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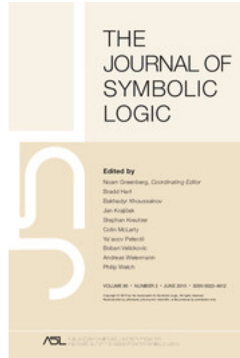

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A DEFINABLE NONSTANDARD MODEL OF THE REALS

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Abstract. We prove, in **ZFC**, the existence of a definable, countably saturated elementary extension of the reals.

§1. **Introduction.** It seems that it has been taken for granted that there is no distinguished, definable nonstandard model of the reals. (This means a countably saturated elementary extension of the reals.) Of course if $V = L$ then there is such an extension (just take the first one in the sense of the canonical well-ordering of L), but we mean the existence provably in **ZFC**. There were good reasons for this: without Choice we cannot prove the existence of *any* elementary extension of the reals containing an infinitely large integer.[1,2] Still there is one.

THEOREM 1.1 (**ZFC**). *There exists a definable, countably saturated extension* ${}^*\mathbb{R}$ *of the reals* $\mathbb{R}$, *elementary in the sense of the language containing a symbol for every finitary relation on* $\mathbb{R}$.

The problem of the existence of a definable proper elementary extension of $\mathbb{R}$ was communicated to one of the authors (Kanovei) by V. A. Uspensky.

A somewhat different, but related problem of *unique existence* of a nonstandard real line ${}^*\mathbb{R}$ has been widely discussed by specialists in nonstandard analysis.[3] Keisler notes in [3, § 11] that, for any cardinal κ, either inaccesible or satisfying $2^\kappa = \kappa^+$, there exists a unique, up to isomorphism, κ-saturated nonstandard real line ${}^*\mathbb{R}$ of cardinality κ, which means that a reasonable level of uniqueness modulo isomorphism can be achieved, say, under GCH. Theorem 1.1 provides a countably saturated nonstandard real line ${}^*\mathbb{R}$, unique in absolute sense by virtue of a concrete definable construction in **ZFC**. A certain modification of this example also admits a reasonable model-theoretic characterization up to isomorphism (see Section 5).

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[1] In fact, from any nonstandard integer we can define a non-principal ultrafilter on $\mathbb{N}$, even a Lebesgue non-measurable set of reals [4], yet it is consistent with **ZF** (even plus Dependent Choices) that there are no such ultrafilters as well as non-measurable subsets of $\mathbb{R}$ [5].

[2] It is worth mentioning that definable nonstandard elementary extensions of $\mathbb{N}$ do exist in **ZF**. For instance, such a model can be obtained in the form of the ultrapower F/U, where F is the set of all arithmetically definable functions $f : \mathbb{N} \to \mathbb{N}$ while U is a non-principal ultrafilter in the algebra A of all arithmetically definable sets $X \subseteq \mathbb{N}$.

[3] "What is needed is an underlying set theory which proves the unique existence of the hyperreal number system [. . .]" (Keisler [3, p. 229]).

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The proof of Theorem 1.1 is a combination of several known arguments. First of all (and this is the key idea), arrange all non-principal ultrafilters over $\mathbb{N}$ in a linear order A, where each ultrafilter appears repetitiously as D_a, $a \in A$. Although A is not a well-ordering, we can apply the iterated ultrapower construction in the sense of [1, 6.5] (which is "a finite support iteration" in the forcing nomenclature), to obtain an ultrafilter D in the algebra of all sets $X \subseteq \mathbb{N}^A$ concentrated on a finite number of axes $\mathbb{N}$. To define a D-ultrapower of $\mathbb{R}$, the set F of all functions $f : \mathbb{N}^A \to \mathbb{R}$, also concentrated on a finite number of axes $\mathbb{N}$, is considered. The ultrapower F/D is **OD**, that is, ordinal-definable, actually, definable by an explicit construction in **ZFC**, hence, we obtain an **OD** proper elementary extension of $\mathbb{R}$. Iterating the D-ultrapower construction ω_1 times in a more ordinary manner, i. e., with direct limits at limit steps, we obtain a definable countably saturated extension.

To make the exposition self-contained and available for a reader with only fragmentary knowledge of ultrapowers, we reproduce several well-known arguments instead of giving references to manuals.

§2. The ultrafilter. As usual, $\mathfrak{c}$ is the cardinality of the continuum.

Ultrafilters on $\mathbb{N}$ hardly admit any definable linear ordering, but maps $a : \mathfrak{c} \to \mathscr{P}(\mathbb{N})$, whose ranges are ultrafilters, readily do. Let A consist of all maps $a : \mathfrak{c} \to \mathscr{P}(\mathbb{N})$ such that the set $D_a = \operatorname{ran} a = \{a(\xi) : \xi < \mathfrak{c}\}$ is an ultrafilter on $\mathbb{N}$. The set A is ordered lexicographically: $a <_{\mathtt{lex}} b$ means that there exists $\xi < \mathfrak{c}$ such that $a \restriction \xi = b \restriction \xi$ and $a(\xi) < b(\xi)$ in the sense of the lexicographical linear order $<$ on $\mathscr{P}(\mathbb{N})$ (in the sense of the identification of any $u \subseteq \mathbb{N}$ with its characteristic function).

For any set u, $\mathbb{N}^u$ denotes the set of all maps $f : u \to \mathbb{N}$.

Suppose that $u \subseteq v \subseteq A$.

If $X \subseteq \mathbb{N}^v$ then put $X \downarrow u = \{x \restriction u : x \in X\}$.

If $Y \subseteq \mathbb{N}^u$ then put $Y \uparrow v = \{x \in \mathbb{N}^v : x \restriction u \in Y\}$.

We say that a set $X \subseteq \mathbb{N}^A$ is *concentrated* on $u \subseteq A$, if $X = (X \downarrow u) \uparrow A$; in other words, this means the following:

$$\forall x, y \in \mathbb{N}^A \, (x \restriction u = y \restriction u \implies (x \in X \iff y \in X)) . \qquad (*)$$

We say that X is a *set of finite support*, if it is concentrated on a finite set $u \subseteq A$. The collection $\mathscr{X}$ of all sets $X \subseteq \mathbb{N}^A$ of finite support is closed under unions, intersections, complements, and differences, i. e., it is an algebra of subsets of $\mathbb{N}^A$. Note that if $(*)$ holds for finite sets $u, v \subseteq A$ then it also holds for $u \cap v$. (If $x \restriction (u \cap v) = y \restriction (u \cap v)$ then consider $z \in \mathbb{N}^A$ such that $z \restriction u = x \restriction u$ and $z \restriction v = y \restriction v$.) It follows that for any $X \in \mathscr{X}$ there is a least finite $u = ||X|| \subseteq A$ satisfying $(*)$.

In the remainder, if U is any subset of $\mathscr{P}(I)$, where I is a given set, then $U i \, \Phi(i)$ (*generalized quantifier*) means that the set $\{i \in I : \Phi(i)\}$ belongs to U.

The following definition realizes the idea of a finite iteration of ultrafilters. Suppose that $u = a_1 < \cdots < a_n \subseteq A$ is a finite set. We put

$$D_u = \{X \subseteq \mathbb{N}^u : D_{a_n} k_n \ldots D_{a_2} k_2 \, D_{a_1} k_1 \, (\langle k_1, k_2, \ldots, k_n \rangle \in X)\} ;$$
$$D = \{X \in \mathscr{X} : X \downarrow ||X|| \in D_{||X||}\}.$$

The following is quite clear.

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PROPOSITION 2.1. (i) *D_u is an ultrafilter on* $\mathbb{N}^u$;
(ii) *if* $u \subseteq v \subseteq A$, v *finite,* $X \subseteq \mathbb{N}^u$, *then* $X \in D_u$ *iff* $X \uparrow v \in D_v$;
(iii) $D \subseteq \mathscr{X}$ *is an ultrafilter in the algebra* $\mathscr{X}$;
(iv) *if* $X \in \mathscr{X}$, $u \subseteq A$ *finite, and* $\|X\| \subseteq u$, *then* $X \in D \iff X \downarrow u \in D_u$.

§3. The ultrapower. To match the nature of the algebra $\mathscr{X}$ of sets $X \subseteq \mathbb{N}^A$ of finite support, we consider the family F of all $f : \mathbb{N}^A \to \mathbb{R}$, *concentrated* on some finite set $u \subseteq A$, in the sense that

$$\forall x, y \in \mathbb{N}^A \ (x \upharpoonright u = y \upharpoonright u \implies f(x) = f(y)). \qquad (\dagger)$$

As above, for any $f \in F$ there exists a least finite $u = \|f\| \subseteq A$ satisfying (†).

Let $\mathscr{R}$ be the set of all finitary relations on $\mathbb{R}$. For any n-ary relation $E \in \mathscr{R}$ and any $f_1, \dots, f_n \in F$, define

$$E^D(f_1, \dots, f_n) \iff D\, x \in \mathbb{N}^A \ E(f_1(x), \dots, f_n(x)).$$

The set $X = \{x \in \mathbb{N}^A : E(f_1(x), \dots, f_n(x))\}$ is obviously concentrated on $u = \|f_1\| \cup \dots \cup \|f_n\|$, hence, it belongs to $\mathscr{X}$, and $\|X\| \subseteq u = \|f_1\| \cup \dots \cup \|f_n\|$.

In particular, $f =^D g$ means that $D\, x \in \mathbb{N}^A \ (f(x) = g(x))$. The following is clear:

PROPOSITION 3.1. *$=^D$ is an equivalence relation on F, and any relation on F of the form E^D is $=^D$-invariant.*

Put $[f]_D = \{g \in F : f =^D g\}$, and ${}^*\mathbb{R} = F/D = \{[f]_D : f \in F\}$. For any n-ary ($n \geq 1$) relation $E \in \mathscr{R}$, let ${}^*E$ be the relation on ${}^*\mathbb{R}$ defined as follows:

$${}^*E([f_1]_D, \dots, [f_n]_D) \ \text{ iff } \ E^D(f_1, \dots, f_n) \ \text{ iff } \ D\, x \in \mathbb{N}^A \ E(f_1(x), \dots, f_n(x)).$$

The independence on the choice of representatives in the classes $[f_i]_D$ follows from Proposition 3.1. Put ${}^*\mathscr{R} = \{{}^*E : E \in \mathscr{R}\}$. Finally, for any $r \in \mathbb{R}$ we put ${}^*r = [c_r]_D$, where $c_r \in F$ satisfies $c_r(x) = r$, $\forall x$.

Let $\mathscr{L}$ be the first-order language containing a symbol E for any relation $E \in \mathscr{R}$. Then $\langle \mathbb{R}; \mathscr{R} \rangle$ and $\langle {}^*\mathbb{R}; {}^*\mathscr{R} \rangle$ are $\mathscr{L}$-structures.

THEOREM 3.2. *The map $r \longmapsto {}^*r$ is an elementary embedding (in the sense of the language $\mathscr{L}$) of the structure $\langle \mathbb{R}; \mathscr{R} \rangle$ into $\langle {}^*\mathbb{R}; {}^*\mathscr{R} \rangle$.*

PROOF. This is a routine modification of the ordinary argument. By $\mathscr{L}[F]$ we denote the extension of $\mathscr{L}$ by functions $f \in F$ used as parameters. It does not have a direct semantics, but if φ is a formula of $\mathscr{L}[F]$ and $x \in \mathbb{N}^A$ then $\varphi[x]$ will denote the formula obtained by the substitution of $f(x)$ for any $f \in F$ which occurs in φ. Thus, $\varphi[x]$ is an $\mathscr{L}$-formula with parameters in $\mathbb{R}$.

LEMMA 3.3 (Łoš). *For any closed $\mathscr{L}[F]$-formula $\varphi(f_1, \dots, f_n)$ (all parameters $f_i \in F$ indicated), we have:*

$$\langle {}^*\mathbb{R}; {}^*\mathscr{R} \rangle \models \varphi([f_1]_D, \dots, [f_n]_D) \iff D\, x\, (\langle \mathbb{R}; \mathscr{R} \rangle \models \varphi(f_1, \dots, f_n)[x]).$$

PROOF. We argue by induction on the logic complexity of φ. For φ an atomic relation $E(f_1, \dots, f_n)$, the result follows by the definition of ${}^*E$. The only notable induction step is $\exists$ in the direction $\Longleftarrow$. Suppose that φ is $\exists y\, \psi(y, f_1, \dots, f_n)$, and

$$D\, x\, (\langle \mathbb{R}; \mathscr{R} \rangle \models \varphi(f_1, \dots, f_n)[x]),$$

that is,

$$D\,x\,(\langle \mathbb{R}\,;\mathscr{R}\rangle \models \exists\, y\; \psi(y,f_1,\dots,f_n)[x])\,.$$

Obviously there exists a function $f \in F$, concentrated on $u = \|f_1\| \cup \dots \cup \|f_n\|$, such that, for any $x \in \mathbb{N}^A$, if there exists a real y satisfying $\langle \mathbb{R}\,;\mathscr{R}\rangle \models \psi(y,f_1,\dots,f_n)[x]$, then $y = f(x)$ also satisfies this formula, i.e., $\langle \mathbb{R}\,;\mathscr{R}\rangle \models \psi(f,f_1,\dots,f_n)[x]$. Formally,

$$\forall\, x \in \mathbb{N}^A\;\big(\exists\, y \in \mathbb{R}\;(\langle \mathbb{R}\,;\mathscr{R}\rangle \models \psi(y,f_1,\dots,f_n)[x]) \implies \\ \langle \mathbb{R}\,;\mathscr{R}\rangle \models \psi(f,f_1,\dots,f_n)[x]\big)\,.$$

This implies $D\,x\,(\langle \mathbb{R}\,;\mathscr{R}\rangle \models \psi(f,f_1,\dots,f_n)[x])$. Then, by the inductive assumption, $\langle {}^*\mathbb{R}\,;{}^*\mathscr{R}\rangle \models \psi([f]_D,[f_1]_D,\dots,[f_n]_D)$, hence $\langle {}^*\mathbb{R}\,;{}^*\mathscr{R}\rangle \models \varphi([f_1]_D,\dots,[f_n]_D)$, as required. ⊣

To accomplish the proof of Theorem 3.2, consider a closed $\mathscr{L}$-formula $\varphi(r_1,\dots,r_n)$ with parameters $r_1,\dots,r_n \in \mathbb{R}$. We have to prove the equivalence

$$\langle \mathbb{R}\,;\mathscr{R}\rangle \models \varphi(r_1,\dots,r_n) \iff \langle {}^*\mathbb{R}\,;{}^*\mathscr{R}\rangle \models \varphi({}^*r_1,\dots,{}^*r_n)\,.$$

Let $f_i = c_{r_i}$, thus, $f_i \in F$ and $f_i(x) = r_i$, $\forall\, x$. Obviously $\varphi(f_1,\dots,f_n)[x]$ coincides with $\varphi(r_1,\dots,r_n)$ for any $x \in \mathbb{N}^A$, hence $\varphi(r_1,\dots,r_n)$ is equivalent to $D\,x\,\varphi(f_1,\dots,f_n)[x]$. On the other hand, by definition, ${}^*r_i = [f_i]_D$. Now the result follows by Lemma 3.3. ⊣

§4. The iteration. Theorem 3.2 yields a definable proper elementary extension $\langle {}^*\mathbb{R}\,;{}^*\mathscr{R}\rangle$ of the structure $\langle \mathbb{R}\,;\mathscr{R}\rangle$. Yet this extension is not countably saturated due to the fact that the ultrapower ${}^*\mathbb{R}$ was defined with maps concentrated on finite sets $u \subseteq A$ only. To fix this problem, we iterate the extension used above ω_1-many times.

Suppose that $\langle M\,;\mathscr{M}\rangle$ is an $\mathscr{L}$-structure, so that $\mathscr{M}$ consists of finitary relations on a set M, and for any $E \in \mathscr{R}$ there is a relation $E^{\mathscr{M}} \in \mathscr{M}$ of the same arity, associated with E. Let F_M be the set of all maps $f : \mathbb{N}^A \to M$ concentrated on finite sets $u \subseteq A$. The structure $F_M/D = \langle {}^*M\,;{}^*\mathscr{M}\rangle$, defined as in Section 3, but with the modified F, will be called *the D-ultrapower* of $\langle M\,;\mathscr{M}\rangle$. Theorem 3.2 remains true in this general setting: the map $x \longmapsto {}^*x$ $(x \in M)$ is an elementary embedding of $\langle M\,;\mathscr{M}\rangle$ in $\langle {}^*M\,;{}^*\mathscr{M}\rangle$.

We define a sequence of $\mathscr{L}$-structures $\langle M_\alpha\,;\mathscr{M}_\alpha\rangle$, $\alpha \le \omega_1$, together with a system of elementary embeddings $e_{\alpha\beta} : \langle M_\alpha\,;\mathscr{M}_\alpha\rangle \to \langle M_\beta\,;\mathscr{M}_\beta\rangle$, $\alpha < \beta \le \omega_1$, so that

(i) $\langle M_0\,;\mathscr{M}_0\rangle = \langle \mathbb{R}\,;\mathscr{R}\rangle$;

(ii) $\langle M_{\alpha+1}\,;\mathscr{M}_{\alpha+1}\rangle$ is the D-ultrapower of $\langle M_\alpha\,;\mathscr{M}_\alpha\rangle$, that is, $\langle M_{\alpha+1}\,;\mathscr{M}_{\alpha+1}\rangle = F_\alpha/D$, where $F_\alpha = F_{M_\alpha}$ consists of all functions $f : \mathbb{N}^A \to M_\alpha$ concentrated on finite sets $u \subseteq A$. In addition, $e_{\alpha,\alpha+1}$ is the associated $*$-embedding $\langle M_\alpha\,;\mathscr{M}_\alpha\rangle \to \langle M_{\alpha+1}\,;\mathscr{M}_{\alpha+1}\rangle$, while $e_{\gamma,\alpha+1} = e_{\alpha,\alpha+1} \circ e_{\gamma\alpha}$ for any $\gamma < \alpha$ (in other words, $e_{\gamma,\alpha+1}(x) = e_{\alpha,\alpha+1}(e_{\gamma\alpha}(x))$ for all $x \in M_\alpha$);

(iii) if $\lambda \le \omega_1$ is a limit ordinal then $\langle M_\lambda\,;\mathscr{M}_\lambda\rangle$ is the direct limit of the structures $\langle M_\alpha\,;\mathscr{M}_\alpha\rangle$, $\alpha < \lambda$. This can be achieved by the following steps:

(a) M_λ is defined as the set of all pairs $\langle\alpha, x\rangle$ such that $x \in M_\alpha$ and $x \notin \operatorname{ran} e_{\gamma\alpha}$ for all $\gamma < \alpha$.
(b) If $E \in \mathscr{R}$ is an n-ary relation symbol then we define an n-ary relation E_λ on M_λ as follows. Suppose that $\boldsymbol{x}_i = \langle\alpha_i, x_i\rangle \in M_\lambda$ for $i = 1, \ldots, n$. Let $\alpha = \sup\{\alpha_1, \ldots, \alpha_n\}$ and $z_i = e_{\alpha_i,\alpha}(x_i)$ for every i, so that $\alpha_i \le \alpha < \lambda$ and $z_i \in M_\alpha$. (Note that if $\alpha_i = \alpha$ then $e_{\alpha_i,\alpha}$ is the identity.) Define $E_\lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ iff $\langle M_\alpha ; \mathscr{M}_\alpha\rangle \models E(z_1, \ldots, z_n)$.
(c) Put $\mathscr{M}_\lambda = \{E_\lambda : E \in \mathscr{R}\}$ – then $\langle M_\lambda ; \mathscr{M}_\lambda\rangle$ is an $\mathscr{L}$-structure.
(d) Define an embedding $e_{\alpha\lambda} : M_\alpha \to M_\lambda$ $(\alpha < \lambda)$ as follows. Consider any $x \in M_\alpha$. If there is a least $\gamma < \alpha$ such that there exists an element $y \in M_\gamma$ with $x = e_{\gamma\alpha}(y)$ then let $e_{\alpha\lambda}(x) = \langle\gamma, y\rangle$. Otherwise put $e_{\alpha\lambda}(x) = \langle\alpha, x\rangle$.

A routine verification of the following is left to the reader.

PROPOSITION 4.1. *If* $\alpha < \beta \le \omega_1$ *then* $e_{\alpha\beta}$ *is an elementary embedding of* $\langle M_\alpha ; \mathscr{M}_\alpha\rangle$ *to* $\langle M_\beta ; \mathscr{M}_\beta\rangle$.

Note that the construction of the sequence of models $\langle M_\alpha ; \mathscr{M}_\alpha\rangle$ is definable, hence, so is the last member $\langle M_{\omega_1} ; \mathscr{M}_{\omega_1}\rangle$ of the sequence. It remains to prove that the $\mathscr{L}$-structure $\langle M_{\omega_1} ; \mathscr{M}_{\omega_1}\rangle$ is countably saturated.

This is also a simple argument. Suppose that, for any k, $\varphi_k(p_k, x)$ is an $\mathscr{L}$-formula with a single parameter $p_k \in M_{\omega_1}$ (the case of many parameters does not essentially differ from the case of one parameter), and there exists an element $x_k \in M_{\omega_1}$ such that $\bigwedge_{i \le k} \varphi_i(p_i, x_k)$ is true in $\langle M_{\omega_1} ; \mathscr{M}_{\omega_1}\rangle$ — in other words, we have $\langle M_{\omega_1} ; \mathscr{M}_{\omega_1}\rangle \models \varphi_i(p_i, x_k)$ whenever $k \ge i$. Fix an ordinal $\gamma < \omega_1$ such that for any k, i there exist (then obviously unique) y_k, $q_i \in M_\gamma$ with $x_k = e_{\gamma\omega_1}(y_k)$ and $p_i = e_{\gamma\omega_1}(q_i)$. Then $\varphi_i(q_i, y_k)$ is true in $\langle M_\gamma ; \mathscr{M}_\gamma\rangle$ whenever $k \ge i$.

Fix $a \in A$ such that D_a is a non-principal ultrafilter, that is, all cofinite subsets of $\mathbb{N}$ belong to D_a. Consider the structure $\langle M_{\gamma+1} ; \mathscr{M}_{\gamma+1}\rangle$ as the D-ultrapower of $\langle M_\gamma ; \mathscr{M}_\gamma\rangle$. The corresponding set F_γ consists of all functions $f : \mathbb{N}^A \to M_\gamma$ concentrated on finite sets $u \subseteq A$. In particular, the map $f(x) = y_k$ whenewer $x(a) = k$ belongs to F_γ. As any set of the form $\{k : k \ge i\}$ belongs to D_a, we have $D_a k\ (\langle M_\gamma ; \mathscr{M}_\gamma\rangle \models \varphi_i(q_i, y_k))$, that is, $D\, x \in \mathbb{N}^A\ (\langle M_\gamma ; \mathscr{M}_\gamma\rangle \models \varphi_i(q_i, f)[x])$, for any $i \in \mathbb{N}$. It follows, by Lemma 3.3, that $\varphi_i({}^*q_i, \boldsymbol{y})$ holds in $\langle M_{\gamma+1} ; \mathscr{M}_{\gamma+1}\rangle$ for any i, where ${}^*q_i = e_{\gamma,\gamma+1}(q_i) \in M_{\gamma+1}$ while $\boldsymbol{y} = [f]_D \in M_{\gamma+1}$ is the D-equivalence class of f in F_γ. Put $\boldsymbol{x} = e_{\gamma+1,\omega_1}(\boldsymbol{y})$; then $\varphi_i(p_i, \boldsymbol{x})$ is true in $\langle M_{\omega_1} ; \mathscr{M}_{\omega_1}\rangle$ for any i because obviously $p_i = e_{\gamma+1,\omega_1}({}^*q_i)$, $\forall\, i$.

§5. **Varia.** By appropriate modifications of the constructions, the following can be achieved:

1. For any given infinite cardinal κ, a κ-saturated elementary extension of $\mathbb{R}$, definable with κ as the only parameter of definition.
2. A *special* elementary extension of $\mathbb{R}$, of as large cardinality as desired. For instance, take, in stage α of the construction considered in Section 4, ultrafilters on $\beth_\alpha$. Then the result will be a definable special structure of cardinality $\beth_{\omega_1}$. Recall that special models of equal cardinality are isomorphic [1, Theorem 5.1.17]. Therefore, such a modification admits an explicit model-theoretical characterization up to isomorphism.

3. A class-size definable elementary extension of $\mathbb{R}$, κ-saturated for any cardinal κ.
4. A class-size definable elementary extension of the whole set universe, κ-saturated for any cardinal κ. (Note that this cannot be strengthened to $\mathtt{Ord}$-saturation, i.e., saturation with respect to all class-size families. For instance, $\mathtt{Ord}^M$-saturated elementary extensions of a minimal transitive model $M \models \mathbf{ZFC}$, definable in M, do not exist — see [2, Theorem 2.8].)

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