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Two cardinals models with gap one revisited

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We succeed to say something on the identities of (μ^+, μ) when $\mu > \theta > \mathrm{cf}(\mu)$ with μ strong limit θ-compact or even μ is limit of compact cardinals.

0 Introduction

There has been much work on κ-compactness of pairs (λ, μ) of cardinals, i. e. when the following holds:

> If T is a set of first order sentences of cardinality $\leq \kappa$ and every finite subset has a (λ, μ)-model M (i. e. a model M of cardinality λ with $|P^M| = \mu$ for a fixed unary P), then T has a (λ, μ)-model.

A particularly important case is $\lambda = \mu^+$ in which case this can be represented as a problem on the κ-compactness of the logic $\mathbb{L}(\mathsf{Q}^{\mathrm{card}}_{\geq\lambda})$, where $(\mathsf{Q}^{\mathrm{card}}_{\geq\lambda} x)\, \varphi$ says that there are at least λ elements x satisfying φ. We deal here only with this case. See Furkhen [1] , Morley and Vaught [6], Keisler [4], Mitchel [5]; for more history see [7].

Now two cardinal theorems can be translated to partition problems so-called identities (Definition 0.2): see [8, 9], lately Shelah and Väänänen [12] or [13].

Restricting ourselves to pairs (μ^+, μ), the identities of $(\aleph_1, \aleph_0)$ were sorted out in [11], but we do not know of the identities of any really different pair (μ^+, μ), i. e. one for which $(\aleph_1, \aleph_0) \nrightarrow (\mu^+, \mu)$. We know that (consistently) some pairs (μ^+, μ) have a different set of identities than $(\aleph_1, \aleph_0)$ but we do not have a characterization in any of those cases. By Mitchel [5] this applies to $(\aleph_2, \aleph_1)$ in the universe gotten by forcing: suitably collapsing of a Mahlo strongly inaccessible to $\aleph_2$. The other such case is when there is a compact cardinal in the interval $(\mathrm{cf}(\mu), \mu)$ by Litman and Shelah. So it would be nice to know (taking the extreme case):

Question 0.1 Assume μ is a singular cardinal the limit of compact and even supercompact cardinals.

1. What are the identities of (μ^+, μ) ?
2. Is (μ^+, μ) $\aleph_0$-compact (equivalently μ-compact) ?

Note that though we already know that there are some identities of (μ^+, μ) which are not identities of $(\aleph_1, \aleph_0)$ we have no explicit example. We give here a partial solution to Question 0.1, 1. by finding families of such identities.

Another problem is consistency of failure of compactness. In [7] we have dealt with the simplest case for pairs (λ, μ) by a reasonable criterion: including no use of large cardinals. From another perspective the simplest case is the consistency of non-compactness of $\mathbb{L}(\mathsf{Q})$, Q a one cardinality quantifier, and the simplest one is $\mathsf{Q} = \exists^{\geq \mu^+}$. So we are again drawn to pairs (μ^+, μ), that is gap one instead of gap two as in [7], so necessarily we need to use large cardinals as if, e. g., $\neg\exists 0^{\#}$, then every such pair is compact.

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Definition 0.2

1. A *partial identity*[1)] $\boldsymbol{s}$ is a pair $(a, e) = (\mathrm{Dom}_{\boldsymbol{s}}, e_{\boldsymbol{s}})$, where a is a finite set and e is an equivalence relation on a subfamily of the family of the finite subsets of a, having the property that $b\, e\, c$ implies $|b| = |c|$. The equivalence class of b with respect to e will be denoted b/e.

1A. We say $\boldsymbol{s}$ is a *full identity* or *identity* if $\mathrm{Dom}(e) = \mathcal{P}(a)$.

1B. We say that partial identities $\boldsymbol{s}_1 = (a_1, e_1)$ and $\boldsymbol{s}_2 = (a_2, e_2)$ are *isomorphic* if there is an isomorphism h from $\boldsymbol{s}_1$ onto $\boldsymbol{s}_2$, which means that h is a one-to-one function from a_1 onto a_2 such that for every $b_1, c_1 \subseteq a_1$ we have $b_1\, e_1\, c_1$ if and only if $h(b_1)\, e_2\, h(c_1)$ (so h maps $\mathrm{Dom}(e_1)$ onto $\mathrm{Dom}(e_2)$). Similarly we define that h is an *embedding of* $\boldsymbol{s}_1$ *into* $\boldsymbol{s}_2$ when $b_1\, e_1\, c_1$ implies $h(b_1)\, e_2\, h(c_1)$.

2. We write $\lambda \to (a, e)_\mu$ if (a, e) is an identity or a partial identity and for every function $f : [\lambda]^{<\aleph_0} \longrightarrow \mu$ there is a one-to-one function $h : a \longrightarrow \lambda$ such that $b\, e\, c$ implies $f(h\text{''}(b)) = f(h\text{''}(c))$. (Instead $\mathrm{Ran}(f) \subseteq \mu$ we may just require $|\mathrm{Ran}(f)| \leq \mu$, this is equivalent.)

3. We define

$$\mathrm{ID}(\lambda, \mu) := \{(n, e) : n < \omega,\ (n, e) \text{ is an identity and } \lambda \to (n, e)_\mu\},$$

and for $f : [\lambda]^{<\aleph_0} \longrightarrow X$ we define $\mathrm{ID}(f)$ as

$$\{(n, e) : (n, e) \text{ is an identity such that for some one-to-one function } h \text{ from } n = \{0, \dots, n-1\} \text{ into } \lambda\text{: for any } b,\ c \subseteq n, \text{ if } b\, e\, c, \text{ then } f(h\text{''}(b)) = f(h\text{''}(c)).$$

Clearly two-place functions are easier to understand; this motivates:

Definition 0.3

1. A *two-identity* or 2-*identity*[2)] is a pair (a, e), where a is a finite set and e is an equivalence relation on $[a]^2$. Let $\lambda \to (a, e)_\mu$ mean $\lambda \to (a, e^+)_\mu$, where $e^+ = e \cup \{(b, b) : b \subseteq a\}$.

2. We define

$$\mathrm{ID}_2(\lambda, \mu) := \{(n, e) : (n, e) \text{ is a 2-identity and } \lambda \to (n, e)_\mu\},$$

and for $f : [\lambda]^2 \longrightarrow X$ we define $\mathrm{ID}_2(f)$ as

$$\{(n, e) : (n, e) \text{ is a 2-identity such that for some one-to-one function } h \text{ from } n = \{0, \dots, n-1\} \text{ into } \lambda\text{: if } \{\ell_1, \ell_2\}\, e\, \{k_1, k_2\}, \text{ then } \ell_1 \neq \ell_2 \in \{0, \dots, n-1\},\ k_1 \neq k_2 \in \{0, \dots, n-1\} \text{ and } f(\{h(\ell_1), h(\ell_2)\}) = f(\{h(k_1), h(k_2)\})\}.$$

3. Let us define

$$\mathrm{ID}_2^{\circledast} := \{({}^n 2, e) : ({}^n 2, e) \text{ is a 2-identity and if } \{\eta_1, \eta_2\} \neq \{\nu_1, \nu_2\} \text{ are subsets of } {}^n 2, \text{ then } \{\eta_1, \eta_2\}\, e\, \{\nu_1, \nu_2\} \text{ implies } \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}.$$

4. In parts 1. and 2. we may replace 2 by $k < \omega$ (only $k < |a|$ is interesting) and also by $(\leq k)$.

Discussion 0.4 By [10] under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $\mathrm{ID}_2(\aleph_\omega, \aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [2, 3] we considered the question of the equality between these $\mathrm{ID}_2(2^{\aleph_0}, \aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be "yes" and may be "no".

Note that $(\aleph_n, \aleph_0) \not\to (\aleph_\omega, \aleph_0)$ so $\mathrm{ID}(\aleph_2, \aleph_0) \neq \mathrm{ID}(\aleph_\omega, \aleph_0)$, but for identities for pairs (i. e. ID_2) the question is meaningful. We can look more at ordered identities.

1) An *identification* in the terminology of [9].

2) A two-identity is not an identity as e is an equivalence relation on a too small set but it is a partial identity.

Definition 0.5

1. An *order-identity* or *ord-identity* is an identity $\boldsymbol{s} = (a, e)$ such that a is an ordered set.
2. For $\boldsymbol{c} : [\lambda]^{<\aleph_0} \longrightarrow \mu$ let

$$\mathrm{OID}(\boldsymbol{c}) := \{(a, e) \,:\, a \text{ is a set of ordinals and there is an order preserving function } f : a \longrightarrow \lambda \text{ such that } b_1 \, e \, b_2 \text{ implies } \boldsymbol{c}(f\text{"}(b_1)) = \boldsymbol{c}(f\text{"}(b_2)) \text{ for any } b_1,\, b_2 \subseteq a\,\}.$$

3. $\mathrm{OID}(\lambda, \mu) := \{(n, e) \,:\, (n, e) \in \mathrm{OID}(\boldsymbol{c}) \text{ for every } \boldsymbol{c} :\, [\lambda]^{<\aleph_0} \longrightarrow \mu\}$.
4. Similarly one defines OID_2, OID_k, and $\mathrm{OID}_{\leq k}$.
5. We write $\lambda \rightarrow_{\mathrm{ord}} (\boldsymbol{s})_\mu$ if $\boldsymbol{s}$ is an ord-identity and for every $\boldsymbol{c} :\, [\lambda]^{<\aleph_0} \longrightarrow \mu$ we have $\boldsymbol{s} \in \mathrm{OID}(\boldsymbol{c})$, see below (equivalently $\mathrm{Dom}(\boldsymbol{c}) = [\lambda]^{<\aleph_0}$, $|\mathrm{Ran}(\boldsymbol{c})| \leq \mu$).

Of course,

Claim 0.6

1. $\mathrm{ID}(\lambda, \mu)$ *can be computed from* $\mathrm{OID}(\lambda, \mu)$.
2. *Let* a *be a finite set of ordinals and* e *an equivalence relation. If* (a, e) *is an identity,* a *a set of ordinals and* $\lambda > \mu$*, then* $(a, e) \in \mathrm{ID}(\lambda, \mu)$ *if and only if for some permutation* π *of* a *we have* $(a, e^\pi) \in \mathrm{OID}(\lambda, \mu)$*, where* $e^\pi = \{(b, c) \,:\, (\pi''(b), \pi''(c)) \in e\}$.
3. *Let* A *be a set of ordinals,* (a, e) *an ord-identity and* $\boldsymbol{c}$ *a function with domain* $[A]^{<\aleph_0}$*. Then* $(a, e) \in \mathrm{ID}(\boldsymbol{c})$ *if and only if for some permutation* π *of* a*,* $(a, e^\pi) \in \mathrm{OID}(\boldsymbol{c})$.
4. *Similarly for* 2*-identities,* k*-identities,* $(\leq k)$*-identities, and partial identities.*

Claim 0.7 *If* $n \in [1, \omega)$ *and* $\boldsymbol{s}$ *is an ordered partial identity, then there is a first order sentence* $\psi_{\boldsymbol{s}}$ *such that* $\psi_{\boldsymbol{s}}$ *has a* (μ^{+n}, μ)*-model if and only if* $\boldsymbol{s} \notin \mathrm{OID}(\mu^{+n}, \mu)$.

Proof. Easy as for some first order ψ sentence, if M is a (μ^{+n}, μ)-model of ψ, then $<^M$ is a linear order of M (of cardinality μ^{+n}) which is μ^{+n}-like (i. e. every initial segment has cardinality $< \mu^{+n}$). □

Definition 0.8

1. For $k < \aleph_0$, we say that (λ, μ) *has* k*-simple identities* if $(a, e) \in \mathrm{ID}(\lambda, \mu)$ implies that $(a, e') \in \mathrm{ID}(\lambda, \mu)$, whenever

$(*)_k$ $a \subseteq \omega$, (a, e) is an identity of (λ, μ) and e' is defined by
$b \, e' \, c$ iff $|b| = |c|$ and for all b', c', if $b' \subseteq b$, $|b'| \leq k$ and $c' = \mathrm{OP}_{c,b}(b')$, then $b' \, e \, c$.

Recall that $\mathrm{OP}_{B,A}(\alpha) = \beta$ if and only if $\alpha \in A$ and $\beta \in B$ and $\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B)$.

2. Similarly we define (λ, μ) *has* k*-simple ordered identities*.

We can ask[3]:

Question 0.9

1. Define reasonably a pair (λ, μ) such that consistently

$\circledast$ $\mathrm{ID}(\lambda, \mu)$ is not recursive,

$\circledast'$ $\mathrm{ID}(\lambda, \mu)$ is not, in a reasonable way, finitely generated.

2. Similarly for $\mathrm{ID}_2(\lambda, \mu)$.
3. Restrict yourself to (μ^+, μ).

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[3] A new e-version of [7] probably is relevant.

1 2-simplicity for gap one

Claim 1.1

1. *If μ is strong limit regular, then* $\mathrm{ID}_2(\mu^+,\mu)$ *is* 2*-simple.*
2. *If $\mu = \mu^{<\mu}$ and $c_0 : [\mu^+]^{<\aleph_0} \longrightarrow \mu$, then we can find $c^* : [\mu^+]^2 \longrightarrow \mu$ such that*
   (a) *if $n \in [2,\omega)$, $\alpha_0,\dots,\alpha_{n-1} < \mu^+$ are without repetitions, $\beta_0,\dots,\beta_{n-1} < \mu^+$ are without repetitions, and $\ell < k < n$ implies $c^*\{\alpha_\ell,\alpha_k\} = c^*\{\beta_\ell,\beta_k\}$, then $c_0\{\alpha_0,\dots,\alpha_{n-1}\} = c_0\{\beta_0,\dots,\beta_{n-1}\}$,*
   (b) *if in addition $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$, then $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2}$ and $\beta_{n-3} < \beta_{n-1}$.*
3. *In 2., if $|\mathrm{Ran}(c)| < \mu$, then $2^{<\mu} = \mu$ (e. g., μ is singular) is enough.*

Remark 1.2 If we know part 1. in Claim 1.1, we may wonder what is the gain in Claim 1.1, 2., as if $\mu = 2^{<\mu}$ is regular, then we know all relevant theory on (μ^+,μ) ?

The answer is that it somewhat clarifies identities of triples (μ^+,μ,κ), e. g.

(a) (μ^+,μ,κ), μ strong limit regular $> \kappa \geq \mathrm{cf}(\mu)$

(b) (μ^+,μ,κ), $\mu = \mu^{\beth_\omega(\kappa)}$.

P r o o f o f C l a i m 1.1. 1. follows from 2., and 2. and 3. are easy by Subclaims 1.3 – 1.7 below (see details at the end).

Subclaim 1.3 *There is $c_1 : [\mu^+]^2 \longrightarrow \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and β_0, β_1, $\beta_2 < \mu^+$ are without repetitions and $c_1\{\beta_\ell,\beta_k\} = c_1\{\alpha_\ell,\alpha_k\}$ for $\ell < k < 3$, then at least two of the following statements hold: $\beta_0 < \beta_1$, $\beta_0 < \beta_2$, $\beta_1 < \beta_2$.*[4)]

P r o o f. Let $\eta_\alpha \in {}^\mu 2$ for $\alpha < \mu^+$ be pairwise distinct and for α, $\beta < \mu^+$ with $\alpha \neq \beta$ let

$$\varepsilon\{\alpha,\beta\} = \min\{\varepsilon : \eta_\alpha \restriction \varepsilon \neq \eta_\beta \restriction \varepsilon\}.$$

Define the function c_1' with domain $[\mu^+]^2$ by

$$c_1'\{\alpha,\beta\} = \{\eta_\alpha \restriction \varepsilon\{\alpha,\beta\},\ \eta_\beta \restriction \varepsilon\{\alpha,\beta\}\}.$$

Then $|\mathrm{Ran}(c_1')| \leq \mu$, because $\mu = 2^{<\mu}$. For $\alpha < \beta$, let

$$c_1''\{\alpha,\beta\} = \begin{cases} 1 & \text{if } \eta_\alpha <_{\mathrm{lex}} \eta_\beta, \\ 0 & \text{otherwise} \end{cases} \qquad \text{(the Sierpiński colouring).}$$

Lastly, define c_1 by $c_1\{\alpha,\beta\} = (c_1'\{\alpha,\beta\}, c_1''\{\alpha,\beta\})$. This is a function with domain $[\mu^+]^2$ and range of cardinality $\leq \mu$, and easily it is as required. (The claim relates to $c_1 : [\mu^+]^2 \longrightarrow \mu$, but clearly you can create such function out of our c_1, since $|\mathrm{Ran}(c_1)| \leq \mu$.) □

Now check

Subclaim 1.4 *For every $c : [\mu^+]^{<\aleph_0} \longrightarrow \mu$ there is a function $c_2 : [\mu^+]^2 \longrightarrow \mu$ such that the following holds: if $n \geq 2$, $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \mu^+$, $\beta_0 < \beta_1 < \dots < \beta_{n-1} < \mu^+$, and $\ell < k < n$ implies that $c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$, then $c\{\alpha_0,\dots,\alpha_{n-1}\} = c\{\beta_0,\dots,\beta_{n-1}\}$.*

P r o o f. We are given $c : [\mu^+]^{<\aleph_0} \longrightarrow \mu$ and for each $\alpha < \mu^+$ let f_α be a one-to-one function from α into the ordinal $|\alpha| \leq \mu$ (we shall use those f_α's also later).

We define an equivalence relation E on $[\mu^+]^2$ by

$(*)$ for $\alpha_1 < \beta_1 < \mu^+$ and $\alpha_2 < \beta_2 < \mu^+$ let $\{\alpha_1,\beta_1\}\,E\,\{\alpha_2,\beta_2\}$ iff

(a) $f_{\beta_1}(\alpha_1) = f_{\beta_2}(\alpha_2)$ and

(b) for any $n < \omega$ and $\gamma_0 < \dots < \gamma_{n-1} < f_{\beta_1}(\alpha_1)$ we have:
if $(\forall \ell < n)\,(\gamma_\ell \in \mathrm{Ran}(f_{\beta_\ell}) \equiv \gamma_\ell \in \mathrm{Ran}(f_{\alpha_\ell}))$ and if $\gamma_\ell \in \mathrm{Ran}(f_{\beta_\ell})$ for $\ell < n$,
then $c\{\alpha_1,\beta_1,f_{\beta_1}^{-1}(\gamma_0),\dots,f_{\beta_1}^{-1}(\gamma_{n-1})\} = c\{\alpha_2,\beta_2,f_{\beta_2}^{-1}(\gamma_0),\dots,f_{\beta_2}^{-1}(\gamma_{n-1})\}$

and similarly if we omit α_1, α_2 and/or β_1, β_2.

4) Notice, that we have only three possibilities (not four): $\beta_0 < \beta_1 < \beta_2$, $\beta_1 < \beta_0 < \beta_2$, $\beta_0 < \beta_2 < \beta_1$.

So $[\mu^+]^2/E$ has cardinality $\leq {}^{\mu>}\mu = \mu$, and let $c_2 : [\mu^+]^2 \longrightarrow \mu$ be such that $c_2\{\alpha_1,\beta_1\} = c_2\{\alpha_2,\beta_2\}$ if and only if $\{\alpha_1,\beta_1\}/E = \{\alpha_2,\beta_2\}/E$. We now check that it is as required in Subclaim 1.4. Let n, $\langle\alpha_\ell : \ell < n\rangle$, $\langle\beta_\ell : \ell < n\rangle$ be as in Subclaim 1.4; so $\ell < k < n$ implies $c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$, hence by $(*)$(a) above (for $k = n-1$) we have that $\ell < n-1$ implies $f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-1}}(\beta_\ell)$, call it γ_ℓ; as $f_{\alpha_{n-1}}$ is one to one, clearly $\langle\gamma_\ell : \ell < n-1\rangle$ is without repetitions. Let $\ell(*) < n$ be such that $\gamma_{\ell(*)}$ is maximal, and for $\ell < n-2$ let γ'_ℓ be γ_ℓ if $\ell < \ell(*)$ and be $\gamma_{\ell+1}$ if $\ell \in [\ell(*), n-2)$. Now apply $(*)$(b) with $\alpha_{\ell(*)}$, α_{n-1}, $\beta_{\ell(*)}$, β_{n-2}, $\langle\gamma'_\ell : \ell < n-2\rangle$ here standing for α_1, β_1, α_2, β_2, $\langle\gamma_\ell : \ell < n-2\rangle$ there, and we get the desired result. □

Subclaim 1.5 *In Subclaim* 1.4*, using $f_\alpha : \alpha \longrightarrow \mu$ as in its proof, we have*

$$c\{\alpha_0,\dots,\alpha_{n-3},\alpha_{n-2},\alpha_{n-1}\} = c\{\beta_0,\dots,\beta_{n-3},\beta_{n-1},\beta_{n-2}\}$$

also when

$(*)$ $n \geq 2$, $\alpha_0 < \alpha_1 < \dots < \alpha_{n-3} < \alpha_{n-2} < \alpha_{n-1} < \mu^+$, $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2} < \mu^+$, $\ell < n-2$ *implies* $f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$, *and* $\ell < k < n$ *implies* $c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$.

Proof. Just the same proof. □

Subclaim 1.6 *There is $c_4 : [\mu^+]^2 \longrightarrow \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$, $\beta_0, \beta_1, \beta_2 < \mu^+$ are without repetitions, and $c_4\{\beta_\ell,\beta_k\} = c_4\{\alpha_\ell,\alpha_k\}$ for $\ell < k < 3$, then $\beta_0 < \beta_1$ and $\beta_0 < \beta_2$.*

Proof. For $\alpha < \beta < \mu^+$ we let $c'\{\alpha,\beta\} = \{f_\beta(\gamma) : \gamma < \alpha \text{ and } f_\beta(\gamma) < f_\beta(\alpha)\}$, and then we let $c_4\{\alpha,\beta\} = (c'\{\alpha,\beta\}, c_1\{\alpha,\beta\}, f_\beta(\alpha))$, where c_1 is from Subclaim 1.3 and $\langle f_\gamma : \gamma < \mu^+\rangle$ is from the proof of Subclaim 1.4. Clearly $|\mathrm{Ran}(c')| \leq \sum_{\zeta<\mu} 2^{|\zeta|} = \mu$, hence $|\mathrm{Ran}(c_4)| \leq \mu^3 = \mu$. If α_ℓ, β_ℓ ($\ell < 3$) form a counterexample, then $c_1\{\alpha_\ell,\alpha_k\} = c_1\{\beta_\ell,\beta_k\}$ for $\ell < k < 3$, hence by Subclaim 1.3 we have three cases according to which one of the inequalities $\beta_\ell < \beta_k$, $\ell < k < 3$ fail.

Case (i): $\beta_0 < \beta_1 < \beta_2$. Trivial: the desired conclusion holds.

Case (ii): $\beta_1 < \beta_0$, so $\beta_1 < \beta_0 < \beta_2$. Let $\zeta_\ell = f_{\alpha_2}(\alpha_\ell)$ for $\ell = 0, 1$, hence $\zeta_0 \neq \zeta_1$ as f_{α_2} is one to one and $\zeta_\ell = f_{\beta_2}(\beta_\ell)$. Now on the one hand, if $\zeta_0 < \zeta_1$, then $c'\{\alpha_1,\alpha_2\} \neq c'\{\beta_1,\beta_2\}$ (as $\zeta_0 \in c'\{\alpha_1,\alpha_2\}$, $\zeta_0 \notin c'\{\beta_1,\beta_2\}$), a contradiction. But on the other hand, if $\zeta_1 < \zeta_0$, then $c'\{\alpha_0,\alpha_2\} \neq c'\{\beta_0,\beta_2\}$ (as $\zeta_1 \in c'\{\beta_0,\beta_2\}$, $\zeta_1 \notin c'\{\alpha_0,\alpha_2\}$), a contradiction, too.

Case (iii): $\beta_2 < \beta_1$. By Subclaim 1.3 we have $\beta_0 < \beta_2 < \beta_1$. This is O.K. for Subclaim 1.6. □

Subclaim 1.7 *For every $c : [\mu^+]^2 \longrightarrow \mu$ there is a function $c_5 : [\mu^+]^2 \longrightarrow \mu$ and $\bar g = \langle g_\alpha : \alpha < \mu^+\rangle$, $g_\alpha : \alpha \longrightarrow \mu$ is one to one, such that*

(a) *$c_5\{\alpha_1,\beta_1\} = c_5\{\alpha_2,\beta_2\}$ implies $c_2\{\alpha_1,\beta_1\} = c_2\{\alpha_2,\beta_2\}$, where c_2 is from Subclaim* 1.4 (*so also from Subclaim* 1.5)*;*

(b) *there are no $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0 < \beta_1 < \beta_2 < \mu^+$ such that $g_{\alpha_2}(\alpha_0) \neq g_{\alpha_1}(\alpha_0)$, $c_5\{\alpha_0,\alpha_1\} = c_5\{\beta_0,\beta_2\}$, $c_5\{\alpha_0,\alpha_2\} = c_5\{\beta_0,\beta_1\}$ and $c_5\{\alpha_1,\alpha_2\} = c_5\{\beta_1,\beta_2\}$;*

(c) *if $c_5\{\alpha_1,\beta_1\} = c_5\{\alpha_2,\beta_2\}$, then $c_4\{\alpha_1,\beta_1\} = c_4\{\alpha_2,\beta_2\}$, where c_4 is from Subclaim* 1.6*;*

(d) *like Subclaims* 1.4 *and* 1.5*, replacing c_2 by c_5 and f_α by g_α.*

Proof. Let $\kappa = \mathrm{cf}(\mu) \leq \mu$ and $\mu = \sum_{i<\kappa} \lambda_i$ be such that if μ is a limit cardinal, then λ_i is (strictly) increasing continuous, and if μ is a successor cardinal, then $\mu = \lambda^+$, $\kappa = \mu$ and $\lambda_i = \lambda$ for $i < \kappa$. We can find $d : [\mu^+]^2 \longrightarrow \kappa$ and $\bar g$ such that

$\circledast_0$ (i) for $\beta < \mu^+$ and $i < \kappa$ the set $A_{\beta,i} := \{\alpha < \beta : d\{\alpha,\beta\} \leq i\}$ has cardinality $\leq \lambda_i$;
(ii) if $\alpha < \beta < \gamma < \mu^+$, then $d\{\alpha,\gamma\} \leq \max\{d\{\alpha,\beta\}, d\{\beta,\gamma\}\}$;
(iii) $\bar g$ is a sequence $\langle g_\alpha : \alpha < \mu^+\rangle$;
(iv) $g_\alpha : \alpha \longrightarrow \mu$ is one to one and if $\lambda_i^+ < \mu$, $i < \kappa$ and $\alpha < \beta$, then $g_\beta(\alpha) < \lambda_i^+$ iff $d\{\alpha,\beta\} \leq i$;
(v) if $\alpha < \beta$, $d\{\alpha,\beta\} = i$ and $\lambda_i^+ = \mu$, then $g_\beta(\alpha) < d\{\alpha,\beta\}$.

(Why we can find them ? By induction on $\beta < \mu^+$ arriving to β, for each $i < \kappa$ let $B_{\beta,i}$ be the minimal subset of A such that: $\alpha_1 \wedge \alpha_2 \wedge \alpha_1 \in A \wedge d\{\alpha_1, \alpha_2\} \le i \rightarrow \alpha_1 \in A$.

So $B_{\beta,i}$ has cardinality $\le \lambda_i$ and is increasing continuous with i and $\beta = \bigcup\{B_{\beta,i} : i < \kappa\}$.

If $(\forall i < \kappa)\,(\lambda_i^+ < \mu)$, then for $\alpha < \beta$ we let $d\{\alpha,\beta\} = \min\{i : \alpha \in B_{\beta,i}\}$ and then choose the function g_β as in clause (iv) by choosing $g_\alpha \restriction B_{\beta,i}$ by induction on $i < \kappa$.

If $(\forall i < \kappa)\,(\lambda_i^+ = \mu)$, we let $g_\beta : \beta \longrightarrow \mu$ be one-to-one and then choose $\langle d\{\alpha,\beta\} : \alpha < \beta\rangle$ as required.)

Define the functions c_6' and c_7' with domain $[\mu^+]^2$ as follows: if $\alpha < \beta$, then

$$\begin{aligned} c_6'\{\alpha,\beta\} = \{(t,\zeta_1,\zeta_2) : {} & \zeta_1,\zeta_2 < \lambda^+_{d\{\alpha,\beta\}} \text{ when } \lambda^+_{d\{\alpha,\beta\}} < \mu \text{ and } \zeta_1,\zeta_2 < d\{\alpha,\beta\} \\ & \text{and } \zeta_1,\zeta_2 \in \mathrm{Ran}(g_\alpha) \text{ when } \lambda^+_{d\{\alpha,\beta\}} = \mu, \\ & t < 2, \\ & \text{and if } t = 0, \text{ then } g_\beta^{-1}(\zeta_1) < g_\beta^{-1}(\zeta_2), \\ & \text{and if } t = 1, \text{ then } g_\beta^{-1}(\zeta_1) > g_\beta^{-1}(\zeta_2)\} \end{aligned}$$

and

$$\begin{aligned} c_7'\{\alpha,\beta\} = \{(t,\zeta,\xi) : {} & \zeta \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Ran}(g_\alpha),\ \xi \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Ran}(g_\beta), \\ & \text{and (if } \lambda^+_{d\{\alpha,\beta\}} = \mu, \text{ then } \zeta < d\{\alpha,\beta\} \text{ and } \xi < d\{\alpha,\beta\}), \\ & \text{and } (g_\alpha^{-1}(\zeta) < g_\beta^{-1}(\xi) \text{ and } t = 0) \text{ or } (g_\alpha^{-1}(\zeta) = g_\beta^{-1}(\xi) \text{ and } t = 1) \\ & \qquad \text{or } (g_\alpha^{-1}(\zeta) > g_\beta^{-1}(\xi) \text{ and } t = 2)\}. \end{aligned}$$

Now for $\alpha < \beta < \mu^+$ we define $c_5'\{\alpha,\beta\} \in \prod\{\lambda_j^+ : j \le d\{\alpha,\beta\}\}$. We do this by induction on β and for a fixed β by induction on $i = d\{\alpha,\beta\}$ and for a fixed β and i by induction on α.

Arriving to $\alpha < \beta$, for each $j \le d\{\alpha,\beta\}$, let $c_5'\{\alpha,\beta\}(j)$ be the first ordinal $\xi < \lambda_j^+$ such that

$\circledast_1$ if $\gamma < \beta$, $d\{\gamma,\beta\} \le j$, and $d\{\gamma,\beta\} = d\{\alpha,\beta\}$ implies $\gamma < \alpha$, then $c_5'\{\alpha,\gamma\}(j) < \xi$.

This is clearly possible. Choose $c_2'' : [\mu^+]^2 \longrightarrow \mu$ as in Subclaim 1.4 using $\bar g$ instead of $\bar f$. The colouring we use is c_5, where for $\alpha < \beta < \mu^+$ we let

$$c_5\{\alpha,\beta\} = (d\{\alpha,\beta\}, g_\beta(\alpha), f_\beta(\alpha), c_2\{\alpha,\beta\}, c_5'\{\alpha,\beta\}, c_6'\{\alpha,\beta\}, c_7'\{\alpha,\beta\}, c_4\{\alpha,\beta\}, c_2''\{\alpha,\beta\}),$$

recalling that c_4 is from Subclaim 1.6 and c_2 is from Subclaim 1.4. Obviously, $|\mathrm{Ran}(c_5)| \le \mu$ and clauses (a), (b) and (d) of Subclaim 1.7 hold. So assume that $\alpha_0 < \alpha_1 < \alpha_2$, $\beta_0 < \beta_1 < \beta_2$ form a counterexample to clause (b) of Subclaim 1.7 and we shall eventually derive a contradiction.

Clearly

$\circledast_2$ (i) $d\{\alpha_0,\alpha_2\} = d\{\beta_0,\beta_1\}, d\{\alpha_0,\alpha_1\} = d\{\beta_0,\beta_2\}, d\{\alpha_1,\alpha_2\} = d\{\beta_1,\beta_2\}$,
(ii) similarly for c_4, c_5', c_6', c_7'.

By clause $\circledast_0$(ii) above we have $d\{\alpha_0,\alpha_2\} \le \max\{d\{\alpha_0,\alpha_1\}, d\{\alpha_1,\alpha_2\}\}$, and applying clause $\circledast_0$(ii) to $\beta_0 < \beta_1 < \beta_2$ and using $\circledast_2$ we have

$$d\{\alpha_0,\alpha_1\} = d\{\beta_0,\beta_2\} \le \max\{d\{\beta_0,\beta_1\}, d\{\beta_1,\beta_2\}\} = \max\{d\{\alpha_0,\alpha_2\}, d\{\alpha_1,\alpha_2\}\}.$$

Hence either $d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$ or $d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}$ for $\ell = 1,2$. We deal with those two cases separately.

Case 1: $\varepsilon = d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$. So (see the definition of c_5' with $\alpha_0,\alpha_2,\alpha_1,\varepsilon$ here standing for α,β,γ,j there, recalling that $\alpha_0 < \alpha_1 < \alpha_2$) we have $\lambda_\varepsilon^+ > c_5'\{\alpha_0,\alpha_2\}(\varepsilon) > c_5'\{\alpha_0,\alpha_1\}(\varepsilon)$. Similarly, $\lambda_\varepsilon^+ > c_5'\{\beta_0,\beta_2\}(\varepsilon) > c_5'\{\beta_0,\beta_1\}(\varepsilon)$. This contradicts $c_5'\{\alpha_0,\alpha_\ell\} = c_5'\{\beta_0,\beta_{3-\ell}\}$ for $\ell = 1,2$.

Case 2: $d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}$ for $\ell = 1,2$. Let $\varepsilon = d\{\alpha_1,\alpha_2\}$. Let $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$ for $\ell = 1,2$, so $\zeta_\ell = g_{\beta_{3-\ell}}(\beta_0)$ for $\ell = 1,2$. By the assumption toward contradiction, i.e. by a demand in Subclaim 1.7(b), we have $\zeta_1 \ne \zeta_2$. Clearly $\zeta_\ell < \lambda^+_{d\{\alpha_0,\alpha_\ell\}} \le \lambda^+_{d\{\alpha_1,\alpha_2\}} = \lambda_\varepsilon^+$ and $\lambda_\varepsilon^+ = \mu$ implies $\zeta_\ell < d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\} = \varepsilon$.

As $c'_7\{\alpha_1, \alpha_2\} = c'_7\{\beta_1, \beta_2\}$ and $g^{-1}_{\alpha_1}(\zeta_1) = \alpha_0 = g^{-1}_{\alpha_2}(\zeta_2)$ clearly $g^{-1}_{\beta_1}(\zeta_1) = g^{-1}_{\beta_2}(\zeta_2)$ and they are well defined.

For $\ell = 1, 2$ as $c_5\{\alpha_0, \alpha_\ell\} = c_5\{\beta_0, \beta_{3-\ell}\}$ by the choice of ζ_ℓ (that is $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$) we have $g_{\beta_\ell}(\beta_0) = \zeta_{3-\ell}$, so $g^{-1}_{\beta_\ell}(\zeta_{3-\ell}) = \beta_0$ for $\ell = 1, 2$, and hence $g^{-1}_{\beta_1}(\zeta_2) = g^{-1}_{\beta_2}(\zeta_1)$. As $c_5\{\alpha_1, \alpha_2\} = c_5\{\beta_1, \beta_2\}$ we have $c'_6\{\alpha_1, \alpha_2\} = c'_6\{\beta_1, \beta_2\}$ but $\zeta_1, \zeta_2 < \lambda^+_\varepsilon$ when $\lambda^+_\varepsilon < \mu$ and $\zeta_1, \zeta_2 < \varepsilon$ when $\lambda^+_\varepsilon = \mu$, hence

$\circledast_3$ $\quad g^{-1}_{\alpha_\ell}(\zeta_1) < g^{-1}_{\alpha_\ell}(\zeta_2)$ iff $g^{-1}_{\beta_\ell}(\zeta_1) < g^{-1}_{\beta_\ell}(\zeta_2)$ for $\ell = 1, 2$.

As $\zeta_1 \neq \zeta_2$ we have $g^{-1}_{\alpha_1}(\zeta_1) \neq g^{-1}_{\alpha_1}(\zeta_2)$.

By symmetry let w. l. o. g. $\zeta_1 > \zeta_2$. We can form an equivalence chain, starting with $g^{-1}_{\beta_1}(\zeta_1) < g^{-1}_{\beta_1}(\zeta_2)$ and arriving to $g^{-1}_{\beta_1}(\zeta_2) < g^{-1}_{\beta_1}(\zeta_1)$, a clear contradiction. Well,

$$\begin{aligned} g^{-1}_{\beta_1}(\zeta_1) < g^{-1}_{\beta_1}(\zeta_2) &\text{ iff } g^{-1}_{\beta_2}(\zeta_2) < g^{-1}_{\beta_2}(\zeta_1) && \text{(by the equalities above)}\\ &\text{ iff } g^{-1}_{\alpha_2}(\zeta_2) < g^{-1}_{\alpha_2}(\zeta_1) && \text{(by } \circledast_3)\\ &\text{ iff } g^{-1}_{\beta_1}(\zeta_2) < g^{-1}_{\beta_1}(\zeta_1) && \text{(by } c'_6\{\alpha_0, \alpha_2\} = c'_6\{\beta_0, \beta_1\} \text{ and use the}\\ &&& \text{parameter } t \text{ in the triple } (t, \zeta_1, \zeta_2)). \end{aligned}$$

So we have proved Subclaim 1.7. □

We can now sum up to obtain a proof of Claim 1.1, 2. from Subclaims 1.3 – 1.7.

We are given $c_0 : [\mu^+]^{<\aleph_0} \longrightarrow \mu$. First we apply Subclaim 1.4 for $c = c_0$ and get $c_2 : [\mu^+]^2 \longrightarrow \mu$ as there and let c_4 be as in Subclaim 1.6. Second, we apply Subclaim 1.7 for $c = c_2$ and get c_5 as there. Let us check that c_5 is as required on c^* in Claim 1.1, 2. So assume $(*)_0$ and $(*)_1$ below and (as the case $n = 2$ is trivial) assume $n \geq 3$, where

$(*)_0$ $\quad \{\alpha_0, \dots, \alpha_{n-1}\} \in [\mu^+]^n$ and $\{\beta_0, \dots, \beta_{n-1}\} \in [\mu^+]^n$,

$(*)_1$ $\quad$ if $\ell < k < n$, then $c_5\{\alpha_\ell, \alpha_k\} = c_5\{\beta_\ell, \beta_k\}$.

Without loss of generality (by renaming)

$(*)_2$ $\quad \alpha_0 < \dots < \alpha_{n-1}$,

and it is enough to prove that $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$. By clause (a) of Sublclaim 1.7 we have

$(*)_3$ $\quad$ if $\ell < k < n$, then $c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$.

By clause (c) of Subclaim 1.7 we have

$(*)_4$ $\quad$ if $\ell < k < n$, then $c_4\{\alpha_\ell, \alpha_k\} = c_4\{\beta_\ell, \beta_k\}$.

Hence by Subclaim 1.6 we have

$(*)_5$ $\quad$ if $\ell < k < n$ and $\ell < n - 2$, then $\beta_\ell < \beta_k$.

(Apply Subclaim 1.6 to α_ℓ, $\alpha_{\ell+1}$, α_k and β_ℓ, $\beta_{\ell+1}$, β_k if $\ell + 1 < k$, and apply Subclaim 1.6 to α_ℓ, $\alpha_{\ell+1}$, $\alpha_{\ell+2}$ and β_ℓ, $\beta_{\ell+1}$, $\beta_{\ell+2}$ if $\ell + 1 = k$.) So

$(*)_6$ $\quad$ (i) $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$ $\quad$ or $\quad$ (ii) $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$.

So clause (b) of Claim 1.1, 2. holds.

Now if (i) of $(*)_6$ holds, then by choice of c'_2, i. e. by Subclaims 1.4, 1.7(d) and $(*)_3$ above, we obtain $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$, so we are done. Otherwise we have (ii) of $(*)_6$, so by clause (b) of Subclaim 1.7 we have

$(*)_7$ $\quad$ if $\ell < n - 2$, then $g_{\alpha_{n-1}}(\alpha_\ell) = g_{\alpha_{n-2}}(\alpha_\ell)$.

(Apply clause (b) of Subclaim 1.7 to α_ℓ, α_{n-2}, α_{n-1} and β_ℓ, β_{n-1}, β_{n-2}.) Therefore, by Subclaim 1.5, i. e. (d) of Subclaim 1.7, we get $c_0\{\alpha_0, \dots, \alpha_{n-1}\} = c_0\{\beta_0, \dots, \beta_{n-1}\}$, finishing the proof of Claim 1.1, 2. □

Claim 1.8 *Defining* $\mathrm{ID}(\lambda, \mu)$, *we can restrict ourselves to* $c : [\lambda]^{<\aleph_0} \longrightarrow \mu$ *such that* $c \restriction [\lambda]^1$ *is constant if* $\mathrm{cf}(\lambda) > \mu$.

Claim 1.9 *Assume* $\mu = 2^{<\mu}$ *and* $n \in [1, \omega)$. *The identities of* $\mathrm{ID}(\mu^{+n}, \mu)$ *are* $(n+1)$*-simple* (*and analogous for* $\mathrm{OID}(\mu^+, \mu)$).

Proof. As for Claim 1.1, we shall give details elsewhere. □

2 Successor of strong limit above compact: 2-identities

So it follows that if μ is strong limit singular and there is a compact cardinal in $(\mathrm{cf}(\mu), \mu)$, then $\mathrm{ID}_2(\mu^+, \mu) \neq \mathrm{ID}_2(\aleph_1, \aleph_0)$. It seems desirable to find explicitly such 2-identities.

The proof of the following does much more.

Claim 2.1 *Assume*

(a) $\boldsymbol{s}_k = (k + \binom{k}{2}, e_{\boldsymbol{s}_k})$[5], *where the non-singleton* $e_{\boldsymbol{s}_k}$*-equivalence classes are the sets*

$$\{\{\ell_0, \ell_2\} : \ell_0 < k \text{ and for some } \ell_1 \in \{\ell_0 + 1, \ldots, k-1\} \text{ we have } \ell_2 = k + \tbinom{\ell_1}{2} + \ell_0\}$$

and

$$\{\{\ell_1, \ell_2\} : \ell_1 < k \text{ and for some } \ell_0 < \ell_1 \text{ we have } \ell_2 = k + \tbinom{\ell_1}{2} + \ell_0\};$$

(b) μ *is strong limit,* θ *is a compact cardinal and* $\mathrm{cf}(\mu) < \theta < \mu$.

Then

1. $\boldsymbol{s}_k \in \mathrm{ID}_2(\mu^+, \mu)$, *moreover* $\boldsymbol{s}_k \in \mathrm{OID}_2(\mu^+, \mu)$;

2. $\boldsymbol{s}_k \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $k \geq 3$, *where for* $k = 3$ *we have* $\boldsymbol{s}_k = (6, e_{\boldsymbol{s}_k})$ *and the non-singleton equivalence classes, after permuting* $\{3, 5\}$ *are* $\{\{1,3\}, \{0,4\}, \{0,5\}\}$ *and* $\{\{1,5\}, \{2,3\}, \{2,4\}\}$.

Proof. Part 1. follows from Claim 2.2, 3. below and part 2. follows from Subclaim 2.3 below. □

Claim 2.2 *Assume*

(a) μ *is strong limit,*

(b) θ *is compact and* $\mathrm{cf}(\mu) < \theta < \mu$,

(c) $\kappa = \mathrm{cf}(\mu)$ *and* $\langle \lambda_i : i < \kappa \rangle$ *is increasing with limit* μ,

(d) $c : [\mu^+]^2 \longrightarrow \mu$,

(e) $d\{\alpha, \beta\} = \min\{i : c\{\alpha, \beta\} < \lambda_i\}$.

1. *We can find* $i(*)$, A, f *such that*

$(*)$ (i) $i(*) < \kappa$, $A \in [\mu^+]^{\mu^+}$ *and* $f : A \longrightarrow \lambda_{i(*)}$,
(ii) *for every set* $B \subseteq A$ *of cardinality* $< \theta$ *there are* μ^+ *ordinals* $\gamma \in A$ *such that for all* $\alpha \in B$, $d\{\alpha, \gamma\} = i(*)$.

2. *In part* 1. *we also have:*

if $A_1 \subseteq A$, $|A_1| \geq \beth_n(\lambda)^+$, *and* $\lambda_{i(*)} \leq \lambda < \mu$, *then there are* $\langle \gamma_\ell : \ell < n \rangle \in {}^n(\lambda_{i(*)})$ *and* $B \in [A_1]^\lambda$ *such that for every* $\alpha_0 < \cdots < \alpha_{n-1}$ *from* B *for arbitrarily large* $\beta < \mu^+$ *we have that* $\ell < n$ *implies* $c\{\alpha_\ell, \beta\} = \gamma_\ell$.

3. $\boldsymbol{s}_k \in \mathrm{ID}_2(c)$, *where* $\boldsymbol{s}_k$ *is from Claim* 2.1(a).

Proof.

1. Let D be a uniform θ-complete ultrafilter on μ^+. Define $f : \mu^+ \longrightarrow \kappa$ by $f(\alpha) = i$ if and only if $\{\gamma < \mu^+ : d\{\alpha, \gamma\} = i\} \in D$. Note that the function f is well defined as D is a θ-complete ultrafilter on μ^+ and $\theta > \kappa \supseteq \mathrm{Ran}(d)$. So for some $i(*)$ the set $A := \{\alpha < \mu^+ : f(\alpha) = i(*)\}$ belongs to D and check that $(*)$ holds, that is (i) and (ii) hold.

[5] We stipulate $\binom{1}{2} = 0$ here.

2. Define $c^* : [A]^n \longrightarrow {}^n(\lambda_{i(*)})$ such that

$\circledast$ if $\alpha_0 < \cdots < \alpha_{n-1}$ are from A, then $\langle c\{\alpha_\ell, \beta\} : \ell < n\rangle = c^*\{\alpha_0, \ldots, \alpha_{n-1}\}$ for μ^+ ordinals $\beta < \mu^+$.

So $\mathrm{Ran}(c^*)$ has cardinality $\leq (\lambda_{i(*)})^n = \lambda_{i(*)}$, hence by the Erdös-Rado Theorem there is an infinite subset B of A_1 (even of cardinality λ^+) such that $c^* \restriction [B]^n$ is constant.

3. Straightforward: In part 2. use $n = 2$, $A_1 = A$ and get B and $\langle \gamma_0, \gamma_1\rangle \in {}^2(\lambda_{i(*)})$ as there and choose $\alpha_0 < \cdots < \alpha_{k-1}$ from B. Next choose α_ℓ for $\ell = 0, 1, \ldots, \binom{k}{2} - 1$, choosing β_ℓ by induction on ℓ. If $\ell = \binom{\ell_1}{2} + \ell_0$ and $\ell_0 < \ell_1 < k$ choose $\beta_\ell \in A$ satisfying $\beta_\ell > \alpha_{k-1}$ and $\beta_\ell > \beta_m$ for $m < \ell$ such that $c\{\alpha_{\ell_0}, \beta_\ell\} = \gamma_0$ and $c\{\alpha_{\ell_1}, \beta_\ell\} = \gamma_1$. Now let $\alpha_{k+\ell} = \beta_\ell$ for $\ell < \binom{k}{2}$, and clearly $\langle \alpha_\ell : \ell < k + \binom{k}{2}\rangle$ realize the identity $\boldsymbol{s}_k$. □

Subclaim 2.3

1. *If* $\boldsymbol{s} \in \mathrm{ID}_2(\aleph_1, \aleph_0)$, *then we can find a function* $h : [\mathrm{Dom}_{\boldsymbol{s}}]^2/e_{\boldsymbol{s}} \longrightarrow \omega$ *respecting* $e_{\boldsymbol{s}}$ (*that is, if* $\{\ell_1, \ell_2\}\, e_{\boldsymbol{s}}\, \{\ell_3, \ell_4\}$, *then* $h\{\ell_1, \ell_2\} = h\{\ell_3, \ell_4\}$) *and there is a linear order* $<$ *of* $\mathrm{Dom}_{\boldsymbol{s}}$ *satisfying*

$\circledast$ *for any equivalence class* $\boldsymbol{a}$ *of* $e_{\boldsymbol{s}}$ *there are* a_0, a_1 *such that*
(i) a_0, a_1 *are disjoint finite subsets of* $\mathrm{Dom}_{\boldsymbol{s}}$;
(ii) *if* $\{\ell_0, \ell_1\} \in \boldsymbol{a}$ *and* $\ell_0 < \ell_1$, *then* $\ell_0 \in a_0$ *and* $\ell_1 \in a_1$;
(iii) *if* $\ell_0 \in a_0$, $\ell_1 \in a_1$, $\{\ell_0, \ell_1\} \notin \boldsymbol{a}$ *and* $\{\ell^0, \ell^1\} \in \boldsymbol{a}$, *then* $h(\{\ell_0, \ell_1\}) > h(\{\ell^0, \ell^1\})$;
(iv) *if* $\{\ell_0, \ell_1\}\, e_{\boldsymbol{s}}\, \{\ell_0, \ell_2\}$ *and* $\ell_1 \neq \ell_2$, *then* $\ell_0 < \ell_1$.

2. *We can add in* $\circledast$:

(v) *if* $\boldsymbol{a}_0$, $\boldsymbol{a}_1$ *are distinct* $\boldsymbol{e_s}$*-equivalence classes, then for some* $m \in \{0, 1\}$, $[\bigcup \boldsymbol{a}_m]^2 \backslash \boldsymbol{a}_m$ *is disjoint to* $\boldsymbol{a}_{1-m}$;
(vi) *in* $\circledast$ *above* a_0, a_1 *can be defined as* $\{\ell_0 : \{\ell_0, \ell_1\} \in \boldsymbol{a},\ \ell_0 < \ell_1\}$, $\{\ell_1 : \{\ell_0, \ell_1\} \in \boldsymbol{a},\ \ell_0 < \ell_1\}$, *respectively.*

3. *If* $k \geq 3$, *then* $\boldsymbol{s}_k$ *from Claim* 2.1(a) *does not belong to* $\mathrm{ID}_2(\aleph_1, \aleph_0)$.

Proof.

1. Remember that by Claim 0.6 we can deal with $\mathrm{OID}(\aleph_1, \aleph_0)$. By [11] we know what is $\mathrm{OID}(\aleph_1, \aleph_0)$: The family of identities in $\mathrm{OID}(\aleph_1, \aleph_0)$ is generated by two operations – duplication and restriction (see below) – from the trivial identity (i. e. $|\mathrm{Dom}_{\boldsymbol{s}}| = 1$), and we prove $\circledast$ by induction on the number of times we need to apply these operations. We may restrict ourselves to ord-identities $\boldsymbol{s}$ with $\mathrm{Dom}_{\boldsymbol{s}}$ being a finite set of natural numbers or even integers.

Recall that (a, e) *is gotten by duplication* if we can find sets a_0, a_1, a_2 and a function g such that

$\circledast_1$ (a) $a_0 < a_1 < a_2$ (i. e. if $\ell_0 \in a_0$, $\ell_1 \in a_1$ and $\ell_2 \in a_2$, then $\ell_0 < \ell_1 < \ell_2$);
(b) $a = a_0 \cup a_1 \cup a_2$;
(c) g is a one-to-one order preserving function from $a_0 \cup a_1$ onto $a_0 \cup a_2$ (so $g \restriction a_0 = \mathrm{id}_{a_0}$); let $g_1 = g$ and $g_2 = g^{-1}$;
(d) for $\ell_0 \neq \ell_1 \in (a_0 \cup a_1)$ we have $\{\ell_0, \ell_1\}\, e\, \{g(\ell_0), g(\ell_1)\}$;
(e) if $\ell_1 \in a_1$ and $\ell_2 \in a_2$, then $\{\ell_1, \ell_2\}/e$ is a singleton;
(f) $\boldsymbol{s}_\ell = (a_0 \cup a_\ell, e \restriction [a_0 \cup a_\ell]^2)$ is from a lower level (up to isomorphism), for $\ell \in \{1, 2\}$.

Recall that (a, e) *is gotten by restriction from* (a', e') if $a \subseteq a'$ and $e = e' \restriction [a]^2$.

Now we prove the existence of h as required by induction on the level. If $|\mathrm{Dom}_{\boldsymbol{s}}| = 1$ this is trivial. If $\boldsymbol{s}$ is gotten by restriction from $\boldsymbol{s}'$ it is trivial too (as if $\boldsymbol{s} = (a, e)$, $\boldsymbol{s}' = (a', e')$, $a \subseteq a'$, $e = e' \restriction [a]^2$ and $h' : [a']^2 \longrightarrow \omega$ is as guaranteed for $\boldsymbol{s}'$, then we let $h(\{\ell_0, \ell_1\}) = h'(\{\ell_0, \ell_1\})$ for $\{\ell_0, \ell_1\} \in [a]^2$; easily h is as required). So assume that $\boldsymbol{s} = (a, e)$ is gotten by duplication. Let a_0, a_1, a_2, g_1, g_2 be as in $\circledast_1$ and let h_1 be as required for $\boldsymbol{s}_1 = (a_0 \cup a_1, e \restriction [a_0 \cup a_1]^2)$, and similarly define h_2 by $h_2\{\alpha,\beta\} = h_1\{g_2(\alpha), g_2(\beta)\}$. Let $n^* = \sup \mathrm{Ran}(h_1)$ and define $h : [a_0 \cup a_1 \cup a_2]^2 \longrightarrow \omega$ by $h \supseteq h_1$, $h \supseteq h_2$ and if $k \in a_1$, $\ell \in a_2$, then we let $h\{k,\ell\} = n^* + 1$. Check that h is as required for $\boldsymbol{s}$.

2. By symmetry, without loss of generality, $h(\boldsymbol{a}_0) < h(\boldsymbol{a}_1)$ and now $m = 1$ satisfies the requirement by applying $\circledast_1$ to the equivalence class $\boldsymbol{a} = \boldsymbol{a}_1$.

3. It is enough to deal with $\boldsymbol{s}_3$. By direct checking the criterion in part 2. fails. $\square$

The following is like Claim 2.1 with μ just limit (not necessarily a strong limit cardinal):

Claim 2.4 *Assume*

(a) $\boldsymbol{s}'_n \in \mathrm{OID}_2$ *is* $(2n + n^2, e_{\boldsymbol{s}'_n})$, *where the non-singleton* $e_{\boldsymbol{s}'_n}$*-equivalence classes are*

$$\{\{\ell_0, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\} \quad \textit{and} \quad \{\{n + \ell_1, 2n + n\ell_0 + \ell_1\} : \ell_0, \ell_1 < n\},$$

(b) μ *is a limit cardinal,* $\mu > \theta > \mathrm{cf}(\mu)$ *and* θ *is a compact cardinal,*

(c) $\boldsymbol{s}''_n \in \mathrm{OID}_2$ *is* $(2^n + 2^{2n}, e_{\boldsymbol{s}''_n})$, *where the non-singleton* $e_{\boldsymbol{s}''_n}$*-equivalence classes are*

$$\boldsymbol{a}^i_\eta = \{\{\ell_i, 2^n + \tbinom{2^n}{\ell_0} + \ell_1\} : \ell_0, \ell_1 < 2^n \text{ and for some } \nu_0, \nu_1 \in {}^n2 \text{ we have } \eta^\frown\langle 0\rangle \trianglelefteq \nu_0,\ \eta^\frown\langle 1\rangle \trianglelefteq \nu_1,\ \ell_0 = \textstyle\sum_{j<n} \nu_0(j)2^j, \text{ and } \ell_1 = \textstyle\sum_{j<n} \nu_1(j)2^j\},$$

where $m < n$, $\eta \in {}^m2$ *and* $i = 0, 1$.

Then

1. $\boldsymbol{s}'_n \in \mathrm{ID}_2(\mu^+, \mu)$, *and moreover* $\boldsymbol{s}'_n \in \mathrm{OID}_2(\mu^+, \mu)$, *and similarly for* $\boldsymbol{s}''_n$.
2. $\boldsymbol{s}'_n \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $n \geq 2$, *and similarly for* $\boldsymbol{s}''_n$.

Proof.

1. Like the proof of Claim 2.2 using [10] instead of the Erdös-Rado Theorem.

2. Otherwise there is $(a, e) \in \mathrm{ID}_2(\aleph_1, \aleph_0)$ and an embedding h of $\boldsymbol{s}'_n$ into (a, e) and by Claim 0.6 without loss of generality $(a, e) \in \mathrm{OID}_2(\aleph_1, \aleph_0)$. Now:

$(*)_1$ If $\ell_0 < n$, $\ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$, then $h(\ell_0) < h(\ell)$.

(Choose $\ell'_1 < n$, $\ell'_1 \neq \ell_1$ and $\ell' = 2n + n\ell_0 + \ell'_1$, so $\ell \neq \ell'$ and $\{\ell_0, \ell\}\, e_{\boldsymbol{s}'_n}\, \{\ell_0, \ell'\}$, hence the pairs $\{h(\ell_0), h(\ell)\}$, $\{h(\ell_0), h(\ell')\}$ are e-equivalent and $h(\ell) \neq h(\ell')$. But on (a, e) we know that if $\{m_0, m_1, m_2\}$ has three members and $\{m_0, m_1\}\, e\, \{m_0, m_2\}$, then necessarily $m_0 < m_2$ and $m_0 < m_1$ (see Observation 2.5, 2. below), so we are done.)

$(*)_2$ If $\ell_0 < n$, $\ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$, then $h(n + \ell_1) < h(\ell)$.

(Like $(*)_1$.) Now we apply Subclaim 2.3., 1. and 2. above, so $\boldsymbol{s}'_n \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$. The conclusion about $\boldsymbol{s}''_n$ follows. $\square$

Observation 2.5

1. *If* $k \geq 2$ *and* $\boldsymbol{s} = (n, e) \in \mathrm{OID}_2(\mu^+, \mu)$, *then we can find* $\boldsymbol{s}' = (n', e')$, *in fact* $n' = 2n - 1$, *such that*

(i) $e' \restriction [n]^2 = e$,

(ii) $\boldsymbol{s}' \in \mathrm{ID}_2(\mu^+, \mu)$,

(iii) *for every* $c : [\mu^+]^{<\aleph_0} \longrightarrow \mu$ *there is a function* $c' : [\mu^+]^{<\aleph_0} \longrightarrow \mu$ *refining* c (*i. e.* $c'(u_1) = c'(u_2)$ *implies* $c(u_1) = c(u_2)$) *such that if* $h : \{0, \ldots, 2n-2\} \longrightarrow \mu^+$ *is one to one and if* $u_1\, e'\, u_2$ *implies* $c'(h''(u_1)) = c'(h''(u_2))$, *then* $h \restriction \{0, \ldots, n-1\}$ *is increasing.*

2. *There is* $c : [\mu^+]^2 \longrightarrow \mu$ *such that if* α, β, γ *are distinct and* $c\{\alpha, \beta\} = c\{\alpha, \gamma\}$, *then* $\alpha < \beta$ *and* $\alpha < \gamma$.

3. *We can replace in* 1. (μ^+, μ) *by* (λ, μ) *if there is* $\boldsymbol{s} = (n, e) \in \mathrm{ID}(\lambda, \mu)$ *such that for some* $c : [\lambda]^{<\aleph_0} \longrightarrow \mu$ *we have*

$\circledast$ *if* $h : n \longrightarrow \lambda$ *induces* $e_{\boldsymbol{s}}$, *then* $h(0) < h(1)$.

Proof.

1. Define e' by

$$u_1\, e'\, u_2 \;:\leftrightarrow\quad u_1\, e\, u_2 \vee u_1 = u_2 \vee \bigvee_{\ell<n-1}(u_1 = \{\ell, n+\ell+1\} \wedge u_2 = \{\ell, \ell+1\}) \vee \bigvee_{\ell<n-1}(u_2 = \{\ell, n+\ell+1\} \wedge u_1 = \{\ell, \ell+1\}).$$

Easily $(2n-1, e') \in \mathrm{OID}(\mu^+, \mu)$. For clause (iii) we use part 2.

2. Let $f_\alpha : \alpha \longrightarrow \mu$ be one to one for $\alpha < \mu^+$ and let $<^*$ a dense linear order on μ^+ with $\{\alpha : \alpha < \mu\}$ a dense subset. Now choose $c_1 : [\mu^+]^2 \longrightarrow \mu$ such that $\alpha <^* \beta$ implies $\alpha <^* c_1\{\alpha, \beta\} <^* \beta$ and define $c_0 : [\mu^+]^2 \longrightarrow \{0, 1\}$ for $\alpha < \beta$ by

$$c_0\{\alpha, \beta\} = \begin{cases} 1 & \text{if } \alpha <^* \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Lastly, define $c : [\mu^+]^2 \longrightarrow \mu$ for $\alpha < \beta$ by

$$c\{\alpha, \beta\} = \mathrm{pr}(2f_\beta(\alpha) + c_0\{\alpha, \beta\}, c_1\{\alpha, \beta\})$$

with some pairing function pr.

3. Similar to part 1., only $|\mathrm{Dom}_{\mathbf{s}'}|$ is larger. □

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