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Reflecting stationary sets and successors of singular cardinals

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Abstract. REF is the statement that every stationary subset of a cardinal reflects, unless it fails to do so for a trivial reason. The main theorem, presented in Sect. 0, is that under suitable assumptions it is consistent that REF and there is a κ which is κ^{+n}-supercompact. The main concepts defined in Sect. 1 are PT, which is a certain statement about the existence of transversals, and the "bad" stationary set. It is shown that supercompactness (and even the failure of PT) implies the existence of non-reflecting stationary sets. E.g., if REF then for many λ's $\neg \mathrm{PT}(\lambda, \aleph_1)$. In Sect. 2 it is shown that Easton-support iteration of suitable Levy collapses yield a universe with REF if for every singular λ which is a limit of supercompacts the bad stationary set concentrates on the "right" cofinalities. In Sect. 3 the use of oracle c.c. (and oracle proper – see [Sh-b, Chap. IV] and [Sh 100, Sect. 4]) is adapted to replacing the diamond by the Laver diamond. Using this, a universe as needed in Sect. 2 is forced, where one starts, and ends, with a universe with a proper class of supercompacts. In Sect. 4 bad sets are handled in ZFC. For a regular λ $\{\delta<\lambda^+ : \mathrm{cf}\delta<\lambda\}$ is good. It is proved in ZFC that if $\lambda=\mathrm{cf}\lambda>\aleph_1$ then $\{\alpha<\lambda^+ : \mathrm{cf}\alpha<\lambda\}$ is the union of λ sets on which there are squares.

0 Introduction

We continue here Magidor-Shelah [MgSh 204] and [Sh 88a] (which is an improved representation of [Sh 108]), generalize the oracle c.c.c. forcing notion [Sh-b, Chap. IV], and solve a problem of Ben David [BD].

In Sect. 3 we rely heavily on the Laver indestructibility of supercompactness (Laver [L]) for $<\kappa$-directed closed forcing notions, which Baumgartner generalized to some not $<\kappa$-directed closed forcing.

For the background and the history of reflection see the recent Mekler and Shelah [MkSh 367] and Jech and Shelah [JSh 387]. For applications of Sect. 4 see [Sh 300, Ch. III, Sect. 6, 7] and Baldwin and Shelah [BSh 387]. Lemma 4.4 improves [Sh 237e]. An argument which shows that assuming the GCH is natural for the problems we deal with is given in [Sh 355].

An innocent reader may wonder if he has read the author's name correctly. The author gave, in the spring of 1988 a handwritten manuscript to A. Levy for having it typed for submission to the Archive for Mathematical Logic. Since then he has been rewriting it, squeezing out of me lengthy replacements for many explicit and implicit "left to the reader", "well known", "clear from ...", etc. All the thanks for the presentation and the much better English (except in this paragraph) should be directed to him.

In the statement of the next theorem REF denotes the assertion that all regular cardinals $>\aleph_1$ are reflecting in the sense of Definition 1.2. Our main results are as follows.

0.1 Theorem. (1) *If V has sufficiently many supercompacts (see* 3.7*) then in some forcing extension:*

$$\mathrm{GCH}+\mathrm{REF}+\textit{there is a } \kappa \textit{ which is } \kappa^{+\omega}\textit{-supercompact}.$$

(2) *If* GCH $+\kappa$ *is* $\kappa^{+(\omega+1)}$*-supercompact then* REF *fails. In fact, there are singular strong limit cardinals* $\lambda_1<\lambda_2$ *of cofinality* ω *and a stationary* $S\subsetneqq\{\delta<\lambda_2^+:\mathrm{cf}\,\delta=\lambda_1^+\}$ *which does not reflect.*

For a class A in which the axiom of extensionality holds we shall denote with j_A the Mostowski collapse of A.

0.2 Definition. 1) κ is a *λ-supercompact for* $A\subseteqq\mathrm{Ord}$ (really a $|H(\lambda)|$-supercompact for A) if: there is a normal κ-complete fine ultrafilter D on $\mathscr{S}_{<\kappa}(H(\lambda))=\{a\subseteqq H(\lambda):|a|<\kappa\}$ such that

$$\{a\in\mathscr{S}_{<\kappa}(H(\lambda)):j_a``(a\cap A)=A\cap j_a``a\}\in D$$

In this case we say that D *preserves* A.

2) $F:\kappa\to H(\kappa)$ is a *Laver diamond [for θ]* if for every $\lambda[\lambda\leqq\theta]$ and $x\in H(\lambda)$ for some normal κ-complete fine ultrafilter D on $\mathscr{S}_{<\kappa}(H(\lambda))$

$$\{a\in\mathscr{S}_{<\kappa}(H(\lambda)):x\in a\wedge j_a(x)=F(a\cap\kappa)\}\in D.$$

In this case we say also that F is a *κ-Laver diamond [for θ].*

0.3 Fact. Forcing by a forcing notion of power $<\kappa$ preserves "κ is *λ-supercompact* for A", for λ with $\mathrm{cf}\,\lambda\geqq\kappa$, and "*there is a Laver diamond for* θ".

Proof. Let P be a forcing notion of power $<\kappa$ and let $\mathrm{cf}\,\lambda\geqq\kappa$. Let κ be λ-supercompact for A and let D be an ultrafilter on $\mathscr{S}_{<\kappa}(H(\lambda))$ witnessing that. We define now an ultrafilter D^G over $\mathscr{S}_{<\kappa}(H(\lambda))^{V[G]}$ in $V[G]$, where G is P-generic over V. For $a\in\mathscr{S}_{<\kappa}(H(\lambda))$ we define

$$a_G=a\cup\{\tau[G]:\tau\in a\wedge\tau \text{ is a } P\text{-name}\wedge 0\Vdash\tau[G]\in H(\lambda)^{V[G]}\}.$$

For $X\subseteqq\mathscr{S}_{<\kappa}(H(\lambda))$ we define $X^G=\{a_G:a\in X\}$. D_G is defined to be the filter generated by $\{X^G:X\in D\}$.

To prove that D^G is a κ-complete filter we shall show that if $T\subseteqq\{X^G:X\in D\}$ and $|T|^{V[G]}<\kappa$ then $\bigcap T\in D^G$. Since $|P|<\kappa$ there is a $T'\in V$, $T'\subseteqq D$, $|T'|<\kappa$ such that $T\subseteqq\{X^G:X\in T'\}$. Let $Y=\bigcap T'$ then $Y^G\subseteqq\bigcap T$, and since D is κ-complete $Y\in D$ and thus $\bigcap T\in D^G$. Since $H(\lambda)^{V[G]}=\{\tau[G]:\tau\in H(\lambda)\wedge\tau \text{ is a } P\text{-name }\wedge 0\Vdash\tau[G]\in H(\lambda)\}$ it follows easily that D^G is fine. Let us see now that D^G is an ultrafilter. Let $\underset{\sim}{\sigma}$ be a

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P-name of a subset of $\mathscr{S}_{<\kappa}(H(\lambda))^{V[G]}$. We define an equivalence relation E on $\mathscr{S}_{<\kappa}(H(\lambda))$ by setting

$$aEb \Leftrightarrow (\forall p \in P)(p \Vdash a_G \in \underset{\sim}{\sigma} \Leftrightarrow p \Vdash b_G \in \underset{\sim}{\sigma}).$$

E has $\leqq |\mathscr{P}(P)| = 2^{|P|} < \kappa$ equivalence classes, hence one of them, which we shall denote with X, is in D. Let $a \in X$. If $a_G \in \underset{\sim}{\sigma}[G]$ then $X^G \subseteq \underset{\sim}{\sigma}[G]$ hence $\underset{\sim}{\sigma}[G] \in D^G$. If $a_G \notin \underset{\sim}{\sigma}[G]$ then $X^G \subseteq \mathscr{S}_{<\kappa}(H(\lambda))^{V[G]} \backslash \underset{\sim}{\sigma}[G]$ hence $\mathscr{S}_{<\kappa}(H(\lambda)) \backslash \underset{\sim}{\sigma}[G] \in D^G$.

In order to show that D^G is normal let us first mention a few simple properties of D. Since $|P| < \kappa$ we can assume that $P \subseteq \kappa$ and since D is fine and κ-complete

$$\{a \in \mathscr{S}_{<\kappa}(H(\lambda)) : P \subseteq a\} \in D.$$

Let F be a function on $\mathscr{S}_{<\kappa}(H(\lambda))$ such that for every $a \in \mathscr{S}_{<\kappa}(H(\lambda))$ $F(a) \subseteq a$. Then, as easily seen by the fineness and normality of D, if

$$\bar{F} = \{u \in H(\lambda) : \{a \in \mathscr{S}_{<\kappa}(H(\lambda)) : u \in F(a)\} \in D\}$$

then $\{a \in \mathscr{S}_{<\kappa}(H(\lambda)) : F(a) = a \cap \bar{F}\} \in D$. Similarly, it is easily seen that

$$\{a \in \mathscr{S}_{<\kappa}(H(\lambda)) : (\forall u, v \in a) \langle u, v \rangle \in a\} \in D.$$

Using these facts it is easy to prove that $\{a \in \mathscr{S}_{<\kappa}(H(\lambda))$: Every partial function from P into a is in $a\} \in D$. The same methods show that for the usual way of defining a P-name of a set in V $\{a \in \mathscr{S}_{<\kappa}(H(\lambda))$: a contains a P-name of every member of $a\} \in D$, hence for almost all a's $a_G = \{\tau[G] \colon \tau \in a \wedge \tau$ is a P-name $\wedge\, 0 \Vdash \tau[G] \in H(\lambda)\}$.

Let $\underset{\sim}{\sigma}$ be the name of a function $\in V[G]$ such that $\underset{\sim}{\sigma}[G](a_G) \in a_G$ for every $a \in \mathscr{S}_{<\kappa}(H(\lambda)) - \{0\}$. For every $a \in \mathscr{S}_{<\kappa}(H(\lambda)) - \{0\}$ let f_a be a function on an open dense subset of P into a such that for $p \in \mathrm{Dom}(f)$ either $p \Vdash \underset{\sim}{\sigma}[G]\,(a_G) \notin a_G$ and $f_a(p)$ is an arbitrary member of a, or $f_a(p)$ is a P-name $\underset{\sim}{\tau}$ such that $\underset{\sim}{\tau} \in a \wedge 0 \Vdash \underset{\sim}{\tau}[G] \in H(\lambda)$ and $p \Vdash \underset{\sim}{\sigma}[G]\,(a_G) = \underset{\sim}{\tau}[G]$. By what we saw above, for almost all a's, in the sense of D, $f_a \in a$, hence there is a function f from an open dense subset Q of P into $\mathscr{S}_{<\kappa}(H(\lambda))$ such that for some $Y \in D$ we have for all $a \in Y$ $f_a = f$. Let $p \in Q \cap G$. Since $p \in G$ it is not the case that $p \Vdash \underset{\sim}{\sigma}[G]\,(a_G) \notin a_G$ hence for $\underset{\sim}{\tau} = f(p)$ we have for every $a \in Y$ $p \Vdash \underset{\sim}{\sigma}[G](a_G) = \underset{\sim}{\tau}[G]$, hence for every $a \in Y^G$ $\underset{\sim}{\sigma}[G](a) = \underset{\sim}{\tau}[G]$, and we have established the normality of D^G.

In order to complete the proof of the preservation of the A-supercompactness we still have to prove the "for A" part of this property. It is easy to see that there is a $Y \in D$ such that each member a of Y has the following properties (i)–(iii):

(i) For every P-name τ which is in a all possible values of τ which are in $H(\lambda)$ are also in a (there are $< \kappa$ such values).

(ii) a is an elementary substructure of $H(\lambda)$ and therefore, since $P \in H(\lambda)$, also a_G is an elementary substructure of $H(\lambda)^{V[G]}$.

(iii) $j_a``(a \cap A) = A \cap j_a``a$ (this is possible by our hypothesis on D).

By (i) $a_G \cap V = a$ and $j_a = j_{a_G} \restriction V$. By (ii) j_a and j_{a_G} map only ordinals to ordinals and thus, by (iii),

$$j_{a_G}``(a \cap A) = j_a``(a \cap A) = A \cap j_a``a = A \cap j_a``(a \cap \mathrm{Ord}) = A \cap j_{a_G}``(a_G \cap \mathrm{Ord}) = A \cap j_{a_G}``a_G.$$

This ends the proof that κ is λ-supercompact for A in $V[G]$.

Now let us assume that $F : \kappa \Rightarrow H(\kappa)$ is a Laver diamond for θ. Let $F' \in V[G]$ be defined by $F'(\alpha) = F(\alpha)\,[G]$ if $F(\alpha)$ is a P-name, and $F'(\alpha) = F(\alpha)$ otherwise. We shall prove that F' is a Laver diamond in V [for θ]. Let $\lambda \leqq \theta$, and assume that for some $p \in G$ and some P-name $\underset{\sim}{\tau}$

$p \Vdash$ *"the value x of τ is in $H(\lambda)$ and is counterexample to F' being a Laver diamond".*

Thus for some $\mu < \theta$ and q, $p \leqq q \in P$, $q \Vdash \underset{\sim}{\tau} \in H(\mu^+)$, and without loss of generality $\lambda = \mu^+$. Since F is a Laver diamond for θ there is a κ-complete normal fine ultrafilter D on $\mathscr{S}_{<\kappa}(H(\lambda))$ such that

$$\{a \in \mathscr{S}_{<\kappa}(H(\chi)) : \underset{\sim}{\tau} \in a \wedge j_a(\underset{\sim}{\tau}) = F(a \cap \kappa)\} \in D.$$

It is easily seen that

$$p \Vdash \text{"}D^G \textit{ is as required for } F' \textit{ to be a Laver diamond in } V[G]\text{"}.$$

1 Reflection of stationary sets versus existence of supercompacts

1.1 Definition. 1) $\mathrm{PT}(\mu, \lambda, \kappa)$ means:

If A is a family of power μ of sets which are of power $< \kappa$ and every $A' \in [A]^{<\lambda}$ has a transversal then also A has a transversal.

2) $\mathrm{PT}(<\mu, \lambda, \kappa)$ is defined similarly except that $|A| < \mu$.

3) $\mathrm{PT}(<\infty, \lambda, \kappa)$ is defined similarly except that $|A|$ can be of any cardinality.

4) $\mathrm{PT}(\lambda, \kappa)$ is $\mathrm{PT}(\lambda, \lambda, \kappa)$.

1.2 Definition. 1) Let λ be a cardinal with $\mathrm{cf}\lambda > \omega$ and let S be a stationary subset of λ. We say that S *reflects* at δ if $\delta < \lambda$, $\mathrm{cf}\delta > \omega$ and $S \cap \delta$ is a stationary subset of δ. Since δ has a club subset consisting entirely of ordinals of cofinality $< \mathrm{cf}\delta$, if S reflects at δ then S must have members of cofinality $< \mathrm{cf}\delta$. We say that S *reflects* if it reflects at some δ.

2) For a regular cardinal λ, we say that λ is *reflecting* if for every regular κ such that $\kappa^+ < \lambda$ and for every $S \subseteq \{\sigma : \sigma < \lambda, \mathrm{cf}\sigma = \kappa\}$ which is a stationary subset of λ S reflects. By the remark above S can reflect only at δ's with $\mathrm{cf}\delta > \kappa$.

3) For a class K of regular cardinals REF(K) denotes the statement that every $\lambda \in K$ is reflecting.

4) REF denotes the statement that every regular cardinal $> \aleph_1$ is reflecting.

1.3 Fact (GCH). *If κ is supercompact then*
(*) there are singular cardinals $\lambda_1 < \lambda_2$ and a stationary subset S of λ_2^+, $S \subseteq \{\delta < \lambda_2^+ : \mathrm{cf}\delta = \lambda_1^+\}$ which does not reflect.

Proof. This follows from 1.4 since if κ is supercompact then, as is well known and is easily seen, $\mathrm{PT}(<\infty, \kappa, \kappa)$, and this implies for all $\theta < \kappa \leqq \lambda$ $\mathrm{PT}(\lambda^+, \lambda^+, \theta)$, which is the hypothesis of 1.4.

1.4 Fact (GCH). *If $\mathrm{cf}\lambda = \theta < \lambda$ and $PT(\lambda^+, \theta^+)$ then*
$(*)_\theta$ for some singular cardinals $\lambda_1 < \lambda_2 \leqq \lambda$ of cofinality θ there is a stationary subset S of $\{\delta < \lambda_2^+ : \mathrm{cf}\delta = \lambda_1^+\}$ which does not reflect.

Proof. Let $\langle \chi_i : i < \theta \rangle$ be an increasing sequence of regular cardinals $< \lambda$ such that $\sup_{i<\theta} \chi_i = \lambda$. As shown, e.g., in [MgSh 204, Lemma 3], one can choose for every $\alpha < \lambda^+$ $f_\alpha \in \bigtimes_{i<\theta} \chi_i$ such that:

(i) For $\alpha < \beta$ $f_\alpha <^* f_\beta$, i.e., for all sufficiently large $i < \theta$ $f_\alpha(i) < f_\beta(i)$.
(ii) For every $f \in \bigtimes_{i<\theta} \chi_i$ there is an $\alpha < \lambda^+$ such that $f <^* f_\alpha$.

Let $\bar{f}=\langle f_\alpha:\alpha<\lambda^+\rangle$. For $A\subseteq\lambda^+$ and $b\subseteq\theta$ we say that $\bar{f}$ is *ascending on A over b* if

$$(\forall\alpha,\beta\in A)(\forall i\in b)(\alpha<\beta\rightarrow f_\alpha(i)<f_\beta(i)).$$

We say that that $\bar{f}$ is *almost ascending on A over b* if for some $i_0<\theta$ $\bar{f}$ is ascending on A over $b\cap[i_0,\theta)$, where $[i,\theta)=\{i:i_0\leqq i<\theta\}$. For $\bar{f}$ we define the bad stationary subsets of λ^+ to be as in [MgSh 204, Sect. 1, preceeding Theorem 11]:

$$\mathrm{bad}(\bar{f})=\{\delta<\lambda^+:\mathrm{cf}(\delta)>\theta\wedge\text{for no unbounded subset } A \text{ of } \delta$$
$$\text{is } \bar{f} \text{ almost ascending on } A \text{ over } \theta\}.$$
$$\mathrm{bad}'(\bar{f})=\{\delta\in\mathrm{bad}(\bar{f}):\mathrm{cf}\,\delta>2^\theta\}.$$

As shown in [MgSh 204, Sect. 1] we have, for every $\delta<\lambda^+$

(Pr_1) $\delta\in\mathrm{bad}'(\bar{f})\Rightarrow\mathrm{cf}\,\delta$ is the successor of a singular cardinal of cofinality θ.

(Pr_2) $\delta\notin\mathrm{bad}(\bar{f})\Rightarrow$ there is a club subset C of δ disjoint from $\mathrm{bad}(\bar{f})$.

By the GCH (Pr_2) holds also for every $\delta\notin\mathrm{bad}'(\bar{f})$.

Now $\mathrm{bad}'(\bar{f})$ is stationary in λ^+, since otherwise, as follows from the proof of [MgSh 204, Theorem 2], our hypothesis $\mathrm{PT}(\lambda^+,\lambda^+,\theta)$ fails.

Let $\delta\leqq\lambda^+$ be the least ordinal δ such that $\delta\cap\mathrm{bad}'(\bar{f})$ is stationary in δ [there is such a δ since $\mathrm{bad}'(\bar{f})$ is stationary in λ^+]. By (Pr_2), and the remark following it, $\delta\in\mathrm{bad}'(\bar{f})$ or $\delta=\lambda^+$. By (Pr_1), or by our choice of λ^+, $\mathrm{cf}\,\delta$ is the successor of a singular cardinal λ_2 of cofinality θ. Let $h:\lambda_2^+\rightarrow\delta$ be strictly increasing and continuous and with the range unbounded in δ, hence

$$S'=\{\alpha<\lambda_2^+:\alpha=\bigcup\alpha\wedge h(\alpha)\in\mathrm{bad}'(\bar{f})\}$$

is stationary, and does not reflect. As $\mathrm{cf}\,\alpha=\mathrm{cf}(h(\alpha))$ for limit ordinals α, the cofinality of each $\alpha\in S'$ is the successor of a singular cardinal λ_1^α of cofinality θ. As $\lambda_1^\alpha<\lambda_2$, there is an ordinal $\lambda_1<\lambda_2$ such that $S=\{\alpha\in S':\lambda_1^\alpha=\lambda_1\}$ is stationary. S,λ_1,λ_2 are as required.

1.4A Remark. In the definition of $\mathrm{bad}(\bar{f})$ we can replace, equivalently, the part "for no unbounded subset A of δ is $\bar{f}$ almost ascending on A over θ" by "some unbounded subset A_0 of δ has no unbounded subset A such that $\bar{f}$ is almost ascending on A over θ".

Proof. To prove the non-trivial direction of the remark let $\mathrm{cf}\,\delta>\theta$ and let A be an unbounded subset of δ such that for some $\gamma^*<\theta$ $\bar{f}$ is ascending on A over $[\gamma^*,\theta)$, and we shall show that every unbounded subset A_0 of δ has an unbounded subset A' on which $\bar{f}$ is almost ascending over θ.

We can assume, without loss of generality, that the order type of A is $\mathrm{cf}\,\delta$. Let $A=\{\alpha_j:j<\mathrm{cf}\,\delta\}$. For every $j<\mathrm{cf}\,\delta$ let $\beta_j<\delta$ and $k_j<\mathrm{cf}\,\delta$ be such that $\alpha_j<\beta_j<\alpha_{k_j}$ and $\beta_j\in A_0$. We clearly have $f_{\alpha_j}<^*f_{\beta_j}<^*f_{\alpha_{k_j}}$, *hence there is an ordinal* $\gamma_j<\theta$ such that for all $\gamma_j<i<\theta$ $f_{\alpha_j}(i)<f_{\beta_j}(i)<f_{\alpha_{k_j}}(i)$. Since $\mathrm{cf}\,\delta>\theta$ there is an unbounded subset X_1 of $\mathrm{cf}\,\delta$ and a $\gamma^{\|}<\theta$ such that for every $j\in X_1$ $\gamma_j=\gamma^{\|}$. Since $\mathrm{cf}\,\delta$ is regular there is a subset X_2 of X_1 such that if $j<j'<\mathrm{cf}\,\delta$ and $j,j'\in X_2$ then $k_j<j'$. Let $\gamma'=\max(\gamma^*,\gamma^{\|})$ and $A'=\{\beta_j:j\in X_2\}$. A' is an unbounded subset of A_0 and $\gamma'<\theta$. We shall now see that $\bar{f}$ is ascending on A' over (γ',θ). Let $\beta,\beta'\in A'$, $\beta<\beta'$, then for some $j<j'<\mathrm{cf}\,\delta$ $j,j'\in X_2$ we have $\beta=\beta_j$ and $\beta'=\beta_{j'}$, and, by the definition of X_2, $k_j<j'$. For $\gamma'<i<\theta$

we have

$$f_\beta(i)=f_{\beta_j}(i)<f_{\alpha_{k_j}}(i), \quad \text{since} \quad i>\gamma'\geqq\gamma^{\|}=\gamma_j.$$

$$f_{\alpha_{k_j}}(i)<f_{\alpha_{j'}}(i), \quad \text{since} \quad k_j<j', \quad \alpha_{k_j},\alpha_{j'}\in A \quad \text{and} \quad i>\gamma'\geqq\gamma^*.$$

$$f_{\alpha_{j'}}(i)<f_{\beta_{j'}}(i)=f_{\beta'}(i), \quad \text{since} \quad i>\gamma'\geqq\gamma^{\|}=\gamma_{j'}.$$

1.4B Remark. The cardinals λ_1,λ_2 and the set S which we have obtained in the proof of 1.4 can be chosen so that $S\subsetneq \mathrm{bad}'(\langle f'_\alpha|\alpha<\lambda_2^+\rangle)$, where for some increasing sequence $\langle\chi_{2,i}:i<\theta\rangle$ of regular cardinals with $\sup_{i<\theta}\chi_{2,i}=\lambda_2$ $\langle f'_\alpha|\alpha<\lambda_2^+\rangle$ is such that for every $\alpha<\lambda_2^+$ $f'_\alpha\in \bigtimes_{i<\theta}\chi_{2,i}$ and

(i) For $\alpha<\beta<\lambda_2^+$ $f'_\alpha<^*f'_\beta$.

(ii) For every $f\in\bigtimes_{i<\theta}\chi_{2,i}$ there is an $\alpha<\lambda_2^+$ such that $f<^*f'_\alpha$.

Proof. For every $\beta<\lambda^+$ such that $\mathrm{cf}\beta>2^\theta$ let $f^*_\beta\in\bigtimes_{i<\theta}\chi_i$ be a least upper bound of $\langle f_\alpha:\alpha<\beta\rangle$, in the sense that $f_\alpha<^*f^*_\beta$ for every $\alpha<\beta$ and for every function g on θ if $g<^*f^*_\beta$ then for some $\alpha<\beta$ $g<^*f_\alpha$. For a proof of the existence of such an f^*_β see [Sh 68, Lemma 11B and (*), p. 61] or [Sh 111, 2.3, p. 269] or [MgSh 204, Lemma 5]. We shall now see that if h is a function on θ such that for every $\alpha<\beta$ $f_\alpha<^*h$ then $f^*_\beta\leqq h$, i.e., for some $i_0<\theta$ $i\geqq i_0\to f^*_\beta(i)\leqq h(i)$. Let g be the function on θ given by

$$g(i)=\begin{cases} h(i) & \text{if} \quad h(i)<f^*_\beta(i) \\ 0 & \text{otherwise}\end{cases}.$$

Since, without loss of generality, $f^*(i)>0$ for every $i<\theta$ we have $g<^*f^*_\beta$, hence for some $\gamma<\beta$ $g<^*f_\gamma$. Since also $f_\gamma<^*h$ the set $\{i<\theta:g(i)=h(i)\}$ is bounded hence, by the definition of g, $\{i<\theta:h(i)<f^*_\beta(i)\}$ is bounded, so $f^*_\beta\leqq^*h$. Now we shall prove the following lemmas and then return to 1.4B.

1.4C Lemma. *For $\delta<\lambda^+$ with $\mathrm{cf}\delta>2^\theta$ let $a_\delta=\{i<\theta:\mathrm{cf}f^*_\delta(i)<\mathrm{cf}\delta\}$. For every subset b of θ there is an unbounded subset A of δ such that $\bar f$ is almost ascending on A over b iff $a_\delta\cap b$ is bounded. In particular, for $b=\theta$, $\delta\in\mathrm{bad}'(\bar f)$ iff $\mathrm{cf}\delta>2^\theta$ and a_δ is unbounded.*

Proof. First we prove that there is an unbounded subset A of δ such that $\bar f$ is almost ascending on A over $\theta\backslash a_\delta$. This implies, trivially, that if $b\subseteq\theta$ and $b\cap a_\delta$ is bounded then $\bar f$ is almost ascending on A over b.

Let $h:\mathrm{cf}\delta\to\delta$ be such that Range(h) is unbounded in δ. We shall define, by induction on $j<\mathrm{cf}\delta$, α_j and ζ_j such that

(1) $h(j)$, $\bigcup_{k<j}\alpha_k<\alpha_j<\delta$,

(2) $\zeta_j<\theta$,

(3) If $k<j$, ζ_k, $\zeta_j<\zeta<\theta$, and $\zeta\notin a_\delta$ then $f_{\alpha_k}(\zeta)<f_{\alpha_j}(\zeta)<f^*_\delta(\zeta)$.

In the j-th step we define a function g_j on θ such that for $\zeta<\theta$

$$g_j(\zeta)=\begin{cases}\sup\{f_{\alpha_k}(\zeta):k<j\wedge f_{\alpha_k}(\zeta)<f^*_\delta(\zeta)\} & \text{if} \quad \zeta\notin a_\delta \\ 0 & \text{if} \quad \zeta\in a_\delta\end{cases}.$$

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For $\zeta \notin a_\delta$ we have, by the definition of a_δ, $\mathrm{cf} f_\delta^*(\zeta) \geqq \mathrm{cf}\delta$, and since $g_j(\zeta)$ is the least upper bound of $j < \mathrm{cf}\delta$ – many ordinals $< f_\delta^*(\zeta)$ we have also $g_j(\zeta) < f_\delta^*(\zeta)$; for $\zeta \in a_\delta$ $g_j(\zeta) < f_\delta^*(\zeta)$, since we can assume, without loss of generality, that $f_\delta^* > 0$ for every $\zeta < \theta$. Thus $g_j <^* f_\delta^*$ and hence, since f_δ^* is a least upper bound of $\{f_\alpha : \alpha < \delta\}$, $g_j < f_\alpha$ for some $\alpha < \delta$; let α_j be such an α which also satisfies (1). Let $\zeta_j < \theta$ be such that

$$\zeta_j \leqq \zeta < \theta \Rightarrow g_j(\zeta) < f_{\alpha_j}(\zeta)) < f_\delta^*(\zeta);$$

clearly α_j and ζ_j satisfy (1)–(3). Since $\mathrm{cf}\delta > 2^\theta$ there is a $\zeta^* < \theta$ such that $\{j < \mathrm{cf}\delta : \zeta_j = \zeta^*\}$ is unbounded in δ. Thus $\bar{f}$ is ascending on

$$A = \{\alpha_j : j < \mathrm{cf}\delta \wedge \zeta_j = \zeta^*\}$$

over $(\theta - a_\delta) \cap [\zeta^*, \theta)$.

To prove the other direction of the lemma it suffices to assume that $\bar{f}$ is ascending on A over b, where A is an unbounded subset of δ, and show that $a_\delta \cap b$ is bounded. Let g be the function on θ given by

$$g(i) = \begin{cases} \sup\{f_\alpha(i) : \alpha \in A\} & \text{if} \quad i \in b \\ f_\delta^*(i) & \text{otherwise} \end{cases}.$$

For $\alpha < \delta$ $f_\alpha <^* f_\delta^*$; let $\zeta_\alpha < \theta$ be such that $\zeta_\alpha \leqq i < \theta \to f_\alpha(i) < f_\delta^*(i)$. Since $\mathrm{cf}\delta > 2^\theta$ A has an unbounded subset A' such that for some $\zeta' < \theta$ $\alpha \in A' \to \zeta_\alpha = \zeta'$. Thus for $\alpha \in A'$ and $\zeta' \leqq i < \theta$ $f_\alpha(i) < f_\delta^*(i)$, hence for $i \in b \cap [\zeta', \theta)$

$$g(i) = \sup_{\alpha \in A} f_\alpha(i) = \sup_{\alpha \in A'} f_\alpha(i) \leqq f_\delta^*(i);$$

thus $g \leqq^* f_\delta^*$. For any $\alpha < \delta$ let $\alpha' \in A \cap (\alpha, \delta)$, then, clearly, $f_\alpha <^* f_{\alpha'} <^* g$. Thus g is an upper bound of $\{f_\alpha : \alpha < \delta\}$, hence, as we have shown above, $f_\delta^* \leqq^* g$. $g \leqq^* f_\delta^*$ and $f_\delta^* \leqq^* g$ imply that for some $i_0 < \theta$ $i_0 < i < \theta \to f_\delta^*(i) = g(i)$, hence $i_0 < i < \theta \to \mathrm{cf} f_\delta^*(i) = \mathrm{cf} g(i)$. For $i_0 < i \in b$ the sequence $\langle f_\alpha(i) : \alpha \in A\rangle$ is ascending and hence $\mathrm{cf} f_\delta^*(i) = \mathrm{cf} g(i) = \mathrm{cf} A = \mathrm{cf}\delta$. Thus $a_\delta \cap b \subseteqq i_0$ and $a_\delta \cap b$ is bounded.

1.4D Lemma. *For $\delta < \lambda^+$ with $\mathrm{cf}\delta > 2^\theta$ and for every cardinal μ such that $\mu^+ < \mathrm{cf}\delta$ the set $b = \{i < \theta : \mathrm{cf} f_b^*(i) \leqq \mu^+\}$ is a bounded subset of θ.*

Proof. For every $i \in b$ let C_i be a cofinal subset of $f_\delta^*(i)$ of cardinality $\mathrm{cf} f_\delta^*(i)$. For every $t \in \bigtimes_{i \in b} C_i$ let g_t be the function on θ given by:

$$g_t(i) = \begin{cases} t(i) & \text{if} \quad i \in b \\ 0 & \text{otherwise} \end{cases}.$$

Clearly $g_t <^* f_\delta^*$, hence for some $\alpha_t < \delta$ $g_t <^* f_{\alpha_t}$.

$$\left|\bigtimes_{i \in b} C_i\right| = \prod_{i \in b} \mathrm{cf} f_\delta^*(i) \leqq (\mu^+)^\theta < \mathrm{cf}\delta$$

(since μ^+, $\theta^+ < \mathrm{cf}\delta$). Therefore

$$\sup\left\{\alpha_t : t \in \bigtimes_{i \in b} C_i\right\} < \mathrm{cf}\delta,$$

and if we denote $\sup\left\{\alpha_t : t \in \bigtimes_{i \in b} C_i\right\}$ with α we have $g_t <^* f_\alpha$ for every $t \in \bigtimes_{i \in b} C_i$. Since $f_\alpha <^* f_\delta^*$ there is an $i_0 < \theta$ such that $i_0 \leqq i < \theta \to f_\alpha(i) < f_\delta^*(i)$; hence there is an $s \in \bigtimes_{i \in b} C_i$

such that $i\in b\setminus i_0 \to f_\alpha(i)\leqq s(i)=g_s(i)$. If b were unbounded this would contradict $g_s<^*f_\alpha$.

Proof of 1.4B (continued). We change now somewhat the construction in the proof of 1.4. By (Pr_1), for every $\alpha\in\mathrm{bad}'(\bar{f})$ $\mathrm{cf}\,\alpha=(\lambda_1^\alpha)^+$ where λ_1^α is a singular limit cardinal of cofinality θ. Since $\lambda_1^\alpha<(\lambda_1^\alpha)^+=\mathrm{cf}\,\alpha\leqq\alpha$ there is, by Fodor's theorem, a stationary subset T of $\mathrm{bad}'(\bar{f})$ and a singular cardinal λ_1 such that $\lambda_1^\alpha=\lambda_1$ for every $\alpha\in T$. For every $\alpha\in T\subseteq\mathrm{bad}'(\bar{f})$ $\mathrm{cf}\,\alpha>2^\theta$ and $a_\alpha\subseteq\theta$, and since $\theta=\mathrm{cf}\,\lambda<\lambda<\lambda^+$ we have $2^\theta=\theta^+\leqq\lambda<\lambda^+$, and therefore there is an $a\subseteq\theta$ and a stationary subset S of T such that for every $\alpha\in S$ $a_\alpha=a$. Let $\delta\leqq\lambda^+$ be the least ordinal δ such that $\delta\cap S$ is stationary in δ, let $\mathrm{cf}\,\delta=\lambda_2^+$, and for $i<\theta$ let $\chi_{2,i}=\mathrm{cf}\,f_\delta^*(i)$. Clearly $\lambda_2>\lambda_1$, since δ has a club subset of ordinals of cofinality $<\mathrm{cf}\,\delta=\lambda_2$. If $\delta=\lambda^+$ then 1.4B holds with $f'_\alpha=f_\alpha$ for $\alpha<\lambda^+$ and $\chi_{2,i}=\chi_i$ for $i<\theta$, so we are left with the case where $\delta\in\mathrm{bad}'(\bar{f})$.

Let us see now that $a\setminus a_\delta$ is bounded in θ. By 1.4C δ has an unbounded subset A such that for some $i_0<\theta$ $\bar{f}$ is ascending in A over $(a-a_\delta)\cap[i_0,\theta)$. Let A' be the set of all accumulation points of A in δ. For every $\beta\in A'$ $A\cap\beta$ is an unbounded subset of β and $\langle f_\alpha:\alpha<\beta\rangle$ is increasing in $A\cap\beta$ over $(a\setminus a_\delta)\cap[i_0,\theta)$. Since A' is a club subset of δ and S is stationary in δ there is a $\beta\in A'\cap S$. Since $\beta\in S$ $a_\beta=a$. By 1.4C $a_\beta\cap(a\setminus a_\delta)\cap[i_0,\theta)$ is bounded, hence also $a\setminus a_\delta$ is bounded.

We shall now see that we can assume, without loss of generality, that $a_\delta=\theta$, i.e., that for $i<\theta$ $\chi_{2,i}<\lambda_2$, and, moreover, that $\langle\chi_{2,i}:i<\theta\rangle$ is an ascending sequence. By 1.4C a is an unbounded subset of θ, and since $a\setminus a_\delta$ is bounded $a\cap a_\delta$ is unbounded. By 1.4D $\{\chi_{2,i}:i\in a\cap a_\delta\}$ is unbounded below λ_2, and since $\mathrm{cf}\,\lambda_2=\theta$ $a\cap a_\delta$ has an unbounded subset b such that $\langle\chi_{2,i}:i\in b\rangle$ is an ascending sequence of infinite cardinals. Let h be the order-preserving function mapping θ on b. For $i<\theta$ let $\chi_i^{\|}=\chi_{h(i)}$, $\chi_{2,i}^{\|}=\chi_{2,h(i)}$, and for every function f on θ let $f^{\|}=fh$. We shall now see that if we replace in 1.4 $\langle\chi_i:i<\theta\rangle$ and $\bar{f}=\langle f_\alpha:\alpha<\lambda^+\rangle$ by $\langle\chi_i^{\|}:i<\theta\rangle$ and $\bar{f}^{\|}=\langle f_\alpha^{\|}:\alpha<\lambda^+\rangle$, respectively, then the hypotheses of 1.4 still hold and $S\subseteq\mathrm{bad}'(\langle f_\alpha^{\|}:\alpha<\lambda^+\rangle)$. By our choice of b $\langle\chi_i^{\|}:i<\theta\rangle$ is an ascending sequence of regular cardinals $<\lambda$, $\sup_{i<\theta}\chi_i^{\|}=\lambda$, and for $\alpha<\beta<\lambda^+$ $f_\alpha^{\|}<^*f_\beta^{\|}$. Routine checking shows that $\langle\chi_i^{\|}:i<\theta\rangle$ is a least upper bound of $\bar{f}^{\|}$ and that for every $\beta<\lambda^+$ such that $\mathrm{cf}\,\beta>2^\theta$ $f_\beta^{*,\|}$ is a least upper bound of $\langle f_\alpha^{\|}:\alpha<\beta\rangle$. Also, for $\beta<\lambda^+$ $a_\beta^{\|}$, which is naturally taken to be $\{i<\theta:\mathrm{cf}\,f_\beta^{*,\|}(i)<\mathrm{cf}\,\delta\}$, satisfies

$$a_\beta^{\|}=\{i<\theta:\mathrm{cf}\,f_\beta^{\|}(h(i))<\mathrm{cf}\,\delta\}=\{i<\theta:h(i)\in a_\beta\}=h^{-1}(a_\beta).$$

For $\beta\in S$ $a_\beta=a$, and since $b\subseteq a$ $a_\beta^{\|}=h^{-1}(a_\beta)=\theta$; by 1.4C $\beta\in\mathrm{bad}'(\bar{f}^{\|})$. Also, clearly, for $i<\theta$ $\chi_{2,i}^{\|}=\mathrm{cf}\,f_\delta^{*,\|}(i)$ and the sequence $\langle\chi_{2,i}^{\|}:i<\theta\rangle$ is ascending. Thus we have established what we claimed at the beginning of the paragraph.

Let C_i be a club subset of $f_\delta^*(i)$ of order type $\chi_{2,i}$. We define, for $\alpha<\delta$ and $i<\theta$, $f_\alpha^0(i)=$ the order type of $C_i\cap f_\alpha(i)$ and

$$f_\alpha^1(i)=\begin{cases}f_\alpha^0(i) & \text{if } f_\alpha^0(i)<\chi_{2,i}\\ 0 & \text{otherwise}\end{cases}.$$

It follows easily from $f_\alpha<^*f_\delta^*$ that $f_\alpha^1=^*f_\alpha^0$, where for functions g,h on θ $g=^*h$ means that for sufficiently large $i<\theta$ $g(i)=h(i)$. Clearly $f_\alpha^1\in\bigtimes_{i<\theta}\chi_{2,i}$. Let $\alpha<\mathrm{cf}\,\delta$; we define a function g on θ by $g(i)=$ the $f_\alpha^1(i)+1$-st member of C_i. Clearly $f_\alpha<^*g<^*f_\delta^*$, therefore there is a $\alpha<\beta<\delta$ such that $g<^*f_\beta$. If $i<\theta$ is such that $f_\alpha^0(i), f_\beta^0(i)<\chi_{2,i}$ then $f_\alpha^1(i)<f_\beta^1(i)<\chi_{2,i}$, hence $f_\alpha^1<f_\beta^1$. Therefore, as easily seen, there is an

ascending function h on λ_2^+ onto a club subset A of δ such that for $\alpha<\beta<\lambda_2^+$ $f^1_{h(\alpha)}<^* f^*_{h(\beta)}$. We denote, for $\alpha<\lambda_2^+$, $f'_\alpha=f^1_{h(\alpha)}$, and

$$S'=\{\gamma<\lambda^+ : \gamma \text{ is a limit ordinal } \wedge h(\gamma)\in S\}.$$

S' is clearly a stationary subset of λ_2 which does not reflect. For $\gamma\in S'$ $h(\gamma)\in S\subseteq \mathrm{bad}'(\bar{f})$, hence $\mathrm{cf}\gamma=\mathrm{cf}h(\gamma)=\lambda_1^+$, and we shall prove that $\gamma\in\mathrm{bad}'(\bar{f}')$. If $\gamma\notin\mathrm{bad}'(\bar{f})$ then γ has an unbounded subset B such that for some $i_0<\theta$ $\langle f'_\alpha:\alpha<\lambda^+\rangle$ is ascending on B over $[i_0,\theta)$. Therefore $\langle f^1_\alpha:\alpha<\delta\rangle$ is ascending on $h(B)$ over $[i_0,\theta)$. Without loss of generality we can assume that for $\alpha\in h(B)$ and $i_0\leq i<\theta$ $f^1_\alpha(i)>0$, since otherwise we can omit the least member of B. Therefore also $\langle f^0_\alpha:\alpha<\delta\rangle$ is ascending on $h(B)$ over $[i_0,\theta)$, and hence $\langle f_\alpha:\alpha<\delta\rangle$ is ascending on B over $[i_0,\theta)$. Since $h(B)$ is unbounded in $h(\gamma)$ this contradicts $h(\gamma)\in\mathrm{bad}'(\bar{f})$. Thus we have shown that $S'\subseteq\mathrm{bad}'(\langle f'_\alpha:\alpha<\lambda_2^+\rangle)$, which ends the proof of 1.4B.

1.5 Theorem (GCH + REF). (1) *For every regular λ less than the first inaccessible* $\neg\mathrm{PT}(\lambda,\aleph_1)$.

(2) *For arbitrarily large regular cardinals λ $\neg\mathrm{PT}(\lambda,\aleph_1)$.*

(3) *Let λ be a regular cardinal and let $\langle\lambda_i:i\leq n\rangle$ be a finite sequence of cardinals such that $\lambda_0=\lambda$ and for every $k<n$ λ_{k+1} is a successor of a cardinal of cofinality λ_k, then* $\neg\mathrm{PT}(\lambda_n,\lambda^++\aleph_1)$.

Proof. (1) and (2) follow immediately from (3). (3) is shown by an easy induction, using [MgSh 204, Sect. 1, Theorem 11] and 1.4.

1.6 Definition. We write $(\lambda_2,<\mu_2)\underset{\kappa}{\longrightarrow}(\lambda_1,<\mu_1)$ if every structure $\mathscr{M}$ with universe λ_2 and with at most κ-many relations and functions, all of which are finitary, has a substructure $\langle N,\dots\rangle$ with $|N|=\lambda_1$ and for all $\alpha\in N\cap\mu_2$ $|N\cap\alpha|<\mu_1$.

1.7 Fact. (1) If $\lambda_1<\lambda_2$ are strong limit singular cardinals of cofinality θ such that

$$(\lambda_2^+,<\lambda_2^+)\underset{\theta}{\longrightarrow}(\lambda_1^+,<\lambda_1^+)$$

then $S=\{\delta<\lambda_2^+:\mathrm{cf}\delta=\lambda_1^+\}\notin I[\lambda_2^+]$, where $I[\lambda_2^+]$ is the ideal on λ_2^+ defined in 2.1.

(2) Under the assumption of (1), if $\langle\chi_{2,i}:i<\theta\rangle$ is an increasing sequence of regular cardinals $<\lambda_2$ such that

$$\sup_{i<\theta}\chi_{2,i}=\lambda_2 \quad\text{and}\quad \bar{f}=\langle f_\alpha:\alpha<\lambda_2^+\rangle,$$

where the f_α's satisfy (i) and (ii) of the proof of 1.4 with respect to $\bigtimes_{i<\theta}\chi_{2,i}$ then $\mathrm{bad}(\bar{f})\cap S$ is a stationary subset of λ_2^+.

(3) Under the same assumptions as in (2), if $A\in I[\lambda_2^+]$ then $\mathrm{bad}(\bar{f})\cap S\cap A$ is a non-stationary subset of λ_2^+.

Proof. (1) We shall show that (1) follows from (2) and (3). Let $\langle\chi_{2,i}:i<\theta\rangle$ and $\bar{f}$ be as in (2). The existence of such an $\bar{f}$ is shown in [Sh 355, Theorem 1.5], and is easily seen if we assume the GCH. By (2) $\mathrm{bad}(\bar{f})\cap S$ is stationary in λ_2^+, hence, by (3), S cannot be in $I[\lambda_2^+]$.

(2) Let $\mathscr{M}$ be a structure with universe λ_2^+ and with the following relation, constants and functions: the order relation $<$, the binary function $\langle f_\alpha(i):\alpha<\lambda_2^+,i<\theta\rangle$, constants for all the members of θ, a constant for λ_2, a unary

predicate C which is a club subset of λ_2^+, and all the Skolem functions for this structure.

Since $(\lambda_2^+, <\lambda_2^+) \xrightarrow[\theta]{} (\lambda_1^+, <\lambda_1^+)$ $\mathcal{M}$ has a substructure $\mathcal{N}=\langle N, \ldots\rangle$ with $|N|=\lambda_1^+$ and $|N\cap\alpha|\leqq\lambda_1$ for every $\alpha\in N$. Since $\mathcal{M}$ contains all its Skolem functions $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$. Thus the order type of N is λ_1^+, and since $\lambda_2\in N$ $|N\cap\lambda_2|\leqq\lambda_1$. Let $\delta=\sup N$ then $\mathrm{cf}\,\delta=\lambda_1^+$. For every $\alpha\in N$ and for all $i\in\theta$ $f_\alpha(i)\in N\cap\lambda_2$, hence

$$|\{f_\alpha(i):\alpha\in N\wedge i<\theta\}|\leqq|N\cap\lambda_2|\leqq\lambda_1<\lambda_1^+=\mathrm{cf}\,\delta\,.$$

Thus for no unbounded subset A of N can $\bar f$ be ascending on A over any non-empty subset of θ (since $\{f_\alpha(i):\alpha\in N\wedge i<\theta\}$ does not have enough members for that). By 1.4A $\delta\in\mathrm{bad}(\bar{\mathrm{f}})$. Since $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$ $C\cap N$ is unbounded in N, hence $\delta=\sup(N\cap C)$, and since C is closed $\delta\in C$. Thus $\delta\in S\cap\mathrm{bad}(\bar{\mathrm{f}})\cap C$, so $S\cap\mathrm{bad}(\bar{\mathrm{f}})\cap C\neq\emptyset$ for an arbitrary club subset C of λ_2^+. Hence $S\cap\mathrm{bad}(\bar{\mathrm{f}})$ is stationary.

(3) We shall now assume that $S_1\subseteq\mathrm{bad}(\bar{\mathrm{f}})\cap S$, S_1 is a stationary subset of λ_2^+ and $S_1\in I[\lambda_2^+]$ and obtain a contradiction. By the definition of $I[\lambda_2^+]$ $S_1\backslash S^{*p}_{\lambda_2^+}(\bar a)$ is a non-stationary subset of λ_2^+, where $\bar a$ is a sequence of length λ_2^+ of bounded subsets of λ_2^+. Without loss of generality we may assume that S_1 has been decreased so that $S_1\subseteq S^{*p}_{\lambda_2^+}(\bar a)$. Using Remark 2.3 and the fact that $I[\lambda_2^+]$ is an ideal (2.4) we can replace S_1 and $\bar a$ by $S_1\cap C$ and $\bar a'$, where C and $\bar a'$ are as in Remark 2.3. Therefore we may assume, without loss of generality, that for $\delta\in S_1$ $\delta=\sup a_\delta$, the order type of a_δ is $\mathrm{cf}\,\delta$ and $(\forall\alpha\in a_\delta)(a_\alpha=a_\delta\cap\alpha)$. Also, since $S_1\subseteq S$, $\mathrm{cf}\,\delta=\lambda_1^+$ for every $\delta\in S_1$. Hence the order type of a'_δ is λ_1^+ and for every $\alpha\in a_\delta$ the order type of a_α is $<\lambda_1^+$.

Since in the hypotheses of (2) any initial segment of θ is negligible and $\langle\chi_{2,i}:i<\theta\rangle$ is increasing with limit λ_2 and $\lambda_1^+<\lambda_2$ we can assume, without loss of generality, that $\lambda_1^+<\chi_{2,i}$ for every $i<\theta$.

We shall now define, by induction on $\alpha<\lambda_2^+$ a function $h:\lambda_2^+\to\lambda_2^+$ and functions

$$g_\alpha\in\mathop{\times}_{i<\theta}\chi_{2,i}$$

as follows. For $i<\theta$

$$g_\alpha(i)=\begin{cases}\sup(\{f_\beta(i):\beta\in a_\alpha\cup\{\alpha\}\}\cup\{g_\beta(i):\beta\in a_\alpha\})+1 & \text{if}\quad |a_\alpha|\leqq\lambda_1^+\\ 0 & \text{otherwise}\end{cases}.$$

Notice that by what we have said above about the order type of a_α the value $g_\alpha(i)$ is defined according to the first case in its definition for every $i\in S_1$ or $i\in a'_\delta$ for $\delta\in S_1$. Since $\chi_{2,i}>\lambda_1^+$ we have $g_\alpha\in\mathop{\times}_{i<\theta}\chi_{2,i}$. Also, if $\delta\in S_1$ and $\alpha,\beta\in a_\delta$ and $\alpha<\beta$ then, by what we have assumed above, $a_\beta=a_\delta\cap\beta$, hence $\alpha\in a_\beta$ and therefore $g_\beta(i)>g_\alpha(i), f_\alpha(i)$ for every $i<\theta$. By our assumption on $\bar f$ there is, for every $\alpha<\lambda_2^+$, a $\beta<\lambda_2^+$ such that $g_\alpha<^*f_\beta$, let $h(\alpha)$ be the least such β. Let

$$C^*=\{\delta<\lambda_2^+:(\forall\alpha<\delta)(h(\alpha)<\delta)\}\,;$$

C^* is clearly club in λ_2^+. Since S_1 is stationary in λ_2^+ there is a $\delta\in S_1\cap C^*$. By our assumption on $\bar a$ a_δ is an unbounded subset of δ. Therefore there is an unbounded subset b of a_δ such that for $\alpha,\beta\in b$ if $\alpha<\beta$ then $h(\alpha)\leqq\beta$. Let h^* be the function on b

defined by $h^*(\alpha)=\min(b\setminus(\alpha+1))$. Let $d=\mathrm{range}(h^*)\subseteq b$; d is an unbounded subset of b. For $\alpha\in d$ there is an $\alpha'\in b$ such that $\alpha=h^*(\alpha')$. By the definitions of h^* and h and since $\alpha=h^*(\alpha')\geqq h(\alpha')$ we have $g_{\alpha'}<^*f_\alpha$. Hence there is an $i_\alpha<\theta$ such that $g_{\alpha'}(i)<f_\alpha(i)$ for every $i\in[i_\alpha,\theta)$. Since $\theta<\lambda_1$ there is an $i^*<\theta$ and an unbounded subset e of α such that $i_\alpha=i^*$ for every $\alpha\in e$. We shall now see that $\bar{f}$ is ascending on e over $[i^*,\theta)$, contradicting $\delta\in\mathrm{bad}(\bar{f})$.

Let $\alpha,\beta\in e$, $\alpha<\beta$. Let $\beta'\in b$ be such that $h^*(\beta')=\beta$, then by our definition of h^* and d, $\alpha\leqq\beta'$. Since $\alpha\in a_\delta$ we have $f_\alpha(i)<g_\alpha(i)$ for all $i<\theta$. Since $\alpha\leqq\beta'$ and $\alpha,\beta'\in a_\delta$ $g_\alpha(i)\leqq g_{\beta'}(i)$ for all $i<\theta$. Since $\beta\in e$ and $\beta\in h^*(\beta')$ $g_{\beta'}(i)<f_\beta(i)$ for all $i\in[i^*,\theta)$. Therefore $f_\alpha<f_\beta(i)$ for all $i\in[i^*,\theta)$, which is what we have to show.

1.7A Remark. 1. If we replace in Def. 1.6

$$(\lambda_2,<\mu_2)\quad\text{by}\quad\{(\lambda_{2,j},<\mu_{2,j}):j<r\}$$

and "has a substructure ... $|N\cap\alpha|<\mu_2$" by "has, for some $j<r$, a substructure $\langle N,\ldots\rangle$ such that $|N|=\lambda_{2,j}$ and for all $\alpha<\mu_1$ $|N\cap\alpha|<\mu_{2,j}$" then one can prove a version of Theorem 1.7 which corresponds to the changed Def. 1.6.

2. The assumption of 1.7 that λ_1,λ_2 are strong limit cardinals can be dropped without changing the conclusion [Sh 355, 2.2].

2 Obtaining REF by repeated Levy collapses

The following definitions and theorems are quoted from [Sh 88a], which gives a better representation of most of [Sh 108].

2.1 Definition [Sh 88a, 1, 2(1)]. Let λ be an uncountable regular cardinal and let $\bar{a}=\langle a_i:i<\lambda\rangle$, where the a_i's are bounded subsets of λ.

1) $S^{*\mathrm{p}}_\lambda(\bar{a})\overset{\mathrm{def}}{=}\{\delta<\lambda$: there is an unbounded subset b of δ of order type $\mathrm{cf}\,\delta<\delta$ such that $(\forall\alpha<\delta)(\exists\beta<\delta)(b\cap\alpha=a_\beta)\}$,

$$S^{*\mathrm{n}}_\lambda(\bar{a})\overset{\mathrm{def}}{=}\lambda\setminus S^{*\mathrm{p}}_\lambda(\bar{a}),$$

where p, n stand for "positive" and "negative", respectively.

2) $I[\lambda]\overset{\mathrm{def}}{=}\{X\subseteq\lambda: X\setminus S^{*\mathrm{p}}_\lambda(\bar{a})$ is a non-stationary subset of λ, for some sequence $\bar{a}$ of length λ of bounded subsets of $\lambda\}$.

3) $S^{*\mathrm{n}}_\lambda$ is a set such that $S^{*\mathrm{n}}_\lambda\subseteq\lambda$, $I[\lambda]=\{S:S\subseteq\lambda\wedge S\cap S^{*\mathrm{n}}_\lambda$ is not stationary in $\lambda\}$. The existence of such a set $S^{*\mathrm{n}}_\lambda$ is shown in [Sh 88a, 3(3)], using $\lambda^{<\lambda}=\lambda$ or some weaker assumptions. It is conjectured that the existence of such a set for every uncountable regular λ cannot be proved in ZFC.

$S^{*\mathrm{p}}_\lambda\overset{\mathrm{def}}{=}\lambda\setminus S^{*\mathrm{n}}_\lambda$. $S^{*\mathrm{p}}_\lambda$ is clearly the maximal member of $I[\lambda]$, up to a non-stationary set.

2.2 Fact [Sh 88a, 3(1)]. $I[\lambda]$ is a normal ideal on λ.

2.3 Lemma. *For all λ and $\bar{a}$ as in 2.1 there is an $\bar{a}'=\langle a'_\alpha:\alpha<\lambda\rangle$ and a subset C of λ such that*

(1) *C is club in λ and contains only limit ordinals.*

(2) *For all $\alpha<\lambda$ $a'_\alpha\subseteq\alpha$.*

(3) *For every limit ordinal* $\alpha<\lambda$ $\sup a'_\alpha=\alpha$ *and the order type of* a'_α *is* $\mathrm{cf}\,\alpha$.
(4) *If* $\delta\in C\cap S^{*\mathrm{p}}_\lambda(\bar a)$ *and* $\alpha\in a'_\delta$ *then* $a'_\alpha=a'_\delta\cap\alpha$.

Proof. Let gd be a one-one mapping of the set of all n-tuples of ordinals $<\lambda$ into λ such that for $\alpha_1,\dots,\alpha_n<\lambda$ $\mathrm{gd}(\alpha_1,\dots,\alpha_n)>\max(\alpha_1,\dots,\alpha_n)$. For every $i<\lambda$ we define a function f_i on a_i as follows. For $\alpha\in a_i$, if there is a $\beta<\lambda$ such that $a_\beta=a_i\cap\alpha$ then $f_i(\alpha)$ is taken to be the least such β, and $f_i(\alpha)=0$ otherwise w.l.o.g. $\bigwedge_i\bigwedge_{\gamma\in a_i}\bigvee_j a_i\cap\gamma=a_j$. We define also a function g_i on a_i into λ by

$$g_i(\alpha)=\mathrm{gd}\Big(\alpha,f_i(\alpha),\bigcup_{\beta\in a_i\cap\alpha}g_i(\beta)\Big)+1\,.$$

(a) f_i and g_i are clearly functions on a_i into λ.
(b) For $i,j<\lambda$, if $a_i\cap\alpha=a_j\cap\alpha$ then $f_i\restriction\alpha=f_j\restriction\alpha$ and hence $g_i\restriction\alpha=g_j\restriction\alpha$.
(c) For $i,j<\lambda$ if $g_i(\alpha)=g_j(\beta)$ then $\alpha=\beta$, $f_i(\alpha)=f_j(\alpha)$, hence

$$a_i\cap(\alpha+1)=a_j\cap(\alpha+1)$$

(since $a_i\cap\alpha=a_{f_i(\alpha)}=a_j\cap\alpha$ and $\alpha\in a_i,a_j$), and by (b)

$$f_i\restriction(\alpha+1)=f_j\restriction(\alpha+1)\,,$$

$g_i\restriction\alpha=g_j\restriction\alpha$. Let $C=\{\delta<\lambda:\delta$ is a limit ordinal closed under the function gd and for all $i<\delta$ $\mathrm{Range}(f_i)$, $\mathrm{Range}(f_j)\subseteq\delta\}$.

We define now a'_ζ for $\zeta<\lambda$ according to the following cases.

Case 1. There are $\alpha,i<\lambda$ such that $\alpha\in a_i$ and $g_i(\alpha)=\zeta$.

We take $a'_\zeta=\{g_i(\beta):\beta\in a_i\cap\alpha\}$. By (c) a'_ζ is well-defined. Since g_i is increasing we have $a'_\zeta\subseteq\zeta$.

Case 2. $\zeta\in S^{*\mathrm{p}}_\lambda(\bar a)\cap C$.

Then ζ is a limit ordinal. By the definition of $S^{*\mathrm{p}}_\lambda(\bar a)$ there is an $a\subseteq\zeta$ unbounded in ζ of order type $\mathrm{cf}\,\zeta$ such that $(\forall\alpha<\zeta)(\exists\beta<\zeta)(a\cap\alpha=a_\beta)$. We define functions f and g on a by

$$f(\alpha)=\min\{\beta:a_\beta=a\cap\alpha\}<\zeta\quad\text{and}\quad g(\alpha)=\mathrm{gd}\Big(\alpha,f(\alpha),\bigcup_{\beta\in a\cap\alpha}g(\beta)\Big)+1\,.$$

Since $\zeta\in C$ and the order type of a is $\mathrm{cf}\,\zeta$ we have $g(\alpha)<\zeta$. We take

$$a'_\zeta=\{g(\beta):\beta\in a\}\subseteq\zeta.$$

Case 3. Otherwise. If ζ is a limit ordinal take a'_ζ to be a club subset of ζ of order type $\mathrm{cf}\,\zeta$; otherwise $a'_\zeta=\emptyset$.

(1) and (2) hold by what was said just now. (3) holds since if ζ is a limit ordinal then one of Cases 2 and 3 holds for ζ. To see that (4) holds let $\delta\in C\cap S^{*\mathrm{p}}_\lambda(\bar a)$ and $\zeta\in a'_\delta$. By the definition of a'_δ we have $a'_\delta=\{g(\beta):\beta\in a\}$, where a,f,g are as in Case 2. Since $\zeta\in a'_\delta\subseteq\delta$ we have $\zeta=g(\gamma)$ for some $\gamma\in a\subseteq\delta$. By the definition of f $a\cap\zeta=a_{f(\zeta)}$. Therefore for every $\alpha\in a\cap\zeta=a_{f(\zeta)}$ $f(\alpha)=f_{f(\zeta)}(\alpha)$, and, as follows immediately by induction, $g(\alpha)=g_{f(\zeta)}(\alpha)$. Since g is an increasing function with non-limit values $\zeta=g(\gamma)>\gamma$, hence $g_{f(\zeta)}(\gamma)=g(\gamma)=\zeta$, and by the definition of a'_ζ in Case 1 we have

$$\begin{aligned}a'_\zeta&=\{g_{f(\zeta)}(\beta):\beta\in a_{f(\zeta)}\cap\gamma\}=\{g(\beta):\beta\in a\cap\gamma\}=\{g(\beta):\beta\in a\wedge g(\beta)<g(\gamma)=\zeta\}\\&=\{g(\beta):\beta\in a\}\cap\zeta=a'_\delta\cap\zeta\,.\end{aligned}$$

2.4 Remark. Let λ, $\bar{a}$, $\bar{a}'$, and C be as in 2.3. Then $S^{*p}_\lambda(\bar{a}) \cap C \subseteq S^{*p}_\lambda(\bar{a}')$, and for every

$$\delta \in S^{*p}_\lambda(\bar{a}) \cap C$$

the subset a of δ which witnesses that $\delta \in S^{*p}_\lambda(\bar{a}')$ can be taken to be a'_δ.

Proof. Let $\delta \in S^{*p}_\lambda(\bar{a}) \cap C$; we shall see that $\delta \in S^{*p}_\lambda(\bar{a}')$, with a'_δ as a witness to it. By (3) a'_δ is an unbounded subset of δ of order type $\mathrm{cf}\,\delta$. Let $\alpha < \delta$ and let γ be the least member of $a'_\delta \setminus \alpha$. Then, clearly, $a'_\delta \cap \alpha = a'_\delta \cap \gamma$. By (4) $a'_\delta \cap \gamma = a'_\gamma$, hence $a'_\delta \cap \alpha = a'_\gamma$ and $\delta \in S^{*p}_\lambda(\bar{a}')$.

2.5 Fact [Sh 88a, Def. 1, Lemma 2(1) and 2(2)]. For $\bar{\mathscr{P}} = \langle \mathscr{P}_i : i < \lambda \rangle$, where for $i < \lambda$ $\mathscr{P}_i$ is a set of bounded subsets of λ and $|\mathscr{P}_i| < \lambda$ let $S^{*p}_\lambda(\bar{\mathscr{P}}) = \Big\{\delta < \lambda$: there is an unbounded subset b of δ such that (i) if δ is singular then the order type of b is $< \delta$, and (ii) $(\forall \alpha < \delta)\Big(b \cap \alpha \in \bigcup_{\beta < \delta} \mathscr{P}_\beta\Big)\Big\}$. Then $I[\lambda] = \{X \subseteq \lambda : X \setminus S^{*p}_\lambda(\bar{\mathscr{P}})$ is a non-stationary subset of λ, for some $\bar{\mathscr{P}}$ as above and $X \cap$ "set of inaccessibles" is non-stationary$\}$.

2.6 Fact. Let λ be a regular uncountable cardinal and let D_λ denote the filter generated by the club subsets of λ.

1) S^{*n}_λ, whose existence is discussed in 2.1(3), is unique modulo D_λ and is equal, modulo D_λ to some $S^{*n}_\lambda(\bar{a})$ (as easily seen). By [Sh 88a, 14(1) and 16(1)] this is the case for every enumeration $\bar{a}$ of all bounded subsets of λ, assuming $\lambda^{<\lambda} = \lambda$ or some weaker assumptions.

2) If λ is a successor of a regular cardinal then

$$\{\delta < \lambda : (\mathrm{cf}\,\delta)^+ < \lambda\} \in I[\lambda]$$

[see (4.4(1))].

3) If λ is strongly inaccessible then S^{*p}_λ can be taken to be $\{\delta < \lambda$: *δ is singular*$\}$ ([Sh 88a, 4(1)]).

4) If $\lambda = \mu^+$, where μ is a strong limit singular cardinal then

(i) If $\delta \in S^{*n}_\lambda$ then $\mathrm{cf}\,\delta$ is not weakly compact.

(ii) If $\mu = \sup_{i < \mathrm{cf}\,\mu} \lambda_i$, where $\langle \lambda_i : i < \mathrm{cf}\,\mu \rangle$ is an ascending sequence of regular cardinals and

(*) $c : \lambda \times \lambda \to \mathrm{cf}\,\mu$ is such that
for all $\alpha, \beta, \gamma < \lambda$ $c(\alpha, \beta) = c(\beta, \alpha)$,
and if $\alpha < \beta < \gamma$ then $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$,
and for all $i < \mathrm{cf}\,\mu$ and $\alpha < \lambda$ $|\{\beta < \alpha : c(\alpha, \beta) = i\}| \leq \lambda_i$

(the existence of such a c is proved in 4.1) and we define

$$\begin{aligned}
S(c) \overset{\mathrm{def}}{=} \{\delta < \lambda &: \textit{for some unbounded } A \subseteq \delta \; c``((A \cap \gamma) \times (A \cap \gamma)) \\
&\quad \textit{is bounded in } \mathrm{cf}\,\mu \textit{ for every } \gamma < \delta\} \\
= \{\delta < \lambda &: \textit{if } \mathrm{cf}\,\delta > \mathrm{cf}\,\mu \textit{ then for some unbounded } A \subseteq \delta \; c``(A \times A) \\
&\quad \textit{is bounded in } \mathrm{cf}\,\mu\} \\
= \{\delta < \lambda &: \textit{if } \mathrm{cf}\,\delta > \mathrm{cf}\,\mu \textit{ then every unbounded subset } A \textit{ of } \delta \\
&\quad \textit{has an unbounded subset } A' \textit{ such that } c``(A' \times A') \textit{ is bounded in } \mathrm{cf}\,\mu\} \\
= \{\delta < \lambda &: \textit{if } \mathrm{cf}\,\delta > \mathrm{cf}\,\mu \textit{ then for some unbounded } A_1, A_2 \subseteq \delta \\
&\quad \textit{and } i < \theta \textit{ if } \alpha < \beta, \alpha \in A_1 \textit{ and } \beta \in A_2 \textit{ then } c(\alpha, \beta) \leq i\}
\end{aligned}$$

then $S(c)$ can be taken to be S^{*p}_λ.

(iii) S^{*n}_λ exists and if $\delta \in S^{*n}_\lambda$ then $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$.

(iv) If c is as in (ii) for V, Q is a forcing notion and $\Vdash_Q$ "$\lambda = \mu^+ \wedge \mu$ *is a strong limit cardinal*" then also $V^Q \models$ "$S(c)$ can be taken to be S^{*n}_λ".

5) If $\langle \lambda_i : i < \theta \rangle$ is a strictly increasing sequence of regular cardinals $> \theta$, $\lambda = \left(\sup_{i<\theta} \lambda_i\right)^+$, $\bar f = \langle f_\alpha : \alpha < \lambda \rangle$, where $f_\alpha \in \mathop{\times}_{i<\theta} \lambda_i$, is $<^*$-increasing (see the proof of 1.4) and

$$\left(\forall f \in \mathop{\times}_{i<\theta} \lambda_i\right)(\exists \alpha < \lambda)(f <^* f_\alpha)$$

then bad$(\bar f)$ includes no stationary set which is in $I[\lambda]$.

6) When the universe is increased S^{*p}_λ can only increase.

Proof. 4) (i) By [Sh 88a, 6(2)] it follows that if θ is a weakly compact cardinal $< \mathrm{cf}\,\mu$ then $\theta \to_p (\theta)^2_{\mathrm{cf}\,\mu}$ (this is a certain partition relation). Therefore, by [Sh 88a, 8(1)], $\{\delta < \lambda : \mathrm{cf}\,\delta = \theta\} \in I[\lambda]$. Hence, by the definition of S^{*n}_λ no δ with $\mathrm{cf}\,\delta = \theta$ is in S^{*n}_λ. This is also proved in 2.9.

(ii) The proof appeared, with a different notation, in [Sh 108]. The proof of $S(c) \in I[\lambda]$is contained in [Sh 88a, 4(3) (c)]. Why is $S(c) = S^{*p}_\lambda$? et $S \subseteq \lambda \setminus S(c) \in I[\lambda]$ and $\bar a$ is such that, without loss of generality, $S^{*p}_\lambda(\bar a) \supseteq S$ (as in 2.3, 2.4).

Also for some $\theta < \lambda$ and $\lambda : \theta \to \mathrm{cf}(\mu)$ we have:

$$S_1 = \{\delta \in S : \mathrm{cf}(\delta) = \theta \text{ and } i \in a_\gamma \Rightarrow c(\delta, i) = \lambda(\mathrm{otp}(a_\gamma \cap i))\}$$

is stationary.

For some $\zeta < \mathrm{cf}(\mu)$, $\theta < \lambda_\zeta$ and $\{\beta = \lambda_j < \theta : \lambda(j) < \zeta\}$ is unbounded in θ.

Now choose by induction on $\alpha < \lambda$

$$\gamma_\alpha = \min\{\gamma : (\forall \beta \in a_\alpha) c(\beta, \gamma) = \lambda(\mathrm{otp}\, a_\alpha)\}.$$

It suffices to prove that for a club of δ's $(\forall \alpha \in a_\delta)\gamma_\alpha < \delta$ and $\delta \in S_1$. This holds by Fodor's theorem; it suffices by the equivalence of the first and fourth definition of $S(c)$.

First let us prove the equivalence of the first two definitions of $S(c)$. Let δ be as in the first definition and $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$. If $\sup c``((A \cap \gamma) \times (A \cap \gamma)) < \mathrm{cf}\,\mu$ for every $\gamma < \delta$ then since $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$, $\{\sup c``((A \cap \gamma) \times (A \cap \gamma)) : \gamma < \delta\}$ is bounded in $\mathrm{cf}\,\mu$. Thus $c``(A \times A)$ is bounded in $\mathrm{cf}\,\mu$ and δ satisfies also the second definition of $S(c)$. Now assume that δ satisfies the second definition of $S(c)$. If $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$ then δ obviously satisfies also the first definition of $S(c)$. If $\mathrm{cf}\,\delta \leqq \mathrm{cf}\,\mu$ then every unbounded subset A of δ of order type $\mathrm{cf}\,\delta$ witnesses that δ satisfies the first definition.

To prove the equivalence of the second, third, and fourth definitions of $S(c)$ it clearly suffices to prove that if $\delta < \lambda$, $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$ and A_0, A_1 are unbounded subsets of δ such that

$$\{c(\alpha_0, \alpha_1) : \alpha_0 \in A_0,\ \alpha_1 \in A_1 \textit{ and } \alpha_0 < \alpha_1\}$$

is bounded in $\mathrm{cf}\,\mu$ then every unbounded subset A of δ has an unbounded subset A' such that $c``(A' \times A')$ is bounded in $\mathrm{cf}\,\mu$. Let

$$\zeta = \sup\{c(\alpha_0, \alpha_1) : \alpha_0 \in A_0,\ \alpha_1 \in A_1 \textit{ and } \alpha_0 < \alpha_1\} < \mathrm{cf}\,\mu.$$

Let A be an unbounded subset of δ. Without loss of generality the order type of A_0 and A_1 and of A is $\mathrm{cf}\,\delta$. Let us define, for $i < \mathrm{cf}\,\delta$, $\alpha_i, \beta_i, \gamma_i < \delta$ by recursion as follows.

$$\alpha_i > \sup\{\alpha_j + 1,\ \beta_j + 1,\ \gamma_j + 1 : j < i\},$$

$$\alpha_i \in A_1, \quad \beta_i \in A, \quad \gamma_i \in A_0, \quad \alpha_i < \beta_i < \gamma_i.$$

Let, for $i < \mathrm{cf}\,\mu$, $\zeta_i = \max\{c(\alpha_i, \beta_i), c(\beta_i, \gamma_i)\} < \mathrm{cf}\,\mu$. Since $\mathrm{cf}\,\delta > \mathrm{cf}\,\mu$ there is a $\gamma^* < \mathrm{cf}\,\mu$ and an unbounded subset T of $\mathrm{cf}\,\delta$ such that for $i \in T$ $\zeta_i < \zeta^*$, and we can take $\zeta^* \geqq \zeta$.

Since the order type of A is $\mathrm{cf}\delta$ $A'=\{\beta_i : i \in T\}$ is an unbounded subset of A. We shall now see that $c``(A' \times A')$ is bounded by γ^*. Let $i,j \in T$, $i<j$. By $(*)_c$

$$c(\beta_i, \beta_j) \leqq \max\{c(\beta_i, \gamma_i), c(\gamma_i, \alpha_j), c(\alpha_j, \beta_j)\} \leqq \max(\zeta_i, \gamma, \zeta_j)$$

(by the definition of ζ_i and since $\gamma_i \in A_0$, $\alpha_j \in A_1$, $\gamma_i < \alpha_j) \leqq \zeta^*$ (since $i,j \in T$).

(iii) $\mu^{<\mathrm{cf}\mu} = \mu$ since μ is a strong limit cardinal. Hence, by [Sh 88a, 4(1)], $\{\delta < \lambda : \mathrm{cf}\delta \leqq \mathrm{cf}\mu\} \in I[\lambda]$. By the definition of S^{*n}_λ no δ with $\mathrm{cf}\delta \leqq \mathrm{cf}\mu$ is in S^{*n}_λ.

(iv) Follows from (ii) by 2.7, since in the transition to V^Q the properties of c are unchanged, except that $\mathrm{cf}\mu$ may decrease and/or λ_i collpase – see 2.7.

5) We essentially repeat here the proof given in [Sh 282]. It clearly suffices to show that for every set $S^{*p}_\lambda(\bar{a})$ there is a club subset C of λ such that $S^{*p}_\lambda(\bar{a}) \cap C \cap \mathrm{bad}(\bar{f}) = \emptyset$. By 2.4 we may assume that $\bar{a}$ satisfies 2.3(2–4). We define by induction functions g_α, $\alpha < \lambda$ such that

(i) $g_\alpha \in \bigtimes_{i<\theta} \lambda_i$,

(ii) $\beta < \alpha < \lambda \rightarrow g_\beta <^* g_\alpha$,

(iii) $f_\alpha <^* g_\alpha$, and

(iv) if $i < \theta$ and $\lambda_i > |a_\alpha|$ then for every $\beta \in a_\alpha$ $g_\beta(i) < g_\alpha(i)$.

The existence of such a g_α follows easily from our assumption on $\bar{f}$; notice that if $|a_\alpha| < \lambda_i$ then $\sup\{g_\beta(i) : \beta \in a_\alpha\} < \lambda_i$ so also (iv) can be satisfied. By our assumption on $\bar{f}$ there is a function $h : \lambda \rightarrow \lambda$ such that for every $\alpha < \lambda$ $g_\alpha <^* f_{h(\alpha)}$. Let $C = \{\delta < \lambda : \delta$ is a limit ordinal closed under $h\}$. For $\delta \in S^{*p}_\lambda(\bar{a}) \cap C$ let A be an unbounded subset of a_δ of order type $\mathrm{cf}\delta$ such that if $\beta, \alpha \in A$, $\beta < \alpha$ then $h(\beta) < \alpha$. For i such that $\lambda_i > \mathrm{cf}\delta$ and $\alpha \in A$ we have, by 2.3, $a_\alpha = a_\delta \cap \alpha$ and $|a_\alpha| = \mathrm{cf}\delta < \lambda$, hence $|a_\alpha| < \lambda_i$. Since $\mathrm{cf}\delta < \lambda = \left(\sup_{i<\theta} \lambda_i\right)^+$ there is an $i_0 < \theta$ such that $\lambda_{i_0} < \mathrm{cf}\delta$. Therefore by (iv) if $\beta \in A$, $\beta < \alpha$ then $g_i(\beta) < g_i(\alpha)$. Thus $\langle g_\alpha(i) : \alpha \in A\rangle$ is strictly ascending. Let us enumerate A by $A = \{\alpha_\nu : \nu < \mathrm{cf}\delta\}$, where $\langle \alpha_\nu : \nu < \mathrm{cf}\delta\rangle$ is strictly ascending. For $\nu < \mathrm{cf}\delta$ we have $g_{\alpha_\nu} <^* f_{\alpha_{\nu+1}} <^* g_{\alpha_{\nu+1}}$ [since $\alpha_{\nu+1} > h(\alpha_\nu)$]. Hence there is a $\zeta_\nu < \theta$ such that for every $\zeta_\nu < i < \theta$

$$g_{\alpha_\nu}(i) < f_{\alpha_{\nu+1}}(i) < g_{\alpha_{\nu+1}}(i).$$

If $\mathrm{cf}\delta \leqq \theta$ then by the definition of $\mathrm{bad}(\bar{f})$ we have $\delta \notin \mathrm{bad}(\bar{f})$. If $\mathrm{cf}\delta > \theta$ then there is a $\zeta^* < \theta$ and an unbounded subset T of $\mathrm{cf}\delta$ such that for $\nu \in T$ $\zeta_\nu = \zeta^*$. Thus for every $i > i_0, \zeta^*$ the sequence $\langle f_{\alpha_{\nu+1}}(i) : i \in T\rangle$ is strictly ascending; hence $\delta \notin \mathrm{bad}(\bar{f})$.

2.7 Lemma. 1) *Let λ, μ be cardinals and ϱ an ordinal such that $\lambda = \mu^+$, $\varrho \leqq \mu$, and $\mathrm{cf}\mu = \mathrm{cf}\varrho < \mu$, and let*

(**) *$c : \lambda \times \lambda \rightarrow \varrho$ be such that*
for all $\alpha, \beta, \gamma < \lambda$ $c(\alpha, \beta) = c(\beta, \alpha)$,
and if $\alpha < \beta < \gamma$ then $c(\alpha, \gamma) \leqq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$,
and for all $i < \varrho$ $\sup_{\alpha<\lambda} |\{\beta < \alpha : c(\alpha, \beta) \leqq i\}| < \mu$.

Then there is a $c' : \lambda \times \lambda \rightarrow \mathrm{cf}\mu$ which satisfies (*) *of* 2.6(ii) *and such that for every $A \subseteq \lambda$ $c``(A \times A)$ is bounded in $\varrho$ iff $c'``(A \times A)$ is bounded in* $\mathrm{cf}\mu$.

2) *If, in addition, $\mu = \sup_{i<\varrho} \lambda_i$, $\langle \lambda_i : i < \varrho\rangle$ is an ascending sequence of regular cardinals and* (*) *holds then $\langle \lambda_i : i < \varrho\rangle$ has a subsequence $\langle \lambda'_i : i < \mathrm{cf}\mu\rangle$ and there is a c' such that* (*) *holds for c'.*

3) *As a consequence, if we define, for c as above, $S(c)$ as in the three definitions in 2.6 with "bounded in ϱ" replacing "bounded in $\mathrm{cf}\mu$" we get $S(c) = S(c')$.*

Proof. By our hypothesis $\mathrm{cf}\varrho=\mathrm{cf}\mu$. Let $d:\mathrm{cf}\mu\to\varrho$ be a strictly increasing and continuous mapping of $\mathrm{cf}\mu$ onto an unbounded subset of ϱ. For (1) let $c'(\alpha,\beta)=\min\{\sigma:d(\sigma)\geqq c(\alpha,\beta)\}$. For (2) we let for $i<\mathrm{cf}\mu$ let λ'_i be the first member of the sequence $\langle\lambda_i:i<\varrho\rangle$ such that

(1) $\lambda'_i\geqq\lambda_{d(i)}$,

(2) $\lambda'_i>\lambda'_j$ for every $j<i$,

(3) $\lambda'_i\geqq|\{\beta<\alpha:c(\alpha,\beta)\leqq d(i)\}|$.

There is such a sequence $\langle\lambda'_i:i<\mathrm{cf}\mu\rangle$ by our assumption about c. Let $c'(\alpha,\beta)$ = the least $\sigma<\mathrm{cf}\mu$ such that $c(\alpha,\beta)<d(\sigma)$, then $c':\lambda\times\lambda\to\mathrm{cf}\mu$, $\langle\lambda'_i:i<\mathrm{cf}\mu\rangle$ is an ascending sequence of regular cardinals and $\sup_{i<\mathrm{cf}\mu}\lambda'_i=\mu$. As easily seen $\langle\lambda'_i:i<\mathrm{cf}\mu\rangle$ and c' satisfy $(*)_c$. The proof of (3) is easy.

2.8 Fact. 1) Let λ be a regular uncountable cardinal such that $\lambda=\lambda^{<\lambda}$, let $S\subseteq S^{*\mathrm{n}}_\lambda$, and let

$$P^{\mathrm{sh}}_S=\{\langle\alpha,D\rangle:\ \alpha<\lambda\wedge D\text{ is a closed subset of }\alpha+1\text{ disjoint from }S\}$$

be ordered by:

$$\langle\alpha_1,D_1\rangle\leqq\langle\alpha_2,D_2\rangle\Leftrightarrow\alpha_1\leqq\alpha_2\wedge D_1=D_2\cap(\alpha_1+1).$$

Then P^{sh}_S is α-strategically closed for every $\alpha<\lambda$ (see Def. 3.1), $|P^{\mathrm{sh}}_S|=\lambda$ and $\Vdash_{P^{\mathrm{sh}}_S}$"$S$ is not stationary", and hence cardinals are not collapsed and cofinalities are not changed by this forcing.

2) If $S\in I[\lambda]$ for a regular cardinal λ, and for every $\delta\in S$ $\mathrm{cf}\delta<\theta$, and Q is a θ-complete forcing notion then

$$\Vdash_Q\text{"}S\text{ is a stationary subset of }\lambda\text{"}$$

(although λ is not necessarily regular in V^Q) [Sh 88a, 18 and 16(3)].

Proof. 1) We shall prove that P^{sh}_S is α-strategically closed for every $\alpha<\lambda$. For the rest of (1), for $\kappa<\lambda$, "cf$>\kappa$" is preserved. For $\kappa\geqq\lambda$ this follows from $|P^{\mathrm{sh}}_S|\leqq\lambda$.

Let $\bar a=\langle a_i:i<\lambda\rangle$ be an enumeration of all bounded subsets of λ, with each set occurring λ times. By [Sh 88a, 2(2)] there is a sequence $\bar a'$ such that for every $\delta<\lambda$ if δ has an unbounded subset b of order type $<\delta$ such that $(\forall\alpha<\delta)(\exists\beta<\delta)(b\cap\alpha=a_\beta)$ then $\delta\in S^{*\mathrm{p}}_\lambda(\bar a')$ – see 2.5. [Notice that by the definition of $S^{*\mathrm{p}}_\lambda(\bar a)$ if the order type of b is $\mathrm{cf}\delta$ then $\delta\in S^{*\mathrm{p}}_\lambda(\bar a)$.] By the definition of $S^{*\mathrm{n}}_\lambda$ $S^{*\mathrm{p}}_\lambda(\bar a')\cap S^{*\mathrm{n}}_\lambda$ is non-stationary, hence there is a club subset C of λ such that $C\cap S^{*\mathrm{p}}_\lambda(\bar a')\cap S^{*\mathrm{n}}_\lambda=\emptyset$. The strategy of Player I in the game $G^\alpha_{P^{\mathrm{sh}}_S,r}$ is to choose in the i-th step a member $\langle\alpha_i,D_i\rangle$ of P^{sh}_S as follows. For $i=0$, let $r=\langle\varrho,D\rangle$ then $\alpha_0=\max(\varrho,\alpha+1)>\alpha$ and $D_0=D$. For $i=j+1$, if the j-th move of Player II was $\langle\beta_j,E_j\rangle$ then $D_j=E_j$ and α_i is such that $\alpha_i>\beta_j$, $(\alpha_j,\alpha_i)\cap C\neq\emptyset$, and $a_{\alpha_i}=\{\alpha_k:k<i\}$. There is such an α_i by our assumption on $\bar a$ and C. If i is a limit ordinal then let $\gamma=\sup_{j<i}\alpha_j$, $d_i=\bigcup_{j<i}D_j\cup\{\gamma\}$, $\alpha_i\geqq\gamma$, and $a_{\alpha_i}=\{\alpha_k:k<i\}$. In this case, the only new thing we have to show is that $\gamma\notin S$. Since for every $j<i$ $(\alpha_j,\alpha_{j+1})\cap C\neq\emptyset$ we have $\gamma\in C$. We shall now see that $\gamma\in S^{*\mathrm{p}}_\lambda(\bar a')$, hence since $\gamma\in C$ $\gamma\notin S$. $b=\{\alpha_j:j<i\}$ is an unbounded subset of γ. The order type of b is $i<\alpha\leqq\alpha_0\leqq\gamma$. For $\beta<\gamma$ let $k<i$ be such that $b\cap\beta=\{\alpha_j:j<k\}$ then $b\cap\beta=a_{\alpha_k}$ and $\alpha_k<\gamma$. Thus, by our choice of $\bar a'$, $\gamma\in S^{*\mathrm{p}}_\lambda(\bar a')$.

2) Left to the reader.

2.9 Lemma. *Let λ,μ,c be as in 2.7 and let $\delta<\lambda$. If $\mathrm{cf}\delta$ is weakly compact then $\delta\in S(c)$.*

Proof. We use the second definition of $S(c)$ and assume $\mathrm{cf}\delta>\mathrm{cf}\mu$. Denote $\mathrm{cf}\delta$ with κ.

Let d be an ascending function from κ onto an unbounded subset B of δ and let $c':\lambda\times\lambda\to\mathrm{cf}\,\mu$ be as in 2.7.

$$|c``(B\times B)|\leqq \mathrm{cf}\,\mu<\mathrm{cf}\,\delta=\kappa\,.$$

Since κ is weakly compact $\kappa\to(\kappa)^2_{<\kappa}$, hence there is an $A\subseteq B$ such that $|A|=\kappa$ and $c'\restriction(A\times A)$ is constant. Thus $\delta\in S(c')$ and by 2.7 $\delta\in S(c)$.

2.10 Fact. Let Q be an arbitrary forcing notion. Let $Q'\lessdot Q$ mean that Q' is a complete sub-forcing of Q, i.e., Q' is a subset of Q with the same partial order, and every maximal antichain of Q' is also a maximal antichain in Q. If λ is the successor of a strong limit singular cardinal μ and this holds also for λ and μ in V^Q, then $V^Q\models\{\delta<\lambda$: there is a $Q'\lessdot Q$ such that $V^{Q'}\models$" $\mathrm{cf}\delta$ is weakly compact"$\}\in I[\lambda]$.

Proof. Let $\mu=\sup\limits_{i<\mathrm{cf}\,\mu}\lambda_i$, where $\langle\lambda_i:i<\mathrm{cf}\mu\rangle$ is a strictly increasing sequence of regular cardinals. By 4.1 there is a function c which satisfies (*) of 2.6(4)(ii). $\mathrm{cf}^{V^Q}\mu$ may be $<\mathrm{cf}\,\mu$ but in any case c satisfies in V^Q (**) of 2.7 and hence, by 2.7, $S(c)^{V^Q}=S(c')^{V^Q}$ where c' satisfies (*) in V^Q. As follows immediately from the first definition of $S(c)$ in 2.6(4)(ii) $S(c)^{V^Q}\supseteq S(c)^V$. By 2.1 and 2.6(4)(ii) in V^Q $S(c)^{V^Q}=S(c')^{V^Q}\in I[\lambda]^{V^Q}$. If $Q'\lessdot Q$ and $\mathrm{cf}^{V^{Q'}}\delta$ is weakly compact in $V^{Q'}$ then by 2.9 $\delta\in S(c)^{V^{Q'}}\subseteq S(c)^{V^Q}\in I[\lambda]^{V^Q}$, which is what we have to show.

2.11 Definition. We say that a cardinal θ is pwccf above κ, where pwccf is an acronym of "potentially of weakly compact cofinality", if for every forcing notion R of power at most 2^κ there is a κ-complete forcing notion Q in V^R with $|Q|\leqq 2^\theta$ such that $\Vdash_{R*Q}$"$\mathrm{cf}\theta$ is a weakly compact cardinal".

2.12 Fact. Suppose $\kappa<\mu<\theta$, where θ is a regular cardinal and μ is 2^θ-supercompact and $F:\mu\to H(\mu)$ is a Laver diamond for θ^+ [the Laver diamond is defined in 0.2(2)]. Then θ is pwccf above κ.

Proof. Repeat the proof of Laver in [L].

2.12A Remark. In 2.12 one can considerably weaken the assumptions of the Laver diamond and the supercompactness – see Gitik-Shelah [GS].

Let Levy$(\kappa,<\lambda)$ denote the usual forcing notion which makes λ the cardinal successor of κ by collapsing all the cardinals between κ and λ to κ by means of function with domains of cardinality $<\kappa$. We shall use letters such as $\underset{\sim}{P},\underset{\sim}{Q},\underset{\sim}{R}$ to denote names for appropriate forcing notions, and $\underline{\mathrm{Levy}}(\kappa,<\lambda)$ will be a name for Levy$(\kappa,<\lambda)$.

We shall now give a sufficient condition for an iteration of the Levy collapse to yield REF. We shall use it later to prove Theorem 0.1 (see 3.8).

2.13 Lemma (GCH). *Suppose*

(A) *$\langle\kappa_i:i<\infty\rangle$ is a strictly increasing continuous sequence of cardinals such that $\kappa_0=\aleph_0$, if κ_δ is singular then $\kappa_{\delta+1}=\kappa_\delta^+$ and if κ_δ is regular then $\kappa_{\delta+1}$ is supercompact.*

(B) *$A_i=\{\theta:\kappa_i<\theta<\kappa_{i+1}\wedge\theta$ is regular $\wedge\,\theta$ is pwccf above $\kappa_i\}$.*

(C) *If κ_δ is singular then $\left\{\delta\in S^{*n}_{\kappa^+_{\delta+1}}:\mathrm{cf}\delta\notin\bigcup\limits_{i<\delta}A_i\right\}$ is not stationary in $\kappa^+_{\delta+1}$.*

(D) *$P_\infty=\lim\langle P_i,\underset{\sim}{Q}_i:i<\infty\rangle$ is the Levy collapse for $\langle\kappa_i:i<\infty\rangle$, i.e., it is Easton-support iterated forcing notion, $\underset{\sim}{Q}_i=\underline{\mathrm{Levy}}(\kappa_i,<\kappa_{i+1})^{V^{P_i}}$ if κ_i is regular, and Q_i is trivial otherwise.*

Then

1) *In* V^{P_∞} REF *holds, as well as* GCH.

2) *If* $\lambda=\kappa_\lambda$ *is* $\kappa_{\lambda+n}$*-supercompact for* $\{\kappa_i : i<\lambda\}$ *then in* V^{P_∞}, λ *is* λ^{+n}*-supercompact.*

Proof. 1) Clearly, $\{\kappa_i : i<\infty\}$ is the class of all infinite cardinals of V^{P_∞}, and $V^{P_\infty}\models$GCH. Also the regularity of the κ_i's is preserved.

To prove REF we show that every regular κ_i with $i>1$ is reflecting. Let θ_0 be any regular cardinal $<\kappa_i$ and assume that in V^{P_∞} $\theta_0^+<\kappa_i$ and $S\subsetneqq\{\delta<\kappa_i:\mathrm{cf}\delta=\theta_0\}$ is stationary. Clearly, for some j $\kappa_j=\theta_0$. κ_j is necessarily regular, and $\kappa_{j+1}=(\theta_0^+)^{V^{P_\infty}}<\kappa_i$. Now for $\delta\in S, (\mathrm{cf}\delta)^V$ can have $\leqq\kappa_{j+1}<\kappa_i$ values, so, without loss of generality, for some θ regular in V, $\kappa_j\leqq\theta<\kappa_{j+1}$, we have for every $\delta\in S$ $\mathrm{cf}^V\delta=\theta$. Now we start working inside $V^{P_{j+1}}$, so we have a (P_∞/P_{j+1})-name $\underset{\sim}{S}$ of S and let p be any member of P_∞/P_{j+1}. We shall prove that p does not force that S does not reflect. As P_∞/P_{i+1} adds no subsets to κ_i we are really working in $V^{P_{i+1}/P_{j+1}}$ [so $\underset{\sim}{S}$ is a P_{i+1}/P_{j+1})-name and $p\in P_{i+1}/P_{j+1}$].

Now comes the main point. In $V^{P_{j+1}}$ $S_\theta=\{\delta<\kappa_i:\mathrm{cf}^V(\delta)=\theta\}\in \mathrm{V}$ is in $I[\kappa_i]$ [i.e., it is disjoint from $(S^{*n}_{\kappa_i})^{V^{P_{j+1}}}$]! This is seen as follows. If κ_i is inaccessible in V then, by 2.6(3), (6) $I[\kappa_i]$ contains all subsets of $\{\delta<\kappa_i: \Vdash_{P_\infty}\delta$ *is singular*$\}$. By the definition of P_∞ the only case left is where i is a successor ordinal, $\kappa_i=\kappa_{i-1}^+$ and κ_{i-1} is singular. Denote $i-1$ with δ. By (C), if $\theta\notin\bigcup_{\zeta<i-1}A_\zeta$ then $S_\theta\cap S^{*,n}_{\kappa_i}=\emptyset$. If $\theta\in\bigcup_{\zeta<i-1}A_\zeta$ then we have that $\theta\in A_j$ hence, by (B), θ is pwccf above κ_j, and by 2.10 $S_\theta\in I[\kappa_i]$. We have, by Def. 2.6 (notice that P_j has power $\leqq 2^{\kappa_j}$ and can therefore be substituted for R in 2.11) there is in V^{P_j} a forcing notion Q, $|Q|\leqq 2^\theta$, Q is κ_j-complete and $\Vdash_Q$"$\mathrm{cf}\theta$ is weakly compact." [One can omit the requirement that $|Q|\leqq 2^\theta$ in the definition 2.11 of pwccf since in order that

$$P_j*\underline{\mathrm{Levy}}(\kappa_j,\theta_1)\cong\mathrm{Levy}(\kappa_j,\theta_1)$$

it suffices that $|Q|=\theta_1<\kappa_{j+1}$. Given any Q as in 2.11, since κ_{j+1} is supercompact there is, by the reflection at a supercompact cardinal, such a Q of cardinality $<\kappa_{j+1}$.] Thus in V^{P_j*Q} $S_\theta\in I[\kappa_i]$. Now by 2.6(4)(iii) and 2.8 (if κ_i is a successor of a singular cardinal) or by 2.6(3) [if κ_i is not a successor of a singular cardinal and therefore, by Assumption (A) it is strongly inaccessible] we know that also in $V^{P_j*Q*\underline{\mathrm{Levy}}(\kappa_j,\theta_1)}$ $S_\theta\cap S^{*n}_{\kappa_i}=\emptyset$, where $\theta_1=|Q|$. However, as Q is κ_j-complete of cardinality $<\kappa_j$, $Q*\mathrm{Levy}(\kappa_j,\theta_1)$ is equivalent, as a forcing notion, to $\mathrm{Levy}(\kappa_j,\theta_1)$. Let

$$G^a\subseteqq P_j*\mathrm{Levy}(\kappa_j,\theta_1)$$

be generic over V, and without loss of generality $p\restriction(j+1)\in G^a$. Now we use the supercompactness of κ_{j+1}, which is preserved in $V[G^a]$ since the forcing notion is of cardinality $<\kappa_j$, to find $N\prec\langle H(\kappa),\in\rangle$, for a sufficiently large κ, such that

$$\{\kappa_i:\kappa_i<\kappa\},\kappa_i,\kappa_j,\theta_1,p\in N$$

and $N\cap\kappa_{j+1}$ is a cardinal, N is isomorphic to some $\langle H(\kappa'),\in\rangle$ for some ordinal κ'. Now let

$$G^b\subseteqq j_N(P_{i+1}/P_j*\mathrm{Levy}(\kappa_j,\theta_1))$$

be generic over j_N" $N[G^a]$ (equivalently, over $V[G^a]$) so that $j_N(p)\in G$. Now in $V[G^a,G^b]$ $j_N(\underset{\sim}{S})$ is interpreted as a stationary subset of $j_N(\kappa_i)$, and in $V[G^a]$ $j_N(\{\delta<\kappa_i:\mathrm{cf}\,\delta=\theta\})$ is in $I[j_N(\kappa_i)]$, hence in

$$V[G^a,G^b]^{\mathrm{Levy}(\kappa_j,<\kappa_{j+1})}$$

it is still so, by 2.6(6). But, again, in V^{P_j}

$$\mathrm{Levy}(\kappa_j,\theta) * j_N(P_{i+1}/P_j * \mathrm{Levy}(\kappa_j,\theta)) * \mathrm{Levy}(\kappa_j,\kappa_{j+1})$$

is equivalent to $\mathrm{Levy}(\kappa_j,\kappa_{j+1})$, and without loss of generality we can compute G^a and G^b in $V^{P_{j+1}}$. Now $j_N^{-1}(G^b)$ is a directed subset of P_{i+1}/P_{j+1} and has an upper bound q, which forces what we need.

2) This is proved by essentially the same proof.

3 Oracle forcing for Laver's diamond

3.1 Definition. 1) A forcing notion P is α-strategically closed if for each $r \in P$ Player I has a winning strategy in the following game $G^{\alpha}_{P,r}$: A play consists of α rounds of moves. In the β-th round Player I chooses a $p_\beta \in P$ such that $r \leqq p_\beta \wedge \bigwedge_{\gamma<\beta} q_\gamma \leqq p_\beta$ and then Player II chooses a $q_\beta \in P$ such that $p_\beta \leqq q_\beta$. Player I wins if he has always a legal move.

2) P is $<\kappa$-strategically closed if P is α-strategically closed for every $\alpha<\kappa$.

3) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i \leqq \alpha, j<\alpha\rangle$ is a $<\kappa$-Easton-support iterated forcing notion if P_i is a forcing notion, $\underset{\sim}{Q}_i$ is a P_i-name in this forcing notion, and P_i is the set of all functions f from subsets of i such that

(i) if λ is inaccessible and $\geqq \kappa$ and for all $j<\lambda$ $|P_j|<\lambda$ then $|\lambda \cap \mathrm{Dom} f| < \lambda$,

(ii) for every $j \in \mathrm{Dom} f$, $\Vdash_{P_j}$ "$f(j) \in \underset{\sim}{Q}_j$".

For $j \notin \mathrm{Dom} f$ we shall identify $f(j)$ with $\emptyset_j$, which is the least member of $\underset{\sim}{Q}_j$. $f \leqq g$ iff for all $j \leqq \mathrm{Dom} f$ $g \restriction j \Vdash_{P_j}$ "$f(j) \leqq g(j)$".

4) $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha, j<\alpha\rangle$ is a κ-Easton-support iterated forcing notion if the same conditions as in (3) are satisfied, except that in (i) we have $\lambda>\kappa$.

3.2 Fact. 1) Suppose that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha, j<\alpha\rangle$ is a $<\kappa$-Easton-supported iterated forcing notion, $\zeta \leqq \kappa$ and each $\underset{\sim}{Q}_j$ is ζ-strategically closed (in V^{P_j}) then so is $P_\alpha = \lim \bar{Q}$. This holds also if we require $\alpha<\kappa$ instead of $\zeta \leqq \kappa$.

2) If P is α-strategically closed forcing then P adds no new sequences of ordinals of length $<\alpha$.

Proof. The proof of this fact is known.

1) For every $i<\alpha$ and, in V^{P_i}, let $s \in \underset{\sim}{Q}_i$ and let $I^{i,\zeta}_s$ be a winning strategy for Player I in the game $G^{\zeta}_{P_i,s}$. The strategy of Player I in the game $G^{\alpha}_{P_\alpha,r}$ is to maintain the following properties of the initial part $\langle p_\beta, q_\beta : \beta<\gamma\rangle$ of the play.

(i) if $\beta<\varrho<\gamma$ then $q_\beta \leqq q_\varrho$, $p_\varrho \leqq q_\varrho$, $r \leqq p_0$,

(ii) for every $j<\alpha$ if there is a β such that $j \in \mathrm{Dom}\, q_\beta$ then let β_j be the least such and $\langle p_{\beta_j+1+\delta}(j), q_{\beta_j+1+\delta}(j) : \delta<\gamma-(\beta_j+1)\rangle$ is (a name of) an initial part of a play of $G^{\zeta}_{Q_j, q_{\beta_j}(j)}$ played by Player I according to the strategy $I^{j,\zeta}_{q_\beta(j)}$.

2) Let $\underset{\sim}{f}$ be a name in P of a function from $\beta<\alpha$ into the ordinals, and let $r \in P$. Pick a winning strategy for Player I in the game $G^{\alpha}_{P,r}$ and let $\langle p_i, q_i : i<\alpha\rangle$ be a play where Player I plays according to this strategy, and for every $i<\beta$ q_i is such that $p_i \leqq q_i$ and q_i forces a value for $\underset{\sim}{f}(i)$. $p_{\beta+1}$ forces all the values of the function hence the function is already in V.

3.3 Definition. For a given supercompact cardinal μ

1) F is called *a direct oracle diamond* (for $<\mu$-strategic closure and $<\mu$-Easton support) if for every sequence $\bar{P} = \langle P_i, Q_j : i \leqq \alpha, j<\alpha\rangle$ $\lessdot$-increasing each P_i

$<\mu$-strategically closed forcing notions and every $x \in V$ and for every sufficiently large ordinal χ so that $\bar{Q}, x \in H(\chi)$ there is a fine normal ultrafilter D on $\mathscr{S}_{<\mu}(H(\chi))$ such that the following set is in D.

$\{a \in \mathscr{S}_{<\mu}(H(\chi)) : a \cap \mu$ *is strong Mahlo cardinal* θ,

$$F(\theta) = \langle j_a(x), j_a``a, j_a(\langle P_i : i \leqq \alpha \rangle, \langle G_i : i \leqq \alpha \rangle)\rangle$$

where j_a *is the Mostowski collapse of* a, $G_i \subseteq P_i$ *is generic over* $j_a``a$, $(j_a``a)[G_\alpha]$ *is of the form* $H(\chi')$ *and* $i<j \Rightarrow G_i \subseteq G_j\}$.

2) F is called an *oracle diamond* if for some $V' \subseteq V$ and $Q \in V'$ Q is $<\mu$-strategically closed in V', F is a direct oracle diamond in V' and $V=(V')^Q$.

3.4 Definition. Let μ be a supercompact cardinal, $F:\mu \to H(\mu)$ a direct oracle diamond, $Q^* \in V$ a forcing notion, $G^* \subseteq Q^*$ generic over V and $Q \in V[G^*]$ a forcing notion. We say that Q satisfies the *F-oracle* condition in $V[G^*]$ if:

(a) Q is $<\mu$-strategically closed.

(b) For all $x \in V$, $p \in Q$, and χ such that $x, p \in H(\chi)$ and for every fine normal filter D on $\mathscr{S}_{<\kappa}(H(\chi))$ (in V!) the set of those $a \in \mathscr{S}_{<\kappa}(H(\chi))$ which satisfy the following implication belongs to D (i.e. to the filter D generated in $V[G^*]$).

If (i) $\kappa_1 = a \cap \kappa$ is a regular cardinal.

(ii) For the Mostowski collapse j_a of a $j_a``a = H \subseteq H(\chi_1)$, $\chi_1 = j_a``\chi_1$.

(iii) $F(\kappa_1)$ is of the form $\langle x, j_a``(a), \langle P_i : i \leqq \alpha \rangle, \langle G_i : i \leqq \alpha \rangle\rangle$.

(iv) For some transitive H such that $\bar{P} = \langle P_i : i \leqq \alpha \rangle \in H \subseteq H(\chi)$, $G_\alpha \subseteq P_\alpha$ is generic over H and $H(\chi) = H[G_\alpha]$.

(v) In H $\bar{P}$ is $\lessdot$-increasing (see 2.10 for the definition of $\lessdot$) and each P_i is a $<\kappa_1$-strategically closed forcing notion. (The main case is where there are Q_i, for $i<\alpha$ such that $\langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha, j<\alpha \rangle$ is an iteration with $<\kappa_1$-Easton support each $\underset{\sim}{Q}_i$ is $<\kappa_1$-strategically closed.

(vi) $j_a(Q^* * Q)$ is $P_\xi * (P_\eta / P_\xi)$ for some $\xi < \eta \leqq \alpha$ and $j_a``(G^* \cap a) = G^\xi$.

Then $\{q \in Q \cap a : j_a(a) \in G_\xi\}$ has an upper bound in Q.

3.5 Theorem. *Suppose that* κ *is supercompact and* $\lambda < \kappa$.

1) *There is a forcing notion* Q, *λ-complete such that* $|Q| = \kappa$ *and* $\Vdash_Q$ "*there is a* μ-*Laver diamond in* V^Q". *(A* μ-*Laver diamond* f *is* $f:\mu \to H(\mu)$ *which is a Laver diamond.)*

2) *Let* F_0 *be a Laver diamond for* κ, *then for some* λ-*strategically closed* κ-c.c. *forcing notion* R *we have in* V^R *that some* $F:\kappa \to H(\kappa)$ *is an oracle diamond.*

3) *Let* $\varphi(-,-)$ *be a formula of set theory (with parameters) any instance of* φ *is satisfied iff it is satisfied in some* $(H(\chi), \in)$ *such that* $\varphi(Q, \lambda)$ *implies that* Q *is a forcing notion which does not add sequences of length* $<\lambda$. *Assume*

(*) *For every* $<\kappa$-*Easton-support iterated forcing notion* $\langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha, j<\alpha \rangle$, *if for every* $j<\alpha$ $\Vdash_{P_j} \varphi(Q_j, \kappa)$ *then also* $\varphi(P_\alpha, \kappa)$.

(**) *If* $\langle P_i, \underset{\sim}{Q}_j : i \leqq \alpha, j<\alpha \rangle$ *is a* $<\kappa$-*Easton-support iterated forcing notion and for every* $i<\alpha$ *there is a strongly Mahlo cardinal* κ_i *such that* $\kappa \leqq \kappa_i$ *and* $\Vdash_{P_i} \varphi(Q_i, \kappa_i)$ *and either* $|P_i| < \kappa_i$ *or* P_i *satisfies the* κ_i-*chain condition*, $|P_i| = \kappa_i$, $\kappa_i = i$ *and for every* $j<i$ $|P_j| < \kappa_j$ *then* $\varphi(P_\alpha, \kappa)$.

Let F_0 *a* κ-*Laver diamond and let the parameters of* φ *be in* $H(\kappa)$ *then there is a forcing notion* R *such that* $\varphi(R, \kappa)$, R *satisfies the* κ-*chain condition and has power* κ *and in* V^R *there is a function* $F:\kappa \to H(\kappa)$ *which is an oracle diamond for the class of all forcing notions* Q *for which* $\varphi(Q, \kappa)$ *(i.e., F satisfies the definition of a direct oracle diamond with all the P_i's in* 3.3(1) *and the Q in* 3.3(2) *required to satisfy* $\varphi(-, \kappa)$*).*

4) *In* (3) *one can drop the requirement that the support be* $<\kappa$-*Easton provided that* (*) *and* (**) *hold as well as*

(***) *For every iteration* $\bar{Q}=\langle P_i, Q_j : i\leqq\alpha, j<\alpha\rangle$ *with the same support and for every* $i<\alpha$ *there is a strongly Mahlo cardinal* κ_i *such that* $|P_i|<\kappa_i$ or $|P_i|=\kappa_i$, P_i *satisfies the* κ*-chain condition and* $\varphi(P_i, \kappa)$.

Proof. 1) See Laver [L].

2) Let $\bar{Q}=\langle P_i, \underset{\sim}{Q}_i : i<\kappa\rangle$ be the λ-Easton-support iterated forcing notion defined by: $\underset{\sim}{Q}_i$ is R_i if i is strongly Mahlo and $>\lambda$ and F_0"$i\subseteqq H(i)$, where R_i is such that $F_0(i)=\langle x, R_i\rangle$ and R_i is a P_i-name of a $<i$-strategically closed forcing notion, and Q_i is trivial otherwise. In V^{P_κ} let $F(i)$ be $y[G_{P_i * R_i}]$ if $F_0(i)=\langle y, R_i\rangle$ and y is a $P_i * R_i$-name, and 0 otherwise. The proof is exactly as in Laver [L].

3), 4) The proof is like that of (2).

Theorem 3.5 yields a universe in which we can iterate many forcing notions preserving supercompactness.

3.6 Lemma. *Let* $F:\kappa\to H(\kappa)$ *be an oracle diamond.*

1) *If* Q_0 *satisfies the F-oracle condition then* $\Vdash_{Q_0}$"*F is an oracle diamond*".

2) *If* Q_0 *satisfies the F-oracle condition and* $\underset{\sim}{Q}_1$ *is a* Q_0*-name satisfying the F-oracle condition then* $Q_0 * \underset{\sim}{Q}_1$ *satisfies the F-oracle condition.*

3) *If* $\langle P_i, \underset{\sim}{Q}_j : i\leqq\alpha, j<\alpha\rangle$ *is a* $<\kappa$*-Easton-support iterated forcing notion (or a* $<\kappa$*-support forcing notion) and each* $\underset{\sim}{Q}_i$ *satisfies the F-oracle condition then* P_α *satisfies the F-oracle condition.*

Proof. The proof is straightforward.

3.7 Fact

(A) Suppose:

(a) $\bar{\kappa}=\langle\kappa_i : i<\infty\rangle$ is a strictly increasing and continuous sequence of cardinals, and for every limit ordinal δ, if κ_δ is singular then $\kappa_{\delta+1}=\kappa_\delta^+$.

(b) $F:\mathrm{Ord}\to V$, $\mathrm{Range}(F\restriction\kappa_i)\subseteqq H(\kappa_i)$.

(c) If κ_i is regular then $F\restriction\kappa_{i+1}$ is a κ_{i+1}-Laver diamond, and thus κ_{i+1} is supercompact.

(d) *Notation.* F_0 and F_1 are defined by $F(\alpha)=\langle F_0(\alpha), F_1(\alpha)\rangle$ when $F(\alpha)$ is a pair, and $F_0(\alpha)=F_1(\alpha)=0$ otherwise.

(B) We define an iterated forcing notion by:

(a) $\langle P_i, \underset{\sim}{Q}_i : i<\infty\rangle$ is an Easton-support iterated forcing notion.

(b) $|P_i|\leqq\kappa_i^+$.

(c) Let $i(\alpha)=\min\{i:\kappa_i>\alpha\}$. We take $F^*(\alpha)$ to be $F_0(\alpha)[G_{P_{i(\alpha)}}]$ if this is defined, and $F^*(\alpha)=0$ otherwise. We shall prove, inductively, that for κ_i regular, in $V^{P_{i+1}}$ $F^*\restriction\kappa_{i+1}$ is a direct oracle diamond.

(d) If κ_i is regular, $Q_i=\lim\langle P_\alpha^i, \underset{\sim}{Q}_\alpha^i : \alpha<\kappa_{i+1}\rangle$ (in V^{P_i}) $\underset{\sim}{Q}_\alpha^i$ is $F_1(\alpha)$ if it is a $P_i * P_\alpha^i$-name of a $<\alpha$-strategically closed forcing notion, α is inaccessible and $>|P_\beta^i|+\kappa_i$ for $\beta<\alpha$ and $F_1(\alpha)$ satisfies the $F^*\restriction\kappa_{j+1}$-oracle condition for $j<i$ with κ_j regular, and otherwise $\underset{\sim}{Q}_\alpha^i$ is trivial.

(e) In $V^{P_{i+1}}$, $A_i\overset{\text{def}}{=}\{\theta:\kappa_i<\theta<\kappa_{i+1} \wedge \theta$ is a pwccf above $\kappa_i\}$ (see 2.6).

(f) If κ_δ is singular then Q_δ is $P_{S_\delta}^{\mathrm{sh}}$ where

$$S_\delta=\left\{\zeta<\kappa_{\delta+1} : \zeta\in S_{\kappa_{i+1}}^{*\mathrm{n}} \text{ but } \mathrm{cf}^V\zeta\notin\bigcup_{j<\delta}A_j\right\}$$

(see 2.1).

Then:

(1) For $\alpha<\beta$ P_β/P_α satisfies the $(F\restriction\kappa_{i+1})$-oracle condition, for $i+1\leqq\alpha$ and a regular κ_i.

(2) P_∞ preserves the supercompactness of κ_{i+1}.

(3) V^{P_∞} is as required in 2.13(D).

(4) If $\lambda=\kappa_\lambda$ is supercompact, $n<\omega$, D is a normal fine ultrafilter on $\mathscr{S}_{<\lambda}(H(\kappa_i))$ and D preserves $\{\kappa_i:i<\infty\}$ [see Def. 0.2(1)] then in V^{P_∞} λ is κ_i-supercompact by an ultrafilter which preserves $\{\kappa_j:j<\infty\}$.

(5) Condition (C) of 2.13 holds.

Proof. The proof is straightforward (using 3.7, 2.8(1)).

3.8 Claim. Suppose $\langle\kappa_i:i<\infty\rangle$ is as in 3.7(A)(a), i.e., for every i if κ_i is regular then κ_{i+1} is supercompact, and $\lambda=\kappa_\lambda$ is supercompact for $\{\kappa_i:i<\infty\}$, $\langle P_i,\underset{\sim}{Q}_i:i<\infty\rangle$ is an Easton-support iterated forcing notion, and $\underset{\sim}{Q}_i$ is such that if i is strongly inaccessible then $\underset{\sim}{Q}_i$ adds a Cohen subset to i and otherwise $\underset{\sim}{Q}_i$ is trivial.

Then, in V^{P_∞} there is an F as required in 3.6 for $\{\kappa_i:i<\infty\}$ such that $\lambda=\kappa_\lambda$ is supercompact.

3.9 Remark. Claim 3.8 yields an F and this F is as required in the assumptions of 3.7. Also 3.7 yields what is required for the assumptions of 2.13. Thus 2.13 completes the proof of 0.1.

Proof. First one obtains $F\restriction[\kappa_i,\kappa_{i+1})$ from G_{κ_i}, the generic subset of Q_{κ_i}. By 3.5(1), (2) this makes F to be as required in 3.7, but what about λ? We correct $F\restriction\{\kappa_i:i<\lambda\}$ by the generic subset of Q_λ.

4 On the ideal $I[\lambda]$

4.1 Lemma. *If $\mu=\sup\limits_{i<\mathrm{cf}\,\mu}\lambda_i$, where $\langle\lambda_i:i<\mathrm{cf}\,\mu\rangle$ is a strictly ascending sequence of uncountable cardinals and $\lambda=\mu^+$ then there is a function $c:\lambda\times\lambda\to\mathrm{cf}\,\mu$ such that for all $\alpha,\beta,\gamma<\lambda$*

(i) *$c(\alpha,\beta)=c(\beta,\alpha)$,*

(ii) *if $\alpha>\beta>\gamma$ then $c(\alpha,\gamma)\leqq\max(c(\alpha,\beta),c(\beta,\gamma))$, and*

(iii) *for all $i<\mathrm{cf}\,\mu$ $|\{\beta<\alpha:c(\alpha,\beta)=i\}|<\lambda_i$.*

Proof [Sh 88a, 4(3)]. We shall define $c(\alpha,\beta)$ for $\beta\leqq\alpha$ by induction on α as follows, and for $\alpha<\beta$ we define $c(\alpha,\beta)=c(\beta,\alpha)$. Let $\langle a_i^\alpha:i<\mathrm{cf}\,\mu\rangle$ be an ascending sequence of sets such that for $i<\mathrm{cf}\,\mu$ $|a_i|<\lambda_i$ and $\bigcup\limits_{i<\mathrm{cf}\,\mu}a_i^\alpha=\alpha$. For $i<\mathrm{cf}\,\mu$ let $a_i'=a_i\cup\{\beta$: *for some $\alpha'\in a_i$ such that $\beta\leqq\alpha'<\alpha$ $c(\alpha',\beta)\leqq i$*$\}$. We define $c(\alpha,\beta)$ to be the least i such that $\beta\in a_i'$. (i) holds trivially, and (iii) is easily seen, by induction on α. Also (ii) is proved by induction on α as follows. Assume $\alpha>\beta>\gamma$ and $c(\alpha,\beta)$, $c(\beta,\gamma)\leqq i$. Since $c(\alpha,\beta)=j\leqq i$ then $\beta\in a_j'\subseteqq a_i'$. By the definition of a_i' one of the following two cases holds.

Case a. $\beta\in a_i$. Then, since $\gamma<\beta<\alpha$ and $c(\beta,\gamma)\leqq i$ we have $\gamma\in a_i'$ and $c(\alpha,\gamma)\leqq i$.

Case b. For some $\alpha'\in a_i$ such that $\beta\leqq\alpha'<\alpha$ $c(\alpha',\beta)\leqq i$. Then, since $c(\alpha',\beta)$, $c(\beta,\gamma)\leqq i$ we have, by the induction hypothesis, $c(\alpha',\gamma)\leqq i$. Now we have $\gamma\leqq\alpha'<\alpha$, $\alpha'\in a_i$, and $c(\alpha',\gamma)\leqq i$, hence $\gamma\in a_i$ and $c(\alpha,\gamma)\leqq i$.

4.2 Fact. 1) Let λ be a strong limit singular cardinal, Q a forcing notion such that $\Vdash_Q$"*λ is a strong limit cardinal and $(\lambda^+)^V=(\lambda^+)^{V^Q}$*", θ a regular ordinal $<\lambda$ and

$$S=\{\delta<\lambda:\mathrm{cf}\,\delta=\theta\}.$$

Each of the following (a)–(c) is a sufficient conditions for

$$\Vdash_Q S\cap(S^{*\mathrm{n}}_{\lambda^+})^V = S\cap(S^{*\mathrm{n}}_{\lambda^+})^{V^Q} (\operatorname{mod} D_{\lambda^+}).$$

(a) Q is θ-complete.
(b) Q adds no new subsets of θ.
(c) Out of any θ members of Q θ are pairwise compatible in Q.
2) We can replace in Part (1) $S^{*\mathrm{n}}_{\lambda^+}$ by $\mathrm{bad}(\bar{f})$.

Proof. Let c be a two-place function from λ^+ to θ as in 4.1, where we take λ for μ, and therefore λ^+ for λ. Thus

$$\begin{aligned} S^{*\mathrm{n}}_{\lambda^+} = S(c) = \{\delta<\lambda^+ &: \textit{if } \operatorname{cf}\delta>\operatorname{cf}\lambda, \textit{ then for some unbounded } A\subseteq\delta \\ & c``(A\times A) \textit{ is bounded in } \operatorname{cf}\lambda\} \\ = \{\delta<\lambda^+ &: \textit{if } \operatorname{cf}\delta>\operatorname{cf}\lambda \textit{ then for any unbounded } A\subseteq\delta \\ & \textit{there is an unbounded } A'\subseteq A \\ & \textit{such that } c``(A'\times A') \textit{ is bounded in } \operatorname{cf}\lambda\}. \end{aligned}$$

and this holds in V^Q too as c, λ, and λ^+ are as required also in V^Q; the cofinality of λ in V^Q may be less than $\operatorname{cf}\lambda$ but this makes no difference, by 2.7.

Clearly $S(c)^V\subseteq S(c)^{V^Q}$. Suppose $\delta\in S(c)^{V^Q}\cap S$, then $(\operatorname{cf}\delta)^V=\theta$, so let A be an unbounded subset of δ of order type θ, $A\in V$. By the second characterization above of $S(c)$ there is an unbounded subset A' of A in V^Q such that $c``(A'\times A')$ is bounded in $\operatorname{cf}\lambda$. By (b) $A'\in V$, hence $\delta\in S(c)^V$. By (c) there is a $p\in Q$ and a Q-name $\underset{\sim}{A}'$such that

$$p\Vdash_Q \text{“}\underset{\sim}{A}' \textit{ is an unbounded subset of } A\wedge c``(\underset{\sim}{A}'\times A') \textit{ is bounded in } \operatorname{cf}^{(V)}\lambda\text{”}.$$

Without loss of generality $p\Vdash$“$c``(\underset{\sim}{A}'\times\underset{\sim}{A}')\subseteq\varepsilon$”, where $\varepsilon<\operatorname{cf}\lambda$.

Let $\underset{\sim}{A}$ be the range of the increasing sequence $\langle\underset{\sim}{\alpha}_i: i<\theta\rangle$. For each $i<\theta$ choose, if possible, a $p\leqq p_i\in Q$, such that $p_i\Vdash\alpha_i\in\underset{\sim}{A}'\wedge\operatorname{otp}(\alpha_i\cap\underset{\sim}{A})\geqq i$. Let $B=\{i<\theta$: p_i *is defined*$\}$. Clearly $|B|=\theta$, and by (c) there is a $B'\subseteq B$ of cardinality θ such that any two p_i's with $i\in B'$ are compatible. Let $A'=\{\alpha_i: i\in B'\}$. Clearly A' is unbounded in A, $A'\in V$, and since any two p_i's with $i\in B'$ are compatible $c``(A'\times A')\subseteq\varepsilon$. Therefore $\delta\in S(c)^V$. Assuming now (a), there is an increasing sequence $\langle p_i: i<\theta\rangle$ such that $p_0=p$, p_i determines the first $i+1$ members of $\underset{\sim}{A}'$. Let α_i be such that $p_i\Vdash\alpha_i$ *is the i-th member of* A'. Since $p_i\geqq p$ and $p\Vdash\underset{\sim}{A}'\subseteq A$ we have $\alpha_i\in A$. Clearly $\langle\alpha_i: i<\theta\rangle$ is an increasing sequence in V of members of A hence $A''=\{\alpha_i: i<\theta\}$ is an unbounded subset of A. It follows easily that $c``(A''\times A'')\subseteq\varepsilon$, hence $\delta\in S(c)^V$.

4.3 Lemma. *Let λ be an uncountable regular cardinal, $B\subseteq A$, $|B|\leqq|A|=\lambda$, $A=\bigcup_{i<\lambda}A_i$, $B=\bigcup_{i<\lambda}B_i$, where the sequences $\langle A_i: i<\lambda\rangle$, $\langle B_i: i<\lambda\rangle$ are increasing and continuous and $|A_i|, |B_i|<\lambda$ for $i<\lambda$. Then*

$$E=\{i<\lambda: A_i\cap B=B_i\}$$

is a club subset of λ.

4.4 Lemma. *Let λ be an uncountable regular cardinal.*

1) *$T\overset{\mathrm{def}}{=}\{\delta<\lambda^+:\operatorname{cf}\delta<\lambda\}$ is in $I[\lambda^+]$.*

2) *T is the union of λ sets which have the square property, i.e., there are sequences $\langle S_i: i<\lambda\rangle$, and $\langle C^i_\delta:\delta\in S_i\rangle$ for $i<\lambda$ such that:*

(a) $\bigcup_{i<\lambda}S_i=T$.

(b) *For* $\delta \in S_i$ C^i_δ *is a subset of* $\delta \cap T$ *of cardinality* $<\lambda$ *closed in* δ, *and if* δ *is a limit ordinal then* C^i_δ *is unbounded in* δ.

(c) *For all* δ_1, δ_2 *if* $\delta_2 \in S_i$ *and* $\delta_1 \in C^i_{\delta_2}$ *then* $\delta_1 \in S_i$ *and* $C^i_{\delta_1} = C^i_{\delta_2} \cap \delta_1$. *(Notice that* δ_1 *may also be a successor ordinal.)*

3) *For each regular* $\theta < \lambda$, *let* $T_\theta = \{\delta < \lambda^+ : \mathrm{cf}\,\delta \leqq \theta\}$. *There are* $\langle S_\zeta : \zeta < \lambda \rangle$ *and* $\langle C^\zeta_\delta : \delta \in S_\zeta \rangle$ *such that*

(a) $\bigcup_{i<\lambda} S_i = T_\theta$.

(b) *For* $\delta \in S_i$ C^i_δ *is a subset of* $\delta \cap T_\theta$ *of order type* $\leqq \theta$ *closed in* δ *and if* δ *is a limit ordinal then* C^i_δ *is unbounded in* δ.

(c) *For all* δ_1, δ_2 *if* $\delta_2 \in S_i$ *and* $\delta_1 \in C^i_{\delta_2}$ *then* $\delta_1 \in S_i$ *and* $C^i_{\delta_1} = C^i_{\delta_2} \cap \delta_1$.

Remark. In (2) and (3) we can add:

(d) *If* $\alpha + 1 \in S_i$ *and* $\mathrm{cf}\,\alpha < \lambda$ *then* $C^i_{\alpha+1} = C^i_\alpha \cup \{\alpha\}$.

Proof. 1) We show that (2) implies (1). For $\alpha < \lambda^+$ let

$$\mathscr{P}_\alpha = \{C^i_\alpha : i < \lambda \wedge \alpha \in S_i\}.$$

Notice that, using the notation of 2.5, $S^{*\mathrm{P}}_{\lambda}(\langle \mathscr{P}_\alpha : \alpha < \lambda^+ \rangle) \supseteqq T \setminus \lambda$. By 2.5 $T \in I[\lambda^+]$.

2) We deal now with the case where $\lambda > \aleph_1$, leaving the case $\lambda = \aleph_1$ to the end of the section (after 4.5). We choose for each $\alpha < \lambda^+$ a sequence $\langle D^i_\alpha : i < \lambda \rangle$ such that:

(i) $D^i_\alpha \subseteqq \alpha$, $|D^i_\alpha| < \lambda$.

(ii) $\langle D^i_\alpha : i < \lambda \rangle$ is increasing and continuous.

(iii) $\alpha = \bigcup_{i<\lambda} D^i_\alpha$.

(iv) The closure in α of D^i_α is included in D^{i+1}_α.

(v) For $\alpha > 0$ $0 \in D^i_\alpha$, and if $\gamma + 1 \in D^i_\alpha$ then $\gamma \in D^i_\alpha$.

(vi) If $\mathrm{cf}\,\alpha < \lambda$ then D^0_α is unbounded in α.

For each $\alpha \in T$ and $i < \lambda$ such that $\mathrm{cf}\,i > \omega$ we define C^i_α as follows. For an ordinal α let $b(\alpha)$ be the ordinal β such that $\alpha = \beta + n$, where β is 0 or a limit ordinal and $n < \omega$.

Case a. If $\mathrm{cf}\,b(\alpha) \neq \mathrm{cf}\,i$ then

$$C^i_\alpha \stackrel{\mathrm{def}}{=} \{\beta \in D^i_\alpha : \mathrm{cf}\,\beta < \lambda\} \cup \{\delta < \alpha : \delta = \sup(D^i_\alpha \cap \delta)\}.$$

Case b. If $\mathrm{cf}\,b(\alpha) = \mathrm{cf}\,i$ $(>\omega)$ then

$$C^i_\alpha \stackrel{\mathrm{def}}{=} \bigcap_{W \text{ is a club subset of } b(\alpha)} \ \bigcup_{\beta \in W \cap T \cap \{\beta < \lambda^+ : \mathrm{cf}\,b(\beta) \neq \mathrm{cf}\,i\}} C^i_\beta \cup [b(\alpha), \alpha).$$

Notice that for $\alpha \in T$, if $\mathrm{cf}\,b(\alpha) \neq \mathrm{cf}\,i$, $i < \lambda$, and $\mathrm{cf}\,i > \omega$ then C^i_α is a closed subset of α and $C^i_\alpha \subseteqq T$. It follows immediately, by induction on α, that for every $\alpha \in T$ $C^i_\alpha \subseteqq T$. For $i < \lambda$, $\mathrm{cf}\,i > \omega$ let

$S_i \stackrel{\mathrm{def}}{=} \{\alpha \in T : C^i_\alpha$ *is closed in* α *and if* α *is a limit ordinal then* C^i_α *is unbounded in* α *and for every* $\beta \in C^i_\alpha$ $C^i_\beta = \beta \cap C^i_\alpha$ *and if* β *is a limit ordinal then* C^i_β *is unbounded in* $\beta\}$.

Thus $\langle S_i : i < \lambda \wedge \mathrm{cf}\,i > \omega \rangle$ and $\langle\langle C^i_\delta : \delta \in S_i \rangle : i < \lambda \wedge \mathrm{cf}\,i > \omega \rangle$ are defined, and we have to prove (a), (b), and (c).

a) We shall prove by induction on $\alpha \in T$ that there is a club subset E of λ such that for every $i < \lambda$ if $i \in E$ and $\mathrm{cf}\,i > \omega$ then $\alpha \in S_i$. The following simple fact will be useful for the inductive proof.

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(1) If $\gamma<\alpha$, $\gamma\in S_i$, and $C^i_\gamma=C^i_\alpha\cap\gamma$ then for every $\beta\leqq\gamma$ if $\beta\in C^i_\alpha$ then $C^i_\beta=C^i_\alpha\cap\beta$ and if β is a limit ordinal then C^i_β is unbounded in β.

Let us prove (1). Since $\gamma\in S_i$ C^i_γ is unbounded in γ. Let $\beta<\gamma$ and $\beta\in C^i_\alpha$ then $\beta\in C^i_\alpha\cap\gamma=C^i_\gamma$ and hence $C^i_\gamma\cap\beta=C^i_\beta$ and if β is a limit ordinal then C^i_β is unbounded in β. Therefore

$$C^i_\alpha\cap\beta=(C^i_\alpha\cap\gamma)\cap\beta=C^i_\gamma\cap\beta=C^i_\beta.$$

For every $\delta\in T\cap\alpha$ there is, by the induction hypothesis, a club subset E^δ_1 of λ such that $i\in E^\delta_1\wedge\mathrm{cf}\,i>\omega\Rightarrow\delta\in S_i$. Also, by (ii) and (iii), for $\delta\in T\cap\alpha$ the set $E^\delta_2=\{i<\lambda:\delta\in D^i_\alpha\}$ is an interval $[\sigma,\lambda)$ for some $\sigma<\lambda$ and hence a club subset of λ. For $\lambda<\delta<\alpha$ take in 4.3 $A=\alpha$, $B=\delta$, $A_i=D^i_\alpha$, and $B_i=D^i_\delta$ then, by 4.3, $E^\delta_3=\{i<\lambda:D^i_\alpha\cap\delta=D^i_\delta\}$ is club in λ. For $\delta<\lambda$, since λ is regular there is, by (ii), a $j_\delta<\lambda$ such that $D^i_\delta=\delta$ for every $i\geqq j_\delta$. Let $j'_\delta=\sup_{\beta\leqq\delta} j_\beta$ then $D^i_\delta=\delta$ for every $i\geqq j$. For $\delta<\lambda$, α we set $E^\delta_3=[j'_\delta,\lambda)$, so if $i\in E^\delta_3$ then $D^i_\alpha\cap\delta=\alpha\cap\delta=\delta=D^i_\delta$. For $\delta\in T\cap\alpha$ let $E^\delta=E^\delta_1\cap E^\delta_2\cap E^\delta_3$.

Now we deal separately with the following cases.

Case (i), $\alpha=0$. We take $E=\lambda$; this case is trivial.

Case (ii), $\alpha=\delta+1$, where $\delta\in T$. We take $E=E^\delta$. Let $i\in E$, $\mathrm{cf}\,i>\omega$, we shall prove that $\alpha\in S_i$. Since $i\in E^\delta\subseteq E^\delta_1,E^\delta_2,E^\delta_3$ we have $\delta\in S_i$, $\delta\in D^i_\alpha$, and $D^i_\alpha=D^i_\delta\cup\{\delta\}$. By the definition of C^i_α either both C^i_δ and C^i_α are defined by Case a, and then $D^i_\alpha=D^i_\delta\cup\{\delta\}$ implies $C^i_\alpha=C^i_\delta\cup\{\delta\}$ and we know also that C^i_α is closed in α or else both are defined by Case b and then $C^i_\alpha=C^i_\delta\cup\{\delta\}$ holds trivially, and hence C^i_α is closed in α since C^i_δ is closed in δ as $\delta\in S_i$. Thus (1) holds for $\gamma=\delta$, and α satisfies the requirements for membership in S_i.

Case (iii), α is a limit ordinal, and we restrict ourselves to $i<\lambda$ with $\mathrm{cf}\,i\neq\mathrm{cf}\,\delta$ when $\alpha=\delta+\omega$. As $\alpha\in T\,\mathrm{cf}\,\alpha<\lambda$. Let $\langle\delta_\sigma:\sigma<\mathrm{cf}\,\alpha\rangle$ be increasing and continuous with limit α and such that for $j<\mathrm{cf}\,\alpha$ which is not a limit ordinal $\mathrm{cf}\,\delta_j\leqq\omega$ and $\mathrm{cf}\,b(\delta_j)\neq\mathrm{cf}\,i$, and if $\mathrm{cf}\,\alpha>\omega$ then $\mathrm{cf}\,\delta_j=\omega$ for every such j. Clearly $\delta_\sigma\in T$ for $\sigma\in\mathrm{cf}\,\alpha$. Let

$$E=\bigcap_{j<\mathrm{cf}\,\alpha}E^{\delta_j}.$$

Suppose $i\in E$, $\mathrm{cf}\,i>\omega$, and $j<\mathrm{cf}\,\alpha$ then, since $i\in E^{\delta_j}$, we have $\delta_j\in S_i$, $\delta_j\in D^i_\alpha$, and $D^i_\alpha\cap\delta_j=D^i_{\delta_j}$.

Subcase (iii$_1$), $\mathrm{cf}\,\alpha\neq\mathrm{cf}\,i$. In this case C^i_α is defined by Case a. Now, for $j<\mathrm{cf}\,\alpha$ $\delta_j\in D^i_\alpha$, hence $\delta_j\in C^i_\alpha$, therefore C^i_α is unbounded in α, and it is clearly closed. For a successor $j<\mathrm{cf}\,\alpha$ $C^i_{\delta_j}$ is defined by Case a, so since $D^i_{\delta_j}=D^i_\alpha\cap\delta_j$ also $C^i_{\delta_j}=C^i_\alpha\cap\delta_j$. Thus (1) holds for arbitrarily large $\gamma<\alpha$ so $\alpha\in S_i$.

Subcase (iii$_2$), $\mathrm{cf}\,\alpha=\mathrm{cf}\,i$ $(>\omega)$. Let $W_0=\{\delta_\sigma:\sigma<\mathrm{cf}\,\alpha\wedge\mathrm{cf}\,\delta_\sigma\neq\mathrm{cf}\,i\}$. We shall see that

$$\text{if}\quad \varepsilon,\zeta\in W_0,\ \varepsilon<\zeta\quad\text{then}\quad C^i_\varepsilon=C^i_\zeta\cap\varepsilon. \tag{2}$$

Let $\varepsilon=\delta_\sigma$, $\zeta=\delta_\tau$, $\sigma<\tau$. We know that

$$D^i_\varepsilon=D^i_{\delta_\sigma}=D^i_\alpha\cap\delta_\sigma=D^i_\alpha\cap\varepsilon$$

and similarly $D^i_\zeta=D^i_\alpha\cap\zeta$. Therefore $D^i_\varepsilon=D^i_\zeta\cap\varepsilon$, and since $\varepsilon,\zeta\in W_0$ both C^i_ε and C^i_ζ are defined by Case a and we have also $C^i_\varepsilon=C^i_\zeta\cap\varepsilon$. Let W' be a club subset of $\{\delta_\sigma:\sigma<\mathrm{cf}\,\alpha\}$ and let $W''=W'\cap W_0$. W'' is clearly an unbounded subset of α. Thus, by (2),

$$\bigcup_{\beta\in W'\cap\{\beta<\alpha:\mathrm{cf}\,b(\beta)\neq\mathrm{cf}\,i\}}C^i_\beta=\bigcup_{\beta\in W''}C^i_\beta=\bigcup_{\beta\in W_0}C^i_\beta.$$

This immediately implies, by Case b,

$$C^i_\alpha = \bigcup_{\beta \in W_0} C^i_\beta . \tag{3}$$

For $\beta \in W_0$ β is a limit ordinal (since in this case $\mathrm{cf}\alpha = \mathrm{cf} i > \omega$) and $\beta \in S_i$ hence C^i_β is club in β, hence by (2) and (3) C^i_β is club in α. Also for $\beta \in W_0$ $C^i_\alpha \cap \beta = C^i_\beta$. Thus (1) holds for every $\gamma \in W_0$, and $\alpha \in S_i$.

Case (iv), $\alpha = \delta + \omega$, and we restrict ourselves to $i < \lambda$ with $\mathrm{cf}\delta = \mathrm{cf} i$. Let $\langle \delta_\sigma : \sigma < \mathrm{cf}\delta \rangle$ be an increasing sequence of ordinals in T with limit δ. We take

$$E = \bigcap_{\sigma < \mathrm{cf}\delta} E^{\delta_\sigma} \cap \bigcap_{n<\omega} E^{\delta+n} .$$

Let $i \in E$, $\mathrm{cf} i > \omega$, we shall prove that $\alpha \in S_i$. $\delta \in S_i$ since $E \subseteq E^\delta_1$, and $D^i_\alpha \cap \delta = D^i_\delta$ since $E \subseteq E^\delta_3$. C^i_α is defined by Case a hence C^i_α is closed in α, and for $n < \omega$ $i \in E \subseteq E^{\delta+n}_2$ hence $\delta + n \in D^i_\alpha \subseteq C^i_\alpha$. Thus C^i_α is unbounded in α. In order to prove that $\alpha \in S_i$ it clearly suffices to show that (1) holds for arbitrarily large $\gamma < \alpha$; we shall now see that it suffices to prove (1) for $\gamma = \delta$. In this case we have for $\delta < \gamma < \alpha$ $C^i_\gamma = C^i_\delta \cup [\delta, \gamma)$. Also, since $i \in E^{\delta+n}_2$ for every $n \in \omega$ we have $D^i_\alpha \supseteq [\delta, \alpha)$ and hence $C^i_\alpha \supseteq [\delta, \alpha]$. Therefore for $\delta < \gamma < \alpha$ we have, since (1) holds for δ,

$$C^i_\delta \cap \gamma = C^i_\alpha \cap \delta \cup [\delta, \gamma) = C^i_\delta \cup [\delta, \gamma) = C^i_\gamma ,$$

and since $i \in E^\gamma_1$ (1) holds for γ.

Now we prove (1) for $\gamma = \delta$. $\delta \in S_i$ since $i \in E^\delta_1$. As we have seen above $\delta \in D^i_\alpha$ and since $\mathrm{cf}\,\delta < \lambda$ and C^i_α is defined by Case a also $\delta \in C^i_\alpha$. For $\sigma < \mathrm{cf}\,\delta$ $D^i_{\delta_\sigma} = \mathrm{D}^i_\alpha \cap \delta_\sigma$, since $i \in E^{\delta_\sigma}_3$, and if $\mathrm{cf}\,\delta_\sigma \neq \mathrm{cf}\, i$ we have also $C^i_{\delta_\sigma} = C^i_\alpha \cap \delta_\sigma$, since both C^i_α and $C^i_{\delta_\sigma}$ are defined by Case a. Let W' be a club subset of $\{\delta_\sigma : \sigma < \mathrm{cf}\,\delta\}$ and let

$$W'' = W' \cap \{\beta < \delta : \mathrm{cf} b(\beta) \neq \mathrm{cf}\, i\} .$$

W'' is clearly an unbounded subset of δ. Thus

$$\bigcup_{\beta \in W' \cap \{\beta < \delta : \mathrm{cf}\, b(\beta) \neq \mathrm{cf}\, i\}} C^i_\beta = \bigcup_{\beta \in W''} C^i_\beta = C^i_\alpha \cap \delta .$$

This immediately implies, by Case b, $C^i_\delta = C^i_\alpha \cap \delta$.

Case (v), $\alpha = \delta + 1$, where $\mathrm{cf}\,\delta = \lambda$. Let $\langle \delta_\sigma : \sigma < \lambda \rangle$ be an increasing continuous sequence converging to δ such that for each $\sigma < \lambda$ if σ is 0 or a successor then $\mathrm{cf}\,\delta_\sigma = \omega$. Since the sequence $\langle D^i_\alpha : i < \lambda \rangle$ is increasing and continuous and for $i < \lambda$ $D^i_\alpha \setminus \{\delta\}$ is a subset of δ of cardinality $< \lambda$ the set $E' = \{i < \lambda : i$ *is a limit ordinal and* $D^i_\alpha \subseteq \delta_i \cup \{\delta\}\}$ is a club subset of λ. Let

$$E = \underset{\sigma < \lambda}{\mathrm{D}} E^{\delta_\sigma} \cap E' ,$$

where D denotes diagonal intersection $\left(\text{i.e., } \underset{j<\lambda}{\mathrm{D}} E_j = \{i < \lambda : (\forall j < i) i \in E_j\}\right)$. We assume now that $i \in E$, $\mathrm{cf}\, i > \omega$ and we shall prove that $\alpha \in S_i$. In this case C^i_α is defined by Case a hence C^i_α is closed in α, i.e., since we know that C^i_α is closed in α as C^i_α is defined by Case a, all we have to show is that for every $\beta \in C^i_\alpha$

$$C^i_\beta = \beta \cap C^i_\alpha \textit{ and if } \beta \textit{ is a limit ordinal then } C^i_\beta \textit{ is unbounded in } \beta . \tag{4}$$

Let $\sigma < i$ and $\mathrm{cf}\sigma \neq \mathrm{cf} i$ then, since $i \in \underset{\varrho < \lambda}{D} E^{\delta_\varrho} \subseteq E^{\delta_\sigma}$ $\delta_\sigma \in S_i$ and $D^i_\alpha \cap \delta_\sigma = D^i_{\delta_\sigma}$, and since both C^i_α and $C^i_{\delta_\sigma}$ are defined by Case a also $C^i_\alpha \cap \delta_\sigma = C^i_{\delta_\sigma}$. Thus (1) holds for

$\gamma=\delta_\sigma$ hence (4) holds for every $\beta<\delta_i=\sup_{\sigma<i}\delta_\sigma$, as i is a limit ordinal. Since $i\in E'$ $D^i_\alpha\subseteq\delta_i\cup\{\delta\}$ hence, since C^i_α is defined by Case a, $C^i_\alpha\subseteq\delta_i+1$. Since we know already that (2) holds for every $\beta<\delta_i$ we still have to prove (2) for $\beta=\delta_i$. Since i is a limit ordinal $\{\delta_\sigma:\sigma<i\}$ is a club subset of δ_i. Let $W_0=\{\delta_\sigma:\sigma<i\wedge \mathrm{cf}\,\delta_0\neq\mathrm{cf}\,i\}$. As in Subcase (iii$_2$) we have (2) and $C^i_{\delta_i}=\bigcup_{\beta\in W_0} C^i_\beta$. Since we know already that for $\beta\in W_0$ $C^i_\beta=C^i_\alpha\cap\beta$ and C^i_β is unbounded in β we have $C^i_{\delta_i}=C^i_\alpha\cap\delta_i$ and $C^i_{\delta_i}$ is unbounded in δ_i. Thus (4) holds also for $\beta=\delta_i$, which ends the proof.

b) By our definition of S_i, if $\alpha\in S_i$ then C^i_α is a closed subset of α, and if α is a limit ordinal it is unbounded in α. We prove now by induction on α that $|C^i_\alpha|<\lambda$. If C^i_α is defined by Case a this obvious. If C^i_α is defined by Case b and α is a limit number then by Subcase (iii$_2$) $C^i_\alpha=\bigcup_{\beta\in W_0} C^i_\beta$, where $W_0\subseteq\alpha$, $|W_0|=|\mathrm{cf}\,\alpha|<\lambda$. By the induction hypothesis $|C^i_\beta|<\lambda$ for $\beta\in W_0$, hence, since λ is regular also $|C^i_\alpha|<\lambda$. If C^i_α is defined by Case b and α is not a limit number then $|C^i_\alpha|<\lambda$ follows immediately from $C^i_\alpha=C^i_{b(\alpha)}\cup[b(\alpha),\alpha)$.

c) follows immediately by the definition of S_i.

3) We shall use (2) and define sets $S_{i,\varepsilon}$ for $i,\varepsilon<\lambda$ such that $\varepsilon=0$ or $\omega\leq\mathrm{cf}\,\varepsilon\leq\theta$ and $\bar{C}^{i,\varepsilon}=\langle C^{i,\varepsilon}_\delta:\delta\in S_{i,\varepsilon}\rangle$. We shall prove that for the set

$$\{\langle i,\varepsilon\rangle : i,\varepsilon<\lambda\wedge(\varepsilon=0\vee\omega\leq\mathrm{cf}\,\varepsilon\leq\theta)\}$$

of indices (a)–(c) of (3) are satisfied. For $\omega\leq\mathrm{cf}\,\varepsilon\leq\theta$ let $D(\varepsilon)$ be a club subset of ε of order type $\mathrm{cf}\,\varepsilon$ such that the members of $D(\varepsilon)$ which are not accumulation points of $D(\varepsilon)$ are successors. For S_i and C^i_α as in (2) we define

$$S_{i,0}=\{\alpha<\lambda^+:\alpha=0 \text{ or } \alpha \text{ is a successor}\},\qquad C^{i,0}_\alpha=\emptyset,$$

and for $\omega\leq\mathrm{cf}\,\varepsilon\leq\theta$

$$S_{i,\varepsilon}=\{\alpha\in S_i:\mathrm{otp}\,C^i_\alpha\in D(\varepsilon)\cup\{\varepsilon\}\},\qquad C^{i,\varepsilon}_\alpha=\{\beta\in C^i_\alpha:\mathrm{otp}\,C^i_\beta\in D(\varepsilon)\}.$$

We shall prove now (a)–(c).

a) Let $\alpha\in T_\theta$, then $\alpha\in S_i$ for some $i<\lambda$, and let $\varepsilon=\mathrm{otp}\,C^i_\alpha$. If $\mathrm{cf}\varepsilon\leq 1$ then α is 0 or a successor and $\alpha\in S_{i,0}$; otherwise clearly $\alpha\in S_{i,\varepsilon}$.

b) For $S_{i,0}$ this is trivial; so we assume now that ε is a limit ordinal. Given $\alpha\in T_\theta$ and $i<\lambda$ let f be the function which counts the members of C^i_α, i.e., f is the increasing function from $\mathrm{otp}\,C^i_\alpha$ onto C^i_α. Since C^i_α is closed f is continuous. For $\beta\in C^i_\alpha$, $C^i_\beta=C^i_\alpha\cap\beta$, hence $f(\mathrm{otp}\,C^i_\beta)=\beta$. By the definition of $C^{i,\varepsilon}_\alpha$ we have $C^{i,\varepsilon}_\alpha=f''(D(\varepsilon)\cap\mathrm{otp}\,C^i_\alpha)$. Since f is continuous and $D(\varepsilon)\cap\mathrm{otp}\,C^i_\alpha$ is closed in $\mathrm{otp}\,C^i_\alpha$, $C^{i,\varepsilon}_\alpha$ is closed in $\sup f''\,\mathrm{otp}\,C^i_\alpha=\sup C^i_\alpha$. Since C^i_α is closed in α also $C^{i,\varepsilon}_\alpha$ is closed in α. Clearly $\mathrm{otp}\,C^{i,\varepsilon}_\alpha\leq\mathrm{otp}\,D(\varepsilon)=\mathrm{cf}\,\varepsilon\leq\theta$.

For $\beta\in C^{i,\varepsilon}_\alpha\subseteq C^i_\alpha$, if β is a limit ordinal then since C^i_β is unbounded in β also $\mathrm{otp}\,C^i_\beta$ is a limit ordinal with the same cofinality and since, by the definition of $C^{i,\varepsilon}_\alpha$, $\mathrm{otp}\,C^i_\beta\in D(\varepsilon)$ we have, by our choice of $D(\varepsilon)$, $\mathrm{cf}\,\mathrm{otp}\,C^i_\beta\leq\theta$, hence $\mathrm{cf}\,\beta\leq\theta$, i.e., $\beta\in T_\theta$.

If α is a limit ordinal then C^i_α is unbounded in δ hence also $\mathrm{otp}\,C^i_\alpha$ is a limit ordinal. Since $\delta\in S_i$, $\mathrm{otp}\,C^i_\alpha\in D(\varepsilon)\cup\{\varepsilon\}$ and the members of $D(\varepsilon)$ which are not accumulation points of $D(\varepsilon)$ are successors $\mathrm{otp}\,C^i_\alpha$ is an accumulation point of $D(\varepsilon)$, i.e., $D(\varepsilon)\cap\mathrm{otp}\,C^i_\alpha$ is unbounded in $\mathrm{otp}\,C^i_\delta$. This is preserved by f hence $C^{i,\varepsilon}_\alpha$ is unbounded in C^i_α which is unbounded in α, hence $C^{i,\varepsilon}_\alpha$ is unbounded in α.

(c) If $\gamma\in S_{i,\varepsilon}$ and $\beta\in C^{i,\varepsilon}_\gamma$ then $\beta<\gamma$ and $\mathrm{otp}\,C^i_\beta\in D(\varepsilon)$. Since $\gamma\in S_{i,\varepsilon}\subseteq S_i$ and $\beta\in C^{i,\varepsilon}_\gamma\subseteq C^i_\gamma$ also $\beta\in S_i$ and $C^i_\beta=C^i_\gamma\cap\beta$. Since $\mathrm{otp}\,C^i_\beta\in D(\varepsilon)$, $\beta\in S_{i,\varepsilon}$. Since $C^i_\beta=C^i_\gamma\cap\beta$ we have, by the definition of $C^{i,\varepsilon}_\beta$ and $C^{i,\varepsilon}_g$, also $C^{i,\varepsilon}_\beta=C^{i,\varepsilon}_\gamma\cap\beta$.

4.5 Lemma. *For every set A of ordinals there is a function $\eta^A = \langle \eta_i^A : i \in A\rangle$ such that for every $i \in A$*

(i) *η_i^A is an ascending sequence of members of $A \cap i$.*

(ii) *If $l <$ length(η_i^A) then $\eta^A_{\eta_i^A(l)} = \eta_i^A \restriction l$.*

(iii) *For every accumulation point δ of A of cofinality ω there is an ascending sequence ξ of length ω of members of A such that $\sup_{n<\omega} \xi_n = \delta$ and for every $n < \omega$, $\xi \restriction n = \eta^A_{\xi(n)}$.*

Proof. By induction on α which is the order type of A.

Case 1. $\alpha = 0$. Take $\eta^\emptyset = \emptyset$.

Case 2. α is a successor. Let $\alpha = \gamma + 1$ and let β be the maximal member of A. Take $\eta^A = \eta^{A \setminus \{\beta\}} \cup \{\langle \beta, \emptyset\rangle\}$.

Case 3. $\alpha = \omega$. Let $\langle \beta_n : n \in \omega\rangle$ be an increasing enumeration of the members of A. Take $\eta^A = \{\langle \beta_n, \langle \beta_0, \ldots, \beta_{n-1}\rangle\rangle : n \in \omega\}$.

Case 4. α is a singular limit ordinal. Let $\langle \beta_i : i < \operatorname{cf}\alpha\rangle$ be an ascending sequence of members of A such that $\beta_0 = 0$ and $\sup_{i<\operatorname{cf}\alpha} \beta_i = \sup A$. Take $\eta^A = \bigcup_{i<\operatorname{cf}\alpha} \eta^{(\beta_i, \beta_{i+1}) \cap A} \cup \eta^{\{\beta_i : i < \operatorname{cf}\alpha\}}$. It is easily seen that η^A satisfies (i)–(iii).

Case 5. α is a regular cardinal $> \omega$. We define for $i < \alpha$, $\alpha_i < \sup A$ by induction on i as follows. $\alpha_0 = 0$, for a limit ordinal i, $\alpha_i = \sup_{j<i} \alpha_j$, and α_{i+1} is chosen so that $\alpha_{i+1} > \alpha_i$ and $\operatorname{otp}[(\alpha_i, \alpha_{i+1}) \cap A] \geqq \omega^2 \cdot \operatorname{otp}[\alpha_i \cap A]$. For $i < \alpha$ let

$$B_i = \{\textit{the } \sigma\textit{-th member of } (\alpha_i, \alpha_{i+1}) \cap A : \sigma \textit{ is a limit ordinal} < \operatorname{otp}[(\alpha_i, \alpha_{i+1}) \cap A]\},$$

and $C_i = ([\alpha_i, \alpha_{i+1}) \cap A) \setminus B_i$. By our choice of $\alpha_{i+1} |B_i| \geqq |\alpha_i \cap A| + \aleph_0$. We define η^A as follows. For $\gamma \in C_i$, $\eta_\gamma^A = \eta_\gamma^{C_i}$, for $\gamma \in B_0$, $\eta_\gamma^A = \emptyset$ and for $\gamma \in B_i$, $i > 0$, we define η_γ^A by induction on i as follows. Let T_i be the set of all finite ascending sequences ϱ of members of $\bigcup_{j<i} B_j$ such that for every $l <$ length(ϱ) $\eta^A_{\varrho(l)} = \varrho \restriction l$, and let F_i be a mapping of B_i onto T_i. We define for $\gamma \in B_i$ $\eta_\gamma^A = F_i(\gamma)$. There is such an F since $\bigcup_{j<i} B_i = \alpha_i \cap A$ and $|B_i| \geqq |\alpha_i \cap A| + \aleph_0$. (i) and (ii) hold by the definition of η^A. To prove (iii) let δ be an accumulation point of A of cofinality ω. If for some $i < \alpha$, $\alpha_i < \delta \leqq \alpha_{i+1}$ then, by our definition of B_i and C_i, δ is also an accumulation point of C_i and hence, by the induction hypothesis concerning η^{C_i}, there is a ϱ as required by (iii). If $\delta = \alpha_i$, where i is a limit ordinal, then let $\langle i_n : n < \omega\rangle$ be an increasing sequence such that $\sup_{n<\omega} i_n = i$. We define by induction on n a sequence $\bar\xi = \langle \xi_n : n \in \omega\rangle$ such that for $n \in \omega$, $\xi_n \in B_i$ as follows. ξ_n is taken to be a member of B_i such that $F_i(\xi_n) = \bar\xi \restriction n$; there is such a ξ_n since $F_i : B_{i_n} \to T_{i_n}$ is onto, $\bar\xi$ is clearly as required in (iii).

Proof of 4.4 (continued). We shall prove that the set $T = \{\delta < \aleph_2 : \operatorname{cf}\delta \leqq \omega\}$ itself has the square property, so $S_i = T$ for all $i < \lambda$. Let $A = \{\delta < \aleph_2 : \operatorname{cf}\delta < \omega\}$ and let η^A be as in 4.5. For $\delta \in T$ if $\operatorname{cf}\delta < \omega$ let $C_\delta^i = \operatorname{Range} \eta_\delta^A$. If $\operatorname{cf}\alpha = \omega$ let $C_\delta^i = \operatorname{Range}\varrho$ where ϱ is an ascending sequence of length ω of members of A with limit α such that for every $n < \omega$, $\varrho \restriction n = \eta^A_{\varrho(n)}$; there is such a ϱ by 4.5. It follows directly that (b) and (c) hold.

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Note added in proof. 1. Note that it is proved in [Sh 420] that for regular cardinals λ and κ satisfying $\lambda>\kappa^+$ there is a stationary $S\subset\{\delta:\delta<\lambda, \mathrm{cf}(\lambda)=\kappa\}$ which belongs to $I[\lambda]$. On the other hand (by a handwritten manuscript) it is consistent with ZFC that no stationary subset of $\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}$ belongs to $I[\aleph_2]$, and it is also consistent that $I[\aleph_2]$ is not generated by the ideal of non stationary subsets of $\aleph_2$ plus one set.

2. We can strengthen REF by demanding: (*) if λ is inaccessible, $\alpha<\lambda^+$, λ is $(\alpha+2)$-Mahlo, $S\subset Mahlo_\alpha(\lambda)-Mahlo_{\alpha+1}(\lambda)$ then $\{\delta\in Mahlo_{\alpha+1}(\lambda)-Mahlo_{\alpha+2}(\lambda): S\cap\delta \text{ stationary}\}$ is stationary, where we define $Mahlo_\alpha(\lambda)$ by induction on α, if $\alpha=0$, $Mahlo_\alpha(\lambda)=\{\mu:\mu<\lambda \textit{ is inaccessible}\}$; if $\alpha=\beta+1$ then $Mahlo_\alpha(\lambda)=\{\mu:\mu<\lambda \text{ is inaccessible, } \mu\cap Mahlo_\beta(\lambda) \text{ is stationary}\}$;
if α is a limit ordinal, let e be a club of α of order type $\mathrm{cf}(\alpha)$ and $Mahlo_\alpha(\lambda)=\{\delta<\lambda:\delta\in Mahlo_\beta(\lambda)$ for every $\beta\in e$, $\mathrm{otp}(e\cap\beta)<\delta\}$.