha_x <_T x$, and set $n = n_x$ to be the least n such that $r^\alpha_x \in T_{\alpha,n}$. Since the set $T \setminus (T \restriction \alpha + 1)$ is in u, it follows from the $\aleph_1$–completeness of u that for some $n = n_\alpha$, $\{x \in T \mid n_x = n\} = X_\alpha \in u$.

Now there is an unbounded $D \subseteq \lambda^+$ and a fixed n such that $n = n_\alpha$ for $\alpha \in D$. If we take any two ordinals $\alpha < \beta$ in D, then the intersection $X_\alpha \cap X_\beta$ is in u, and any x in this intersection is such that $a = r^\alpha_x$ and $b = r^\beta_x$ are comparable (being both below x), and in the nth part, as required.

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Step two: Every spine has a cofinal branch. Suppose that D and n define a spine of T as above. That is, assume $\{T_{\alpha,n} \mid \alpha \in D\}$, where $D \subseteq \lambda^+$ is unbounded, is a collection such that for every $\alpha < \beta$ in D there are $a \in T_{\alpha,n}$, $b \in T_{\beta,n}$ such that $a <_T b$. Find a λ_{n+1}–complete ultrafilter, υ, over λ^+ containing D and the co-bounded subsets. Fix any $\alpha \in D$. For every $\beta > \alpha$ in D find $a(\beta) \in T_{\alpha,n}$ and $b(\beta) \in T_{\beta,n}$ such that $a(\beta) <_T b(\beta)$. Use the completeness of υ, and the fact that the cardinality of each level of the spine is only λ_n, to find $a_\alpha \in T_{\alpha,n}$ and $\xi_\alpha \in \lambda_n$ such that for a set of β's in υ, $a_\alpha = a(\beta)$ and $b(\beta) = \langle \xi_\alpha, \beta\rangle$ (which is the ξ_α–th element of $T_{\beta,n} = \lambda_n \times \{\beta\}$). For an unbounded $D' \subseteq D$ the ordinal ξ_α has the fixed value ξ for $\alpha \in D'$. Now the collection $\{a_\alpha \mid \alpha \in D'\}$ is a branch of T, because if $\alpha_1, \alpha_2 \in D'$ then for some β (in fact for a set of βs in υ) both a_{α_1} and a_{α_2} are below the ξth point of $T_{\beta,n}$.

In a very direct way, one can generalize this to find that if λ is singular and a limit of strongly compact cardinals as above, then any strong system over λ^+ with index-set of size $< \lambda$ has a branch of size λ^+.

4 The narrowing property

Definition 4.1. *Let $\mu > \chi$ be two cardinals, where $\mu = \lambda^+$. A poset Q has the narrow derived-system property for (μ, χ) if whenever $\mathscr{S}$ is a Q-name of a strong μ-system, with $\leq \chi$ relations, then $Derived_Q(\mathscr{S})$ has a narrow subsystem.*

Theorem 4.1. *Suppose that κ is $\kappa^{+(\omega+1)}$-huge. That is (see Definition 2.3), κ is the critical point of an elementary embedding $j : V \to M$, where M is a transitive class such that $M^\mu \subseteq M$, for $\mu = \lambda^+ = j(\kappa)^{+(\omega+1)}$. Let $Q = Coll(\kappa^{+(\omega+1)}, < j(\kappa))$ (or any other $\kappa^{+(\omega+1)}$-closed poset of size $< \lambda$). Then Q has the narrow derived-system property for $(\mu, \kappa^{+\omega})$.*

Proof. Suppose $\mathscr{S}_0$ is forced by every condition in Q to be a strong system on μ, and $\kappa^{+\omega}$ is its index set. Let $\mathscr{S}_1 = Derived_Q(\mathscr{S}_0)$ be its derived-system; we must find a narrow subsystem of $\mathscr{S}_1$ (that is, one of width and index set of size $< \lambda$). The αth level of $\mathscr{S}_0$ (and of $\mathscr{S}_1$) is $\lambda \times \{\alpha\}$, and we denote it by $(S_0)_\alpha$. The nth part of this level, which has size $j(\kappa)^{+n}$, is denoted $(S_0)_{\alpha,n}$.

Observe that $j(Q)$ is μ–closed in M since Q is $\kappa^{+(\omega+1)}$–closed and $\mu = j(\kappa^{+(\omega+1)})$. In fact, $j(Q)$ is μ–closed in V since M is sufficiently closed.

$j(\mathscr{S}_0)$ is in $V^{j(Q)}$ a strong $j(\mu)$-system with relations indexed by $j(\kappa^{+\omega}) = j(\kappa)^{+\omega} = \lambda$. The αth level of $j(\mathscr{S}_0)$ is $j(\lambda) \times \{\alpha\}$. It is more convenient to denote this level by $(jS_0)_\alpha$. Similarly, the nth part of this level is denoted $(jS_0)_{\alpha,n}$ $(= j^2(\kappa)^{+n} \times \{\alpha\})$.

It follows from the closure of M under μ-sequences (and the fact that $\mu < j(\mu)$) that $j''\mu$ is a bounded subset of $j(\mu)$ in M, and we let $\beta^* < j(\mu)$ be a bound of $j''\mu$. Let b^* be any fixed ordinal in $j(\lambda) \times \{\beta^*\}$ (so b^* is a node of level β^* in $j(\mathscr{S}_0)$).

Inductively, define—in M—a $j(Q)$ increasing sequence of conditions $\{s_\alpha \mid \alpha < \mu\}$, starting with any condition, as follows:

1. At limit stages, the μ-closure of $j(Q)$ is used to find an upper bound to the sequence of length $< \mu$ so far constructed.
2. If s_α is defined, then $s_{\alpha+1}$ is defined as follows: Since $j(\mathscr{S}_0)$ is forced to be a strong system, there exist $a \in (jS_0)_{j(\alpha)}$ and $\zeta < \lambda$ such that $\langle a, b^* \rangle$ is forced by some extension of s_α to stand in the ζth relation of $j(\mathscr{S}_0)$. So we pick $s_{\alpha+1}$, extending s_α, $a_\alpha \in (jS_0)_{j(\alpha)}$, and ζ_α such that

$$s_{\alpha+1} \Vdash_{j(Q)} \langle a_\alpha, b^* \rangle \text{ stands in the } \zeta_\alpha\text{th relation.}$$

Since $\lambda < \mu$ there is a fixed $\zeta^0 < \lambda$, and $n \in \omega$, such that for some unbounded set $D \subseteq \mu$, $\zeta_\alpha = \zeta^0$ and $a_\alpha \in (jS_0)_{\alpha,n}$ for all $\alpha \in D$.

Now the derived-system $\mathscr{S}_1$ has width λ and relations indexed by $Q \times \kappa^{+\omega}$. We claim that the narrow substructure of $\mathscr{S}_1$ defined by $\{(S_0)_{\alpha,n} \mid \alpha \in D\}$ is a system, thereby proving the theorem. If this is not the case, then for some $\alpha_1 < \alpha_2$ in D,

> there are no $a_1 \in (S_0)_{\alpha_1,n}$ and $a_2 \in (S_0)_{\alpha_2,n}$ such that $\langle a_1, a_2 \rangle$ stands in a relation of $\mathscr{S}_1$,

or specifically, "there are no $a_1 \in (S_0)_{\alpha_1,n}$, $a_2 \in (S_0)_{\alpha_2,n}$, $q \in Q$, and $\zeta \in \kappa^{+\omega}$ such that $q \Vdash_Q \langle a_1, a_2 \rangle$ stand in the ζth relation." But then, applying j to this statement we get a contradiction to:

> $s_{\alpha_2+1} \Vdash \langle a_{\alpha_1}, b^* \rangle$, $\langle a_{\alpha_2}, b^* \rangle$ and hence $\langle a_{\alpha_1}, a_{\alpha_1} \rangle$ as well stand in the ζ^0th relation.

Corollary 4.2. *Let κ be $\kappa^{+(\omega+1)}$-huge (as in the theorem). Then it is possible to collapse $j(\kappa)^{+(\omega+1)}$ to be $\aleph_{\omega+1}$ with a forcing poset that has the narrow derived-system property for $(j(\kappa)^{+(\omega+1)}, \omega)$. In other words, there is a forcing poset P such that*

1. *$j(\kappa)^{+(\omega+1)}$ becomes $\aleph_{\omega+1}$ in V^P, and*
2. *the derived-system $Derived_P(\mathscr{S})$ of every strong system on $j(\kappa)^{+(\omega+1)}$ with countably many relations in V^P has a narrow subsystem (in V).*

Proof. The desired poset is simply the collapse of $j(\kappa)$ to become $\aleph_2$, but not in the most direct way. It is rather the product of two collapses that works: the collapse of $\kappa^{+\omega}$ to $\aleph_0$, and the one that makes $j(\kappa)$ the double successor of $\kappa^{+\omega}$ (both posets are defined in V). Let $Q = Coll(\kappa^{+(\omega+1)}, < j(\kappa))$, $K = Coll(\aleph_0, \kappa^{+\omega})$, and then $P = Q \times K$ is the desired collapse. In V^P, $\kappa^{+\omega}$ is countable, $\kappa^{+(\omega+1)}$ is $\aleph_1$, $j(\kappa)$ is $\aleph_2$, and $\mu = \lambda^+ = j(\kappa)^{+(\omega+1)}$ becomes $\aleph_\omega^+$.

So let $\mathscr{S}_0$ be in V^P any strong μ-system with a countable set of relations; then $Derived_P(\mathscr{S}_0)$ can be obtained in two stages, corresponding to the product $P = Q \times K$ and to the decomposition $V^P = (V^Q)^K$. First, in V^Q, form $\mathscr{S}_1 = Derived_K(\mathscr{S}_0)$. Then $\mathscr{S}_1$ is in V^Q a strong system on μ, with $|K| \times \aleph_0 = \kappa^{+\omega}$ relations, one relation for each pair formed with a condition in K and a relation (index) in $\mathscr{S}_0$. Hence, by the theorem, $Derived_Q(\mathscr{S}_1) \sim Derived_P(\mathscr{S}_0)$ has (in V) a narrow subsystem as required.

5 The potential branching property and a model with no Aronszajn trees

Suppose $\mu = \lambda^+$. The potential branching property for μ is the following statement:

> If $\mathscr{S}$ is a narrow system on μ, then for every $\chi < \lambda$ there is a χ-complete forcing poset that introduces an unbounded branch to $\mathscr{S}$.

Recall that a branch of a system is a set of nodes and a relation in the system which includes every increasing pair from the set. In the following section we will see how to obtain the potential branching property, but here we use it to obtain a model with no Aronszajn trees on $\aleph_{\omega+1}$.

Theorem 5.1. *Let κ be $\kappa^{+(\omega+1)}$-huge and suppose that the potential branching property holds for μ (where $\mu = j(\kappa)^{+(\omega+1)}$). Then there is a generic extension in which μ becomes $\aleph_\omega^+$ and it carries no Aronszajn trees.*

Proof. The poset P of Corollary 4.2 works. Recall that $Q = Coll(\kappa^{+(\omega+1)}, < j(\kappa))$, $K = Coll(\aleph_0, \kappa^{+\omega})$, and then $P = Q \times K$. The cardinality of P is $\chi = j(\kappa)$. We know by Corollary 4.2 that $j(\kappa)^{+(\omega+1)}$ is $\aleph_{\omega+1}$ in V^P and that the derived-system (via P) of any strong system over $j(\kappa)^{+(\omega+1)}$ with countably many relations has a narrow subsystem in V. We will prove that there are no μ–Aronszajn trees in V^P.

Suppose that $\mathbf{T}$ is in V^P a name of a μ-tree. We will first show that there is in V a χ^+–complete poset R such that, in $(V^P)^R$, $\mathbf{T}$ acquires an unbounded branch. Then the preservation theorem (2.1, applied with $\lambda = j(\kappa)^{+\omega}$) shows that $\mathbf{T}$ has a branch already in V^P.

To see how R is obtained, apply Corollary 4.2 to $\mathbf{T}$, considered as a single-relation strong system, and find in V a narrow subsystem $\mathscr{S}$ to $Derived_P(\mathbf{T})$. But then for $\chi = |P| < \lambda$ there is (by the potential branching property) a χ^+-complete forcing poset R that introduces an unbounded branch to $\mathscr{S}$. This can be shown to give an unbounded branch to the tree $\mathbf{T}$ in $V^{P\times R}$. But then the preservation theorem shows that $\mathbf{T}$ already has a branch in V^P.

6 The final model

For the consistency of *no Aronszajn trees on* $\aleph_{\omega+1}$ we must show how to obtain the assumptions of Theorem 5.1, namely how to get a cardinal κ which is $\kappa^{+(\omega+1)}$-huge with the potential branching property for $j(\kappa)^{+(\omega+1)}$. The main point is to prove that whenever an ω sequence of supercompact cardinals converging to λ is collapsed, then the potential branching property for λ^+ holds. When this is combined with assumptions A_1 and A_2 described in Sect. 2.5, then the assumptions for Theorem 5.1 are obtained.

Theorem 6.1. *Suppose $\langle \lambda_i \mid i < \omega \rangle$, with $\lambda = \bigcup_{i<\omega} \lambda_i$ and $\mu = \lambda^+$, is an increasing ω–sequence of μ–supercompact cardinals (except for λ_0 which is just* *a regular cardinal). Let $C = Coll(\langle\lambda_i \mid i < \omega\rangle)$ be the full support iteration that makes λ_i to be λ_0^{+i}. Then, in V^C, the potential branching property holds for μ:*

> *If $\mathscr{S}$ is a narrow system on μ, then, for every $k < \omega$, there is a λ_k–complete forcing that introduces an unbounded branch to $\mathscr{S}$.*

Proof. We will actually prove the following combinatorial statement in V^C:

> For every $n < \omega$ and function $F : [\mu]^2 \to \chi$, where $\chi < \lambda_n$, there is a λ_n–complete forcing C^* such that in $(V^C)^{C^*}$ the following holds: For some $\nu \in \chi$ there is an unbounded set $U \subseteq \mu$ such that for every $\alpha_1 < \alpha_2$ in U there is $\beta > \alpha_2$ such that
>
> $$F(\alpha_1, \beta) = F(\alpha_2, \beta) = \nu. \tag{2}$$

(We call such a set U "a branch" of F.)

First, we argue that this statement suffices to prove the theorem. Let $\mathscr{S} = (T, R) \in V^C$ be a narrow system over μ, and let $\chi < \lambda$ be such that the width of $\mathscr{S}$ and the cardinality of its index set are $\leq \chi$. Suppose that $\chi < \lambda_n$ and we will find a λ_n–complete forcing that introduces an unbounded branch to $\mathscr{S}$. For this, define in V^C a function $F : [\mu]^2 \to \chi^3$ by

> $F(\alpha_1, \alpha_2) = (\zeta, \tau_1, \tau_2)$ iff the τ_1 member of T_{α_1} and the τ_2 member of T_{α_2} stand in the ζ relation R_ζ.

Then, by the assumed combinatorial principle, there is an unbounded set $U \subseteq \mu$, and fixed ordinals $\nu = (\zeta, \tau_1, \tau_2)$ as in Eq. (2). This implies that the τ_1-th points of T_α (that is $\langle\tau_1, \alpha\rangle$) for $\alpha \in U$ form an R_ζ branch of $\mathscr{S}$ (use item *2* in the definition of systems).

We now explain why it suffices to prove the combinatorial principle for $n = 0$. Given $F : [\mu]^2 \to \chi$ where $\chi < \lambda_n$ in V^C (suppose for simplicity that *every* condition in C forces that F is into χ), decompose $C \simeq C_n * C^n$ where $C_n = Coll(\langle\lambda_i \mid i \leq n\rangle)$, and C^n is the name in V^{C_n} of $Coll(\langle\lambda_i \mid n \leq i < \omega\rangle)$. In V^{C_n} define $\lambda'_m = \lambda_{n+m}$. Then each λ'_m, for $m > 0$ is μ–supercompact. (Indeed the embedding $j : V \to M$ with critical point λ_k, for $k > n$, can be extended in V^{C_n} to an embedding of V^{C_n} into M^{C_n}, where M^{C_n} possesses the same μ–closure properties.) Thus, if we know case $n = 0$ of the theorem in $V_0 = V[C_n]$, we could apply it there to $C^n = Coll(\langle\lambda'_i \mid i < \omega\rangle)$ and get in $V_0^{C^n} = V^C$ the desired λ'_0–complete (λ_n–complete) poset that adds a branch to F. To save ourselves from too many superscripts, we denote V^{C_n} by V and M^{C_n} by M and assume $n = 0$.

So, returning to the theorem, assume that G is a V-generic filter over C, and F is in $V[G]$ a function from $[\mu]^2$ into $\chi < \lambda_0$. In the following lemma, we will describe a λ_0-complete forcing P in $V[G]$ that introduces a μ-branch to F. Let $j : V \to M$ be an elementary embedding with critical point λ_1, such that $j(\lambda_1) > \mu$ and M is closed in V under μ-sequences. The following lemma will be proved later on.

Lemma 6.2. *There is in $V[G]$ a λ_0-complete poset P such that in $V[G]^P$ there is an extension of the embedding j to an elementary embedding of $V[G]$ into $N = M[j(G)]$.*

Accepting the lemma for a moment, $j(F) \in N$ can be defined; it is a function on $[j(\mu)]^2$ and into $\chi < \lambda_0$ (j is the identity below λ_1).

Since $j(\mu) > \mu$, there is an ordinal $\beta < j(\mu)$ above all the ordinals in $j''\mu$. Now, for each $\alpha < \mu$ we can find some $\zeta < \chi$ such that

$$j(F)(j(\alpha), \beta) = \zeta.$$

Since $\chi < \lambda_0$, and as μ is regular in V and no new sequences of length $< \lambda_0$ are added to V in $V[G]^P$, we may find a single ζ such that for unboundedly many α's the equality $j(F)(j(\alpha), \beta) = \zeta$ holds. Since j is elementary, it follows that for any $\alpha_1 < \alpha_2$ in this unbounded set $F(\alpha_1, b) = F(\alpha_2, b) = \zeta$ holds for some $b > \alpha_2$, $b < \mu$. Thus an unbounded μ–branch for ζ was found in $V[G]^P$, which is a λ_0–closed extension of $V[G]$.

We turn now to the proof of Lemma 6.2. The collapsing poset $C_1 = Coll(\lambda_0, < \lambda_1)$ is a factor of $C = Coll(\langle \lambda_i \mid i < \omega \rangle)$, and for simplicity of expression, we identify $c \in C_1$ with the condition $\langle c_1 \mid i < \omega \rangle \in C$ defined by $c_0 = c$ and $c_i = \emptyset$ for $i > 0$.

Denote each $j(\lambda_i)$ with λ_i^*. Then $\lambda_0^* = \lambda_0$, but $\lambda_1^* > \mu$. In M, $j(C)$ is $[Coll(\langle \lambda_i^* \mid i \in \omega \rangle)]^M$, and $C_1^* = Coll(\lambda_0, < \lambda_1^*)$ (which is the same—defined in V or in M) is a factor of $j(C) = [Coll(\langle \lambda_i^* \mid i < \omega \rangle)]^M$.

Let $G \subseteq C$ be a V-generic filter over C. Observe that if $\langle c_i \mid i < \omega \rangle \in G$, then $c \in G$ as well. In order to extend j on $V[G]$ and to prove the lemma, we should find in $V[G]$ a λ_0-complete poset P such that in $V[G]^P$ there is a V-generic filter G^* over $j(C)$ such that

$$\text{If } g \in G, \text{ then } j(g) \in G^*.$$

If we do so, then an embedding of $V[G]$ into $M[G^*]$ can be defined as follows: For any $x \in V[G]$, let $\mathbf{x}$ be a name of x in V^C. Then $j(\mathbf{x})$ is a name in $M^{j(C)}$ and we define $j'(x)$ to be its interpretation in $M[G^*]$. We trust the reader to check that j' is a well defined elementary extension of j.

Instead of writing down P, we will describe it as an iteration of two extensions, each one λ_0-complete.

Since the cardinality of C is μ, $\mu < \lambda_1^*$ is collapsed to λ_0 in $V^{C_1^*}$, Lemma 2.7 implies that there is a projection, Π, of C_1^* onto C, which can be used to find a generic extension of $V[G]$ which has the form $V[H]$ for a V-generic filter H over C_1^* such that:

1. The passage from $V[G]$ to $V[H]$ is done by forcing with a λ_0-closed forcing.
2. For every $c \in C_1$, $c \in G$ iff $c \in H$.

Thus, for every $g \in G$, $j(g) \in j(C)/H$. Indeed, any $g \in G$ has the form $g = \langle c, \bar{r} \rangle$ where $c \in C_1 \cap G$, and $\bar{r}$ is the remaining part of the sequence. Then $j(g) = \langle c, j(\bar{r}) \rangle$ where $c \in C_1^* \cap H$, and thus $j(g) \in j(C)/H$.

It follows that $j''G$ (the image of G under the restriction of j to C) is in $M[H]$ a pairwise compatible collection of conditions in $j(C)/H$. Since $j(C)/H$ is isomorphic to $Coll(\langle\lambda_i^* \mid i \geq 1\rangle)$ in $M[H]$, it is λ_1^*-complete, and a supremum, denoted $s \in j(C)/H$ can be found for $j''G$. This is our "master condition": If G^* is any $V[H]$-generic filter over $j(C)/H$, containing s, then:

1. G^* is in fact V-generic over $j(C)$.
2. The forcing $j(C)/H$ is λ_1^*-complete in $M[H]$, and it is thence λ_0-complete in $V[H]$ (because $M[H]$ is λ_0-closed in $V[H]$).

7 Conclusion

We have proved the following theorem:

Theorem 7.1. *Assume a cardinal κ and sequence $L = \langle\lambda_i \mid i < \omega\rangle$ such that*

B_1: *For $\lambda = \sup\{\lambda_i \mid i < \omega\}$ and $\mu = \lambda^+$, each λ_i, $i > 0$, is μ-supercompact.*
B_2: *κ is the critical point of an embedding $j : V \longrightarrow M$ where $j(\kappa) = \lambda_0$ and $M^\mu \subseteq M$.*

Then there is a generic extension in which there are no $\aleph_{\omega+1}$ Aronszajn trees.

Indeed, in Sect. 2.5 we saw that by making a preparatory extension we may assume that κ is such that if $C = Coll(L)$, then in V^C κ is $\kappa^{+(\omega+1)}$-huge. So, we go to V^C, and find that the potential branching property for $\mu = j(\kappa)^{+(\omega+1)}$ holds (by Theorem 6.1). But now, in V^C, all the assumptions for theorem 5.1 hold. Thus, in a final extension, obtained as a product of $Coll(\kappa^{+(\omega+1)}, < j(\kappa))$ and $Coll(\aleph_0, \kappa^{+\omega})$, there are no Aronszajn trees on $\aleph_{\omega+1}$.

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