JOURNAL OF ALGEBRA 142, 492–510 (1991)

On Whitehead Modules

PAUL C. EKLOF

University of California, Irvine, California 92717

AND

SAHARON SHELAH*

Hebrew University, Jerusalem, Israel, and Rutgers University, New Brunswick, New Jersey 08903

Communicated by Peter M. Neumann

Received January 23, 1990

It is proved that it is consistent with $ZFC + GCH$ that, for any reasonable ring R, for every R-module K there is a non-projective module M such that $\mathrm{Ext}^1_R(M, K) = 0$; in particular, there are Whitehead R-modules which are not projective. This is generalized to show that it is consistent that, for certain rings R, there are Whitehead R-modules which are not the union of a continuous chain of submodules so that all quotients are small Whitehead R-modules. An application to Baer modules is also given: it is proved undecidable in $ZFC + GCH$ whether there is a single test module for being a Baer module. © 1991 Academic Press, Inc.

INTRODUCTION

An R-module M is called a Whitehead module if $\mathrm{Ext}^1_R(M, R) = 0$. Clearly any projective R-module is Whitehead. Whitehead's Problem asked if every Whitehead group (i.e., $\mathbb{Z}$-module) is projective, i.e., free. The second author (in [S1, S2]) showed that this problem is not solvable in ZFC: the answer is affirmative assuming $V = L$ and negative assuming $MA + \neg CH$. Later [S3] he showed that it is consistent with $ZFC + GCH$ that the answer is negative; the treatment of this in [S4] uses the notion of uniformization.

In view of this, a natural question (posed by J. Adamek and J. Rosický) is whether it can be proved in ZFC (or $ZFC + GCH$) that there is *some* group K such that $\mathrm{Ext}^1_{\mathbb{Z}}(M, K) = 0$ only if M is free. A negative answer to this question will be given here (see 4.2). We also show that the cardinality of the first non-free Whitehead group can be arbitrarily large (see 2.8).

* Partially supported by the United States–Israel Binational Science Foundation. Pub. No. 379.

0021-8693/91 $3.00

Recent work, notably in [BFS], investigates Whitehead R-modules for other domains R. "Positive" structural results are obtained (for certain R) assuming $V=L$; in general these do not assert that the Whitehead module is projective, but rather that it is the union of a continuous chain of submodules so that quotients of successive members of the chain are "small" (i.e., of cardinality $\leqslant$ the cardinality of R) Whitehead modules. The "negative" results (which say that it is consistent with $ZFC+GCH$ that the positive result fails) use the methods of proper forcing and apply only to countable R. In this paper we obtain "negative" results for rings of arbitrary size (see 4.3, 4.4, and 4.5).

Other recent papers have studied Baer R-modules for arbitrary domains R. The module B is a Baer module if $\mathrm{Ext}^1_R(B, T)=0$ for all torsion modules T.) In this case, the characterization theorem is proved in ZFC (cf. [EF, EFS]). But here, in conjunction with results in [EFS], we are able to show that it is independent of $ZFC+GCH$ whether there is a fixed torsion module $\mathscr{T}$ such that a module B of projective dimension $\leqslant 1$ is Baer if and only if $\mathrm{Ext}^1_R(B, \mathscr{T})=0$ (see 4.7). Even for $\mathbb{Z}$-modules, this is new.

The main theorem (2.1) of this paper asserts the consistency of the existence of a tree with certain properties including a uniformization property. Sections 1 and 3 describe how to construct (generalized) Whitehead modules from this tree. Section 4 gives the applications to specific families of rings, as described above.

The proof of the main theorem in Section 2 assumes a knowledge of forcing. The rest of the paper assumes only a knowledge of basic algebraic notions, except that parts of Section 4 assume familiarity with [BFS] or [EFS].

We thank Alan Mekler for his comments, in particular for informing us of Adamek's question; Mekler independently answered Adamek's question.

1. The Construction

Throughout this paper, R will denote an infinite ring, and ρ the cardinality of R. In this section, R will be a ring which is not left perfect; that is, it has an infinite descending chain of principal right ideals. (For example, an integral domain is left perfect if and only if it is a field: see 4.1 below.) All modules are left R-modules. The following lemma is due essentially to Bass ([Ba, Lemma 1.1]; see also Lemma 5.7 of [EM1] or Proposition VII.1.1 of [EM2]).

1.1. Lemma. *There is an increasing chain $\langle F_i\colon i\leqslant\omega\rangle$ of free R-modules of cardinality ρ such that for all $i<j\leqslant\omega$, F_j/F_i is free, but $F_\omega/(\bigcup_{i<\omega}F_i)$ is not projective.*

1.2. Definitions. If λ and θ are ordinals, let ${}^{\leqslant\theta}\lambda$ denote the set of all functions from ν into λ, for some $\nu \leqslant \theta$. A *tree* is a subset T of ${}^{\leqslant\theta}\lambda$ which is closed under restriction; i.e., if $\eta \in T$, then $\eta \restriction \nu \in T$ for any $\nu \in \mathrm{dom}(\eta)$. We identify an ordinal with the set of its predecessors; e.g., $n = \{0, 1, ..., n-1\}$. If $\eta \in T$, the *length* of η—denoted $l(\eta)$—is defined to be the domain, $\mathrm{dom}(\eta)$, of η; it is an ordinal $\leqslant \theta$. Let $T_\nu = \{\eta \in T : l(\eta) = \nu\}$.

We make T into a poset by defining $\zeta \leqslant \eta$ if and only if $\zeta = \eta \restriction l(\zeta)$. We will also assume that for any tree T there exists (a unique) θ such that $T \subseteq {}^{\leqslant\theta}\lambda$ and for all $\zeta \in T$ there exists $\eta \in T_\theta$ such that $\zeta \leqslant \eta$; in that case we say that T has *height* θ.

Fix a chain $\langle F_i : i \leqslant \omega \rangle$ as in Lemma 1.1. Given a tree T of height ω, for each $\eta \in T$, let M_η^T equal $F_{l(\eta)}$. For each $\zeta \leqslant \eta$ in T, let $\iota_{\zeta\eta}$ be the inclusion map: $M_\zeta^T \to M_\eta^T$. Then let M^T be the direct limit of $(M_\eta^T, \iota_{\zeta\eta} : \zeta \leqslant \eta \in T)$. More concretely, M^T equals $\oplus\{M_\eta^T : \eta \in T\}/K$ where K is the submodule generated by all elements of the form $x_\eta - y_\zeta$ where $y_\zeta \in M_\zeta^T$, $x_\eta \in M_\eta^T$, $\zeta \leqslant \eta$, and $\iota_{\zeta\eta}(y_\zeta) = x_\eta$. There is a canonical embedding ι_η of M_η^T into M^T; in the future we shall identify M_η^T with $\iota_\eta(M_\eta^T)$, and regard M_η^T as a submodule of M^T.

In this section we shall deal mainly with trees of height ω, but we make some definitions in greater generality for the purposes of the generalizations in Section 3. For the remainder of this section we will be investigating the properties of M^T under various assumptions on T.

1.3. Definitions. Let κ be a regular cardinal $> \rho$.

(i) A tree $T \subseteq {}^{\leqslant\theta}\lambda$ is called *κ-free* if for every $S \subseteq T_\theta$ such that $|S| < \kappa$, there is a function $\psi : S \to \theta$ such that

$$\{\{\eta \restriction \nu : \psi(\eta) < \nu \leqslant \theta\} : \eta \in S\}$$

is a family of pairwise disjoint sets.

(ii) An R-module M is called *κ-free* if there is a set $\mathscr{C}$ of submodules of M such that:

(1) every element of $\mathscr{C}$ is a free module of cardinality $< \kappa$;

(2) every subset of M of cardinality $< \kappa$ is contained in an element of $\mathscr{C}$; and

(3) $\mathscr{C}$ is closed under unions of well-ordered chains of length $< \kappa$. (Compare [EM2, IV.1.1].)

1.4. Lemma. *Let κ be a regular cardinal $> \rho$. If T is a κ-free tree of height ω, then M^T is a κ-free module.*

Proof. We will let $\mathscr{C}$ consist of all submodules of M^T of the form $\sum \{M^T_\eta : \eta \in S\}$ where S is a subset of T_ω of cardinality $<\kappa$. It suffices to prove that each such element of $\mathscr{C}$ is free. Given S, let ψ be as in the definition of κ-free. Well-order the elements of S as $\{\eta_\alpha : \alpha < \tau\}$. For simplicity of notation, let $M^T_{\alpha,\psi}$ denote $M^T_{\eta_\alpha \restriction (\psi(\eta_\alpha)+1)}$. For each α we can write

$$M^T_{\eta_\alpha} = M^T_{\alpha,\psi} \oplus N_\alpha$$

for some (free) submodule N_α. Choose a basis B_α of N_α. Now inductively define a basis, Y_α, of $\sum \{M^T_{\beta,\psi} : \beta < \alpha\}$ for each $\alpha \leqslant \tau$ so that the Y_α form a continuous chain. If Y_α has been chosen, let n be maximal with respect to the properties that is is $\leqslant \psi(\eta_\alpha) + 1$ and that $M^T_{\eta_\alpha \restriction n} \subseteq \sum \{M^T_{\beta,\psi} : \beta < \alpha\}$ (i.e., that $\eta_\nu \restriction n = \eta_\beta \restriction n$ for some $\beta < \alpha$). Then

$$M^T_{\alpha,\psi} = M^T_{\eta_\alpha \restriction n} \oplus D_\alpha$$

for some free submodule D_α (which will be zero if $n = \psi(\eta_\alpha) + 1$). Choose a basis of D_α and add it to Y_α to form $Y_{\alpha+1}$. If σ is a limit ordinal $\leqslant \tau$, we let $Y_\sigma = \bigcup \{Y_\alpha : \alpha < \sigma\}$. Then

$$\bigcup \{B_\alpha : \alpha < \tau\} \cup Y_\tau$$

will be a basis of $\sum \{M^T_\eta : \eta \in S\}$. ∎

1.5. Lemma. *Suppose λ is a regular cardinal $>\rho$ and $T \subseteq {}^{\leqslant\theta}\lambda$ is a tree of height θ and cardinality λ such that*

> *there is a stationary subset E of λ such that we can enumerate $T_\theta = \{\eta_\alpha : \alpha < \lambda\}$ so that for all $\delta \in E$, there exists $\nu \geqslant \delta$ such that for all $\mu \in \theta$, $\eta_\nu \restriction \mu \in \{\eta_\alpha \restriction \mu : \alpha < \delta\}$.* (†)

If $\theta = \omega$ (so M^T is defined), then M^T is not projective.

Proof. M^T is generated by $\bigcup \{M^T_{\eta_\nu} : \nu < \lambda\}$, so M^T has cardinality λ. For each $\tau < \lambda$ let A_τ be the submodule of M^T generated by $\bigcup \{M^T_{\eta_\alpha} : \alpha < \tau\}$. Then $\{A_\tau : \tau < \lambda\}$ is a λ-filtratrion of M^T, and it suffices to prove that for all $\delta \in E$,

$$\{\tau > \delta : A_\tau / A_\delta \text{ is not projective}\} \text{ is stationary in } \lambda. \qquad (*)$$

(Compare [EM2, Sect. IV.1].) Indeed, if M^T were projective, then by a result of Kaplansky [K, Theorem 1], M^T would be the direct sum of countably generated projectives, so there would be a cub $C \subseteq \lambda$ such that for all $\nu, \tau \in C$ with $\nu < \tau$, A_τ / A_ν is projective. But then there exists $\delta \in E \cap C$ and for all τ in the cub $C \cap \{\nu \in \lambda : \nu > \delta\}$, A_τ / A_δ is projective, which contradicts $(*)$.

Now given $\delta \in E$, let $\nu \leqslant \delta$ be as in the hypothesis of the lemma. We claim that for all $\tau > \nu$, A_τ / A_δ is not projective, which will prove (*). First we observe that

$$A_\tau / A_\delta = (M^T_{\eta_\nu} + A_\delta)/A_\delta \oplus \left(\sum \{M^T_{\eta_\alpha} : \alpha < \tau,\ \alpha \neq \nu\} + A_\delta \right) \Big/ A_\delta. \qquad (**)$$

Indeed, A_τ / A_δ is clearly the sum of the modules on the right. To see that the sum is direct, suppose $x \in M^T_{\eta_\nu}$ and $y \in \sum \{M^T_{\eta_\alpha} : \alpha < \tau,\ \alpha \neq \nu\}$ such that $x \equiv y \pmod{A_\delta}$; if $x \notin A_\delta$, then by choice of ν, $x \in M^T_{\eta_\nu} \setminus \bigcup_{n \in \omega} M^T_{\eta_\nu \restriction n}$; but then x can't equal y since x is not identified with anything in the construction of M^T.

Now $(M^T_{\eta_\nu} + A_\delta)/A_\delta \cong M^T_{\eta_\nu} / (M^T_{\eta_\nu} \cap A_\delta) \cong F_\omega / (\bigcup_{n<\omega} F_n)$ so it is not projective; therefore A_τ / A_δ is not projective by (**). ▌

1.6. Definitions. If μ is a cardinal, we say that a tree $T \subseteq {}^{\leqslant \theta}\lambda$ of height θ has *μ-uniformization* if for every family $\{\psi_\eta : \eta \in T_\theta\}$ where $\psi_\eta : \theta \to \mu$, there exists $\Psi : T \to \mu$ and $\Psi^* : T_\theta \to \theta$ such that for all $\eta \in T_\theta$, $\Psi(\eta \restriction \nu) = \psi_\eta(\nu)$ whenever $\Psi^*(\eta) \leqslant \nu < \theta$.

Recall that M is called a Whitehead module if $\mathrm{Ext}^1_R(M, R) = 0$. If χ is a cardinal, we say that M is *χ-Whitehead* if $\mathrm{Ext}^1_R(M, K) = 0$ whenever K is a module of cardinality $\leqslant \chi$. So if M is χ-Whitehead for some $\chi \geqslant \rho$, then M is Whitehead.

Connections between uniformization properties and Whitehead properties have been investigated for example in [S4; S5; MS2; EM2, Sect. XII.3]. Here, the main result is the following:

1.7. Theorem. *If $\lambda > \chi \geqslant \rho$ and $T \subseteq {}^{\leqslant \omega}\lambda$ is a tree of height ω which has 2^{χ^ρ}-uniformization, then M^T is a χ-Whitehead module.*

We will break the proof into two parts. The theorem follows immediately from the following two propositions.

1.8. Proposition. *If $\lambda > \chi \geqslant \rho$ and $T \subseteq {}^{\leqslant \omega}\lambda$ is a tree of height ω, then M^T is a χ-Whitehead module provided that T satisfies the following property:*

> *given a set A of cardinality $\leqslant \chi^\rho$ and for each $\eta \in T$ a non-empty set P_η of functions from $\{\eta \restriction n : n \leqslant l(\eta)\}$ to A satisfying:*
>
> (a) *$g \in P_\eta$ and $m < l(\eta)$ implies $g \restriction \{\eta \restriction n : n \leqslant m\} \in P_{\eta \restriction m;}$ and* $\qquad (*)_{\chi,\rho}$
>
> (b) *$g \in P_\zeta$ and $\zeta < \eta$ implies that there exists $g' \in P_\eta$ such that $g \subseteq g'$;*
>
> *then there is a function $G: T \to A$ such that for every $\eta \in T$, $G \restriction \{\eta \restriction n : n \leqslant l(n)\}$ belongs to P_η.*

1.9. PROPOSITION. *If $\lambda > \chi \geqslant \rho$ and $T \subseteq {}^{\leqslant\omega}\lambda$ is a tree of height ω which has 2^{χ^ρ}-uniformization, then T satisfies property $(*)_{\chi,\rho}$ of* 1.8.

Proof of 1.8. Suppose we are given a short exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} N \xrightarrow{\pi} M^T \longrightarrow 0, \tag{E}$$

where $|K| \leqslant \chi$. We need to show that (E) represents the zero element in $\mathrm{Ext}^1_R(M, K)$, i.e., that there is a *splitting* of π, i.e., a homomorphism $\varphi: M^T \to N$ such that $\pi \circ \varphi =$ the identity on M^T. Choose a set function $u: M^T \to N$ such that $\pi \circ u =$ the identity on M^T. (It exists because π is surjective.) Then the splittings, φ, of π (if any) are in one–one correspondence with set mappings $h: M^T \to K$ such that $h(0) = 0$ and for all $x, y \in M^T$ and $r \in R$,

(i) $rh(x) - h(rx) = ru(x) - u(rx)$; and

(ii) $h(x) + h(y) - h(x+y) = u(x) + u(y) - u(x+y)$.

(The correspondence is given by $h = u - \varphi$. Compare [F, Sect. 49].) For any submodule M' of M^T we will denote by $\mathrm{Trans}(M', K)$ the set of all set mappings $h: M' \to K$ which satisfy the conditions above for all $x, y \in M'$ and $r \in R$. So (E) splits if and only if $\mathrm{Trans}(M^T, K)$ is non-empty.

Now for each $\eta \in T$, there is an isomorphism $\theta_\eta: F_{l(\eta)} \to M^T_\eta$; moreover we can choose these isomorphisms so that $\theta_\zeta \subseteq \theta_\eta$ if $\zeta \leqslant \eta$. By using these isomorphisms we shall identify the elements of $\mathrm{Trans}(M^T_\eta, K)$ with functions from $F_{l(\eta)}$ to K. For any $h \in \mathrm{Trans}(M^T_\eta, K)$, let $\mathrm{seq}(h)$ be the function with domain $\{\eta \restriction \nu : \nu \leqslant l(\eta)\}$ such that $\mathrm{seq}(h)(\eta \restriction \nu) = h \restriction F_\nu$.

Let A be the set of all functions from F_ν (for some $\nu \leqslant \omega$) to K; so A has cardinality $\leqslant \chi^\rho$. For each $\eta \in T$, let $P_\eta = \{\mathrm{seq}(h) : h \in \mathrm{Trans}(M^T_\eta, K)\}$. Now we can verify that conditions (a) and (b) of $(*)_{\chi,\rho}$ hold: (a) is easy, and (b) and the fact that P_η is non-empty hold because F_ν and F_ν/F_m are free for all $m < \nu \leqslant \omega$.

Let $G: T \to A$ be as in $(*)_{\chi,\rho}$. Then we can define $h: M^T \to K$ as follows: $h \restriction M^T_\eta = G(\eta): M^T_\eta \to K$ (under the identification, θ_η, of $F_{l(\eta)}$ with M^T_η). Clearly h is well-defined and belongs to $\mathrm{Trans}(M^T, K)$, so (E) splits. ▮

Proof of 1.9. Suppose we are given P_η ($\eta \in T$) as in $(*)_{\chi,\rho}$. Let $\eta \in T_\omega$ and $n \in \omega$. For each $g \in P_{\eta \restriction n}$, choose one element, g^η, in P_η such that $g \subseteq g^\eta$. (This is possible by (b).) Having done this for all $n \in \omega$, for any $g \in P_{\eta \restriction n}$, let $\mathrm{ord}_\eta(g)$ be the minimal $m \leqslant n+1$ such that $g \subseteq [g \restriction \{\eta \restriction i : i < m\}]^\eta$. (It exists because $n+1$ has this property.) By an abuse of notation we shall write $g \restriction k$ for $g \restriction \{\eta \restriction i : i < k\}$; thus $g \in P_{\eta \restriction n}$ equals $g \restriction n+1$ and is always contained in $[g \restriction \mathrm{ord}_\eta(g)]^\eta$.

Sh:379

For each $\eta \in T_\omega$, define a function ψ_η on ω as

$$\psi_\eta(n) = \{(g \upharpoonright n, [g \upharpoonright \mathrm{ord}_\eta(g \upharpoonright n)]^\eta) : g \in P_{\eta \upharpoonright n}\}.$$

(By (b), we have also

$$\psi_\eta(n) = \{(g, [g \upharpoonright \mathrm{ord}_\eta(g)]^\eta) : g \in P_{\eta \upharpoonright n-1}\}.)$$

Then the ψ_η take values in the power set of (a set isomorphic to) ${}^{<\omega}A \times {}^{\omega}A$; that is, in a set of cardinality 2^κ where $\kappa = \chi^\rho \cdot (\chi^\rho)^{\aleph_0} = \chi^\rho$ (since ρ is $\geqslant \aleph_0$). So by hypothesis there is a function Ψ on T taking values in $\bigcup \{\mathrm{ran}(\psi_\eta) : \eta \in T_\omega\}$ so that for all $\eta \in T_\omega$, $\Psi(\eta \upharpoonright n) = \psi_\eta(n)$ for sufficiently large $n \in \omega$.

Now we want to define $G: T \to A$ as required. We define $G(\zeta)$ for $\zeta \in T \setminus T_\omega$ by induction on the length, $l(\zeta)$, of ζ so that

$$G \upharpoonright \{\zeta \upharpoonright i : i \leqslant l(\zeta)\} \text{ belongs to } P_\zeta. \tag{¶}$$

Suppose that we have done this for all ζ of length $<n$, and suppose $l(\eta) = n$. Then

$$g \stackrel{\mathrm{def}}{=} G \upharpoonright \{\eta \upharpoonright i : i \leqslant n-1\} \tag{¶¶}$$

belongs to $P_{\eta \upharpoonright n-1}$. We let $G(\eta) = \Psi(\eta)(g)(\eta)$, if this is defined; otherwise, we let $G(\eta)$ be chosen arbitrarily so that (¶) holds. (This is possible by (b).)

Now we must define G on T_ω. Given $\eta \in T_\omega$, there exists n $(= \Psi^*(\eta))$ so that for $m \geqslant n$, $\Psi(n \upharpoonright m) = \psi_\eta(m)$. Let g be defined as (¶¶). Then for all m, $G(\eta \upharpoonright m) = [g \upharpoonright \mathrm{ord}_\eta(g)]^\eta\ (\eta \upharpoonright m)$. (This is trivial for $m < n$ and is proved by induction for $m \geqslant n$ using the definitions of n and G.) Hence $G \upharpoonright \{\eta \upharpoonright m : m \in \omega\} = [g \upharpoonright \mathrm{ord}_\eta(g)]^\eta \upharpoonright \{\eta \upharpoonright m : m \in \omega\}$, so we define $G(\eta)$ to be $[g \upharpoonright \mathrm{ord}_\eta(g)]^\eta\ (\eta)$.

Then G has the desired property by (¶) and the fact that $G \upharpoonright \{\eta \upharpoonright n : n \leqslant \omega\} = [g \upharpoonright \mathrm{ord}_\eta(g)]^\eta$ for $\eta \in T_\omega$ (where g is as in (¶¶)). ∎

2. The Consistency Result

In this section we prove the following:

2.1. Theorem. *There is a model of ZFC + GCH such that for every uncountable cardinal κ of cofinality $\theta < \kappa$,*

> *for $\lambda = \kappa^+$, there is a tree $T \subseteq {}^{\leqslant \theta}\lambda$ of height θ and cardinality λ such that T is λ-free, satisfies (†) of 1.5, and has μ-uniformization for every $\mu < \kappa$.* $(*)_\kappa$

Since for any χ and ρ there is a cardinal κ of cofinality ω which is $>2^{\chi^\rho}$, as a consequence of the results of Section 1, we then have the following:

2.2. COROLLARY. *It is consistent with ZFC + GCH that for every non-left perfect ring R and every cardinal χ, there is a regular cardinal $\kappa^+ > |R| + \chi$ such that there is an R-module M of cardinality κ^+ which is κ^+-free but not projective and is a χ-Whitehead module.*

In this section we assume a knowledge of forcing. Our notation generally follows that of [J], except that we use $p \leqslant q$ to mean that q is a stronger condition than p. Thus, for example, a subset D of P is dense in P if for all $p \in P$ there exists $d \in D$ such that $p \leqslant d$.

If κ is a cardinal, a poset P is said to be *κ-closed* if for every $\tau \leqslant \kappa$, every ascending sequence

$$p_0 \leqslant p_1 \leqslant \cdots \leqslant p_\alpha \qquad (\alpha < \tau)$$

has an upper bound. P is said to be *κ-strategically closed* if for every $\tau \leqslant \kappa$, Player I has a winning strategy in the following game of length τ. Players I and II alternately choose an ascending sequence

$$p_0 \leqslant p_1 \leqslant \cdots p_\alpha \qquad (\alpha < \tau)$$

of elements of P, where Player I chooses at the even ordinals; Player I wins if and only if at each stage there is a legal move and the whole sequence, $(p_\alpha : \alpha < \tau)$, has an upper bound. (In some sources, κ-strategically closed is called κ^+-strategically complete.) In many applications, κ-strategically closed posets serve the same purposes as κ-closed ones. For example, if P is κ-strategically closed, then $V[G]$ has no new functions from κ into V, and hence cardinals $\leqslant \kappa^+$ and their cofinalities are preserved.

As an example, and as a step in the proof of the theorem, we prove the following. (Compare [MS1, p. 142].)

2.3. LEMMA. *Assume $2^\kappa = \kappa^+$. For any regular cardinal $\mu \leqslant \kappa$, there is a poset Q^0 of cardinality $\leqslant \kappa^+$ which is κ-strategically closed (and hence preserves all cardinals and preserves cofinalities $\leqslant \kappa^+$) and is such that in $V[G]$ there is a non-reflecting stationary subset E of κ^+ such that every member of E has cofinality μ. (Here, "non-reflecting" means that for every limit $\delta < \kappa^+$, $E \cap \delta$ is not stationary in δ.)*

Proof. Let Q^0 be the set of all functions $q: \alpha \to 2 = \{0, 1\}$ $(\alpha < \kappa^+)$ such that $q(\nu) = 1$ implies that $cf(\nu) = \mu$ and such that for all limit $\delta \leqslant \alpha$, the intersection of $q^{-1}[1]$ $(= \{\nu < \alpha : q(\nu) = 1\})$ with δ is not stationary in δ.

Then

$$E = \bigcup \{q^{-1}[1] : q \in G\}$$

will be the desired set. To see that E is stationary in κ^+, suppose that q forces **f** is the name of a continuous increasing function $f: \kappa^+ \to \kappa^+$; choose an ascending chain

$$q_0 \leqslant q_1 \leqslant \cdots \leqslant q_\alpha \qquad (\alpha < \mu)$$

such that for each α there are β_α, γ_α such that $q_\alpha \Vdash f(\beta_\alpha) = \gamma_\alpha$ and

$$\mathrm{dom}(q_\alpha) \geqslant \gamma_\alpha > \mathrm{dom}(q_\nu)$$

for all $\nu < \alpha$. Let $\delta = \sup\{\gamma_\alpha : \alpha < \mu\} = \sup\{\mathrm{dom}(q_\alpha) : \alpha < \mu\}$ and let

$$q = \bigcup \{q_\alpha : \alpha < \mu\} \cup \{(\delta, 1)\}.$$

Then $q \in Q^0$ since $q^{-1}[1]$ is not stationary in δ, because δ has cofinality μ. Moreover, $q \Vdash \delta \in \mathrm{rge}(\mathbf{f}) \cap E$ (since f is continuous).

Since Q^0 has cardinality $\leqslant \kappa^+$, it satisfies the κ^{++}-chain condition, and so preserves cardinals $> \kappa^+$. To show that all cardinals $\leqslant \kappa^+$ and their cofinalities are preserved in $V[G]$, it suffices to show that Q^0 is κ-strategically closed. For a limit ordinal $\tau \leqslant \kappa$ we describe Player I's winning strategy. I chooses q_α for even α such that $\mathrm{dom}(q_\alpha)$ is a successor ordinal, $\sigma_\alpha + 1$, and $q_\alpha(\sigma_\alpha) = 0$. Moreover, at limit ordinals α he chooses q_α to have domain $= \sup\{\sigma_\beta : \beta < \alpha\} + 1$. Then $q = \bigcup \{q_\alpha : \alpha < \tau\}$ is a member of Q^0 because $\{\sigma_\alpha : \alpha < \tau,\ \alpha \text{ even}\}$ is a cub in $\mathrm{dom}(q)$ which misses $q^{-1}[1]$. (Compare [J, p. 255, Exercise 24.1.3].) ∎

2.4. Definition. Suppose E is a stationary subset of κ^+ consisting of ordinals of cofinality θ. A *ladder system* on E is a family of functions $\langle \zeta_\delta : \delta \in E \rangle$ such that $\zeta_\delta : \theta \to \delta$ is strictly increasing with $\sup(\mathrm{rge}(\zeta_\delta)) = \delta$. We say that it *has μ-uniformization* if for every family $\langle g_\delta : \delta \in E \rangle$ such that $g_\delta : \mathrm{rge}(\zeta_\delta) \to \mu$, there exists $g: \kappa^+ \to \kappa^+$ such that for every $\delta \in E$, there exists $\beta_\delta < \theta$ so that for $\nu \geqslant \beta_\delta$,

$$g(\zeta_\delta(\nu)) = g_\delta(\zeta_\delta(\nu)).$$

2.5. By Theorem 2.10 of [S6], if κ is a singular strong limit cardinal and $2^\kappa = \kappa^+$ and E is a non-reflecting stationary subset of κ^+ consisting of ordinals of cofinality $= cf(\kappa)$, then there is a poset Q^1 of cardinality 2^{κ^+} satisfying the κ^{++}-chain condition which adds no new sequences of length

$\leqslant\kappa$ and is such that in V^{Q^1}, E is stationary in κ^+, and there exists a ladder system $\langle\zeta_\delta : \delta\in E\rangle$ which has μ-uniformization for every $\mu<\kappa$.

Q^1 is an iteration which has the following properties:

every member of Q^1 is a function, p, with domain a subset of 2^{κ^+} of power $\leqslant\kappa$ such that $p(i)$ is a function from some ordinal $<\kappa^+$ to κ^+; moreover, for each $\gamma<\kappa^+$, $\{p\in Q^1 : \gamma\leqslant \mathrm{dom}[p(i)]$ for all $i\in\mathrm{dom}(p)\}$ is a dense subset of Q^1; (#)

for any limit ordinal σ, any sequence $p_0<p_1<\cdots<p_\nu$ $(\nu<\sigma)$ of elements of Q^1 has an upper bound provided that there are ordinals γ_ν $(\nu<\sigma)$ such that for every limit $\sigma'\leqslant\sigma$ $\sup\{\gamma_\nu : \nu<\sigma'\}\notin E$ and for every $i\in\mathrm{dom}(p_\nu)$, $p_\nu(i)$ is a function with domain an ordinal $\leqslant\gamma_\nu$ and $>\gamma_\beta$ for all $\beta<\nu$. (# #)

Now given a ground model V which satisfies *GCH* and given a singular cardinal $\aleph_\delta$, let $Q^0_\delta\in V$ and $Q^1_\delta\in V^{Q^0_\delta}$ be as above for $\kappa=\aleph_\delta$. Let $\dot{Q}^1_\delta$ be a name for Q^1_δ in V.

2.6. LEMMA. *The forcing $Q^0_\delta * \dot{Q}^1_\delta$ is $\aleph_\delta$-strategically closed.*

Proof. There is a dense subset D of $Q^0_\delta * \dot{Q}^1_\delta$ consisting of pairs (q,p) where p is a function in V and $q \Vdash p\in\dot{Q}^1_\delta$. Player I chooses moves of the form $(q,p)\in D$ where $q\in Q^0_\delta$ and has domain of the form $\sigma_\alpha+1$ where $q(\sigma_\alpha)=0$ and $p\in Q^1_\delta$ is such that for all i in the domain of p, the domain of $p(i)$ includes $\bigcup_{\beta<\alpha}\sigma_\beta$ and is included in σ_α; moreover, for limit δ, $\sigma_\delta=\sup\{\sigma_\alpha : \alpha<\delta\}$. (Player I can do this because of (#).) Then using (# #) and the proof of 2.3, we see that this is a winning strategy for Player I; in particular, Player I has a move at limit ordinals. ∎

Proof of 2.1. We start with a ground model V which satisfies *GCH*. For any α, $P_\alpha=\langle P_j,\dot{Q}_i : j\leqslant\alpha,\ i<\alpha\rangle$ will be an iteration with Easton support; i.e., we take direct limits when $\aleph_\alpha$ is regular and inverse limits elsewhere. For each ordinal i, let $\dot{Q}^0_1$ be the forcing notion in V^{P_i} described in Lemma 2.3 for the cardinal $\kappa=\aleph_i$ if $\aleph_i$ is singular, and otherwise let it be 0. Let $\dot{Q}^1_i$ be a name for the forcing in $V^{P_i*\dot{Q}^0_i}$ described in 2.5 for the cardinal $\kappa=\aleph_i$, if $\aleph_i$ is singular and let it be 0 otherwise. Let $\dot{Q}_i=\dot{Q}^0_i*\dot{Q}^1_i$, a forcing in V^{P_i}. Let P be the direct limit of the P_α $(\alpha\in\mathrm{Ord})$. We claim that V^P has the desired properties. We shall use freely the following facts:

2.7. (a) For every κ and every Easton support iteration $\langle P_j,\dot{Q}_i : \kappa\leqslant j\leqslant\alpha,\ \kappa\leqslant i<\alpha\rangle$, if each $\dot{Q}_i$ is κ-strategically closed, then so is P_α.

(b) $P = P_\alpha * P_{\geqslant\alpha}$, where, in V^{P_α}, $P_{\geqslant\alpha}$ is the direct limit of P^α_β ($\beta \in \mathrm{Ord}$), with P^α_β the Easton support iteration $\langle P^\alpha_j, \dot{Q}^\alpha_i : j \leqslant \beta,\ i < \beta\rangle$ where $\dot{Q}^\alpha_i = \dot{Q}_{\alpha+i}$.

(c) $|P_n| = 1$ (for $n \in \omega$); if $\aleph_\delta$ is singular, $|P_\delta| \leqslant \aleph_{\delta+1}$ and $|P_{\delta+n+1}| \leqslant \aleph_{\delta+2}$; if $\aleph_\delta$ is inaccessible, $|P_\delta| = \aleph_\delta$.

(d) $P_{\geqslant\alpha}$ is $\aleph_\alpha$-strategically closed, and $P_{\geqslant\alpha+1}$ is even $\aleph_{\alpha+n}$-strategically closed for all $n \in \omega$.

Now we show that no regular cardinal λ becomes singular in V^P. For $\lambda \leqslant \aleph_\omega$ this is because P_ω is trivial and $P_{\geqslant\omega}$ is $\aleph_\omega$-strategically closed. Now consider $\lambda = \aleph_{\delta+n}$ where δ is a limit ordinal. For each $\alpha < \delta$, $|P_\alpha| < \aleph_\delta$, and hence P_α satisfies the $\aleph_\delta$-chain condition. Thus in V^{P_α}, λ is still regular. Since $P_{\geqslant\alpha}$ is $\aleph_\alpha$-strategically closed, λ has cofinality $> \aleph_\alpha$ in V^P. As this holds for each $\alpha < \delta$, after the forcing λ has cofinality $\geqslant \aleph_\delta$. If $n = 0$, or $n = 1$ and $\aleph_\delta$ is singular, we are done. (If $cf(\lambda) < \lambda$ in V^P, then $cf(\lambda) < \chi_\delta$, as all ordinals in $[\aleph_\delta, \aleph_{\delta+1})$ are singular already in V.) If $n \geqslant 2$, then since $|P_\delta| \leqslant \aleph_{\delta+1}$, and $\dot{Q}_\delta$ satisfies the $\aleph_{\delta+2}$-chain condition, λ is regular in $V^{P_{\delta+1}}$; since $P_{\geqslant(\delta+1)}$ is λ-strategically closed (by 2.7(d)), λ is regular in V^P. If $n = 1$ and $\aleph_\delta$ is regular, we argue similarly using the facts that $|P_\delta| \leqslant \aleph_\delta$ and that $P_{\geqslant\delta}$ is λ-strategically closed.

Next we prove that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ in V^P for all α. If $\alpha < \omega$, this is because we add no new subsets of $\aleph_\omega$. If $\alpha = \delta + n$ where δ is a limit ordinal and $n \geqslant 1$, it is enough to show that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds in $V^{P_{\delta+1}}$ since $P_{\geqslant\delta+1}$ is $\aleph_\alpha$-strategically closed. We use the fact (see [J, Lemma 19.4]) that in $V^{P_{\delta+1}}$, $2^{\aleph_\alpha} \leqslant$ the cardinality *in* V of $(r.o.\ (P_{\delta+1}))^{\aleph_\alpha}$; but $|r.o.(P_{\delta+1})| \leqslant \aleph_{\delta+2}^{\aleph_{\delta+1}}$ by 2.7(c) and because $P_{\delta+1}$ satisfies the $\aleph_{\delta+2}$-chain condition (cf. [J, Exercise 17.22, p. 158]). Since $V \models GCH$, we can conclude that in $V^{P_{\delta+1}}$, $2^{\aleph_\alpha} \leqslant (\aleph_{\delta+2}^{\aleph_{\delta+1}})^{\aleph_\alpha} = \aleph_{\alpha+1}$. For $n = 0$ and $\aleph_\delta$ regular, the proof is similar, so assume $n = 0$ and $\aleph_\delta$ is singular. We must show that $\aleph_\delta^\kappa = \aleph_{\delta+1}$ where $\kappa = cf(\aleph_\delta)$. Now if $\kappa = \aleph_\beta$ $(< \aleph_\delta)$, it is enough to prove that $\aleph_\delta^\kappa = \aleph_{\delta+1}$ in V^{P_β} because $P_{\geqslant\beta}$ adds no new functions on $\aleph_\beta$. But calculating as in [J, Lemma 19.4], we have that in V^{P_β}, $\aleph_\delta^\kappa \leqslant$ the cardinality *in* V of $|r.o.(P_\beta)|^{(\aleph_\beta \times \aleph_\delta)}$ which is $\leqslant ((\aleph_{\beta+2})^{\aleph_{\beta+2}})^{(\aleph_\beta \cdot \aleph_\delta)} = \aleph_{\delta+1}$ since δ is a limit ordinal $> \beta$ and GCH holds in V.

Now we must define the tree $T \subseteq {}^{\leqslant\theta}(\kappa^+)$ when κ is a singular cardinal and $\theta = cf(\kappa)$. By construction there is a non-reflecting stationary subset E of κ^+ and a ladder system $\langle \zeta_\delta : \delta \in E\rangle$ which has μ-uniformization for all $\mu < \kappa$. Let $T = \{\zeta_\delta \restriction \nu : \delta \in E,\ \nu \leqslant \theta\}$. Then $T_\theta = \{\zeta_\delta : \delta \in E\}$. We claim that T has μ-uniformization (in the sense of 1.6) for every $\mu < \kappa$. Given $\{\psi_{\zeta_\delta} : \delta \in E\}$, where $\psi_{\zeta_\delta} : \theta \to \mu$, define $g_\delta : \mathrm{rge}(\zeta_\delta) \to \mu$ by $g_\delta(\zeta_\delta(\nu)) = \psi_{\zeta_\delta}(\nu)$. Then according to 2.4 there exists $g : \kappa^+ \to \kappa^+$ such that for all $\delta \in E$ there exists $\beta_\delta < \theta$ such that $g(\zeta_\delta(\nu)) = g_\delta(\zeta_\delta(\nu))$ for $\nu \geqslant \beta_\delta$. Define $\Psi : T \to \mu$ by

Sh:379

$\Psi(\zeta_\delta \upharpoonright \nu) = g(\zeta_\delta(\nu))$ (if $<\mu$ and 0 otherwise) and let $\Psi^*(\zeta_\delta) = \beta_\delta$ for all $\delta \in E$, and we see that Definition 1.6 is satisfied.

Next, we prove that T is κ^+-free. In fact, we show—by induction on $\sup S$—that for any subset S of E such that $\sup S < \kappa^+$, there exists $\psi: S \to \theta$ such that $\{\{\zeta_\delta \upharpoonright \nu : \psi(\delta) < \nu \leqslant \theta\} : \delta \in S\}$ is a family of pairwise disjoint sets. Without loss of generality, $\sup S$ is a limit ordinal; let C be a cub in $\sup S$ which misses E. (Remember, E is non-reflecting.) Without loss of generality $0 \in C$. For any $\beta \in C$, let β^+ denote the successor of β in C, and let $S_\beta = \{\delta \in S : \beta < \delta < \beta^+\}$. By induction, there exists $\psi_\beta: S_\beta \to \theta$ such that $\{\{\zeta_\delta \upharpoonright \nu : \psi_\beta(\delta) < \nu \leqslant \theta\} : \delta \in S_\beta\}$ is a family of disjoint sets. Moreover, we can assume that $\zeta_\delta(\psi_\beta(\delta)) > \beta$ for each $\delta \in S_\beta$. Since each element of S belongs to a unique S_β, we can then define ψ to be the union of the ψ_β.

Finally, we must verify that T satisfies (†) of 1.5. Because $\kappa^{<\theta} = \kappa$, there is a cub C in κ^+ so that for all $\delta \in C$, if there exists $\alpha \in E$, $\mu < \theta$ such that $\zeta_\alpha \upharpoonright \mu : \mu \to \delta$, then there exists $\beta \in E \cap \delta$ such that $\zeta_\beta \upharpoonright \mu = \zeta_\alpha \upharpoonright \mu$. We claim that $C \cap E$ is the desired stationary set. Indeed, for $\delta \in C \cap E$, we can choose $\nu = \delta$: since ζ_δ takes values in δ, $\zeta_\delta \upharpoonright \mu \in \{\zeta_\beta \upharpoonright \mu : \beta < \delta, \beta \in E\}$ for all μ. (Here the elements of T_θ are enumerated by the elements of E; to enumerate by κ^+, use the strictly increasing function $f: \kappa^+ \to E$ which enumerates E, and let $\eta_\alpha = \zeta_{f(\alpha)}$, then $f^{-1}[C \cap E]$ is the desired stationary set.) ∎

2.8. The method of proof of 2.1 shows that is consistent with GCH that the first non-free Whitehead group can be arbitrarily large. More precisely, for any ordinal β there is a generic extension L^P of L which preserves the cardinals aind cofinalities of L and satisfies GCH and is such that the smallest non-free Whitehead group in L^P has cardinality $> \aleph_\beta$. Indeed, for any model V of GCH, and any ordinal α, let $\dot{Q}_i = 0$ for $i < \alpha$, and otherwise define P as before. Then P adds no new subsets of $\aleph_\alpha$, and if μ is a singular cardinal $\geqslant \aleph_\alpha$ and of cofinality ω, the proof of 2.1 and 2.2 shows that there is a non-free Whitehead group of cardinality μ^+ in V^P. Now in L^P, $\diamondsuit_\lambda(E)$ continues to hold for al regular $\lambda < \aleph_\alpha$ and all stationary $E \subseteq \lambda$ (because no new subsets of $\aleph_\alpha$ are added), so every Whitebread group of cardinality $\leqslant \aleph_\alpha$ is free (by [S1, S2]). Thus given β, if we choose $\alpha \geqslant \beta$ to be an ordinal of cofinality ω and define P as above, then in L^P the first non-free Whitehead group has cardinality $\aleph_{\alpha+1}$.

In fact, by using other uniformization results from [S6] and appropriately modifying the constructions herein, one can show that for any λ which is regular in L, there is a generic extension L^P of L with the same cardinals and cofinalities as L and satisfying GCH in which λ is the cardinality of the smallest non-free Whitehead group. First of all, by doing an initial forcing, we can assume that λ is not the successor of a weakly compact cardinal. Now, if $\lambda = \kappa^+$, where κ is singular, then the model described in the previous paragraph (for $\aleph_\alpha = \kappa$) has the desired properties. If κ has cofinality $\aleph_0$, this is clear; but when κ has uncountable cofinality, θ (not weakly compact), we need to show that there is a non-free Whitehead group of cardinality λ. Now in L, θ-free does not imply θ^+-free (cf. [EM2, Theorem VII.1.4]), so—in L or L^P—we can find an increasing chain $\langle F_i : i \leqslant \theta \rangle$ such that $\langle F_i : i < \theta \rangle$ is continuous and for all $i < j \leqslant \theta$, $|F_i| = \theta$, F_j and F_j/F_i are free, but $F_\theta/(\bigcup_{i<\theta} F_i)$ is not free (cf [EM2, Lemma VII.2.2]). Then Hypothesis 3.3a (following) is satisfied (with $\mathscr{F}$ = the class of free groups), so as in Section 3, we can construct a non-free Whitehead group.

If λ is inaccessible, we can use [S6, 2.8] to force a suitable $\langle \zeta_\delta : \delta \in E \rangle$, $E \subseteq \{\delta \in \lambda : cf(\delta) = \theta\}$, where θ is $\aleph_0$ or any regular cardinal $< \lambda$.

If λ is the successor of a regular cardinal κ, then we use [S6, 2.12], but not only for the uniformization (as the relevant parallel of 1.9 fails), but directly to get an analog of $(*)_{\chi, \rho}$ as follows:

> we can find $\langle \zeta_\delta : \delta < \lambda,\ cf(\delta) = \kappa \rangle$, where $\zeta_\delta : \kappa \to \delta$ is an increasing function with unbounded range which is tree-like (i.e., $\zeta_{\delta_1}(\alpha) = \zeta_{\delta_2}(\beta)$ implies $\alpha = \beta$ and $\zeta_{\beta_1} \upharpoonright \alpha = \zeta_{\delta_2} \upharpoonright \beta$) and such that given $\langle \Psi_\delta : \delta < \lambda,\ cf(\delta) = \kappa \rangle$, a family of functions, $\mathrm{rge}(\zeta_\delta) \to \kappa$ so that for every α, $\beta < \lambda$ of cofinality κ, and every $i < \kappa$ with $\zeta_\alpha(i) = \zeta_\beta(i)$ and every $f \in \Psi_\beta$, there exists $g \in \Psi_\alpha$ such that $g \upharpoonright \mathrm{rge}(\zeta_\alpha \upharpoonright i+1) = f \upharpoonright \mathrm{rge}(\zeta_\beta \upharpoonright i+1)$, then there exists a function $G: \lambda \to \kappa$ such that for all $\delta < \lambda$ with $cf(\delta) = \kappa$, $G \upharpoonright \mathrm{rge}(\zeta_\delta) \in \Psi_\delta$. $(*)_\kappa$

We continue as in the case of successors of singulars of uncountable cofinality. We can change the forcing so that $(*)_\kappa$ holds for all regular κ (or all regular $\kappa \geqslant \aleph_\alpha$). Alternately, for successors of regulars, we can imitate the proper forcing proof of the existence of non-free Whitehead groups (as in [S3]), which is what Mekler did.

3. Generalization

In the spirit of [BFS, EFS], we shall consider a generalization of the results of Section 1 where "free" is replaced by the property of being the union of a continuous chain whose quotients belong to a fixed class of modules.

3.1. Definition. If $\mathscr{F}$ is a fixed family of R-modules, a module M is called *$\mathscr{F}$-free* if M is the union of an increasing continuous chain $\langle M_\nu : \nu < \alpha \rangle$ of submodules such that $M_0 = 0$ and for all α, $M_{\alpha+1}/M_\alpha$

belongs to $\mathscr{F}$. (Here, "continuous" means for every limit ordinal $\sigma < \alpha$, $M_\sigma = \bigcup_{\nu < \sigma} M_\nu$.) For example, if $\mathscr{F} = \{R\}$, $\mathscr{F}$-free is just free.

We say that M is *κ-$\mathscr{F}$-free* if it satisfies the definition in 1.3 with "free" replaced by "$\mathscr{F}$-free."

We shall be interested especially in the following choices for $\mathscr{F}$, where μ is a cardinal:

$\mathscr{W}(\mu)$ = the class of all Whitehead R-modules which are generated by $\leqslant \mu$ elements.

$\mathscr{W}_\chi(\mu)$ = the class of all χ-Whitehead modules which are generated by $\leqslant \mu$ elements. (For the definition of χ-Whitehead, see 1.6.)

$\mathscr{B}(\mu)$ = the class of Baer modules which are generated by $\leqslant \mu$ elements. (Recall that M is a Baer module if $\mathrm{Ext}^1_R(M, T) = 0$ for all torsion modules T; here R is an integral domain.)

3.2. It should be noted that when $\mathscr{F} = \mathscr{W}(\mu)$ (respectively, $\mathscr{W}_\chi(\mu)$, or $\mathscr{B}(\mu)$), and $\langle M_\nu : \nu < \beta \rangle$ is an increasing continuous chain of modules such that $M_0 \in \mathscr{F}$ and $M_{\nu+1}/M_\nu$ belongs to $\mathscr{F}$ for all $\nu < \beta$, then $\bigcup \{M_\nu : \nu < \beta\}$ is Whitehead (respectively, χ-Whitehead or Baer). (See, for example, [E1, Theorem 1.2].) Thus, $\mathscr{W}(\rho)$-free implies Whitehead.

Suppose that the following holds:

3.3. HYPOTHESIS. *$\mathscr{F}$ is $\mathscr{W}(\mu)$, $\mathscr{W}_\chi(\mu)$, or $\mathscr{B}(\mu)$ for some cardinal μ, and there is an increasing chain $\langle F_i : i \leqslant \omega \rangle$ of modules such that for all $i < j \leqslant \omega$, F_j and F_j/F_i belong to $\mathscr{F}$, but $F_\omega/(\bigcup_{i<\omega} F_i)$ does not.*

In that case, given a tree $T \subseteq {}^{\leqslant\omega}\lambda$, let M^T be defined as in 1.2, but using the chain $\langle F_i : i \leqslant \omega \rangle$ in 3.3. Let $\rho = |F_\omega| + |R| + \mu + \chi$. We claim that the analogs of 1.4, 1.5, and 1.7 hold. We will state the analogs and briefly comment on the changes needed in their proofs.

1.4 (bis). *Let κ be a regular cardinal $> \rho$. If $T \subseteq {}^{\leqslant\omega}\lambda$ is a κ-free tree of height ω, then M^T is κ-$\mathscr{F}$-free.*

To prove this, we inductively describe how to write $\sum \{M^T_{\beta, \psi} : \beta < \alpha\}$ as the union of a continuous chain with quotients which belong to $\mathscr{F}$— using the fact that $D_\alpha \in \mathscr{F}$. Then we extend this to a chain whose union is $\sum \{M^T_\eta : \eta \in S\}$, by induction on α, using the fact that the $N_\alpha \in \mathscr{F}$.

1.5 (bis). *Suppose λ is a regular cardinal $> \rho$ and $T \subseteq {}^{\leqslant\omega}\lambda$ is a tree of height ω and cardinality λ satisfying (†). Then M^T is not $\mathscr{F}$-free.*

Sh:379

(We could, in fact, prove that M^T is not the direct summand of an $\mathscr{F}$-free module.) It suffices to prove that for $\delta \in E$

$$\{\tau > \delta : A_\tau / A_\delta \text{ is not } \mathscr{F}\text{-free}\} \text{ is stationary in } \lambda.$$

Now if ν is as in the hypothesis of (†), we have (**) as in the proof of 1.5; if A_τ/A_ν were $\mathscr{F}$-free, then by 3.2, A_τ/A_ν would be Whitehead (or χ-Whitehead, or Baer, depending on what $\mathscr{F}$ is), which yields a contradiction, using (**), since $(M^T_{\eta_\nu} + A_\delta)/A_\delta \cong F_\omega/(\bigcup_{n<\omega} F_n)$ is not.

1.7 (bis). *Suppose $\mathscr{F}$ is $\mathscr{W}_\chi(\mu)$. If $\lambda > \chi \geqslant \rho$ and $T \subseteq {}^{\leqslant\omega}\lambda$ is a tree of height ω which has 2^{χ^ρ}-uniformization, then M^T is a χ-Whitehead module.*

The proof is the same as before, using the fact that $F_j/F_i \in \mathscr{F}$ to get (b) of $(*)_{\chi,\rho}$ (cf. [E2, Lemma 2.3]). (Note that we cannot conclude that M^T is Baer if $\mathscr{F}$ is $\mathscr{B}(\mu)$, since the definition of Baer requires $\mathrm{Ext}^1_R(M^T, T) = 0$ for arbitrarily large T—or at least for T of cardinality $\geqslant |M^T|$.)

We can generalize even further, by using trees of height $>\omega$. Suppose that we have for some cardinal $\theta \geqslant \omega$:

3.3a. HYPOTHESIS. *$\mathscr{F}$ is $\mathscr{W}(\mu)$ or $\mathscr{W}_\chi(\mu)$ for some cardinal $\mu \geqslant \theta$, and there is an increasing chain $\langle F_i : i \leqslant \theta \rangle$ of modules such that $\langle F_i : i < \theta \rangle$ is continuous and for all $i < j \leqslant \theta$, F_j and F_j/F_i belong to $\mathscr{F}$, but $F_\theta/(\bigcup_{i<\theta} F_i)$ does not.*

Given a tree $T \subseteq {}^{\leqslant\theta}\lambda$, we can define M^T as in 1.2. Then 1.4(bis), 1.5(bis), and 1.7(bis) hold with ω replaced by θ. In order to prove 1.7(bis), we need to add to $(*)_{\chi,\rho}$ in 1.8 an additional hypothesis:

(c) if $\eta \in T$, $l(\eta)$ is a limit ordinal $<\theta$, and $g_\nu \in P_{\eta \restriction \nu}$ for each $\nu \in l(\eta)$, so that $g_\nu \subseteq g_\tau$ if $\nu \leqslant \tau < l(\eta)$, then there exists $g \in P_\eta$ such that $\bigcup \{g_\nu : \nu < l(\eta)\} \subseteq g$.

This condition holds in the proof of 1.8, because the chain $\langle F_i : i < \theta \rangle$ is continuous. There are some changes in the proof of 1.9. For each $\eta \in T_\theta$, each $\nu < \theta$, and each $g \in P_\eta \restriction \nu$, choose one element $[g \restriction \nu] \in P_\eta$ which extends $g \restriction \nu$; it exists by (b) and (c). (Recall that $g = g \restriction \nu + 1$; if ν is a limit ordinal, then $g \restriction \nu$ is not an element of any P_ζ.) Then let $\mathrm{ord}_\eta(g \restriction \nu)$ be the minimal ordinal $\tau \leqslant \nu$ such that $g \restriction \nu \subseteq [g \restriction \tau]^\eta$. Define

$$\psi_\eta(\nu) = \{(g \restriction \nu,\ [g \restriction \mathrm{ord}_\eta(g \restriction \nu)]^\eta) : g \in P_{\eta \restriction \nu}\}$$

for all ordinals $\nu < \theta$. Now continue as before.

We can generalize further still by replacing the variety of R-modules with an arbitrary variety or semi-variety, V, of algebras in a fixed vocabulary L.

(A semi-variety is a class of algebras satisfying a set of universal Horn sentences.) The notions of free and κ-free make sense in this setting.

3.4. DEFINITION. $M \in V$ to said to be χ-*Whitehead* if for every $N \in V$ and every homomorphism $\varphi: N \to M$ such that for every $x \in M$, $\varphi^{-1}[\{x\}]$ has cardinality $\leqslant \chi$, there is a homomorphism $\sigma: M \to N$ such that $\varphi \circ \sigma$ is the identity on M.

3.5. If for some $\kappa > |L| + \theta_0$, there is a non-free $L_{\infty\kappa}$-free algebra of cardinality κ, then there is a chain $\langle F_i : i \leqslant \theta \rangle$ for some θ such that for all $i < j \leqslant \theta$, F_i is a free factor of F_j, but $\bigcup_{i<\theta} F_i$ is not a free factor of $F_\theta * K$ for any free algebra K. (See [EM1, Lemma 3.1].) Then we can continue as before.

4. APPLICATIONS

First, let us note the following.

4.1. LEMMA. *If R is an integral domain, then R is left perfect if and only if R is a field.*

Proof. Clearly a field is left perfect. Conversely, suppose R is not a field. Let x be a member of R which is not a unit. Then $R \supset xR \supset x^2R \supset \cdots \supset x^nR \supset \cdots$ is an infinite descending chain of principal ideals. ∎

The following is an immediate consequence of 4.1 and 2.2 and answers Adamek's question. (See the Introduction.)

4.2. PROPOSITION. *It is consistent with $ZFC + GCH$ that for every integral domain R which is not a field and every R-module K, there is an R-module M such that* $\operatorname{Ext}^1_R(M, K) = 0$ *but M is not projective.*

As another consequence, we obtain the proof of one-half of the following independence result:

4.3. COROLLARY. *Let R be a slender P.I.D. of cardinality $\aleph_1$ which is not a field. Then it is undecidable in $ZFC + GCH$ whether every Whitehead R-module is free.*

Proof. It is proved in [BFS, Theorem 5.1], using a result of Gerstner, Kaup, and Weidner, that assuming $V = L$, every Whitehead R-module (for R as given) is free. Proposition 4.2 (with $K = R$) shows that this result is not provable in $ZFC + GCH$. (The consistency proof in [BFS] applies only to countable domains.) ∎

For integral domains in general, we would not expect that every Whitehead module is free (or even projective), but it is proved in [BFS], Theorem 3.1] that it is consistent with ZFC that for domains R (of cardinality ρ) such that RD-submodules of torsion-free Whitehead R-modules are Whitehead, a torsion-free R-module M is Whitehead only if it is $\mathscr{W}(\rho)$-free; i.e., it is the union of a continuous chain $\langle M_\nu : \nu < \alpha \rangle$ of submodules such that $M_0 = 0$ and for every ν, $M_{\nu+1}/M_\nu$ is Whitehead and of cardinality $\leqslant \rho$. It is also proved there that it is consistent that this result fails for *countable* domains which are not fields. We are going to prove a similar result for some uncountable domains. Let Q denote the quotient-field of R. First, let us note that if R is a cotorsion domain—i.e., $\operatorname{Ext}^1_R(A, R) = 0$ whenever A is Q or an ideal of R— then every torsion-free R-module is Whitehead, so certainly every torsion-free module is $\mathscr{W}(\rho)$-free. (See [FSa, p. 243],) For example, a maximal valuation domain is cotorsion. The hypotheses on R in the following are satisfied if, for example, R is an almost-maximal valuation domain which is not maximal (cf. [BFS, Sect. 7]).

4.4. THEOREM. *It is consistent with $ZFC + GCH$ that whenever R is a Prüfer domain which is not cotorsion and is such that RD-submodules of Whitehead modules are Whitehead, then there is a Whitehead R-module which is not $\mathscr{W}(\rho)$-free.*

Proof. It suffices to show that Hypothesis 3.3a holds for $\mathscr{F} = \mathscr{W}(\rho)$ and some $\theta \leqslant \rho$. Since R is not cotorsion, there is some $A \subseteq Q$ which is not Whitehead. Choose such an A with the minimal number, θ, of generators. Since every finitely generated torsion-free module over a Prüfer domain is projective, θ must be an infinite cardinal. Then $A = \bigcup_{\nu<\theta} A_\nu$ (continuous) where each A_ν is a submodule generated by $< \theta$ elements. Thus each A_ν is Whitehead. For each $\nu < \theta$ choose a free module L_ν such that there is a homomorphism $\psi_\nu : L_\nu \to A$ with $\psi_\nu(L_\nu) = A_{\nu+1}$. Let $F_\theta = \bigoplus_{\nu<\theta} L_\nu$ and let $\varphi: F_\theta \to A$ be such that $\varphi \restriction L_\nu = \psi_\nu$; then φ is surjective. For $i < \theta$, let $F_i = \ker(\varphi \restriction \bigoplus_{\nu<i} L_\nu)$. Clearly $\langle F_i : i < \theta \rangle$ is a continuous chain, and $F_\theta / \bigcup_{i<\theta} F_i \cong A$, which is not Whitehead. Moreover, $F_\theta/F_i \cong A_i \oplus \bigoplus_{\nu \geqslant i} L_\nu$, which is Whitehead for all $i < \theta$. Also, for $i < j < \theta$, F_j/F_i is an RD-submodule of F_θ/F_i because $(F_\theta/F_i)(F_j/F_i) \cong F_\theta/F_j$ is torsion-free; thus F_j/F_i is Whitehead by hypothesis. Similarly, F_j is Whitehead since it is an RD-submodule of F_θ, which is free. ∎

4.5. COROLLARY. *It is consistent with $ZFC + GCH$ that whenever R is an almost-maximal valuation domain which is not maximal, there is a Whitehead R-module which is not $\mathscr{W}(\rho)$-free.*

As another application, we consider Baer modules. The question is whether there is a single test module for being Baer, i.e., a single torsion module $\mathscr{T}$ such that M (of arbitrary cardinality) is Baer if and only if it is of projective dimension $\leqslant 1$ and $\mathrm{Ext}^1_R(M, \mathscr{T}) = 0$. If there is one, then there is one of the form $\mathscr{T}_\kappa =^{\mathrm{def}} \bigoplus_{r \in R} (R/Rr)^{(\kappa)}$ (cf. [EF, Lemma 4]).

For $R = \mathbb{Z}$, in [E1] it was shown that $V = L$ implies that $\mathscr{T}_\omega$ is a test group for being a Baer group (and $MA + \neg CH$ implies that it is not). Now Griffith showed (in ZFC) that every Baer group is free. Thus, as an immediate consequence of 4.2, it is independent of $ZFC + GCH$ whether or not there is a test group for being a Baer group.

Since for arbitrary domains R, Baer modules are not necessarily (as far as we know) projective, we must do more work. In [EFS, Theorem A], it was proved (in ZFC) that every Baer module is $\mathscr{B}(\aleph_0)$-free. Also, in [EFS, Sect. 5] the proof in [E1] was generalized to show that $V = L$ implies that $\mathscr{T}_\omega$ is a test module for Baer if R is of cardinality $\leqslant \aleph_1$. This proof can be extended further to show that $V = L$ implies that $\mathscr{T}_\kappa$ is a test module if $|R| \leqslant \kappa$

4.6. THEOREM. *It is consistent with $ZFC + GCH$ that for every integral domain R which is not a field, and every cardinal κ, there is an R-module M of projective dimension $\leqslant 1$ such that* $\mathrm{Ext}^1_R(M, \mathscr{T}_\kappa) = 0$, *but M is not a Baer module.*

Proof. We use the model of Theorem 2.1. Given R (of cardinality ρ) and κ, choose $\chi > \max\{\kappa, \rho\}$. Then there is a regular cardinal $\lambda > \chi$ and a tree $T \subseteq {}^{\leqslant \omega}\lambda$ which is λ-free, of cardinality λ, satisfies (†) of 1.5, and has 2^{χ^ρ}-uniformization. There is a chain $\langle F_i : i \in \omega \rangle$ such that each F_i is free and countably generated and for $i < j \leqslant \omega$, F_j/F_i is free, but $F_\omega/(\bigcup_{i<\omega} F_i)$ is not a Baer module (cf. [EFS, Sect. 5]). Now let M^T be constructed as in 1.2 using this chain. Then M^T is χ-Whitehead, so $\mathrm{Ext}^1_R(M, \mathscr{T}_\kappa) = 0$. Moreover, M^T is λ-free of cardinality λ, so it is the union of a λ-filtration consisting of free modules, and hence by Auslander's Lemma (cf. [FSa, p. 73]) has projective dimension $\leqslant 1$ (because all the quotients have projective dimension $\leqslant 1$). Since Hypothesis 3.3 holds with $\mathscr{F} = \mathscr{B}(\aleph_0)$, M^T is not $\mathscr{B}(\aleph_0)$-free, so it is not a Baer module, by Theorem A of [EFS]. ∎

4.7. COROLLARY. *For any domain R which is not a field, it is undecidable in $ZFC + GCH$ whether there is a torsion module $\mathscr{T}$ such that an R-module M is a Baer module if and only if $pd(M) \leqslant 1$ and* $\mathrm{Ext}^1_R(M, \mathscr{T}) = 0$.

Proof. By the remarks before 4.6, $V = L$ implies that there is a test module for Baer. On the other hand, 4.6 implies that it is consistent that there is no test module. ∎

Note added in proof. J. Trlifaj (*Comment. Math. Univ. Carolinae* **31** (1990), 621–625) used a uniformization principle like that proved consistent in Section 2 to show that it is consistent that every regular Ext-ring is completely reducible. In a later preprint he extends his argument to arbitrary non-left perfect rings, and also shows that the hypothesis of non-left perfect is essential in 2.2.

References

[Ba] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466–488.

[BFS] T. Becker, L. Fuchs, and S. Shelah, Whitehead modules over domains, *Forum Math.* **1** (1989), 53–68.

[E1] P. C. Eklof, Homological algebra and set theory, *Trans. Amer. Math. Soc.* **227** (1977), 207–225.

[E2] P. C. Eklof, "Set-Theoretic Methods in Homological Algebra and Abelian Groups," Univ. of Montreal Press, 1980.

[EF] P. C. Eklof and L. Fuchs, Baer modules over valuation domains, *Ann. Mat. Pura Appl.* **150** (1988), 363–374.

[EFS] P. C. Eklof, L. Fuchs, and S. Shelah, Baer modules over domains, *Trans. Amer. Math. Soc.* **322** (1990), 547–560.

[EM1] P. C. Eklof and A. H. Mekler, Categoricity results for L_{∞} κ-free algebras, *Ann. Pure Appl. Logic* **37** (1988), 81–99.

[EM2] P. C. Eklof and A. H. Mekler, "Almost Free Modules: Set-Theoretic Methods," North-Holland, Amsterdam, 1990.

[F] L. Fuchs, "Infinite Abelian Groups," Academic Press, Orlando, FL, 1970, 1973.

[FSa] L. Fuchs and L. Salce, "Modules over Valuation Domains," Lecture Notes in Pure and Applied Math., Vol. 97, Dekker, New York, 1985.

[J] T. Jech, "Set Theory," Academic Press, Orlando, FL, 1978.

[K] I. Kaplansky, Projective modules, *Ann. of Math. (2)* **68** (1958), 372–377.

[MS1] A. H. Mekler and S. Shelah, When κ-free implies strongly κ-free, *in* "Abelian Group Theory," pp. 137–148, Gordon & Breach, New York, 1987.

[MS2] A. H. Mekler and S. Shelah, Uniformization principles, *J. Symbolic Logic* **54** (1989), 441–459.

[S1] S. Shelah, Infinite abelian groups, Whitehead problem and some constructions, *Isreal J. Math.* **18** (1974), 243–256.

[S2] S. Shelah, A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals, *Isreal J. Math.* **21** (1975), 319–340.

[S3] S. Shelah, Whitehead groups may not be free even assuming CH, I, *Israel J. Math.* **28** (1977), 193–203.

[S4] S. Shelah, Whitehead groups may not be free even assuming CH, II, *Israel J. Math.* **35** (1980), 257–285.

[S5] S. Shelah, "Proper Forcing," Lecture Notes in Mathematics, Vol. 940, Springer-Verlag, New York/Berlin, 1982.

[S6] S. Shelah, Diamonds uniformization, *J. Symbolic Logic* **49** (1984), 1022–1033.