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What majority decisions are possible

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ABSTRACT

Suppose we are given a family of choice functions on pairs from a given finite set (with at least three elements) closed under permutations of the given set. The set is considered the set of alternatives (say candidates for an office). The question is, what are the choice functions c on pairs of this set of the following form: for some (finite) family of "voters", each having a preference, i.e. a choice from each pair from the given family, $\mathbf{c}\{x, y\}$ is chosen by the preference of the majority of voters. We give full characterization.

0. Introduction

Condorcet's "paradox" demonstrates that given three candidates A, B and C, the majority rule may result in the society preferring A to B, B to C and C to A. McGarvey [3] proved a far-reaching extension of Condorcet's paradox: for every asymmetric relation R on a finite set M of candidates there is a strict-preferences (linear orders, no ties) voter profile that has the relation R as its strict simple majority relation. In other words, for every asymmetric relation (equivalently, a tournament) R on a set M of m elements there are n linear order relations on M, $R_1, R_2, \ldots, R_n$ such that for every $a, b \in M$, aRb if and only if

$$|\{i : aR_ib\}| > n/2.$$

McGarvey's proof gave $n = m(m-1)$. Stearns [5] found a construction with $n = m$ and noticed that a simple counting argument implies that n must be at least $m/\log m$. Erdős and Moser [2] were able to give a construction with $n = O(m/\log m)$. Alon [1] showed that for some constant $c_1 > 0$ we can find $R_1, \ldots, R_n$ with

$$|\{i : aR_ib\}| > (1/2 + c_1/\sqrt{n})n,$$

and that this is no longer the case if c_1 is replaced with another constant $c_2 > c_1$.

Gil Kalai asked to what extent the assertion of McGarvey's theorem holds if we replace the set of order relations by an arbitrary isomorphism class of choice functions on pairs of elements (see Definition 0.4). Namely, the question is to characterize under which conditions clause (A) of 0.1 below holds (i.e. Question 1.4).

Instead of choice functions we can speak on tournaments, see Observation 0.5.

The main result is (follows from 2.1), (the cases $n = 1$; 2 are trivial).

Theorem 0.1. *Let X be a finite set and $\mathfrak{D}$ be a non-empty family of choice functions for $\binom{X}{2}$ closed under permutations of X. Then the following conditions are equivalent:*

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(A) *for any choice function c on* $\binom{X}{2}$ *we can find a finite set J and* $c_j \in \mathfrak{D}$ *for* $j \in J$ *such that for any* $x \neq y \in X$:

$$c\{x, y\} = y \quad \underline{\text{iff}}\ |J|/2 < |\{j \in J : c_j\{x, y\} = y\}|$$

(so equality never occurs)

(B) *for some* $c \in \mathfrak{D}$ *and some* $x \in X$ *we have* $|\{y : c\{x, y\} = y\}| \neq (|X| - 1)/2$.

Gil Kalai further asks

Question 0.2. (1) In 0.1 can we bound $|J|$ reasonably?

(2) What is the result of demanding a "non-trivial majority"? (say 51%?)

Under 0.1 it seems reasonable to characterize what can be $\{c : c$ a choice function for pairs from X gotten as in clause (A) of 0.1 using $c_j \in \mathfrak{D}\}$, when we vary $\mathfrak{D}$, so 0.1 tells us for which sets $\mathfrak{D}$ the resulting family is maximal.

We then give in 3.7 a complete solution also to the question: what is the closure of a set of choice functions by majority; in fact, there are just two.

We also may allow each "voter" to abstain; this means that his choice function is only partial. We hope to deal with this elsewhere, but there are more cases, e.g. of course, if all voters have no opinion on any pair, majority decision will always be a draw (giving a third possibility). Note that now we consider also majority decisions which give a draw in some of the cases. The present work was present in the conference in honour of Michael O. Rabin, Summer 2005.

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An earlier version is [4].

Notation 0.3. Let n, m, k, ℓ, i, j denote natural numbers.

Let r, s, t, a, b denote real numbers.

Let x, y, z, u, v, w denote members of the finite set X.

Let $\binom{X}{k}$ be the family of subsets of X with exactly k members.

Let c, d denote partial choice functions on $\binom{X}{2}$.

Let $\text{conv}(A)$ be the convex hull of A, here for $A \subseteq \mathbb{R} \times \mathbb{R}$.

Let $\text{Per}(X)$ be the set of permutations of X.

The "translation" to tournaments is not really used, still we explain it.

Definition 0.4. (1) We say that c is a choice [partial choice] function for $\binom{X}{k}$ if c is a function with domain $\binom{X}{k}$ $\left[\text{domain} \subseteq \binom{X}{k}\right]$ such that $x \in \text{Dom}(c) \Rightarrow c(x) \in x$.

(2) We say c is a choice function for pairs from X <u>when</u> c is a choice function for $\binom{X}{2}$.

(3) If c is a partial choice function for pairs from X, let $\text{Tor}[c]$ be the following directed graph:

(a) the set of nodes is X

(b) the set of edges is $\{(x, y) : x \neq y$ are from X and $c\{x, y\} = y\}$.

(3A) Let $\mathbf{G}_c$ be the non-directed graph derived from $\text{Tor}[c]$.

(4) Let c_1, c_2 be choice functions for pairs from X. We say π is an isomorphism from c_1 onto c_2 <u>if</u> π is a permutation of X such that for every $x, y \in X$ we have $c_1\{x, y\} = y \Leftrightarrow c_2\{\pi(x), \pi(y)\} = \pi(y)$.

Observation 0.5. (1) For any set X, the mapping $c \mapsto \text{Tor}[c]$ is a one-to-one mapping from the set of choice functions for pairs from X onto the set of tournaments on X.

(1A) It is also a one to one map from the set of partial choice functions onto the set of directed graphs on X (so for $x \neq y$ maybe (x, y) is an edge maybe (y, x) is an edge but not both and maybe none).

(2) For choice functions c_1, c_2 of pairs from X; we have c_1, c_2 are isomorphic iff $\text{Tor}[c_1], \text{Tor}[c_2]$ are isomorphic tournaments.

(2A) Similarly for partial choice functions and directed graphs.

1. Basic definitions and facts

Hypothesis 1.1. Assume

(a) X is a (fixed) finite set with $\mathbf{n} \geq 3$ members, i.e. $\mathbf{n} = |X|$.

(b) $\mathfrak{C} = \mathfrak{C}^1 = \mathfrak{C}^1_X$ is the set of partial choice functions on $\binom{X}{2}$, see Definition 0.4(1); when $c\{x, y\}$ is not defined it is interpreted as abstaining or having no preference.

Let $\mathfrak{C}^0 = \mathfrak{C}^{\text{full}} = \mathfrak{C}^{\text{full}}_X = \mathfrak{C}^0_X$ be the set of $c \in \mathfrak{C}^1_X$ which are full, i.e., $\text{Dom}(c) = \binom{X}{2} = \{\{x, y\} : x \neq y \in X\}$.

(c) $\mathscr{C}, \mathscr{D}$ vary on subsets of $\mathfrak{C}$
(d) $\mathfrak{D}$ vary on non-empty subsets of $\mathfrak{C}$ which are symmetric where

Definition 1.2. (1) $\mathscr{C} \subseteq \mathfrak{C}$ is symmetric if it is closed under permutations of X (i.e. for every $\pi \in \text{Per}(X)$ the permutation $\hat{\pi}$ maps $\mathscr{C}$ onto itself where π induces $\hat{\pi}$, a permutation of $\mathfrak{C}$, that is $c_1 = c_2^{\pi}$ or $c_1 = \hat{\pi}c_2$ mean that: $x_1 = \pi(x_2), y_1 = \pi(y_2)$ implies $c_1\{x_1, y_1\} = y_1 \Leftrightarrow c_2\{x_2, y_2\} = y_2$).

(2) For $\mathscr{D} \subseteq \mathfrak{C}$ and $x \neq y \in X$ let $\mathscr{D}_{x,y} = \{d \in \mathscr{D} : d\{x, y\} = y\}$.

Definition 1.3. For $\mathscr{D} \subseteq \mathfrak{C}$ let maj-cl($\mathscr{D}$) be the set of $d \in \mathfrak{C}$ such that for some real numbers $r_c = r_c[d] \in [0, 1]_{\mathbb{R}}$ for $c \in \mathscr{D}$ satisfying $\Sigma_{c\in\mathscr{D}} r_c = 1$ we have[1]

$$d\{x, y\} = x \Leftrightarrow \frac{1}{2} < \Sigma\{r_c : c\{x, y\} = x \text{ and } c \in \mathscr{D}\} + \Sigma\{r_c/2 : c\{x, y\} \text{ is undefined and } c \in \mathscr{D}\}.$$

Remark. (1) Clearly maj is for majority. At first glance this is not the same as the problem stated in the introduction but easily they are equivalent (see clause (c) of 2.1).

(2) Note that if we deal with full choice functions only, as originally, then we require that the sum is never $\frac{1}{2}$.

(3) Modulo the equivalence above, Kalai's original question was

Question 1.4. If $|X|$ is sufficiently large and $\mathfrak{D} \subseteq \mathfrak{C}^{\text{full}}$ (is symmetric), when is it true that maj-cl($\mathfrak{D}$) $= \mathfrak{C}^{\text{full}}$?

Definition 1.5. (1) Let Dis $=$ Dis(X) $= \{\mu : \mu$ a distribution on $\mathfrak{C}_X\}$; of course, "μ a distribution on $\mathfrak{C}$" means μ is a function from $\mathfrak{C}$ into $[0, 1]_{\mathbb{R}}$ such that $\Sigma\{\mu(c) : c \in \mathfrak{C}\} = 1$.

(2) For $\mathscr{C} \subseteq \mathfrak{C}$ and $\mu \in \text{Dis}(\mathfrak{C})$ let $\mu(\mathscr{C}) = \Sigma\{\mu(c) : c \in \mathscr{C}\}$ so $\mu(\mathscr{C}) \geq 0, \mu(\mathfrak{C}) = 1$.

(3) For $\mathscr{D} \subseteq \mathfrak{C}$ let $\text{Dis}_{\mathscr{D}} = \{\mu \in \text{Dis} : \mu(\mathscr{D}) = 1\}$.

(4) Let pr($\mathfrak{C}$) $= \{\bar{t} : \bar{t} = \langle t_{x,y} : x \neq y \in X\rangle$ such that $t_{x,y} \in [0, 1]_{\mathbb{R}}$ and $t_{y,x} = 1 - t_{x,y}\}$, we may write $\bar{t}(x, y)$ instead of $t_{x,y}$; pr stands for probability.

(5) For $T \subseteq$ pr($\mathfrak{C}$) let pr-cl(T) be the convex hull of T.

(6) For $d \in \mathfrak{C}$ let $\bar{t}[d] = \langle t_{x,y}[d] : x \neq y \in X\rangle$ be defined by $[t_{x,y}[d] = 1 \Leftrightarrow d\{x, y\} = y \Leftrightarrow t_{x,y}[d] \neq 0]$ when $\{x, y\} \in$ Dom(d) and $t_{x,y}[d] = \frac{1}{2} = t_{y,x}[d]$ if $x \neq y \in X$, $\{x, y\} \notin$ Dom(d).

(7) Let pr-cl($\mathscr{D}$) for $\mathscr{D} \subseteq \mathfrak{C}$ be pr-cl($\{\bar{t}[d] : d \in \mathscr{D}\}$) and let prd($\mathscr{D}$) $= \{\bar{t}[c] : c \in \mathscr{D}\}$.

(8) For $\mathscr{C} \subseteq \mathfrak{C}$ we let sym-$c\ell(\mathscr{C})$ be the minimal $\mathscr{D} \subseteq \mathfrak{C}$ which is symmetric and includes $\mathscr{C}$. For $T \subseteq$ pr($\mathfrak{C}$) let maj(T) $= \{c \in \mathfrak{C}$: for some $\bar{t} \in T$ we have $c =$ maj($\bar{t}$)$\}$, see below, and for $\mathscr{D} \subseteq \mathfrak{C}$ let maj-cl($\mathscr{D}$) $=$ maj(pr-cl($\mathscr{D}$)).

(9) For $\bar{t} \in$ pr($\mathfrak{C}$) we define maj($\bar{t}$) as the $c \in \mathfrak{C}$ such that $c\{x, y\} = y \Leftrightarrow t_{x,y} > \frac{1}{2}$.

Claim 1.6. (1) *For* $d \in \mathfrak{C}$ *we have* $\bar{t}[d] \in$ pr($\mathfrak{C}$).

(2) *For* $\mathscr{D} \subseteq \mathfrak{C}$ *we have* $\text{Dis}_{\mathscr{D}} \subseteq$ Dis.

(3) prd($\mathfrak{C}$) $=$ pr($\mathfrak{C}$) *and if* $\mathscr{D} \subseteq \mathfrak{C}$ *then* prd($\mathscr{D}$) $\subseteq$ pr-cl($\mathscr{D}$) $\subseteq$ Dis.

(4) *If* $\mathscr{C} \subseteq \mathfrak{C}$ *then* $\mathscr{C} \subseteq \text{sym} - c\ell(\mathscr{C}) \subseteq \mathfrak{C}$.

(5) *If* $T \subseteq$ pr($\mathfrak{C}$) *then* maj(T) $\subseteq \mathfrak{C}$.

(6) *For* $\mathscr{D} \subseteq \mathfrak{C}$ *the two definitions of* maj-$c\ell(\mathscr{D})$ $=$ maj(pr-cl($\mathscr{D}$)) *in* 1.5(8) *and* 1.3 *are equivalent.*

Proof. Obvious. □

Kalai showed that not everything is possible.

Claim 1.7 (*G. Kalai*). *If* $\mathscr{C} \subseteq \mathfrak{C}^{\text{full}}$ *and for every* $c \in \mathscr{C}$ *and* $x \in X$, *the in-valency and out-valency are equal, (i.e.,* $\text{val}_c(x) = (|X| - 1)/2$, *see below) then every* $d \in \mathfrak{C}^{\text{full}} \cap$ maj-cl($\mathscr{C}$) *satisfies:*

$(*)$ *if* $\emptyset \neq Y \subsetneq X$ *then the directed graph* Tor(d) *satisfies: there are edges from* Y *to* $X \setminus Y$ *and from* $X \setminus Y$ *to* Y.

Proof. See 3.2(1), (2) (and not used earlier). □

Definition 1.8. (1) For $d \in \mathfrak{C}$ and $x \in X$ let $\text{val}_d(x)$, the valency of x for d be $|\{y : y \in X, y \neq x, d\{x, y\} = y\}| + |\{y : y \in X, y \neq x, d\{x, y\}$ not defined$\}|/2$, so if d is full the second term disappears. Let $\text{val}_d^+(x) = |\{y : y \in X, y \neq x$ and $d\{x, y\} = y\}|$ so $\text{val}_d^+(x) \in \{0, \ldots, \mathbf{n} - 1\}$.

We also call $\text{val}_d^+(x)$ the out-valency[2] of x in d and also call it $\text{val}_d^{+1}(x)$ and we let $|\{y : y \in X, y \neq x$ and $d\{y, x\} = x\}|$ be the in-valency of x and denote it by $\text{val}_d^{-1}(x)$; note that if d is full (i.e., $\in \mathfrak{C}_X^{\text{full}}$), then $\text{val}_d^{-1}(x) = \mathbf{n} - \text{val}_d^{+1}(x) - 1$ and $\text{val}_d(x) = \text{val}_d^+(x)$.

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[1] Note that there is no a priori reason to assume that $\mathscr{D}_2 =$ maj-$c\ell(\mathscr{D}_1)$ implies $\mathscr{D}_2 =$ maj-$c\ell(\mathscr{D}_2)$.

[2] Natural under the tournament interpretation.

(2) For $d \in \mathfrak{C}$ let $\mathrm{Val}(d) = \{\mathrm{val}_d(x) : x \in X\}$.
(3) For $d \in \mathfrak{C}$ and $\ell \in \{0, 1\}$ let $V_\ell(\mathrm{d}) = \{(\mathrm{val}_d(x_0), \mathrm{val}_d(x_1)) : x_0 \neq x_1 \in X$ and $d\{x_0, x_1\} = x_\ell\}$.
(4) For $d \in \mathfrak{C}$ and $\ell \in \{0, 1\}$ let $V^*_\ell(\mathrm{d}) = \{\bar{k} - (\ell, 1 - \ell) : \bar{k} \in V_\ell(\mathrm{d})\}$ and let $V^*(\mathrm{d}) = V^*_0(\mathrm{d}) \cup V^*_1(\mathrm{d})$.
(5) For $c \in \mathfrak{C}$ let $\mathrm{dual}(c) \in \mathfrak{C}$ have the same domain as c and satisfy $\mathrm{dual}(c)\{x, y\} \in \{x, y\} \setminus \{c\{x, y\}\}$ when defined; similarly $\bar{t}' = \mathrm{dual}(\bar{t})$ for $\bar{t} \in \mathrm{pr}(\mathfrak{C})$ means that $t'_{x,y} = 1 - t_{x,y}$.
(6) Let $V_{1/2}(\mathrm{d}) = \{(\mathrm{val}_d(x_0), \mathrm{val}_d(x_1)) : x_0 \neq x_1 \in X$ and $d\{x_0, x_1\}$ is not defined$\}$ and $V^*_{1/2}(\mathrm{d}) = V_{1/2}(\mathrm{d})$.

Claim 1.9. (1)

(α) $c_1 \in \text{sym-cl}\{c_2\}$ <u>iff</u> $\mathrm{dual}(c_1) \in \text{sym-cl}\{\mathrm{dual}(c_2)\}$
(β) $c_1 \in \text{maj-cl}(\text{sym-cl}\{c_2\})$<u>iff</u> $\mathrm{dual}(c_1) \in \text{maj-cl}(\text{sym-cl}\{\mathrm{dual}(c_2)\})$.

(2) $(k_0, k_1) \in V_0(\mathrm{d}) \Leftrightarrow (k_1, k_0) \in V_1(\mathrm{d})$ *and* $(k_0, k_1) \in V^*_0(\mathrm{d}) \Leftrightarrow (k_1, k_0) \in V^*_1(\mathrm{d})$.
(3) *"$c_1 \in \text{sym-cl}\{c_2\}$" is an equivalence relation on $\mathfrak{C}$ and it implies $V_\ell(c_1) = V_\ell(c_2)$ for $\ell = 0, 1$.*

Proof. Easy. □

2. When every majority choice is possible: A characterization

The following is the main part of the solution (probably (c) ⇔ (g) is the main conclusion here).

Main Claim 2.1. *Assume that $\mathfrak{D} \subseteq \mathfrak{C}^{\text{full}}$ which is symmetric and non-empty, (i.e., $\mathfrak{D}$ is a non-empty set of choice functions on $\binom{X}{2}$ closed under permutation on X) and for simplicity assuming that $\mathfrak{D} = \text{sym-cl}(d^*)$ for any $d^* \in \mathfrak{D}$. <u>Then</u> the following conditions on $\mathfrak{D}$ are equivalent, where x, y vary on distinct members of X:*

(a) $\text{maj-cl}(\mathfrak{D}) \supseteq \mathfrak{C}^{\text{full}}_X$
(a)′ $\text{maj-cl}(\mathfrak{D}) = \mathfrak{C}$
$(\mathrm{b})_{\mathrm{x,y}}$ *there is* $\bar{t} \in \text{pr-cl}(\mathfrak{D}) \subseteq \mathrm{pr}(\mathfrak{C})$ *such that*
  (i) $t_{x,y} > \frac{1}{2}$
  (ii) $\{x, y\} \neq \{u, v\} \in \binom{X}{2} \Rightarrow t_{u,v} = \frac{1}{2}$
(c) *for any* $c \in \mathfrak{C}^{\text{full}}$ *we can find a finite set J and sequence $\langle d_j : j \in J\rangle$ such that $d_j \in \mathfrak{D}$ and: if $u \neq v \in X$ then* $c\{u, v\} = v \Leftrightarrow |\{j \in J : d_j\{u, v\} = v\}| > |J|/2$
(c)′ *like clause* (c) *for* $c \in \mathfrak{C}$
(d) $(\frac{1}{2}, \frac{1}{2})$ *belongs to* $\mathrm{Pr}_{>\frac{1}{2}}(\mathfrak{D})$*, see Definition 2.2*
(e) $(\frac{1}{2}, \frac{1}{2}) \in \mathrm{Pr}_{\neq 1/2}(\mathfrak{D})$
(f) $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ *can be represented as* $r^*_0 \times \bar{s}_0 + r^*_1 \times \bar{s}_1$ *where*
  $(*)$(i) $r^*_0, r^*_1 \in [0, 1]_{\mathbb{R}} \setminus \{\frac{1}{2}\}$
   (ii) $1 = r^*_0 + r^*_1$
   (iii) *for* $\ell = 0, 1$ *the pair* $\bar{s}_\ell \in \mathbb{R} \times \mathbb{R}$ *belongs to the convex hull of* $V^*_\ell(d^*)$ *for some* $d^* \in \mathfrak{D}$*, see Definition 1.8 (4), but recall that by a hypothesis of the claim, the choice of* d^* *is immaterial*
(g) *for some* ($d^* \in \mathfrak{D}$ *and)* $x \in X$ *we have* $\mathrm{val}_{d^*}(x) \neq \frac{\mathbf{n}-1}{2}$.

Proof. $(\mathrm{b})_{\mathrm{x,y}} \Leftrightarrow (\mathrm{b})_{\mathrm{x',y'}}$:
(So $x, y, x', y' \in X$ and $x \neq y, x' \neq y'$). Trivial as $\mathfrak{D}$ is closed under permutations of X hence so is $\text{pr-cl}(\mathfrak{D})$.
$(\mathrm{b})_{\mathrm{x,y}} \Rightarrow (\mathrm{a})'$:
Let $c \in \mathfrak{C}$.
Let $\{(u_i, v_i) : i < i(*)\}$ without repetitions list the pairs (u, v) of distinct members of X such that $c\{u, v\} = v$; clearly $i(*) \leq \binom{|X|}{2}$ and $c \in \mathfrak{C}^{\text{full}} \Rightarrow i(*) = \binom{|X|}{2}$. For each $i < i(*)$ as $(b)_{x,y} \Rightarrow (b)_{u_i,v_i}$ clearly there is $\bar{t}^i \in \text{pr-cl}(\mathfrak{D})$ such that

$$t^i_{u_i,v_i} > \frac{1}{2} \quad \text{so } t^i_{v_i,u_i} = 1 - t_{u_i,v_i} < \frac{1}{2}$$

$$\{u_i, v_i\} \neq \{u, v\} \in \binom{X}{2} \Rightarrow t^i_{u,v} = \frac{1}{2}.$$

Let $\bar{t}^* = \langle t^*_{u,v} : u \neq v \in X\rangle$ be defined by

$$t^*_{u,v} = \Sigma\{t^i_{u,v} : i < i(*)\}/i(*).$$

As $\text{pr-cl}(\mathfrak{D})$ is convex and $i < i(*) \Rightarrow \bar{t}^i \in \text{pr-cl}(\mathfrak{D})$ clearly $\bar{t}^* \in \text{pr-cl}(\mathfrak{D})$. Now for each $j < i(*)$, $t^i_{u_j,v_j}$ is $\frac{1}{2}$ if $i \neq j$ and is $> \frac{1}{2}$ if $i = j$. Hence $t^*_{u_j,v_j}$ being the average of $\langle t^i_{u_j,v_j} : i < i(*)\rangle$ is $> \frac{1}{2}$. Hence $t^*_{v_j,u_j} = 1 - t^*_{u_j,v_j} < \frac{1}{2}$. So by the choice of $\langle (u_i, v_i) : i < i(*)\rangle$ we have

$c\{u,v\} = v \Rightarrow t^*_{u,v} > \frac{1}{2}$ hence $c\{u,v\} = u \Rightarrow t^*_{u,v} < \frac{1}{2}$. Now lastly $c\{u,v\}$ undefined $\Rightarrow \bigwedge_i t^i_{u,v} = \frac{1}{2} \Rightarrow t^*_{u,v} = \frac{1}{2}$. So $\bar{t}^*$ witness $c \in$ maj-cl$(\mathfrak{D})$ as required in clause (a)$'$.

(a)$' \Rightarrow$ (a):

Trivial.

(a) $\Rightarrow$ (b)$_{x,y}$:

By clause (a), for every $d \in \mathfrak{C}^{\rm full}$ there is $\langle r_c : c \in \mathfrak{D}\rangle$ as in Definition 1.3, hence for some $\varepsilon_d > 0$, $u \neq v \in X \wedge d\{u,v\} = v \Rightarrow \frac{1}{2} + \varepsilon_d < \Sigma\{r_c : c \in \mathfrak{D}$ and $c\{u,v\} = v\}$. Hence $\varepsilon = \operatorname{Min}\{\varepsilon_d : d \in \mathfrak{D}\}$ is a real > 0.

Let $T = \{\bar{t} : \bar{t} \in$ pr-cl$(\mathfrak{D})$ and $t_{x,y} \geq \frac{1}{2} + \varepsilon\}$, so

$(*)_1$ $T \neq \emptyset$

[Why? By the choice of ε and recall that $\mathfrak{D}$ is symmetric]

$(*)_2$ T is convex and closed.

[Why? Trivial.]

For $\bar{t} \in T$ define

⊠ $\operatorname{err}(\bar{t}) = \max\{|t_{u,v} - \frac{1}{2}| : u \neq v \in X$ and $\{u,v\} \neq \{x,y\}\}$

$(*)_3$ if $\bar{t} \in T$, $\operatorname{err}(\bar{t}) > 0$ then we can find $\bar{t}' \in T$ such that $\operatorname{err}(\bar{t}') \leq \operatorname{err}(\bar{t})(1 - \operatorname{err}(\bar{t}))$ and $t'_{x,y} \geq (t_{x,y} + \frac{1}{2} + \varepsilon)/2 \geq \frac{1}{2} + \varepsilon$.

Why? Choose $d \in \mathfrak{C}^{\rm full}$ such that $d\{x,y\} = y$ and

$$u \neq v \in X \quad \& \quad \{u,v\} \neq \{x,y\} \quad \& \quad t_{u,v} > \frac{1}{2} \Rightarrow d\{u,v\} = u$$

(so if $t_{u,v} = t_{v,u} = \frac{1}{2}$ it does not matter what is $d\{u,v\}$; such d exists trivially).

So d is "a try to correct $\bar{t}$".

As we are assuming clause (a) and by the choice of ε_d, we can find $\bar{r}^* = \langle r^*_c : c \in \mathfrak{D}\rangle$ with $r^*_c \in [0,1]_{\mathbb{R}}$ and $1 = \Sigma\{r^*_c : c \in \mathfrak{D}\}$ such that

$$\frac{1}{2} + \varepsilon_d < \Sigma\{r^*_c : c \in \mathfrak{D} \text{ and } c\{x,y\} = y\}$$

and if $u \neq v$ are from X and $\{u,v\} \neq \{x,y\}$ then

$$d\{u,v\} = v \Rightarrow \frac{1}{2} < \Sigma\{r^*_c : c \in \mathfrak{D} \text{ and } c\{u,v\} = v\}$$

hence

$$d\{u,v\} = u \Rightarrow \frac{1}{2} > \Sigma\{r^*_c : c \in \mathfrak{D} \text{ and } c\{u,v\} = v\}.$$

By the choice of ε without loss of generality $\frac{1}{2} + \varepsilon < \Sigma\{r^*_c : c \in \mathfrak{D}$ and $c\{x,y\} = y\}$. Let $\bar{s} = \langle s_{u,v} : u \neq v \in X\rangle$ be defined by $s_{u,v} = \Sigma\{r^*_c : c\{u,v\} = v\}$, so

$\circledast_1$ (i) $\bar{s} \in$ pr-cl$(\mathfrak{D})$

(ii) $s_{x,y} > \frac{1}{2} + \varepsilon$ (so $s_{y,x} < \frac{1}{2}$)

(iii) if $t_{u,v} > \frac{1}{2}$ and $u \neq v \in X$, $\{u,v\} \neq \{x,y\}$ then $d\{u,v\} = u$ hence $s_{u,v} < \frac{1}{2}$

(iv) if $t_{u,v} < \frac{1}{2}$ and $u \neq v \in X$, $\{u,v\} \neq \{x,y\}$ then $d\{u,v\} = v$ hence $s_{u,v} > \frac{1}{2}$.

Choose $\delta \in (0,1)_{\mathbb{R}}$ as $\operatorname{err}(\bar{t})$. Let $\bar{t}' = (1-\delta)\bar{t} + \delta\bar{s}$, i.e. $t'_{u,v} = ((1-\delta)t_{u,v} + \delta s_{u,v})$ so clearly

$\circledast_2$ (i) $\bar{t}' \in$ pr-cl$(\mathfrak{D})$

(ii) $t'_{x,y} \geq \frac{1}{2} + \varepsilon$

(iii) if $u \neq v \in X$, $\{u,v\} \neq \{x,y\}$ then $|t'_{u,v} - \frac{1}{2}| \leq \operatorname{err}(\bar{t})(1 - \operatorname{err}(\bar{t}))$

(iv) $\bar{t}' \in T$.

[Why? Clause (i) as pr-cl$(\mathfrak{D})$ is convex. Clause (ii) as easily $t'_{x,y} = ((1-\delta)t_{x,y} + \delta s_{x,y})$, but $t_{x,y} \geq \frac{1}{2} + \varepsilon$ as $\bar{t} \in T$ and $s_{x,y} \geq \frac{1}{2} + \varepsilon$ by $\circledast_1$(ii). Now the main point, for clause (iii) note that $t'_{u,v} - \frac{1}{2} = -(t'_{v,u} - \frac{1}{2})$ so as $d \in \mathfrak{C}^{\rm full}$ without loss of generality $d\{u,v\} = u$ hence $t_{u,v} \geq \frac{1}{2}$, hence by the choice of d we have $s_{u,v} \leq \frac{1}{2}$ and both are in $[0,1]_{\mathbb{R}}$ and:

$$\left|t'_{u,v} - \frac{1}{2}\right| = \left|(1-\delta)\left(t_{u,v} - \frac{1}{2}\right) + \delta\left(s_{u,v} - \frac{1}{2}\right)\right| \leq \max_{s \in [0,\frac{1}{2}]_{\mathbb{R}}} \left|(1-\delta)\left(t_{u,v} - \frac{1}{2}\right) + \delta\left(s - \frac{1}{2}\right)\right|$$

$$= \operatorname{Max}\left\{\left|(1-\delta)\left(t_{u,v} - \frac{1}{2}\right) + \delta\left(\frac{1}{2} - \frac{1}{2}\right)\right|, \left|(1-\delta)\left(t_{u,v} - \frac{1}{2}\right) + \delta\left(0 - \frac{1}{2}\right)\right|\right\}$$

$$= \text{Max}\left\{\left|(1-\delta)\left(t_{u,v}-\frac{1}{2}\right)\right|, \left|(1-\delta)\left(t_{u,v}-\frac{1}{2}\right)-\frac{1}{2}\delta\right|\right\}$$
$$\leq \text{Max}\left\{(1-\delta)\left(t_{u,v}-\frac{1}{2}\right), (1-\delta)\left(t_{u,v}-\frac{1}{2}\right), \frac{1}{2}\delta\right\}$$
$$\leq \text{Max}\left\{(1-\delta)\,\text{err}(\bar{t}), (1-\delta)\,\text{err}(\bar{t}), \frac{1}{2}\delta\right\} \leq \text{err}(\bar{t})(1-\text{err}(\bar{t}))$$

(recalling $\delta = \text{err}(\bar{t}) \in [0, \frac{1}{2}]_{\mathbb{R}}$ so $\frac{1}{2}\delta \leq \text{err}(\bar{t})(1-\text{err}(\bar{t}))$ as required), so clause (iii) holds.

Clause (iv) follows. So $\circledast_2$ holds.]

So we are done proving $(*)_3$.

As T is closed (and is included in a $\{\bar{t} : \bar{t} = \langle t_{u,v} : u \neq v \in X\rangle$ and $0 \leq t_{u,v} \leq 1\}$ which is compact), clearly there is $\bar{t} \in T$ such that $u \neq v \in X$ & $\{u, v\} \neq \{x, y\} \Rightarrow t_{u,v} = \frac{1}{2}$ as required in part (ii) of $(\text{b})_{x,y}$.

(c)′ ⇒ (c):

Trivial.

(c) ⇒ (a):

Let $d^* \in \mathfrak{C}^{\text{full}}$ and let $\langle c_j : j \in J\rangle$ witness clause (c) for d^*.

Let $r_c = |\{j \in J : c_j = c\}|/|J|$ now $\langle r_c : c \in \mathfrak{D}\rangle$ witness clause (a), i.e., witness that $d^* \in \text{maj-cl}(\mathfrak{D})$.

(a) ⇒ (c):

Let $d^* \in \mathfrak{C}^{\text{full}}$ and let $\langle r_c : c \in \mathfrak{D}\rangle$ be as guaranteed for d^* by clause (a). Let $n(*) > 0$ be large enough such that $|\frac{1}{2} - r_c| > \frac{1}{n(*)}$ for $c \in \mathfrak{K}$ and for $c \in \mathfrak{D}$ let $k_c \in \{0, \ldots, n(*) - 1\}$ be such that $c \in \mathfrak{D} \Rightarrow k_c \leq n(*) \times r_c < k_c + 1$; note that k_c exists as $r_c \in [0, 1]_{\mathbb{R}}$. As $\sum_c \frac{k_c}{n(*)} \leq 1 \leq \sum_c \frac{k_c+1}{n(*)}$, we can choose $m_c \in \{k_c, k_c + 1\}$ such that $r'_c = \frac{m_c}{n(*)}$ satisfies $\Sigma\{r'_c : c \in \mathfrak{D}\} = 1$. Let $J = \{(c, m) : c \in \mathfrak{D}$ and $m \in \{1, \ldots, m_c\}\}$ and we let $c_{(d,m)} = d$ for $(d, m) \in J$. Now the "majority" of $\langle c_t : t \in J\rangle$, see Definition 1.3, choose d^* so clause (c) holds.

(a)′ ⇒ (c)′:

Similar, using: if a finite set of equalities and inequalities with rational coefficients is solvable in $\mathbb{R}$ then it is solvable in $\mathbb{Q}$.

Before we deal with clauses (d), (e), (f) and (g) of 2.1, we define

Definition 2.2. (1) For $\mathscr{D} \subseteq \mathfrak{C}^{\text{full}}$ and $A \subseteq [0, 1]_{\mathbb{R}}$ let $\text{Pr}_A(\mathscr{D})$ be the set of pairs (s_0, s_1) of real numbers $\in [0, 1]_{\mathbb{R}}$ such that for some $\bar{t} \in \text{pr-cl}(\mathscr{D})$ and $x \neq y \in X$ and $a \in A$ we have $\bar{t} = \bar{t}\langle x, y, a, s_0, s_1\rangle$ where

(2) $\bar{t} = \bar{t}\langle x, y, a, s_0, s_1\rangle$ where $x \neq y \in X$, $a \in [0, 1]_{\mathbb{R}}$ and $s_0, s_1 \in [0, 1]_{\mathbb{R}}$ and $\bar{t} = \langle t_{u,v} : u \neq v \in X\rangle \in \text{pr}(\mathfrak{C})$ is defined by

(α) $t_{x,y} = a$
(β) if $z \in X \setminus \{x, y\}$ <u>then</u> $t_{y,z} = s_1$ (hence $t_{z,y} = 1 - s_1$)
(γ) if $z \in X \setminus \{x, y\}$ <u>then</u> $t_{x,z} = s_0$ (hence $t_{z,x} = 1 - s_0$)
(δ) if $z_1 \neq z_2 \in X \setminus \{x, y\}$ <u>then</u> $t_{z_1,z_2} = \frac{1}{2}$.

(3) In $\text{Pr}_A(\mathscr{D})$ we may replace A by $1, 0, \neq \frac{1}{2}, > \frac{1}{2}, < \frac{1}{2}$ if A is $\{1\}, \{0\}, [0, 1]_{\mathbb{R}} \setminus \{\frac{1}{2}\}, (\frac{1}{2}, 1]_{\mathbb{R}}, [0, \frac{1}{2})_{\mathbb{R}}$ respectively.
(4) For $\ell \in \{0, 1\}$ and $\mathscr{D} \subseteq \mathfrak{C}^{\text{full}}$ let $\text{Prd}_\ell(\mathscr{D})$ be the set of pairs $\bar{s} = (s_0, s_1)$ of real (actually rational) numbers $\in [0, 1]_{\mathbb{R}}$ such that for some $c \in \mathscr{D}$ and $x \neq y \in X$ we have $\bar{s} = \bar{s}^{c,x,y} = (s_0^{c,x,y}, s_1^{c,x,y})$ where

(i) $s_1^{c,x,y} = |\{z : z \in X \setminus \{x, y\}$ and $c\{y, z\} = z\}|/(\mathbf{n} - 2)$
(ii) $s_0^{c,x,y} = |\{z : z \in X \setminus \{x, y\}$ and $c\{x, z\} = z\}|/(\mathbf{n} - 2)$
(iii) $\ell = 1 \Leftrightarrow \ell \neq 0 \Leftrightarrow c\{x, y\} = y$.

(5) For $\mathscr{D} \subseteq \mathfrak{C}^{\text{full}}$ let $\text{Prd}(\mathscr{D})$ be $\text{Prd}_0(\mathscr{D}) \cup \text{Prd}_1(\mathscr{D})$.

Claim 2.3. *Let* $\mathscr{D} \subseteq \mathfrak{C}^{\text{full}}$

(1) $\text{Pr}_{A_1}(\mathscr{D}_1) \subseteq \text{Pr}_{A_2}(\mathscr{D}_2)$ *if* $A_1 \subseteq A_2 \subseteq [0, 1]_{\mathbb{R}}$ *and* $\mathscr{D}_1 \subseteq \mathscr{D}_2 \subseteq \mathfrak{C}^{\text{full}}$
(2) $\text{Pr}_A(\mathscr{D})$ *is a convex subset of* $[0, 1]_{\mathbb{R}} \times [0, 1]_{\mathbb{R}}$ *when A is a convex subset of* $[0, 1]_{\mathbb{R}}$
(3) $\text{Prd}_\ell(\mathscr{D})$ *is finite and its convex hull is* $\subseteq \text{Pr}_\ell(\mathscr{D})$, *increasing with* $\mathscr{D}(\subseteq \mathfrak{C}^{\text{full}})$ *for* $\ell = 0, 1$
(4) *For* $x \neq y \in X$ *and* $c \in \mathfrak{C}^{\text{full}}$ *satisfying* $\ell = 1 \Rightarrow c\{x, y\} = y$ *and* $\ell = 0 \Rightarrow c\{x, y\} = x$ *we have (see* Definition 1.8(1)*)*

$$s_0^{c,x,y} = (\text{val}_c(x) - \ell)/(\mathbf{n} - 2)$$
$$s_1^{c,x,y} = (\text{val}_c(y) - (1 - \ell))/(\mathbf{n} - 2).$$

(5) $\text{Pr}_{A_1 \cup A_2}(\mathscr{D}) = \text{Pr}_{A_1}(\mathscr{D}) \cup \text{Pr}_{A_2}(\mathscr{D})$, *in fact* $\text{Pr}_A(\mathscr{D}) = \cup\{\text{Pr}_{\{a\}}(\mathscr{D}) : a \in A\}$.

Proof. Immediate; part (3) holds by 2.4(1) below concerning part (4) recall Definition 1.8(1). □

Claim 2.4. (1) *If $x \neq y \in X$ and $c \in \mathfrak{C}$ and $\ell \in \{0,1\}$ satisfies $\ell = 1 \Rightarrow c\{x,y\} = y$ and $\ell = 0 \Rightarrow c\{x,y\} = x$ recalling Definitions 1.5(6) and 1.2(1) we have: $\bar t\langle x, y, \ell, s_0^{c,x,y}, s_1^{c,x,y}\rangle = \frac{1}{|\Pi_{x,y}|}\Sigma\{\bar t[\hat\pi(c)] : \pi \in \Pi_{x,y}\}$ where $\Pi_{x,y} := \{\pi \in \mathrm{Per}(X) : \pi(x) = x$ and $\pi(y) = y\}$ hence $|\Pi_{x,y}| = (\mathbf{n}-2)!$.*

(2) *If $\mathscr{D} \subseteq \mathfrak{C}$ is symmetric, $x \neq y \in X$ and $\bar t \in \text{pr-cl}(\mathscr{D})$ and $\bar t^* = \Sigma\{\bar t^\pi : \pi \in \Pi_{x,y}\}/|\Pi_{x,y}|$ where $\bar t^\pi = \langle t^\pi_{u,v} : u \neq v \in X\rangle$, $t^\pi_{u,v} = t_{\pi(u),\pi(v)}$ then $\bar t^* \in \text{pr-cl}(\mathscr{D})$ and $(s_0, s_1) \in \mathrm{Pr}_{\{a\}}(\mathscr{D})$ where $a = t_{x,y}$, $s_0 = \Sigma\{t_{x,z} : z \in X \setminus \{x,y\}\}/(\mathbf{n}-2)$ and $s_1 = \Sigma\{t_{y,z} : z \in X \setminus \{x,y\}\}/(\mathbf{n}-2)$.*

Proof. Easy (in part (2), $\bar t^*$ witness $(s_0, s_1) \in \mathrm{Pr}_{\{a\}}(\mathscr{D})$). □

Claim 2.5. *For any symmetric non-empty $\mathscr{D} \subseteq \mathfrak{C}^{\mathrm{full}}$ (i.e., closed under permutations of X):*

(1) *For $\ell \in \{0,1\}$, the set $\mathrm{Pr}_\ell(\mathscr{D})$ is the convex hull of $\mathrm{Prd}_\ell(\mathscr{D})$ in $\mathbb{R} \times \mathbb{R}$.*

(2) *$\mathrm{Pr}_{[0,1]}(\mathscr{D})$ is the convex hull of $\mathrm{Prd}(\mathscr{D})$.*

(3) *Let $a \in [0,1]_\mathbb{R}$ and $s_0^*, s_1^* \in [0,1]_\mathbb{R}$. Then $\bar t^* = \bar t\langle x, y, a, s_0^*, s_1^*\rangle \in \text{pr-cl}(\mathscr{D})$ iff we can find $\langle r_{\bar s,\ell} : \ell \in \{0,1\}$ and $\bar s \in \mathrm{Prd}_\ell(\mathscr{D})\rangle$ such that $r_{\bar s,\ell} \in [0,1]_\mathbb{R}$ and $1 = \Sigma\{r_{\bar s,\ell} : \ell \in \{0,1\}, \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\}$ and $(s_0^*, s_1^*) = \Sigma\{r_{\bar s,\ell} \times \bar s : \ell \in \{0,1\}, \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\}$ and $a = \Sigma\{r_{\bar s,1} : \bar s \in \mathrm{Prd}_1(\mathscr{D})\}$.*

Proof. (1) By 2.3(3) we have one inclusion.

For the other direction assume $(s_0^*, s_1^*) \in \mathrm{Pr}_\ell(\mathscr{D})$ and we should prove that the pair (s_0^*, s_1^*) belongs to the convex hull of $\mathrm{Prd}_\ell(\mathscr{D})$. Fix $x \neq y \in X$ and let $\bar t^* = \bar t\langle x, y, \ell, s_0^*, s_1^*\rangle$ so

$\boxtimes_1$ $\bar t^* = \langle t^*_{u,v} : u \neq v \in X\rangle$ is defined as follows $t^*_{x,y} = \ell$, $t^*_{u,v} = \frac12$ if $u \neq v \in X \setminus \{x,y\}$, $t^*_{y,z} = s_1^*$ if $z \in X \setminus \{x,y\}$, $t^*_{x,z} = s_0^*$ if $z \in X \setminus \{x,y\}$.

As $(s_0^*, s_1^*) \in \mathrm{Pr}_\ell(\mathscr{D})$ by Definition 2.2 we know that $\bar t^* \in \text{pr-cl}(\mathscr{D})$ and let $\bar r = \langle r_c : c \in \mathscr{D}\rangle$ be such that

$\boxtimes_2$ $x, y, \bar r$ witness that $\bar t^* \in \text{pr-cl}(\mathscr{D})$, so $r_c \geq 0$ and $1 = \Sigma\{r_c : c \in \mathscr{D}\}$ and

$$\bar t^* = \Sigma\{r_c \times \bar t[c] : c \in \mathscr{D}\}.$$

As $t^*_{x,y} = \ell$, necessarily

$\boxtimes_3$ $r_c \neq 0 \Rightarrow c \in \mathscr{D}^\ell_{x,y} := \{c \in \mathscr{D} : (\ell = 1 \Rightarrow c\{x,y\} = y)$ and $(\ell = 0 \Rightarrow c\{x,y\} = x)\}$.

To make the rest of the proof also a proof of part (3) let $a = \ell$ (as the real number a may be $\neq 0, 1$; in any case we use $m \in \{0,1\}$ below).

Let $\Pi_{x,y} = \{\pi \in \mathrm{Per}(X) : \pi(x) = x, \pi(y) = y\}$ and recall that for $\pi \in \mathrm{Per}(X)$, $\hat\pi$ is the permutation of $\mathfrak{C}$ which π induces, defined in 1.2, so $\hat\pi$ maps $\mathscr{D}^\ell_{x,y}$ onto $\mathscr{D}^\ell_{x,y}$ if $\pi \in \Pi_{x,y}$ recalling we have assumed that $\mathscr{D}$ is symmetric. Clearly $|\Pi_{x,y}| = (\mathbf{n}-2)!$; recall that $\bar t^{**} = \bar t^*[\hat\pi[c]]$ if $t^{**}_{u,v} = t^*_{\pi(u),\pi(v)}$.

For $(s_0, s_1) \in \mathrm{Prd}_\ell(\mathscr{D}) \subseteq \{(\frac{m_1}{(\mathbf{n}-2)!}, \frac{m_2}{(\mathbf{n}-2)!}) : m_1, m_2 \in \{0, 1, \ldots, (\mathbf{n}-2)!\}\}$ let

$\boxtimes_4$ $r^*_{(s_0,s_1)} = \Sigma\{r_c : \bar s^{c,x,y} = (s_0, s_1)\}$ and for $m \in \{0,1\}$ we let $r^*_{(s_0,s_1),m} = \Sigma\{r_c : \bar s^{c,x,y} = (s_0,s_1)$ and $m = 1 \Rightarrow c\{x,y\} = y$ and $m = 0 \Rightarrow c\{x,y\} = x\}$.

Clearly $\pi \in \Pi_{x,y} \Rightarrow \langle t^*_{\pi(u),\pi(v)} : u \neq v \in X\rangle = \bar t^*$, just check the definition, hence (by the beginning of this sentence; by the equation in $\boxtimes_2$; by arithmetic; by 2.4(1); as $\langle\{c \in \mathscr{D} : m = 1 \Rightarrow c\{x,y\} = y$ and $m = 0 \Rightarrow c\{x,y\} = x$ and $\bar s^{c,x,y} = (s_0, s_1)\} : m \in \{0,1\}$ and $(s_0, s_1) \in \mathrm{Prd}_m(\mathscr{D})\rangle$ is a partition of $\mathscr{D}$; by the choice of $r^*_{(s_0,s_1)}$ in $\boxtimes_4$) we have:

$\boxtimes_5$

$$\begin{aligned}
\bar t^* &= \frac{1}{|\Pi_{x,y}|}\sum_{\pi\in\Pi_{x,y}}\langle t^*_{\pi(u),\pi(v)} : u \neq v \in X\rangle = \frac{1}{|\Pi_{x,y}|}\sum_{\pi\in\Pi_{x,y}}\sum_{c\in\mathscr{D}} r_c \times \bar t^*[\hat\pi(c)] \\
&= \sum_{c\in\mathscr{D}} r_c\left(\frac{1}{|\Pi_{x,y}|}\sum_{\pi\in\Pi_{x,y}}\bar t^*[\hat\pi(c)]\right) = \sum_{c\in\mathscr{D}} r_c \times \bar t\langle x, y, a, s_0^{c,x,y}, s_1^{c,x,y}\rangle \\
&= \sum_{m\in\{0,1\}}\sum_{(s_0,s_1)\in\mathrm{Prd}_m(\mathscr{D})}(\Sigma\{r_c : c \in \mathscr{D}, m = 1 \Rightarrow c\{x,y\} \\
&= y \text{ and } m = 0 \Rightarrow c\{x,y\} = x \text{ and } \bar s^{c,x,y} = (s_0,s_1)\}) \times \bar t\langle x, y, m, s_0, s_1\rangle \\
&= \sum_{m\in\{0,1\}}\sum_{(s_0,s_1)\in\mathrm{Prd}_m(\mathscr{D})} r^*_{(s_0,s_1),m}\bar t\langle x, y, m, s_0, s_1\rangle.
\end{aligned}$$

Now concentrate again on the case $a = \ell \in \{0,1\}$, so $r^*_{\bar s,1-\ell} = 0$ by $\boxtimes_3$ and $r^*_{\bar s,\ell} = r^*_{\bar s}$. So clearly

$\circledast_1$ $r^*_{\bar s} \geq 0$

[Why? As the sum of non-negative reals]

$\circledast_2$ $1 = \Sigma\{r^*_{\bar s} : \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\}$

[Why? As by Definition 2.2(2), $c \in \mathscr{D}$ & $r_c > 0 \Rightarrow c \in \mathscr{D}^\ell_{x,y} \Rightarrow \bar s^{c,x,y} \in \mathrm{Prd}_\ell(\mathscr{D})$ and the definition of $r^*_{\bar s}$]

$\circledast_3$ we have

(α) $z \in X \setminus \{x, y\} \Rightarrow s^*_1 = t^*_{y,z} = \Sigma\{r^*_{(s_0,s_1)} \times s_1 : (s_0, s_1) \in \mathrm{Prd}_\ell(\mathscr{D})\}$,

(β) $z \in X \setminus \{x, y\} \Rightarrow s^*_0 = t^*_{x,z} = \Sigma\{r^*_{(s_0,s_1)} \times s_0 : (s_0, s_1) \in \mathrm{Prd}_\ell(\mathscr{D})\}$.

[Why? By $\boxtimes_5$ and the definition of $\bar t\langle x, y, m, s_0, s_1\rangle$.]

So $\langle r^*_{\bar s} : \bar s \in \mathrm{Prd}_\ell(\mathscr{D})\rangle$ witness that $(s^*_0, s^*_1) \in$ convex hull of $\mathrm{Prd}_\ell(\mathscr{D})$.

(2) Similar proof (and not used).

(3) One direction is as in 2.3(3). For the other, by the hypothesis, $\boxtimes_1$ in the proof of part (1) with ℓ replaced by a holds. So by the part of the proof of part (1) from $\boxtimes_2$ till (and including) $\boxtimes_5$ we know that $r^*_{\bar s,m}$ are defined and $\boxtimes_5$ holds. So

$\boxdot_1$ $r^*_{\bar s,m} \ge 0$

$\boxdot_2$ $1 = \Sigma\{r^*_{\bar s,m} : m \in \{0, 1\}$ and $\bar s \in \mathrm{Prd}_m(\mathscr{D})\}$

$\boxdot_3$ $(s^*_0, s^*_1) = \Sigma\{r^*_{\bar s,m} \times \bar s : m \in \{0, 1\}, \bar s \in \mathrm{Prd}_m(\mathscr{D})\}$

[Why? By $\boxtimes_5$.]

$\boxdot_4$ $a = \Sigma\{r^*_{\bar s,1} : \bar s \in \mathrm{Prd}_1(\mathscr{D})\}$.

[Why? Use $\boxtimes_5$, noting that $t^*_{x,y} = a$.]

So we are done. □

Continuation of the proof of 2.1:

(d) ⇔ (b)$_{x,y}$:

Read Definition 2.2(1) (and the symmetry). □

(d) ⇒ (e):

By 2.3(1).

(e) ⇒ (d):

Why? If clause (e) holds, for some $a \in [0, 1]_\mathbb{R} \setminus \{\frac12\}$ and $x \ne y \in X$ we have $\bar t^* =: \bar t\langle x, y, a, \frac12, \frac12\rangle \in$ pr-cl($\mathfrak{D}$). If $a > \frac12$ this witness $(\frac12, \frac12) \in \mathrm{Pr}_{>1/2}(\mathscr{D})$, so assume $a < \frac12$. But trivially $\bar t\langle y, x, 1 - a, \frac12, \frac12\rangle$ is equal to $\bar t^*$ hence (as in 1.9) is in pr-cl($\mathfrak{D}$) and by symmetry we are done.

(e) ⇔ (f):

Clearly (e) means that

$(*)_0$ there are $r_c \in [0, 1]_\mathbb{R}$ for $c \in \mathfrak{D}$ such that $1 = \Sigma\{r_c : c \in \mathfrak{D}\}$ and $a \in [0, 1]_\mathbb{R} \setminus \{\frac12\}$ such that $\bar t\langle x, y, a, \frac12, \frac12\rangle = \Sigma\{r_c \times \bar t[c] : c \in \mathfrak{D}\}$.

By 2.5(3) we know that $(*)_0$ is equivalent to

$(*)_1$ there are $r_{\bar s,\ell} \in [0, 1]_\mathbb{R}$ for $\bar s \in \mathrm{Prd}_\ell(\mathfrak{D})$, $\ell \in \{0, 1\}$ such that

(i) $1 = \Sigma\{r_{\bar s,\ell} : \bar s \in \mathrm{Prd}_\ell(\mathfrak{D}), \ell \in \{0, 1\}\}$

(ii) $(\frac12, \frac12) = \Sigma\{r_{\bar s,\ell} \times \bar s : \bar s \in \mathrm{Prd}_\ell(\mathfrak{D}), \ell \in \{0, 1\}\}$

(iii) $\frac12 \ne a = \Sigma\{r_{\bar s,1} : \bar s \in \mathrm{Prd}_1(\mathfrak{D})\}$.

But by 2.3(4) and Definition 2.2(4) for $\ell \in \{0, 1\}$:

$$\mathrm{Prd}_\ell(\mathfrak{D}) = \left\{\left(\frac{\mathrm{val}_c(x) - \ell}{\mathbf{n} - 2}, \frac{\mathrm{val}_c(y) - (1 - \ell)}{\mathbf{n} - 2}\right) : c \in \mathfrak{D} \text{ and } x \ne y \text{ and } (\ell = 1 \Rightarrow c\{x, y\} = y \text{ and } \ell = 0 \Rightarrow c\{x, y\} = x)\right\}.$$

Let $d^* \in \mathfrak{D}$, recall that $\mathfrak{D} =$ sym-cl$(\{d^*\})$ by a hypothesis of 2.1 and recall $V_\ell(d^*) = \{(k_1, k_2) :$ for some $x_1 \ne x_2 \in X, k_1 = \mathrm{val}_{d^*}(x_1), k_2 = \mathrm{val}_{d^*}(x_2)$ and $d^*\{x_1, x_2\} = x_{\ell+1}\}$ for $\ell = 0, 1$. So $(*)_1$ means (recalling the definition of $\mathrm{Prd}_\ell(\mathfrak{D})$)

$(*)_2$ there is a sequence $\langle r_{\bar k,\ell} : \bar k \in V_\ell(d^*)$ and $\ell \in \{0, 1\}\rangle$ such that

(i) $r_{\bar k,\ell} \in [0, 1]_\mathbb{R}$ and

(ii) $1 = \Sigma\{r_{\bar k,\ell} : \bar k \in V_\ell(d^*)$ and $\ell \in \{0, 1\}\}$ and

(iii) $(\frac12, \frac12) = \Sigma\{r_{(k_1,k_2),\ell} \times \left(\frac{k_1-\ell}{\mathbf{n}-2}, \frac{k_2-(1-\ell)}{\mathbf{n}-2}\right) : \ell \in \{0, 1\}$ and $(k_1, k_2) \in V_\ell(d^*)\}$

(iv) $\frac12 \ne \Sigma\{r_{\bar k,1} : \bar k \in V_1(d^*)\}$.

Let us analyze $(*)_2$. Let $r^*_\ell = \Sigma\{r_{\bar k,\ell} : \bar k \in V_\ell(d^*)\}$ for $\ell \in \{0, 1\}$. So

$\circledast_1$ $r^*_\ell \in [0, 1]_\mathbb{R}$ and $1 = r^*_0 + r^*_1$.

Now clause (iii) of $(*)_2$ means (iii)$_1$ + (iii)$_2$ where (iii)$_1$ $\frac{1}{2} = \frac{1}{\mathbf{n}-2}(\Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\}) - \frac{1}{\mathbf{n}-2}(\Sigma\{r_{\bar{k},\ell} \times \ell : \bar{k} \in V_\ell(d^*), \ell \in \{0,1\}\}) = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\} - \frac{r_1^*}{\mathbf{n}-2}$, equivalently i.e.,

(iii)$'_1$ $\frac{\mathbf{n}}{2} - (1 - r_1^*) = \frac{\mathbf{n}-2}{2} + r_1^* = \Sigma\{r_{\bar{k},\ell} \times k_1 : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\}$ equivalently and

(iii)$_2$ $\frac{1}{2} = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\} - \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times (1-\ell) : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\} = \frac{1}{\mathbf{n}-2}\Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*), \ell \in \{0,1\}\} - \frac{r_0^*}{\mathbf{n}-2}$, i.e.,

(iii)$'_2$ $\frac{\mathbf{n}}{2} - (1 - r_0^*) = \frac{\mathbf{n}-2}{2} + r_0^* = \Sigma\{r_{\bar{k},\ell} \times k_2 : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\}$.

Together (iii) of $(*)_2$ is equivalent to

(iii)$^+$ $\left(\frac{\mathbf{n}}{2} - (1-r_1^*), \frac{\mathbf{n}}{2} - (1-r_0^*)\right) = \Sigma\{r_{\bar{k},\ell} \times \bar{k} : \bar{k} \in V_\ell(d^*)$ and $\ell \in \{0,1\}\}$.

Let $\bar{s}_\ell = \Sigma\{r_{\bar{k},\ell} \times \bar{k} : \bar{k} \in V_\ell(d^*)\}/r_\ell^*$ if $r_\ell^* > 0$ and any member of $\mathrm{conv}(V_\ell(d^*))$ if $r_\ell^* = 0$, so $(*)_2$ is equivalent to ($V_\ell(d^*)$ is from Definition 1.8)

$(*)_3$ there are $\bar{s}_0, \bar{s}_1, r_0^*, r_1^*$ such that
- (i) $\bar{s}_\ell \in \mathrm{conv}(V_\ell(d^*))$ for $\ell = 0, 1$
- (ii) $r_0^*, r_1^* \in [0,1]_{\mathbb{R}}$ and $1 = r_0^* + r_1^*$
- (iii) $(\frac{\mathbf{n}}{2} - (1-r_1^*), \frac{\mathbf{n}}{2} - (1-r_0^*))$ is $r_0^* \times \bar{s}_0 + r_1^* \times \bar{s}_1$
- (iv) $r_\ell^* \neq \frac{1}{2}$ (by clause (iv) in $(*)_2$ above).

Clearly $(*)_3$(iii) is equivalent to

(iii)$'$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ is $r_0^* \times (\bar{s}_0 - (0,1)) + r_1^* \times (\bar{s}_1 - (1,0))$.

So $(*)_3$ is equivalent to

$(*)_4$ clauses (i), (ii), (iv) of $(*)_3$ and (iii)$'$ above holds.

But recalling Definition 1.8(4) of $V_\ell^*(d^*)$, this is clause (f), so we are done proving (e) $\Leftrightarrow$ (f).

(g) $\Rightarrow$ (f): By 2.6, 2.8 and 2.9 (i.e., they show (g) + $\neg$(f) lead to a contradiction).

(f) $\Rightarrow$ (g): It suffices to prove $\neg$(g) $\Rightarrow$ $\neg$(f). This holds trivially as $\neg$(g) implies $(s_0, s_1) \in \mathrm{conv}(V_\ell(d^*)) \Rightarrow s_0 = s_1$.

We have proved $(b)_{x,y} \Leftrightarrow (b)_{x',y'}$, $(b)_{x,y} \Rightarrow (a)' \Rightarrow (a) \Rightarrow (b)_{x,y}$, $(c)' \Rightarrow (c) \Rightarrow (a) \Rightarrow (c)$, $(a)' \Rightarrow (c)' \Rightarrow (c)$, $(d) \Leftrightarrow (b)_{x,y}$, $(d) \Rightarrow (e) \Rightarrow (d)$, $(e) \Leftrightarrow (f)$, $(g) \Rightarrow (f) \Rightarrow (g)$, so we are done proving 2.1. □

Claim 2.6. *Assume that clause* (f) *of* 2.1 *fails,* $d = d^* \in \mathfrak{D}$ *but clause* (g) *of* 2.1 *holds (equivalently* $\langle \mathrm{val}_d(x) : x \in X\rangle$ *is not constant).* *Then the following holds:*

$\boxdot_1$ *there are no* $\bar{s}^0 \in \mathrm{conv}(V_0^*(d^*))$, $\bar{s}^1 \in \mathrm{conv}(V_1^*(d^*))$ *such that* $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ *lie on* $\mathrm{conv}\{\bar{s}^0, \bar{s}^1\}$ *and for some* $\ell \in \{0,1\}$ *this set,* $\mathrm{conv}\{\bar{s}^0, \bar{s}^1\}$, *contains an interior point of* $\mathrm{conv}(V_\ell^*(d^*))$

$\boxdot_2$ *the lines* $L_0^* = \{(\frac{\mathbf{n}}{2} - 1, y) : y \in \mathbb{R}\}$, $L_1^* = \{(x, \frac{\mathbf{n}}{2} - 1) : x \in \mathbb{R}\}$ *divides the plane; and* $\mathrm{conv}(V^*(d^*))$ *is*
- (i) *included in one of the four closed half planes or*
- (ii) *is disjoint to at least one of the closed quarters minus* $\{(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)\}$.

Remark 2.7. (1) Recall $V^*(d^*) = V_0^*(d^*) \cup V_1^*(d^*)$ and $V_\ell^*(d^*) = \{\bar{s} - (\ell, 1-\ell) : \bar{s} \in V_\ell(d^*)\}$.
(2) So

- (i) $(k_1, k_2) \in V_0^*(d^*) \Leftrightarrow (k_1, k_2) + (0,1) \in V_0(d^*) \Leftrightarrow (k_1, k_2 + 1) \in V_0(d^*)$
  [Why? By Definition 1.8(4).]
- (ii) $(k_2, k_1) \in V_1^*(d^*) \Leftrightarrow (k_2, k_1) + (1,0) \in V_1(d^*) \Leftrightarrow (k_2 + 1, k_1) \in V_1(d^*)$ (see 1.8(4))
  hence
- (iii) $(k_1, k_2) \in V_0^*(d^*) \Leftrightarrow (k_1, k_2 + 1) \in V_0(d^*) \Leftrightarrow (k_2 + 1, k_1) \in V_1(d^*) \Leftrightarrow (k_2, k_1) \in V_1^*(d^*)$.
  [Why? By the above (i) + (ii) and 1.9(2).]

Proof. Toward contradiction assume that $\boxdot_2$ or $\boxdot_1$ in the claim fails. So necessarily

$(*)_0$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin V^*(d^*)$
[Why? If it belongs to $V_\ell^*(d^*)$ let $r_\ell^* = 1, r_{1-\ell}^* = 0$ and we get clause (f) of 2.1 which we are assuming fails]

$(*)'_0$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin \mathrm{conv}(V_\ell^*(d^*))$
[Why? As in the proof of $(*)_0$.]

$(*)_1$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ belongs to the convex hull of $V_0^*(d^*) \cup V_1^*(d^*)$ hence of $\mathrm{conv}(V_0^*(d^*)) \cup \mathrm{conv}(V_1^*(d^*))$
[Why? Otherwise $\boxdot_1$ trivially holds; also there is a line L through $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ such that $V^*(d^*)$ lie in one open half plane of L, so easily clause (ii) of $\boxdot_2$ holds hence $\boxdot_2$ holds recalling $(*)_0$. But we are assuming toward contradiction that $\boxdot_1$ fails or $\boxdot_2$ fails.]

Let $E = \{(\bar{s}_0, \bar{s}_1) : \bar{s}_\ell \in \operatorname{conv}(V^*_\ell(d^*))$ for $\ell = 0, 1$ and $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ belongs to the convex hull of $\{\bar{s}_0, \bar{s}_1\}\}$

$(*)_2$ $E \neq \emptyset$
[Why? By $(*)_1$]

$(*)_3$ if $r_0, r_1 \in [0, 1]_{\mathbb{R}}$, $1 = r_0 + r_1$, $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) = r_0 \times \bar{s}_0 + r_1 \times \bar{s}_1$ and $\bar{s}_\ell \in \operatorname{conv}(V^*_\ell(d^*))$ for $\ell = 0, 1$ then $r_0 = r_1 = \frac{1}{2}$
[Why? Otherwise clause (f) holds contradicting an assumption of 2.6.]

$(*)_4$ if $(\bar{s}_0, \bar{s}_1) \in E$ then $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) = \frac{1}{2}(\bar{s}_0 + \bar{s}_1)$ and $\bar{s}_0 \neq \bar{s}_1$
[Why? By $(*)_3$ and the definition of E and $(*)'_0$]

$(*)_5$ if $(\bar{s}_0, \bar{s}_1) \in E$, $\ell \in \{0, 1\}$, then $\bar{s}_\ell$ is the unique member of $\operatorname{conv}(V^*_\ell(d^*))$ which lies on the line through $\{\bar{s}_0, \bar{s}_1\}$.
[Why? Otherwise let $\bar{s}'_\ell$ be a counterexample. If $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \operatorname{conv}\{\bar{s}_\ell, \bar{s}'_\ell\}$ then it belongs to $\operatorname{conv}(V^*_\ell(d^*))$ contradicting $(*)'_0$. So letting $\bar{s}'_{1-\ell} = \bar{s}_{1-\ell}$ we know that $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \operatorname{conv}\{\bar{s}'_{1-\ell}, \bar{s}'_\ell\}$ hence by the definition of E we get $(\bar{s}'_0, \bar{s}'_1) \in E$ so by $(*)_4$ we deduce $\frac{1}{2}(\bar{s}'_0 + \bar{s}'_1) = (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) = \frac{1}{2}(\bar{s}_0 + \bar{s}_1)$ hence subtracting the two equations, $\bar{s}_{1-\ell}$ is cancelled and we get $\bar{s}'_\ell = \bar{s}_\ell$, contradiction]

$(*)_6$ $\boxdot_1$ holds (so by the assumption towards contradiction $\boxdot_2$ fails).
[Why? Assume $\bar{s}_0, \bar{s}_1$ are as there hence (by the definition of E), $(\bar{s}_0, \bar{s}_1) \in E$, now by $(*)_5$ the set (the line through $\bar{s}_0, \bar{s}_1) \cap \operatorname{conv}(V^*_\ell(d^*))$ is equal to $\{\bar{s}_\ell\}$. So the line through $\bar{s}_0, \bar{s}_1$ cannot contain an interior point of $\operatorname{conv}(V^*_\ell(d^*))$.]

Easily (by $(*)_4$ and the definition of E):

$(*)_7$ $E_\ell := \{\bar{s}_\ell : (\bar{s}_0, \bar{s}_1) \in E\}$ is a convex subset of $\operatorname{conv}(V^*_\ell(d^*)) \subseteq \mathbb{R}^2$.

Also

$(*)_8$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin E_\ell$
[Why? By $(*)'_0 + (*)_7$; also because if $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in E_\ell$ then by $(*)_4$ we have $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \operatorname{conv}(V^*_0(d^*)) \cap \operatorname{conv}(V^*_1(d^*))$ contradiction to $(*)'_0$.] $\square$

Now we split the rest of the proof to three cases which by $(*)_2$ trivially exhausts all the possibilities.

Case 1: E is not a singleton.

This implies by $(*)_4$ that E_ℓ (defined in $(*)_7$) has at least two members, so by $(*)_7$ the set $V^*_\ell(d^*)$ is not a singleton for $\ell = 0, 1$. As $|E| \geq 2$ by $(*)_4$ clearly $|E_1| \geq 2$. Also by $(*)_5$ if $\bar{s}_1 \in E_1$ so $\bar{s}_1 \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ by $(*)_8$, then $\bar{s}_1$ is the unique member of $\operatorname{conv}(V^*_1(d_1)) \cap$ (the line through $\bar{s}_1, (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$). Also E_1 is convex (by $(*)_7$) so necessarily

$(*)_9$ E_1 lies on a line L_1 to which by $(*)_5$, the point $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ does not belong.

Let

$(*)_{10}$ L_0 is the line $\{(a_0, a_1) : (a_1, a_0) \in L_1\}$.

As $E_1 \subseteq \operatorname{conv}(V^*_1(d^*)) \cap L_1$ is a convex set with ≥ 2 members and $(*)_5$ it follows that $\operatorname{conv}(V^*_1(d^*))$ is included in this line L_1 and as $V^*_0(d^*) = \{(k_2, k_1) : (k_1, k_2) \in V_1(d^*)\}$, (by clause (iii) of Remark 2.7(2)) it follows that $\operatorname{conv}(V^*_0(d^*))$ is included in the line $L_0 = \{(a_0, a_1) : (a_1, a_0) \in L_1\}$ to which $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ does not belong.

But $E_0 = \{\bar{s}_0 : (\bar{s}_0, \bar{s}_1) \in E\}$ is necessarily an interval of L_0 and by $(*)_4$ we have

$\odot_0$ $L_0 = \{(a_0, a_1) : 2(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) - (a_0, a_1) \in L_1\}$.

As L_1 is a line, for some reals r_0, r_1, r_2 we have

$\odot_1$ $L_1 = \{(a_0, a_1) \in \mathbb{R}^2 : r_0 a_0 + r_1 a_1 + r_2 = 0\}$

and

$\odot_2$ $(r_0, r_1) \neq (0, 0)$.

Hence by the definition of L_0 in $(*)_{10}$ above we have

$\odot_3$ $L_0 = \{(a_0, a_1) \in \mathbb{R}^2 : r_1 a_0 + r_0 a_1 + r_2 = 0\}$

and by $(*)_4$ the line L_0 includes the interval $\{2(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) - \bar{s}_1 : \bar{s}_1 \in E_1\}$ so

$$L_0 = \{(a_0, a_1) : (-r_0)a_0 + (-r_1)a_1 + r'_2 = 0\}$$

where $r'_2 = 2r_0(\frac{\mathbf{n}}{2} - 1) + 2r_1(\frac{\mathbf{n}}{2} - 1) + r_2$.

So for some $s \in \mathbb{R}$ we have $r_0 = -sr_1$, $r_1 = -sr_0$, $r_2 = sr'_2$ but $(r_0, r_1) \neq (0, 0)$ hence $s \in \{1, -1\}$ hence $r_0 \in \{r_1, -r_1\}$, so without loss of generality $r_0 = 1$, $r_1 \in \{1, -1\}$.

Subcase 1A: $r_1 = -1$.

So $d^*\{x, y\} = y \Rightarrow (\operatorname{val}_{d^*}(x), \operatorname{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow (\operatorname{val}_{d^*}(x) - 1, \operatorname{val}_{d^*}(y)) \in V^*_1(d^*) \Rightarrow (\operatorname{val}_{d^*}(x) - 1, \operatorname{val}_{d^*}(y)) \in L_1 \Rightarrow \operatorname{val}_{d^*}(x) - \operatorname{val}_{d^*}(y) = -r_2 + 1$, i.e., is constant, is the same for any such pair (x, y). If the directed graph $\operatorname{Tor}(d^*) = (X, \{(u, v) : d^*\{u, v\} = v\})$ contains no cycle, or just no cycle of length 3, then for some list $\{x_\ell : \ell < \mathbf{n}\}$ of X, we have $d^*\{x_{\ell_1}, x_{\ell_2}\} = x_{\max\{\ell_2, \ell_1\}}$ for $\ell_1 \neq \ell_2 < \mathbf{n}$. This implies $V_0(d^*) = \{(\ell_2, \ell_1) : \ell_1 < \ell_2 < \mathbf{n}\}$, easy contradiction to $(*)_0$ as $\mathbf{n} \geq 3$.

So the directed graph $\mathrm{Tor}(d^*) = (X, \{(u, v) : d^*\{u, v\} = v\})$ necessarily contains a cycle, so necessarily $-r_2 + 1 = 0$. Recall that when $\mathfrak{D} \subseteq \mathfrak{C}^{\mathrm{full}}$, the graph is connected so the $\mathrm{val}_{d^*}(x)$ is the same for all $x \in X$ hence is necessarily $(\frac{\mathbf{n}}{2} - 1)$, which is not an "allowable" case, in particular, contradict clause (g) of 2.1 which we are assuming.

Subcase 1B: $r_1 = 1$.

Clearly $x \neq y \in X \;\&\; d^*\{x, y\} = y \Rightarrow (\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y)) \in V_1(d^*) \Rightarrow (\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*) \Rightarrow (\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in L_1 \Rightarrow \mathrm{val}_{d^*}(x) + \mathrm{val}_{d^*}(y) = -r_2 + 1$. As $\mathbf{n} \geq 3$ and recall that $\mathfrak{D} \subseteq \mathfrak{C}^{\mathrm{full}}$, so there are distinct $x_0, x_1, x_2 \in X$ so $\mathrm{val}_{d^*}(x_{\ell_1}) + \mathrm{val}_{d^*}(x_{\ell_2}) = -r_2 + 1$ for $\{\ell_1, \ell_2\} \in \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}$, the order is not important as $r_1 = r_0$ hence $\mathrm{val}_{d^*}(x_1), \mathrm{val}_{d^*}(x_2)$ are equal and $-r_2+1$ is twice their value. So for $y \in X \setminus \{x_1\}$, we have $\mathrm{val}_{d^*}(x_1) + \mathrm{val}_{d^*}(y) = -r_2+1$ so $\mathrm{val}_{d^*}(y) = \mathrm{val}_{d^*}(x_2)$, so we are done as in case 1A.

Case 2: $E = \{(\bar{s}_0^*, \bar{s}_1^*)\}$ and $\bar{s}_0^* \neq \bar{s}_1^*$.

Let L be the line through $\{\bar{s}_0^*, \bar{s}_1^*\}$ and let the real r_0, r_1, r_2 be such that $L = \{(a_0, a_1) : r_0 a_0 + r_1 a_1 + r_2 = 0\}$ and $(r_0, r_1) \neq (0, 0)$.

So by $(*)_5$ the set $\mathrm{conv}(V_\ell^*(d^*))$ intersect L in the singleton $\{\bar{s}_\ell^*\}$

$\odot_4$ no one (closed) half plane for the line L contains $V_0^*(d^*) \cup V_1^*(d^*)$.

[Why? As then $\boxdot_2$ of 2.6 holds (if L is not parallel to the x-axis and the y-axis (i.e., $r_0, r_1 \neq 0$) then $\boxdot_2$(ii) holds, otherwise $\boxdot_2$(i) holds); so by $(*)_6$ this is against our assumption toward contradiction.]

$\odot_5$ There are $\bar{k}_0 \in V_0^*(d^*) \setminus \{\bar{s}_0^*\}$ and $\bar{k}_1 \in V_1^*(d^*) \setminus \{\bar{s}_1^*\}$ such that they are outside L in different sides.

[Why? First, if there are $\ell \in \{0, 1\}$ and $\bar{k}', \bar{k}'' \in V_\ell^*(d^*) \setminus L$ on different sides of L then also $V_{1-\ell}^*(d^*)$ has a member outside L (by clause (iii) of 2.7(2) and $(*)_5$) and call it $\bar{k}_{1-\ell}$, so the choice $\bar{k}_\ell = \bar{k}'$ or the choice $\bar{k}_\ell = \bar{k}''$ is as required. Second, if there is no such $\ell \in \{0, 1\}$ by $\odot_4$ there are $\ell \in \{0, 1\}$ and $\bar{k}_\ell \in V_\ell^*(d^*) \setminus L$, by 2.7(2) there is $\bar{k}_{1-\ell} \in V_{1-\ell}^*(d^*) \setminus \{\bar{s}_\ell\}$ but by $(*)_5$ we have $\bar{k}_{1-\ell} \notin L$. By $\odot_4$ the pairs $\bar{k}_0, \bar{k}_1$ are from different sides of L.

As $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ lie in the open interval spanned by $\bar{s}_0^*$ and $\bar{s}_1^*$, necessarily $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ is an interior point of $\mathrm{conv}\{\bar{s}_0^*, \bar{k}_0, \bar{s}_1^*, \bar{k}_1\}$, easy contradiction to the case assumption.

Case 3: $E = \{(\bar{s}_0^*, \bar{s}_1^*)\}$ and $\bar{s}_0^* = \bar{s}_1^*$.

So by $(*)_4$ clearly $\bar{s}_\ell^* = (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$, but this contradicts $(*)_0'$ above. □

Claim 2.8. *In 2.6, clause* (i) *of* $\boxdot_2$ *is impossible.*

Proof. So toward contradiction assume clause (i) of $\boxdot_2$ holds. We know that $\langle \mathrm{val}_{d^*}(x) : x \in X \rangle$ is not constant (as we assume clause (g) of 2.1 holds). As the average of $\mathrm{val}_{d^*}(x), x \in X$ is $\frac{\mathbf{n}-1}{2} = \frac{\mathbf{n}}{2} - \frac{1}{2}$ clearly

$(*)_1$ for some points $x \in X$ we have $\mathrm{val}_{d^*}(x)$ is $< \frac{\mathbf{n}-1}{2} = \frac{\mathbf{n}}{2} - \frac{1}{2}$ and for some point $x \in X$ we have $\mathrm{val}_{d^*}(x)$ is $> \frac{\mathbf{n}}{2} - \frac{1}{2}$.

The assumption (i.e. (i) of $\boxdot_2$ of 2.6) leaves us with four possibilities, so we have 4 cases (according to which half plane). □

Case 1: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_0 > \frac{\mathbf{n}}{2} - 1$.

It follows that by clause (i) of 2.7(2)

$$(k_0, k_1) \in V_0(d^*) \Rightarrow (k_0, k_1 - 1) \in V_0^*(d^*) \subseteq V^*(d^*) \Rightarrow k_0 \leq \frac{\mathbf{n}}{2} - 1.$$

So if $x \in X$ and for some $y \in X \setminus \{x\}$ we have $d^*\{x, y\} = x$ (this means just that, $\mathrm{val}_{d^*}(x) < \mathbf{n} - 1$) then $\mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$, so

$(*)_2$ if $x \in X$ and $\mathrm{val}_{d^*}(x) < \mathbf{n} - 1$ <u>then</u> $\mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

We shall show that this is impossible (this helps also in case 3). There can be at most one $x \in X$ with $\mathrm{val}_{d^*}(x) = \mathbf{n} - 1$; if there is none then we have:

if $x \in X$ then $\mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

But the average valency is $\frac{\mathbf{n}-1}{2}$ which is $> \frac{\mathbf{n}}{2} - 1$, contradiction. So there is $x^* \in X$ such that $\mathrm{val}_{d^*}(x^*) = \mathbf{n} - 1$, of course, it is unique. Now $\mathrm{Tor}^-(d^*) := (X \setminus \{x^*\}, \{(y, z) : y \neq z \in X \setminus \{x^*\}, d^*\{y, z\} = z\})$ is a directed graph with $\mathbf{n} - 1$ points and every $y \in X \setminus \{x^*\}$ has the same out-valency in $\mathrm{Tor}(d^*)$ and in $\mathrm{Tor}^-(d^*)$, hence each $y \in X \setminus \{x^*\}$ has (in $\mathrm{Tor}(d^*)$ and in $\mathrm{Tor}^-(d^*)$) out-valency $\leq \frac{\mathbf{n}}{2} - 1 = \frac{(\mathbf{n}-1)-1}{2}$, so necessarily $\mathbf{n}$ is even and every node in $\mathrm{Tor}^-(d^*)$ has out-valency exactly $\frac{(\mathbf{n}-1)-1}{2} = \frac{\mathbf{n}}{2} - 1$; as $\mathbf{n} \geq 3$ we can choose $y \neq z \in X \setminus \{x^*\}$ and without loss of generality $d^*(y, z) = z$. Now $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1), (\mathbf{n} - 1, \frac{\mathbf{n}}{2} - 1) \in V_1(d^*)$ as witnessed by the pairs $(y, z), (x^*, y)$ respectively, hence $(\frac{\mathbf{n}}{2} - 2, \frac{\mathbf{n}}{2} - 1), (\mathbf{n} - 2, \frac{\mathbf{n}}{2} - 1) \in V_1^*(\mathrm{d})$, again by Definition 1.8(4). Hence (as $\mathbf{n} \geq 3$ so $\mathbf{n} - 2 \geq \frac{\mathbf{n}}{2} - 1$) we have $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \mathrm{conv}(V_1^*(d^*))$, so $r_0^* = 0$ given contradiction to "clause (f) of 2.1 fails" assumed in 2.6.

Case 2: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_0 < \frac{\mathbf{n}}{2} - 1$.

By clause (i) of 2.7(2) it follows that

$$(k_0, k_1) \in V_0(d^*) \Rightarrow (k_0, k_1 - 1) \in V_0^*(d^*) \subseteq V^*(d^*) \Rightarrow k_0 \geq \frac{\mathbf{n}}{2} - 1.$$

So if $x \in X$ and for some $y \in X \setminus \{x\}$ we have $d^*\{x, y\} = x$ (equivalently $\mathrm{val}_{d^*}(x) < \mathbf{n} - 1$) then $(\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y)) \in V_0(d^*) \Rightarrow (\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y) - 1) \in V_0^*(d^*)$ hence $\mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$, but $\mathbf{n} - 1 \geq \frac{\mathbf{n}}{2} - 1$ so in any case

$(*)_3$ if $x \in X$ then $\mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$.

For the rest of the proof of case 2, we shall use only $(*)_3$. This serves us also in case 4. So $x \in X \Rightarrow \mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1$.

If $\mathbf{n}$ is odd we have $x \in X \Rightarrow \mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - \frac{1}{2} = \frac{\mathbf{n}-1}{2}$, impossible by $(*)_1$ so $\mathbf{n}$ is even. Let $k = \frac{\mathbf{n}}{2} - 1$. The average $\mathrm{val}_{d^*}(x)$ is necessarily $k + \frac{1}{2}$ hence $Y =: \{x \in X : \mathrm{val}_{d^*}(x) \leq k$ (equivalently $= k$)$\}$ has at least $k + 1 = \frac{\mathbf{n}}{2}$ members. If $x \in X$,$\mathrm{val}_{d^*}(x) = k + 1$ then $x \notin Y$ and $|\{y : d\{x, y\} = y\}| = k + 1 = \frac{\mathbf{n}}{2} > |X \setminus (Y \cup \{x\})|$ so there is $y \in Y$ such that $d\{x, y\} = y$, hence $(k, k) = (\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*)$ so clause (f) of 2.1 holds with $r_0^* = 0$, contradiction. So

$(*)_4$ $x \in X \Rightarrow \mathrm{val}_{d^*}(x) \neq k + 1$.

Now $|Y| = \mathbf{n}$ is impossible by $(*)_1$. Also if $|Y| = \mathbf{n} - 1$ let x^* be the unique element of X outside Y so in the tournament $\mathrm{Tor}^- := (Y, \{(x, y) : x \neq y$ are from Y and $d^*\{x, y\} = y\})$ each x has out-valency $\leq \mathrm{val}_{d^*}(x) = k = \frac{(|Y|-1)}{2}$, but this is the average so equality holds. Now if $x \in Y$ then x has out-valency k in $\mathrm{Tor}^-(d^*)$ and has out-valency k in $\mathrm{Tor}(d^*)$ hence $d^*\{x, x^*\} \neq x^*$ hence $\mathrm{val}_{d^*}(x^*) = \mathbf{n} - 1$ and we get contradiction as in Case 1. Hence

$(*)_5$ $|Y| \leq \mathbf{n} - 2$.

Clearly we can find $x_1 \in Y$ such that $|\{y \in Y : y \neq x_1, d^*\{x_1, y\} = y\}| \leq \frac{|Y|-1}{2}$ (as if we average this number on the $x_1 \in Y$ we get $\frac{|Y|-1}{2}$) but $\frac{|Y|-1}{2} \leq \frac{\mathbf{n}}{2} - \frac{3}{2} = k - \frac{1}{2} < |\{y \in X : d\{x_1, y\} = y\}|$ hence there is $x_2 \in X \setminus Y$ such that $d^*\{x_1, x_2\} = x_2$. Now let $m = \mathrm{val}_{d^*}(x_2)$ so $m > k$ as $x_2 \notin Y$ and $m \neq k + 1$ by $(*)_4$ hence $m > k + 1$ and (x_1, x_2) witness $(k - 1, m) \in V_1^*(d^*)$. As $\mathbf{n} \geq 3$ and (see the paragraph before $(*)_4$) $|Y| \geq \frac{\mathbf{n}}{2}$ obviously $|Y| \geq 2$ hence (as any pair of $y_1 \neq y_2$ from Y witness) also $(k - 1, k) \in V_1^*(d^*)$. As $|Y| \geq \frac{\mathbf{n}}{2} = k + 1$, $\mathrm{val}_{d^*}(x_2) > k + 1 \geq \mathbf{n} - |Y|$ easily there is $x_3 \in Y$ such that $d\{x_2, x_3\} = x_3$ hence (x_2, x_3) witness $(m - 1, k) \in V_1^*(d^*)$. Now $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) = (k, k) \in \mathrm{conv}\{(k - 1, k), (m - 1, k)\}$ recalling $m > k + 1$. But $(k - 1, k), (m - 1, k)$ belong to $V_1^*(d^*)$ hence $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \in \mathrm{conv}(V_1^*(d^*))$, contradiction to "not clause (f) of 2.1" with $r_0^* = 0$.

Case 3: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_1 > \frac{\mathbf{n}}{2} - 1$.

So by clause (ii) of 2.7 it follows that

$$(k_0, k_1) \in V_1(d^*) \Rightarrow (k_0 - 1, k_1) \in V_1^*(d^*) \subseteq V^*(d^*) \Rightarrow k_1 \leq \frac{\mathbf{n}}{2} - 1.$$

So if $y \in X$ and for some $x \in X \setminus \{y\}$ we have $d\{x, y\} = y$ then $(\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*)$ hence $\mathrm{val}_{d^*}(y) \leq \frac{\mathbf{n}}{2} - 1$. But there is such x iff $\mathrm{val}_{d^*}(y) \neq \mathbf{n} - 1$, that is

$(*)_6$ if $y \in X$ and $\mathrm{val}_{d^*}(y) \neq \mathbf{n} - 1$ then $\mathrm{val}_{d^*}(y) \leq \frac{\mathbf{n}}{2} - 1$.

We continue as in Case 1, (after $(*)_2$ which uses only $(*)_2$ or dualize see 1.9(1)).

Case 4: For no $(k_0, k_1) \in V^*(d^*)$ do we have $k_1 < \frac{\mathbf{n}}{2} - 1$.

So $(k_0, k_1) \in V_1(d^*) \Rightarrow (k_0 - 1, k_1) \in V_1^*(d^*) \subseteq V^*(d^*) \Rightarrow k_1 \geq \frac{\mathbf{n}}{2} - 1$. So if $y \in X$ and for some $x \in X \setminus \{y\}$ we have $d\{x, y\} = y$ then $(\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*) \Rightarrow \mathrm{val}_{d^*}(y) \geq \frac{\mathbf{n}}{2} - 1$. So if $\mathrm{val}_{d^*}(y) < \mathbf{n} - 1$ then there is such x hence $\mathrm{val}_{d^*}(y) \geq \frac{\mathbf{n}}{2} - 1$, but if $\mathrm{val}_{d^*}(y) \geq \mathbf{n} - 1$ we get the same conclusion, so

$(*)_7$ $\mathrm{val}_{d^*}(y) \geq \frac{\mathbf{n}}{2} - 1$

and we can continue as in case 2 after $(*)_3$. □

Claim 2.9. *In 2.6, clause* (ii) *of* $\boxdot_2$ *is impossible.*

Proof. Note that as we are assuming the failure of clause (f) of 2.1

$(*)_0$ $(\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1) \notin V^*(d^*)$.

Again we have four cases.

Case 1: If $a_0 \geq \frac{\mathbf{n}}{2} - 1$, $a_1 \geq \frac{\mathbf{n}}{2} - 1$ but $(a_0, a_1) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ then $(a_0, a_1) \notin \mathrm{conv}(V^*(d^*))$.

So

$(*)_1$ for at most one $x \in X$ we have $\mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2}$.

[Why? If $x \neq y \in X$ and $\mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2}$, $\mathrm{val}_{d^*}(y) \geq \frac{\mathbf{n}}{2}$ then $(\mathrm{val}_{d^*}(x) - 1, \mathrm{val}_{d^*}(y)) \in V_1^*(d^*) \subseteq V^*(d^*)$ or $(\mathrm{val}_{d^*}(x), \mathrm{val}_{d^*}(y) - 1) \in V_0^*(d^*) \subseteq V^*(d^*)$, a contradiction to the case assumption in both cases.]

If there is no $x \in X$ with $\mathrm{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2}$ then $x \in X \Rightarrow \mathrm{val}_{d^*}(x) < \frac{\mathbf{n}}{2}$ and so $x \in X \Rightarrow \mathrm{val}_{d^*}(x) \leq \frac{\mathbf{n}-1}{2}$ but this is the average valency, so always equality holds, contradicting an assumption of 2.6.

So assume

$(*)_2$ $x_0 \in X$, $\mathrm{val}_{d^*}(x_0) \geq \frac{\mathbf{n}}{2}$.

Now

$(*)_3$ if $y \in X \setminus \{x_0\}$ and $d^*\{x_0, y\} = y$ then $\text{val}_{d^*}(y) < \frac{\mathbf{n}}{2} - 1$ (hence $\leq \frac{\mathbf{n}}{2} - \frac{3}{2}$).
[Why? As $d^*\{x_0, y\} = y$ then $(\text{val}_{d^*}(x_0) - 1, \text{val}_{d^*}(y)) \in V_1^*(d^*) \subseteq V^*(d^*)$, now $\text{val}_{d^*}(x_0) - 1 \geq \frac{\mathbf{n}}{2} - 1$ hence by the case assumption $+(*)_0$ we have $\text{val}_{d^*}(y) < \frac{\mathbf{n}}{2} - 1$.]

$(*)_4$ if $y \in X \setminus \{x_0\}$ and $d^*\{x_0, y\} = x_0$ then $\frac{\mathbf{n}}{2} - 1 > |\{z \in X \setminus \{x_0, y\} : d^*\{y, z\} = z\}|$.
[Why? As $d^*\{x_0, y\} = x_0$ clearly $(\text{val}_{d^*}(x_0), \text{val}_{d^*}(y) - 1) \in V_0^*(d^*) \subseteq V^*(d^*)$, $\text{val}_{d^*}(x_0) \geq \frac{\mathbf{n}}{2} > \frac{\mathbf{n}}{2} - 1$, hence by the case assumption $\text{val}_{d^*}(y) - 1 < \frac{\mathbf{n}}{2} - 1$ so $\text{val}_{d^*}(y) < \frac{\mathbf{n}}{2}$, i.e. $\frac{\mathbf{n}}{2} > |\{z \in X \setminus \{y\} : d^*\{y, z\} = z\}|$ and as $d^*\{x_0, y\} = x_0$ this gives the desired inequality.]

So letting $Y = X \setminus \{x_0\}$ we have $(Y, \{(y, z), y \neq z \in Y, d^*(y, z) = z\})$ is a tournament satisfying each node has out-valency $\leq \frac{\mathbf{n}-3}{2} < \frac{(\mathbf{n}-1)-1}{2} = \frac{|Y|-1}{2}$ (why? by $(*)_3 + (*)_4$), contradiction.

Case 2: If $a_1 \leq \frac{\mathbf{n}}{2} - 1$ and $a_2 \leq \frac{\mathbf{n}}{2} - 1$ and $(a_1, a_2) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ then $(a_1, a_2) \notin \text{conv}(V^*(d^*))$.
Clearly, as above in the proof of $(*)_1$

$(*)'_1$ there is at most one $x \in X$ with $\text{val}_{d^*}(x) \leq \frac{\mathbf{n}}{2} - 1$.

If there is none then $x \in X \Rightarrow \text{val}_{d^*}(x) \geq \frac{\mathbf{n}}{2} - 1 + \frac{1}{2} = \frac{\mathbf{n}-1}{2}$, so considering the average of $\text{val}_{d^*}(y)$ equality always holds so clause (g) of 2.1 fails contradicting an assumption of 2.6. So assume

$(*)'_2$ $x_0 \in X, \text{val}_{d^*}(x_0) \leq \frac{\mathbf{n}}{2} - 1$

and by $(*)'_1 + (*)'_2$ clearly

$(*)'_3$ if $y \in X \setminus \{x_0\}$ then $\text{val}_{d^*}(y) > (\frac{\mathbf{n}}{2} - 1) = \frac{\mathbf{n}-2}{2}$ so $\text{val}_{d^*}(y) \geq \frac{\mathbf{n}-1}{2}$.

The directed graph $\mathbf{G} = (X \setminus \{x_0\}, \{(y, z) : d\{y, z\} = z\})$ has $\mathbf{n} - 1$ nodes and let $Y_0 = \{y \in X : y \neq x_0 \text{ and } d^*\{y, x_0\} = x_0\}$ and $Y_1 = \{y \in X : y \neq x_0 \text{ and } d^*\{x_0, y\} = y\}$.
Clearly $Y_0, Y_1, \{x_0\}$ is a partition of X, so Y_0, Y_1 is a partition of the set of nodes in $\mathbf{G}$. Also

$\boxdot$ (i) $y \in Y_0 \Rightarrow d^*(x_0, y) = x_0 \Rightarrow (\text{val}_{d^*}(x_0), \text{val}_{d^*}(y) - 1) \in V_0^*(d^*) \subseteq V^*(d^*) \Rightarrow$ (by the case assumption $+(*)'_2 + (*)_0$) $\text{val}_{d^*}(y) - 1 > \frac{\mathbf{n}}{2} - 1 \Rightarrow \text{val}_{d^*}(y) >$
$\frac{\mathbf{n}}{2} \Rightarrow$ the valency of y in $\mathbf{G}$ is $> \frac{\mathbf{n}}{2} - 1 \Rightarrow$
the valency of y in $\mathbf{G}$ is $\geq \frac{\mathbf{n}-1}{2}$
(ii) $y \in Y_1 \Rightarrow d^*\{x_0, y\} = y$ (by $(*)_3$) $\Rightarrow \text{val}_{d^*}(y) \geq \frac{\mathbf{n}-1}{2} \Rightarrow$ the valency of y in $\mathbf{G}$ is $\geq \frac{\mathbf{n}-1}{2} - 0 = \frac{\mathbf{n}-1}{2}$.

So every node in $\mathbf{G}$ has out-valency (in $\mathbf{G}$) at least $\frac{\mathbf{n}-1}{2}$, a contradiction as the average out-valency is $\frac{\mathbf{n}-2}{2}$.

Case 3: If $a_1 \geq \frac{\mathbf{n}}{2} - 1$ and $a_2 \leq \frac{\mathbf{n}}{2} - 1$ and $(a_1, a_2) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$, then $(a_1, a_2) \notin \text{conv}(V^*(d^*))$.
So (as in the proof of $(*)_1$ using $(*)_0$)

$\odot_1$ there cannot be $x_0, x_1 \in X$ such that $\text{val}_{d^*}(x_0) \geq \frac{\mathbf{n}}{2}$ and $\text{val}_{d^*}(x_1) \leq \frac{\mathbf{n}}{2} - 1$ (the $x_0 \neq x_1$ follows)

so one of the following two sub-cases hold.

Subcase 3A: $x \in X \Rightarrow \text{val}_{d^*}(x) < \frac{\mathbf{n}}{2}$.
So $x \in X \Rightarrow \text{val}_{d^*}(x) \leq \frac{\mathbf{n}-1}{2}$ and (looking at average valency) equality holds, contradicting clause (g) of 2.1 which we are assuming.

Subcase 3B: $x \in X \Rightarrow \text{val}_{d^*}(x) > \frac{\mathbf{n}}{2} - 1$.
So $x \in X \Rightarrow \text{val}_{d^*}(x) \geq \frac{\mathbf{n}-1}{2}$, and we finish as above.

Case 4: If $a_1 \leq \frac{\mathbf{n}}{2} - 1$ and $a_2 \geq \frac{\mathbf{n}}{2} - 1$ and $(a_1, a_2) \neq (\frac{\mathbf{n}}{2} - 1, \frac{\mathbf{n}}{2} - 1)$ then $(a_1, a_2) \notin \text{conv}(V(d^*))$.
Similar to case 3 (or dualize the situation by 1.9(1)). $\square$

3. Balanced choice functions

Here we analyze the case clause (g) of 2.1 fail and give a complete answer (and show the equivalence of relatives of "balance".

Definition 3.1. (1) $c \in \mathfrak{C}$ is called balanced if $x \in X \Rightarrow \text{val}_c(x) = (\mathbf{n} - 1)/2$, let $\mathfrak{C}^{\text{bl}} = \{c \in \mathfrak{C} : c \text{ is balanced}\}$.
(2) $\bar{t} \in \text{pr}(\mathfrak{C})$ is called balanced if $x \in X \Rightarrow \Sigma\{t_{x,y} : y \in X \setminus \{x\}\} = (\mathbf{n} - 1)/2$. Let $\text{pr}^{\text{bl}}(\mathfrak{C})$ be the set of balanced $\bar{t} \in \text{pr}(\mathfrak{C})$.
(2A) $\bar{t} = \text{pr}(\mathfrak{C})$ is super-balanced if $t_{x,y} = \frac{1}{2}$ for $x \neq y \in X$.
(3) We say $c \in \mathfrak{C}$ is pseudo-balance iff every edge of $\text{Tor}(c)$ belongs to a directed cycle, see Definition 0.4(2).
(3A) $c \in \mathfrak{C}$ is called partition$^+$-balanced when: if $\emptyset \neq Y \subsetneqq X$ then for some $x \in X \setminus Y$ and $y \in Y$ we have $c\{x, y\} = y$; $c \in \mathfrak{C}$ is called partition-balanced when "if $\emptyset \subsetneqq Y \subsetneqq X$" then: for some $x \in X \setminus Y, y \in Y$ we have $c\{x, y\} = y$ iff for some $x \in X \setminus Y, y \in Y$ we have $c(x, y) = x$.

(3B) $c \in \mathfrak{C}$ is called weight-balanced <u>when</u>: for some balanced $\bar{t} \in \operatorname{pr}(\mathfrak{C})$ we have

$$c\{x, y\} = y \Leftrightarrow t_{x,y} > \frac{1}{2}.$$

(4) We call $\mathscr{D} \subseteq \mathfrak{C}$ balanced <u>if</u> every $c \in \mathscr{D}$ is balanced, similarly $T \subseteq \operatorname{pr}(\mathfrak{C})$ is called balanced if every $\bar{t} \in T$ is. Similarly for the other properties.

(5) If $x, y, z \in X$ are distinct, let $\bar{t} = \bar{t}^{\langle x,y,z\rangle}$ be defined by:

$t_{u,v}$ is 1 if $(u, v) \in \{(x, y), (y, z), (z, x)\}$

$t_{u,v}$ is 0 if $(u, v) \in \{(y, x), (z, y), (x, z)\}$

$t_{u,v}$ is $\frac{1}{2}$ if otherwise.

(6) For $k \geq 3$ a sequence $\bar{x} = (x_0, \ldots, x_{k-1})$ with $x_\ell \in X$ and no repetitions and $a \in [0, 1]_{\mathbb{R}}$ let $\bar{t} = \bar{t}_{\bar{x},a} \in \operatorname{pr}(\mathfrak{C})$ be defined by $t_{x_i,x_j} = a$, $t_{x_j,x_i} = 1 - a$ if $j = i + 1 \bmod k$ and $t_{x,y} = \frac{1}{2}$ for $x \neq y \in X$ otherwise. If $a = 1$ we may omit it.

(7) Let $c_* \in \mathfrak{C}$ be the empty function. We call $\mathscr{D}$ trivial if $\mathscr{D} = \{c_*\}$ or $\mathscr{D} = \emptyset$.

Fact 3.2. (0) $\operatorname{pr}^{\mathrm{bl}}(\mathfrak{C})$ is a convex subset of $\operatorname{pr}(\mathfrak{C})$ and it is preserved by the permutations of $\operatorname{pr}(\mathfrak{C})$ induced by permutations of X.

(1) If $c \in \mathfrak{C}^{\mathrm{bl}}$ then $\bar{t}[c]$ belongs to $\operatorname{pr}^{\mathrm{bl}}(\mathfrak{C})$ but is not necessarily super-balanced; if $\bar{t} \in \operatorname{pr}(\mathfrak{C})$ is balanced and $c = \operatorname{maj}(\bar{t})$, <u>then</u> c is pseudo-balanced.

(2) If $c \in \mathfrak{C}^{\mathrm{full}}$ is weight-balanced <u>then</u> it is partition$^+$-balanced.

(2A) If $c \in \mathfrak{C}$ is weight-balanced <u>then</u> it is partition-balanced (similar to (*) of 1.7; note that not every $c \in \mathfrak{C}$ is pseudo-balanced and even some $c \in \mathfrak{C}^{\mathrm{full}}$ is not pseudo-balanced).

(3) If $c \in \mathfrak{C}^{\mathrm{full}}$ <u>then</u> c is partition$^+$-balaned iff c is partition-balanced...

(4) If $c \in \mathfrak{C}$ then $\operatorname{maj}(\bar{t}[c]) = c$. If $\mathscr{D} \subseteq \mathfrak{C}$ is balanced, <u>then</u> pr-cl$(\mathscr{D})$ is balanced hence every member of maj-cl$(\mathscr{D})$ is pseudo-balanced.

Proof. (0), (3), (4) Check, on part (1) see 3.4.

(2) By (2A) and (3).

(2A) Toward contradiction assume (Y, x, y) is a counterexample so $\emptyset \subsetneqq Y \subsetneqq X$, $x \in X \setminus Y$, $y \in Y$ and $c\{x, y\} = y$ and there are no $u \in Y$, $v \in X \setminus Y$ such that $c\{u, v\} = v$. Let $Y_0 = X \setminus Y$, $Y_1 = X$.

As c is pseudo-balanced, there is $\bar{t} \in \operatorname{pr}^{\mathrm{bl}}(\mathfrak{C})$ such that $c = \operatorname{maj}(c)$. So necessarily $t_{x,y} > \frac{1}{2}$ but $u \in Y_0$, $v \in Y_1 \Rightarrow t_{v,u} \leq \frac{1}{2} \Rightarrow t_{u,v} \geq \frac{1}{2}$.

Let $Y_0 = X \setminus Y$, $Y_1 = Y$, so $\Sigma\{t_{u,v} - \frac{1}{2} : u \in Y_0, v \in Y\} \geq t_{x,y} - \frac{1}{2} > 0$.

For $u \in Y_0$ let $s^0_u = \Sigma\{t_{u,v} - \frac{1}{2} : v \in Y_0 \setminus \{u\}\}$, $s^1_u = \Sigma\{t_{u,v} - \frac{1}{2} : v \in Y_1\}$ so $s^0_u + s^1_u = 0$, hence $0 = \Sigma\{s^0_u + s^1_u : u \in Y_0\} = \Sigma\{s^0_u : u \in Y_0\} + \Sigma\{s^1_u : u \in Y_0\}$ but by the previous sentence the second summand is positive. Hence $\Sigma\{s^0_u : u \in y_0\}$ is negative, but it is zero because $u_1 \neq u_2 \in y \Rightarrow (t_{u_1,u_2} - \frac{1}{2}) + (t_{u_2,u_1} - \frac{1}{2}) = 0$.

Claim 3.3. *If $\mathfrak{D} \subseteq \mathfrak{C}^{\mathrm{full}}$ is non-empty, symmetric and not balanced, <u>then</u>* maj-cl$(\mathfrak{D}) = \mathfrak{C}$.

Proof. Choose $d^* \in \mathfrak{D}$ which is not balanced, and let $\mathfrak{D}' =:$ sym-cl$(\{d^*\})$, so $\mathfrak{D}'$ is as in 2.1 and it satisfies clause (g) there hence it satisfies clause (a)$'$ there. This means that maj-cl$(\mathfrak{D}') = \mathfrak{C}$ but $\mathfrak{D}' \subseteq \mathfrak{D} \subseteq \mathfrak{C}$ hence $\mathfrak{C} =$ maj-cl$(\mathfrak{D}') \subseteq$ maj-cl$(\mathfrak{D}) \subseteq \mathfrak{C}$ so we are done. □

Fact 3.4. (1) If $c \in \mathfrak{C}^{\mathrm{bl}}$ or just $c \in \mathfrak{C}$ is weight-balanced <u>then</u> c is partition-balanced and is pseudo-balanced, i.e. every edge of Tor$[c]$ belongs to some directed cycle.

(2) Assume that $\bar{t} \in \operatorname{pr}(\mathfrak{C})$ is balanced, <u>then</u> $\operatorname{maj}(\bar{t})$ is pseudo-balanced, i.e. if $t_{x,y} > \frac{1}{2}$ then we can find $k \geq 3$ and $x_0, \ldots, x_{k-1} \in X$ with no repetitions such that $(x_0, x_1) = (x, y)$ and $j = i + 1 \bmod k \Rightarrow t_{x_i,x_j} > \frac{1}{2}$.

Proof. (1) As c is weight balanced, it is $\operatorname{maj}(\bar{t})$ for some balanced $\bar{t} \in \operatorname{pr}(\mathfrak{C})$, i.e. $\bar{t} \in \operatorname{pr}^{\mathrm{bl}}(\mathfrak{C})$, so now the second conclusion "every edge of Tor(c) belongs to a directed cycle", follows from part (2) by the definition of maj(c). The first conclusion "partition-balanced" follows from the first; assume $\emptyset \subsetneqq Y \subsetneqq X$, $x \in X \setminus Y$, $y_1 \in Y$ and (x, y) is an edge of Tor(c) then $t_{u,v} > \frac{1}{2}$ hence there is $k \geq 3$ and $x_0, \ldots, x_{k-1} \in X$ as in part (2).

Let i_* be the maximal $i \in \{1, \ldots, k-1\}$ such that $x_i \in Y$, (well defined as $i = 1$ is O.K.) and $j_* = i_* + 1 \bmod k$, so $t_{x_{i_*},x_{j_*}} > \frac{1}{2}$, $x_{i_*} \in Y$, $x_{j_*} \in X \setminus Y$ as required in Definition 3.1(3A).

(2) Assume $\bar{t} \in \operatorname{pr}(\mathfrak{C})$ is balanced and (x, y) is an edge of $\mathbf{G}_{\operatorname{maj}(\bar{t})}$, so $t_{x,y} > \frac{1}{2}$ and there are no $\langle x_0, \ldots, x_{k-1}\rangle$ as promised. Let Y_1 be the set of $z \in X$ such that there are k and $z_0, \ldots, z_k \in X \setminus \{y\}$ such that $z_0 = z$, $z_k = x$, $t_{z_i,z_{i+1}} > \frac{1}{2}$ for $i < k$.

Let $Y_2 = X \setminus Y_1$ so

$\odot_1$ $\{x\} \subseteq Y_1 \subseteq X \setminus \{y\}$ so (Y_1, Y_2) is a partition of X to non-empty sets.

Now

$\odot_2$ if $u \in Y_1, v \in Y_2$ then $t_{u,v} \geq \frac{1}{2}$.

[Why? Toward contradiction assume $t_{u,v} < \frac{1}{2}$ so $t_{v,u} > \frac{1}{2}$. Now by the definition of $u \in Y_1$ there are $z_0 = u, z_1, \ldots, z_k = x$ as there; so $\{z_0, \ldots, z_k\} \subseteq Y_1$ by the definition of Y_1 and without loss of generality $\langle z_0, \ldots, z_k \rangle$ is without repetitions. If $v = y$ then $y, z_0, z_1, \ldots, z_k$ is a cycle as required. If $v \neq y$ then the sequence $v, z_0, \ldots, z_k$ shows that $v \in Y_1$ contradicting the assumption $v \in Y_2$ of $\odot_2$.]

$\odot_3$ for some $u \in Y_1, v \in Y_2$ we have $t_{u,v} > \frac{1}{2}$.

[Why? Choose $u = x, v = y$.]

By $\odot_2 + \odot_3$ we get a contradiction to 3.2(2). □

Claim 3.5. *Assume $|X| \geq 3, \mathfrak{D} \subseteq \mathfrak{C}^{\rm full}$ is symmetric, non-empty and balanced. Then, for any distinct $x, y, z \in Z$ we have $\bar{t}^{\langle x,y,z \rangle} \in$ pr-cl$(\mathfrak{D})$.*

Proof. Let $d \in \mathfrak{D}$ be non-trivial, now Tor(d) =: $(X, \{(u, v) : d\{u, v\} = v\})$ is a directed graph with equal out-valance and in-valance for every node, it has a directed cycle. As $\mathfrak{D} \subseteq \mathfrak{C}^{\rm full}_X$, it follows that this graph has a triangle, i.e., $x, y, z \in X$ distinct such that

$(*)_1$ $d\{x, y\} = y, d\{y, z\} = z, d\{z, x\} = x$.

Let $\Pi_{x,y,z} = \{\pi \in \mathrm{Per}(X) : \pi \restriction \{x, y, z\}$ is the identity$\}$. Let $\bar{t} = \Sigma\{\bar{t}[d^\pi] : \pi \in \Pi_{x,y,z}\}/|\Pi_{x,y,z}|$.

Clearly $d^\pi \in \mathfrak{D}$ for $\pi \in \Pi_{x,y,z}$ hence $\bar{t} \in$ pr-cl$(\mathfrak{D})$. Also by $(*)_1$ and the definition of $\Pi_{x,y,z}$

$(*)_2$ $t_{x,y} = t_{y,z} = t_{z,x} = 1$.

Also

$$\begin{aligned}&|\{w : w \in X \setminus \{x, y, z\} \text{ and } d\{x, w\} = w\}|\\ &\quad = |\{w : w \in X \setminus \{x\} \text{ and } d\{x, w\} = w\}| - |\{w : w \in \{y, z\} \text{ and } d\{x, w\} = w\}|\\ &\quad = (|X| - 1)/2 - 1 = (|X| - 3)/2\end{aligned}$$

so

$$\begin{aligned}&|\{w : w \in X \setminus \{x, y, z\} \text{ and } d\{x, w\} = x\}|\\ &\quad = (|X| - 3) - |\{w : w \in X \setminus \{x, y, z\} \text{ and } d\{x, w\} = w\}| = (|X| - 3) - (|X| - 3)/2 = (|X| - 3)/2\end{aligned}$$

hence

$(*)_3$ $t_{x,w} = 1/2 = t_{w,x}$ for $w \in X \setminus \{x, y, z\}$.

Similarly

$(*)_4$ $t_{y,w} = 1/2 = t_{w,y}$ for $w \in X \setminus \{x, y, z\}$

$(*)_5$ $t_{z,w} = 1/2 = t_{w,z}$ for $w \in X \setminus \{x, y, z\}$

and even easier (and as in Section 2)

$(*)_6$ $t_{u,v} = 1/2$ if $u \neq v \in X \setminus \{x, y, z\}$.

So, by the definition of $\bar{t}^{\langle x,y,z \rangle}$, we are done. □

Claim 3.6. *Assume $\mathfrak{D} \subseteq \mathfrak{C}^{\rm full}$ is symmetric non-empty and $c \in \mathfrak{C}$ is pseudo-balanced then $c \in$ maj-cl$(\mathfrak{D})$.*

Proof. Without loss of generality $\mathfrak{D}$ is balanced (otherwise use 3.3). So by 3.5

⊛ if $x, y, z \in X$ are distinct then $\bar{t}^{\langle x,y,z \rangle} \in$ pr-cl$(\mathfrak{D})$.

Let $\langle \bar{x}^i : i < i(*) \rangle$ list the set cyc(c) of tuples $\bar{x} = \langle x_\ell : \ell \leq k \rangle$ such that:

$\odot$ (a) $k \geq 2, x_\ell \in X$

(b) $\ell_1 < \ell_2 \leq k \Rightarrow x_{\ell_1} \neq x_{\ell_2}$

(c) $c\{x_\ell, x_{\ell+1}\} = x_{\ell+1}$ for $\ell < k$

(d) $c\{x_k, x_0\} = x_0$.

For a tuple $\bar{x} = \langle x_\ell : \ell < m \rangle$ let $\ell g(\bar{x})$ be the length of $\bar{x}$, m.

Note

⊗ for every $\bar{x} \in$ cyc(c) for some $\bar{t} = \bar{t}^{\bar{x}} \in$ pr-cl$(\mathfrak{D})$ we have

(a) $t_{u,v} = \frac{1}{2} + \frac{1}{2(\ell g(\bar{x})-2)}$ <u>if</u> $(u, v) \in \{(x_{\ell_1}, x_{\ell_2}) : \ell_1 < \ell g(\bar{x}) - 1 \;\&\; \ell_2 = \ell_1 + 1 \text{ or } \ell_1 = \ell g(\bar{x}) - 1 \;\&\; \ell_2 = 0\}$

(b) $t_{u,v} = \frac{1}{2} - \frac{1}{2(\ell g(\bar{x})-2)}$ <u>if</u> (v, u) is as above

(c) $t_{u,v} = \frac{1}{2}$ if otherwise.

[Why? If $\bar{x} = \langle x_\ell : \ell \leq k\rangle$, let $\bar{t}$ be the arithmetic average of $\langle \bar{t}^{\langle x_0,x_1,x_2\rangle}, \bar{t}^{\langle x_0,x_2,x_3\rangle}, \ldots, \bar{t}^{\langle x_0,x_{k-1},x_k\rangle}\rangle$.]

Now let

$$\bar{t} = \Sigma \left\{ \frac{1}{i(*)} \bar{t}^{\bar{x}^i} : i < i(*) \right\}.$$

(In fact we just need that $c\{y_0, y_1\} = y_1 \Rightarrow (y_0, y_1)$ appears in at least one cycle $\bar{x}^i$, $i < i(*)$). As every edge of Tor(c) belongs to a directed cycle easily $c =$ maj($\bar{t}$). □

So now we can give a complete answer.

Conclusion 3.7. Assume

(a) $\mathfrak{D} \subseteq \mathfrak{C}^{\text{full}}$ is symmetric, non-empty
(b) $c \in \mathfrak{C}$.

Then $c \in$ maj-cl($\mathfrak{D}$) iff $\mathfrak{D}$ has a non-balanced member or c is pseudo-balanced.

Proof. If $\mathfrak{D}$ has no non-balanced member and c is not pseudo-balanced, by 3.2(3) we know $c \notin$ maj-cl($\mathfrak{D}$).

For the other direction, if $d^* \in \mathfrak{D}$ is not balanced use 3.3 that is (a)$' \Leftrightarrow$ (g) of claim 2.1 for sym-cl$\{d^*\}$. Otherwise $\mathfrak{D}$ is balanced non-empty, c is pseudo-balanced and we use 3.6. □

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