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Masas in the Calkin algebra without the Continuum Hypothesis

Saharon Shelah and Juris Steprāns

Abstract. Methods for constructing masas in the Calkin algebra without assuming the Continuum Hypothesis are developed.

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1 Introduction

In [2] Anderson began a study of the extension property of C^*-subalgebras that he continued in [3]. A C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is said to have the extension property if every pure state on $\mathcal{A}$ has a unique extension to a state on $\mathcal{B}$. In his concluding remarks in [4] Anderson expresses the following view: *"In order to make further progress on the extension problem for atomic masas in $\mathcal{B}(\mathcal{H})$ it appears that a clearer understanding of the structure of the masas in the Calkin algebra would be useful."* As a test question, Anderson asks whether every masa in the Calkin algebra which is generated by its projections lifts to a masa in the algebra of all bounded operators on separable Hilbert space. He later provides a negative answer to this question assuming the Continuum Hypothesis [4].

The starting point for the present investigation is a potential argument producing the same negative conclusion as Anderson's in [4] but not relying on the Continuum Hypothesis. It will be seen that this argument runs into a serious difficulty but, even if it did work, the argument would not produce masas that can be tested for the sort of properties Anderson had in mind. For example, another test question he asks in [2] is whether every masa in the Calkin algebra that is generated by its

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projections is permutable[1]. This concept will not be explored further here, but it serves to illustrate that the lack of a classification of masas in the Calkin algebra should not be misunderstood to mean that the structure of these objects can not be further investigated.

Before continuing, some notation will be established. Let $\mathbb{H}$ denote a fixed separable Hilbert space with inner product $\langle x, y\rangle$ and let $\{e_n\}_{n\in\omega}$ be a fixed basis for $\mathbb{H}$. For $x \in \mathbb{H}$ define $\text{supp}(x) = \{i \in \omega \mid \langle x, e_i\rangle \neq 0\}$. Define $S(\mathbb{H}) = \{x \in \mathbb{H} \mid \|x\| = 1\}$. For $X \subseteq \mathbb{N}$ define $\mathbb{H}(X)$ to be the subspace of $\mathbb{H}$ generated by $\{e_i \mid i \in X\}$ and define P_X to be the orthogonal projection onto $\mathbb{H}(X)$. When thinking of $\mathbb{H}$ as ℓ^2 then the e_i will be identified with characteristic functions of singletons. Moreover, P_X can be identified with multiplication by the characteristic function of X, an element of ℓ^∞.

Let $\mathfrak{B}(\mathbb{H})$ denote the algebra of bounded operators on $\mathbb{H}$ and let $\mathfrak{C}(\mathbb{H})$ be the compact operators. For any orthonormal family $\mathcal{X} \subseteq \mathbb{H}$ define $\mathfrak{D}(\mathcal{X})$ to be the subalgebra of $\mathfrak{B}(\mathbb{H})$ diagonal with respect to $\mathcal{X}$; in other words, $T \in \mathfrak{D}(\mathcal{X})$ if and only if there is bounded function $D : \mathcal{X} \to \mathbb{C}$ such that $T(z) = \sum_{x\in\mathcal{X}}\langle x, z\rangle D(x)x$. Let $\mathcal{C}$ be the Calkin algebra $\mathfrak{B}(\mathbb{H})/\mathfrak{C}(\mathbb{H})$ and $\pi : \mathfrak{B}(\mathbb{H}) \to \mathcal{C}$ be the quotient map. A masa in a C^*-algebra is a maximal, abelian, self-adjoint subalgebra.

There are two important facts about masas in $\mathcal{C}$ relevant to the present investigation. The first is the following result due to Johnson and Parrott [10]:

Theorem 1.1. *If $\mathfrak{A} \subseteq \mathfrak{B}(\mathbb{H})$ is a masa then $\pi(\mathfrak{A})$ is a masa in $\mathcal{C}$.*

The second is that the masas in $\mathfrak{B}(\mathbb{H})$ can be characterized as those algebras of the form $L^\infty(\mu)$ acting on $L^2(\mu)$ where μ is a regular probability measure. The following result states this more precisely (see, e.g., [7, pp. 48 and 53]).

Theorem 1.2. *If $\mathfrak{A}$ is a masa in $\mathfrak{B}(\mathbb{H})$ then there is a locally compact subset $X \subseteq \mathbb{R}$ and a regular Borel probability measure μ on X such that $\mathfrak{A}$ is $*$-isomorphic to $L^\infty(\mu)$ acting on $L^2(\mu)$ and, moreover, the $*$-isomorphism is a homeomorphism with respect to the weak operator topology.*

The two key cases are provided when μ is an atomic measure on a countable set or when it is an atomless measure on a set without isolated points. Since $L^\infty(\mu)$ is not separable, this result on its own does not guarantee that the number of masas in $\mathfrak{B}(\mathbb{H})$ is not greater than $2^{\aleph_0}$. However, given a Borel probability measure μ on

[1] A C^*-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$ is permutable in $\mathcal{B}$ if there are mutually orthogonal projections $\{p_n\}_{n\in\omega}$ in $\mathcal{A}$ and unitaries $u_n \in \mathcal{B}$ such that $u_n\mathcal{A}u_n^* = \mathcal{A}$ and $u_n p_0 u_n^* = p_n$ for each n. A masa is permutable if it is permutable in the Calkin algebra.

X, the $*$-isomorphism from $L^\infty(\mu)$ to $\mathfrak{B}(\mathbb{H})$ is determined by it values on $\mathcal{C}(X)$ which is separable (see [7, p. 49]). This yields the following.

Corollary 1.3. *There are no more than* $2^{\aleph_0}$ *masas in* $\mathfrak{B}(\mathbb{H})$.

To begin, it is worth noting why Anderson is concerned in [4] only with masas in $\mathcal{C}$ that are generated by their projections. In [2] he points out he has obtained in [3] a characterization of when a C^* subalgebra has the extension property; in other words, each pure state has a unique extension to the larger algebra. However, the characterization only applies if the subalgebra is generated by its projections.

Let S be the uni-lateral shift defined by $S(e_n) = e_{n+1}$. Then $\pi(S)$ is normal – in other words, $\pi(S)\pi(S)^* = \pi(S)^*\pi(S)$ – and so there is some masa in $\mathcal{C}$ containing it. On the other hand, $SS^* \neq S^*S$ so this algebra is not the quotient of any masa in $\mathfrak{B}(\mathbb{H})$ containing S – indeed, by an argument using the Fredholm index, it can be shown that there is no normal $T \in \mathfrak{B}(\mathbb{H})$ such that $\pi(T) = S$ and hence there is no abelian algebra containing $\pi(S)$ which is the quotient of a masa in $\mathfrak{B}(\mathbb{H})$. However, by virtue of containing $\pi(S)$ this algebra will not be generated by its projections. To see this, one can rely on a proposition found in [8] showing that every projection in $\mathcal{C}$ is the image of a projection under π. Hence, if $\pi(S)$ were approximated by commuting projections, then tail ends of S would also be approximated by projections that were as close to commuting as desired. This would contradict that $SS^* \neq S^*S$. Nevertheless, Anderson showed in [4], that, assuming the Continuum Hypothesis, there is a masa in $\mathcal{C}$ which is generated by its projections but is not the quotient of any masa in $\mathfrak{B}(\mathbb{H})$.

There is a simple strategy for constructing a masa in $\mathcal{C}$ which is not the quotient of any masa in $\mathfrak{B}(\mathbb{H})$. (See [8] for a description of an argument of Akemann and Weaver [1] using this.) Let $\mathcal{A}$ be a family of almost disjoint subsets of $\mathbb{N}$ of size $2^{\aleph_0}$. For each $A \in \mathcal{A}$ choose a pair of projections Q_A^0 and Q_A^1 such that

- $P_A Q_A^0 P_A = Q_A^0$ and $P_A Q_A^1 P_A = Q_A^1$,
- $Q_A^0 Q_A^1 - Q_A^1 Q_A^0$ is not compact.

Note that for any function $F : \mathcal{A} \to \{0, 1\}$ the family

$$\mathcal{A}(F) = \left\{\pi\left(Q_A^{F(A)}\right) \mid A \in \mathcal{A}\right\}$$

is a commuting family of projections. Let $\mathcal{A}_c(F)$ be the C^*-algebra generated by $\mathcal{A}(F)$. It is immediate that the algebras $\mathcal{A}_c(F)$ and $\mathcal{A}_c(F')$ are distinct if F and F' are. If it can be shown that each $\mathcal{A}_c(F)$ can be extended to a masa this will complete the proof since it follows that there are $2^{2^{\aleph_0}}$ distinct masas contradicting Corollary 1.3 if each of them lifts to a masa in $\mathfrak{B}(\mathbb{H})$.

Each $\mathcal{A}_c(F)$ is clearly abelian, self-adjoint and generated by projections, but extending to a maximal such family poses a problem. To see this, the following simple fact will be useful:

If $\mathfrak{A}$ is an abelian C^*-subalgebra of $\mathcal{C}$ generated by projections then $\mathfrak{A}$ is maximal abelian if and only if for every self-adjoint operator $S \in \mathcal{C} \setminus \mathfrak{A}$ there is some projection $Q \in \mathfrak{A}$ such that $SQ \neq QS$.

In order to establish this, let $T \in \mathcal{C} \setminus \mathfrak{A}$ be arbitrary and suppose that T commutes with every element of $\mathfrak{A}$. Let $T = A + iB$ where A and B are self-adjoint. Each element of $\mathfrak{A}$ must be normal since $\mathfrak{A}$ is an abelian C^* algebra. Therefore the Fuglede Lemma [9] implies that T^* also commutes with every element of $\mathfrak{A}$ and, hence, so do both A and B. Hence both A and B belong to $\mathfrak{A}$ and, therefore, so does T.

Therefore, in order to extend $\mathcal{A}_c(F)$ to a masa it suffices to add to $\mathcal{A}_c(F)$ all self-adjoint $T \in \mathcal{C}$ which commute with each member of $\mathcal{A}_c(F)$. The catch is that in order for the extended family to be generated by projections it is necessary to also add to $\mathcal{A}_c(F)$ some projections generating T. Before presenting the following example showing that this might not be possible the following definition is needed.

Definition 1.4. The cardinal $\mathfrak{p}$ is defined to be the least cardinal of a family of $\mathcal{F}$ of subsets of $\mathbb{N}$ such that

- $A \cap B \in \mathcal{F}$ for any A and B in $\mathcal{F}$,
- there is no infinite $X \subseteq \mathbb{N}$ such that $X \setminus A$ is finite for all $A \in \mathcal{F}$.

Theorem 1.5 (Bell [6]). *If $\mathfrak{c} = \mathfrak{p}$ then Martin's Axiom for σ-centred partial orders holds; in other words, given*

- *a partial order $\mathbb{P}$ such that $\mathbb{P} = \bigcup_{n=1}^{\infty} \mathbb{P}_n$ where each $\mathbb{P}_n$ is centred (that is, any finite subfamily has a lower bound), and*
- *a family $\{D_\xi\}_{\xi \in \kappa}$ such that $\kappa < \mathfrak{c}$ and each D_ξ is a dense subset of $\mathbb{P}$ (that is, for all $p \in \mathbb{P}$ there is $d \in D_\xi$ below p),*

there is a centred $G \subseteq \mathbb{P}$ such that $G \cap D_\xi \neq \emptyset$ for each ξ.

Example 1.6. If $\mathfrak{p} = \mathfrak{c}$ then there is a self-adjoint operator $A \in \mathfrak{B}(\mathbb{H})$ and an abelian subalgebra $\mathfrak{C}$ of $\mathcal{C}$ such that

- $\pi(A)$ commutes with each member of $\mathfrak{C}$,
- $\mathfrak{C}$ is generated by projections,
- if $\|A - \sum_{i=1}^{k} d_i P_i + K\| < 1/4$ where K is compact and each P_i is a projection and each $d_i \in \mathbb{R}$ then there is some $j \leq k$ such that $\pi(P_j)$ does not commute with $\mathfrak{C}$,

- if Q is a projection not commuting with A modulo a compact operator then there is $C \in \mathfrak{C}$ such that $\pi(Q)$ and C do not commute.

In other words, if $\mathfrak{C}$ is extended to this masa then this will not be generated by projections.

Proof. Recall that the *essential spectrum* of an operator T is denoted $\sigma_{\text{ess}}(T)$ and is defined to be the set of all complex numbers in the closure of the spectrum which are either non-isolated or have infinite multiplicity. For an operator $f \in \ell^\infty$ the essential spectrum satisfies the equation

$$\sigma_{\text{ess}}(f) = \bigcap_{n\in\omega} \overline{\{f(j) \mid j \geq n\}}$$

and so this describes the essential spectrum for self-adjoint operators.

Let A be any self-adjoint operator such that $\sigma_{\text{ess}}(A) = [0, 1]$. Without loss of generality A can be identified with some $\alpha \in \ell^\infty$ such that $A(e_n) = \alpha(n)e_n$. In this case the range of α is dense in $[0, 1]$. The family $\mathfrak{C} = \{\pi(C_\xi)\}_{\xi\in\mathfrak{c}}$ will be constructed by an induction of length $\mathfrak{c}$. Let

$$\left\{\sum_{i=1}^{k_\xi} \gamma_\xi^i P_\xi^i\right\}_{\xi\in\mathfrak{c} \text{ and } \xi \text{ even}}$$

enumerate all finite linear combinations of pairwise orthogonal projections in $\mathcal{C}$ such that

$$\left\| A - \sum_{i=1}^{k_\xi} \gamma_\xi^i P_\xi^i - K \right\| < \frac{1}{4} \tag{1}$$

for some compact K. Let $\{Q_\xi\}_{\xi\in\mathfrak{c} \text{ and } \xi \text{ odd}}$ enumerate all projections that do not commute with A modulo a compact set. The required induction hypothesis is that for each ξ the following hold:

- C_ξ is a projection commuting modulo a compact operator with A,
- $\sigma_{\text{ess}}(AC_\xi)$ is nowhere dense,
- $\pi(C_\eta)$ commutes with $\pi(C_\xi)$ for each $\eta \in \xi$,
- if ξ is even then there is some $j \leq k_\xi$ and $\zeta \leq \xi$ such that $\pi(P_\xi^j)$ does not commute with $\pi(C_\zeta)$,
- if ξ is odd then there is some $\zeta \leq \xi$ such that $\pi(Q_\xi)$ does not commute with $\pi(C_\zeta)$.

It should be clear that this suffices. The one point the reader may question is the utility of $\sigma_{\text{ess}}(AC_\xi)$ since if AC_ξ is not normal the usual spectral theory does not apply. However, one should observe that if Q is a projection commuting with A modulo a compact set then, letting $\{w_n\}_{n\in\omega}$ be an orthonormal basis for the range of Q, it follows that for any $\epsilon > 0$ for all but finitely many n there is a subset S_n^ϵ of the support of w_n such that

- the projection of w_n onto the subspace spanned by $\{e_j\}_{j\in S_n^\epsilon}$ has norm greater than $1-\epsilon$,
- $|\alpha(i)-\alpha(j)| < \epsilon$ for any i and j in S_n^ϵ.

The essential spectrum of AQ can then be calculated to be

$$\bigcap_{\epsilon>0}\bigcap_{k\in\omega}\overline{\{\alpha(e_j)\mid j\in S_k^\epsilon\}}$$

and this will be used in the following argument.

In order to perform the induction, assume first that ξ is even. It may as well be assumed that each P_ξ^I commutes modulo a compact operator with A and each C_ζ for $\zeta\in\xi$. The following claim applies only the case that ξ is even.

Claim 1. *There is some $t\in[\frac{1}{3},1]$ and distinct integers $\bar{i}$ and $\bar{j}$ such that for each $\epsilon>0$ both of the sets $(t-\epsilon,t+\epsilon)\cap\sigma_{\text{ess}}(AP_\xi^{\bar{i}})$ and $(t-\epsilon,t+\epsilon)\cap\sigma_{\text{ess}}(AP_\xi^{\bar{j}})$ have non-empty interior.*

To see this, note first that

$$\bigcup_{i=1}^{k_\xi}\sigma_{\text{ess}}(AP_\xi^i)\supseteq\left[\tfrac{1}{3},1\right]$$

because otherwise there is an open $U\subseteq[\frac{1}{3},1]$ such that the corresponding spectral projection P – namely, the characteristic function of $\alpha^{-1}U$ – has the property that $\|AP\|>\frac{1}{3}$ and P_ξ^iP is compact for each $i\le k_\xi$. This contradicts that (1) holds for some compact K. Similar reasoning shows that the diameter of each $\sigma_{\text{ess}}(AP_\xi^i)$ is less than $1/2$. Hence there are at least two distinct i and j such that $\sigma_{\text{ess}}(AP_\xi^i)\cap[\frac{1}{3},1]$ has non-empty interior. If the claim fails then an elementary compactness argument yields a contradiction to the connectedness of $[\frac{1}{3},1]$. In particular, if the claim fails then it is possible to choose an $i(t)\le k_\xi$ and a neighborhood V_t of each $t\in[\frac{1}{3},1]$ such that the interior of $\sigma_{\text{ess}}(AP_\xi^{i(t)})$ is dense in V_t. Note that if $V_t\cap V_s\ne\emptyset$ and $i(s)\ne i(t)$ for some s and t then the claim is proved. Hence, each of the sets

$$U_j=\bigcup\{V_s\mid i(s)=j\}$$

is clopen in $[\frac{1}{3},1]$. Since there are at least two distinct i and j such that

$\sigma_{\mathrm{ess}}(AP^i_\xi) \cap [\frac{1}{3}, 1]$ has non-empty interior, this yields two non-empty U_j and a contradiction.

Let t, $\bar{i}$ and $\bar{j}$ be as in Claim 1 and define the partial order $\mathbb{P}$ to consist of conditions

$$p = \left(B^p, \{z^p_i\}_{i=1}^{k^p}\right)$$

where

(1) B^p is a finite subset of ξ,

(2) each z^p_i has finite support and rational range – as an element of ℓ^2 – and $\|z^p_i\| = 1$,

(3) $2/3 < \|P^J_\xi(z^p_n)\| < 3/4$ for each $n \leq k^p$,

(4) $\|A(z^p_n) - tz^p_n\| < 2^{-n}$ for each $n \leq k^p$.

Define $p \leq q$ if and only if

- $B^p \supseteq B^q$,
- $k^p \geq k^q$,
- $z^p_n = z^q_n$ for $n \leq k^q$,
- if $\eta \in B^q$ and $n > k^q$ then $\|C_\eta(z^p_n)\| < 2^{-n}$.

It is immediate from Condition 2 in its definition that $\mathbb{P}$ is σ-centred and that the set of all p such that $\eta \in B^p$ is dense for any $\eta \in \xi$. It will be shown that the set of all p such that $k^p \geq m$ is also dense for each m. Given this, let $G \subseteq \mathbb{P}$ meet all these dense sets and define C_ξ to be the orthogonal projection onto the subspace spanned by $\{z^p_n \mid p \in G \text{ and } n \leq k^p\}$. It follows from Condition 3 that C_ξ and P^J_ξ do not commute modulo a compact set. From Condition 4 it follows that $\sigma_{\mathrm{ess}}(AC_\xi) = \{t\}$. The definition of $\leq$ guarantees that C_ξ commutes with each C_η modulo a compact if $\eta \in \xi$.

In order to establish the required density assertion it suffices to show that for any p there is some $q \leq p$ such that $k^q = k^p + 1$. To this end let p be given. From the Weyl–von Neumann–Berg Theorem it follows that it is possible to find an orthonormal basis $\{z_n\}_{n\in\omega}$ such that there are compact operators K, $\{K_\zeta\}_{\zeta\in B^p}$ and $\{K^i\}_{i=1}^{k_\xi}$ as well as functions $\{X_\zeta\}_{\zeta\in B^p}$ and $\{X^i\}_{i=1}^{k_\xi}$ from ω to $\{0, 1\}$ and a function $\psi : \omega \to (0, 1)$ such that

- $A(z_n) = \psi(n)z_n + K(z_n)$,
- $C_\zeta(z_n) = X_\zeta(n)z_n + K_\zeta(z_n)$ for $\zeta \in B_p$,
- $P^i_\xi(z_n) = X^i(n)z_n + K^i(z_n)$ for $1 \leq i \leq k_\xi$.

Let $\epsilon = 2^{-k^p-1}$ and let M be such that $\|K(z_n)\| < \epsilon$, $\|K^i(z_n)\| < \epsilon$ and $\|K_\zeta(z_n)\| < \epsilon$ for $n > M$ and $i \leq k_\xi$ and $\zeta \in B^p$.

Using the fact that $\sigma_{\text{ess}}(AP_{\bar{i}}) \cap (t-\epsilon, t+\epsilon)$ and $\sigma_{\text{ess}}(AP_{\bar{j}}) \cap (t-\epsilon, t+\epsilon)$ both have non-empty interior together with the fact that $\sigma_{\text{ess}}(AC_\zeta)$ is nowhere dense for each $\zeta \in B^p$ it follows that there are integers i' and j' greater than M such that

- $X^{\bar{i}}(i') = 1$ and $X^{\bar{j}}(j') = 1$,
- $X^{\bar{i}}(j') = 0$ and $X^{\bar{j}}(i') = 0$,
- $X_\zeta(i') = X_\zeta(j') = 0$ for $\zeta \in B^p$,
- $|\psi(i') - t| < \epsilon$ and $|\psi(j') - t| < \epsilon$.

Let $z = (z_{i'} + z_{j'})/\sqrt{2}$ and note that $2/3 < \|P_\xi^{\bar{j}}(z)\| < 3/4$. Also $\|A(z) - tz\| < 2\epsilon$. Moreover, $\|C_\zeta(z)\| = \|K_\zeta(z)\| < 2\epsilon$ for $\zeta \in B^p$. Hence, it is easy to define z' to satisfy Condition 2 and yet be so close to z that all these inequalities are still satisfied with z' in place of z. Let

$$p = \left(B^p, \{z_i^p\}_{i=1}^{k^p+1}\right)$$

so that $z^{k^p+1} = z'$.

Now assume that ξ is odd. It may be assumed that $A = \psi \in \ell^\infty$ and acts on ℓ^2 by multiplication where $\psi : \mathbb{N} \to (0, 1)$ has its range dense in $(0, 1)$. If $Q_\xi C_\zeta - C_\zeta Q_\xi$ is not compact for some ζ then there is nothing to do, so assume that $Q_\xi C_\zeta - C_\zeta Q_\xi$ is compact for all $\zeta \in \xi$. Since $\psi Q_\xi - Q_\xi \psi$ is not compact a pigeonhole argument produces $a < b$, $\delta > 0$ and a sequence $\{\varphi_n\}_{n=1}^\infty$ such that

- $\varphi_n \in \ell^2$ for each n,
- $\|P_{\varphi_n^{-1}(b,1)} \varphi_n\| > \delta$,
- $\|P_{\varphi_n^{-1}(0,a)} \varphi_n\| > \delta$,
- the supports of the φ_n are pairwise disjoint finite sets,
- $\lim_{n\to\infty} Q_\xi(\varphi_n) = 1$,

where projections are being identified with characteristic functions thought of as elements of ℓ^∞. Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}$ and note that for each $\zeta \in \xi$ there is $\Delta_\zeta \in \{0, 1\}$ such that $\lim_{\mathcal{F}} C_\zeta(\varphi_n) = \Delta_\zeta$. Using that $\mathfrak{p} = \mathfrak{c}$ there is a set $X \subseteq \mathbb{N}$ such that $\lim_{n\in X} C_\zeta(\varphi_n) = \Delta_\zeta$ for each $\zeta \in \xi$.

Now let $\{I_n\}_{n\in\omega}$ enumerate all intervals with rational endpoints in $(0, 1)$. Then construct J_n, A_n^i, k_n and Z_n such that

- J_n is a rational interval such that $J_n \subseteq I_n$,
- $Z_n \subseteq \mathbb{N}$ is infinite and $Z_{n+1} \subseteq Z_n$,

- $Z_0 = X$,
- $A_n^i \subseteq \varphi_n^{-1}(0, a)$ are finite sets such that $A_{n+1}^i \subseteq A_n^i$ for each $i \in Z_{n+1}$,
- $\|\varphi_i P_{A_n^i}\| > \delta(1 - \sum_{j=0}^n 2^{-j-2})$ for $i \in Z_n$,
- $k_n \in Z_n$,
- the image of A_n^j under φ_j is disjoint from J_n for every $j \in Z_n$ including k_j,
- if $j \neq i$ then $A_j^{k_j}$ is disjoint from the support of φ_{k_i}.

If this can be done, then letting C_ξ be the orthogonal projection onto the subspace of ℓ^2 spanned by

$$\{\varphi_{k_j} P_{A_j^{k_j}}\}_{j \in \omega}$$

it is immediate that $C_\xi Q_\xi - Q_\xi C_\xi$ is not compact. Also immediate is the fact that $\sigma_{\text{ess}}(C_\xi)$ is disjoint from $\bigcup_{j \in \omega} J_j$ and hence is nowhere dense. Since $Z_0 = X$ it follows that C_ξ commutes with each C_ζ for $\zeta \in \xi$.

To carry out the induction suppose that Z_n is given. Choosing k_n to satisfy the last clause is then easy. Let $I_{n+1} = \bigcup_{i=1}^L I^i$ be a partition of I_{n+1} into intervals of length less than $\delta 2^{-n-3}$. For one of these intervals it must be that there is an infinite set Z_{n+1} and some $k \leq L$ such that

$$\left\|\varphi_i P_{A_n^i} P_{\varphi^{-1}\{I_{n+1} \setminus I^k\}}\right\| > \delta\left(1 - \sum_{j=0}^{n+1} 2^{-j-2}\right)$$

for every $i \in Z_{n+1}$. Then let $A_{n+1}^i = A_n^i \cap \varphi^{-1}(I_{n+1} \setminus I^k)$ for $i \in Z_{n+1}$ and let $J_{n+1} = I^k$. □

Question 1. Is the hypothesis $\mathfrak{p} = \mathfrak{c}$ necessary for the example? In other words, is it consistent with set theory that every abelian, self adjoint subalgebra of the Calkin algebra generated by projections can be extended to a masa generated by projections?

It must be noted at this point that a successful implementation of the strategy outlined before the preceding example would not signal any progress towards gaining the *"clearer understanding of the structure of the masas in the Calkin algebra"* seen to be useful by Anderson. Since the Axiom of Choice is invoked to obtain maximality, very little can be said about the structure of the masa produced this way. For example, one can still ask whether there is a masa in the Calkin algebra which is not locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$. The following definition makes this precise:

Definition 1.7. A masa $\mathcal{A} \subseteq \mathcal{C}$ will be said to be *locally the quotient of a masa in* $\mathfrak{B}(\mathbb{H})$ if there is a non-trivial projection $p \in \mathcal{A}$, a Hilbert subspace $\mathbb{H}_0 \subseteq \mathbb{H}$ and a masa $\mathfrak{A} \subseteq \mathfrak{B}(\mathbb{H}_0)$ such that $\pi(\mathfrak{A}) = \{pap \mid a \in \mathcal{A}\}$.

It is not possible to say whether or not a masa produced by invoking Zorn's Lemma is locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$. The next section will provide a method for constructing masas with some control over their structure. In particular, it will be immediate that these masas are locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$.

Before continuing, it is worth remarking that Anderson's construction of masa in [4] can easily be modified to produce a masa which is not locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$. The idea Anderson exploited is that for any algebra of the form $L^\infty(\mu)$ there is an operator which commutes with only countably many projections in $L^\infty(\mu)$. With the assistance of the Continuum Hypothesis, Anderson constructed a masa in $\mathcal{C}$ which contains an uncountable set of projections which he called *almost central*, namely every element of $\mathcal{C}$ commutes with all but countably many of them. Clearly this can not be the quotient of any $L^\infty(\mu)$. It is routine to modify the transfinite construction used by Anderson to ensure that not only does his masa contain a central family $\mathcal{P}$, but also that for any non-trivial projection $q \in \mathcal{C}$ the family $\{qpq \mid p \in \mathcal{P}\}$ is central. This guarantees that Anderson's masa is not locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$.

Question 2. Is there a masa in the Calkin algebra which is not locally the quotient of a masa in $\mathfrak{B}(\mathbb{H})$?

Question 3. Is there a masa in the Calkin algebra such that for any non-trivial projection $q \in \mathcal{C}$ the family $\{qpq \mid p \in \mathcal{P}\}$ is central?

Of course, it has already been noted that a positive answer to Question 3 will yield a positive answer to Question 2 and, assuming the Continuum Hypothesis, the answer to both is positive. It would also be interesting to know whether the two questions are, in fact, equivalent.

2 Masas from almost disjoint families

The goal of this section is to provide a construction, using only a very weak set theoretic hypothesis, of a masa generated by its projections which is not equal to $\pi(\mathfrak{A})$ for any masa $\mathfrak{A} \subseteq \mathbb{H}$. The important point to emphasize here is that the masa to be constructed is generated by its projections; without this restriction such examples can be produced by simple arguments as discussed in §1. The set theoretic hypothesis used is described in Definition 2.1. The description of the hypothesis as *very weak* is justified by the fact that it is not known to be consistently false and, in fact, the consistency of the failure of a similar hypothesis – to be discussed later – is a long standing open problem. Hence, it is possible that the hypothesis used in the construction is not an extra set theoretic hypothesis at all.

Definition 2.1. If $\mathcal{I}$ is an ideal on $\mathbb{N}$ then define an ideal $\mathcal{I}_*$ on the family of finite, non-empty subsets of $\mathbb{N}$ to be generated by $\{h \in [\mathbb{N}]^{<\aleph_0} \mid h \cap X \neq \emptyset\}$ for $X \in \mathcal{I}$. For any ideal $\mathcal{I}$ the positive sets with respect to $\mathcal{I}$ are defined to be $\mathcal{P}(\cup\mathcal{I}) \setminus \mathcal{I}$ and are denoted by $\mathcal{I}^+$. An almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ will be said to be *strongly separable* if and only if for every $\mathcal{H} \in \mathcal{I}(\mathcal{A})^+_*$ there are $2^{\aleph_0}$ sets $X \in \mathcal{A}$ such that for any $n \in \mathbb{N}$ there is $h \in \mathcal{H}$ such that $h \subseteq X \setminus n$.

As a guide to understanding this definition, it is worth noting that $\mathcal{H} \in \mathcal{I}(\mathcal{A})^+_*$ if and only if for every finite subfamily $\{A_i\}_{i=1}^k \subseteq \mathcal{A}$ there is $h \in \mathcal{H}$ such that $h \cap \bigcup_{i=1}^k A_i = \emptyset$.

Definition 2.2. For any ideal $\mathcal{I}$ on $\mathbb{N}$ define a subset $Z \subseteq S(\mathbb{H})$ to be $\mathcal{I}$-large if and only if

$$\{\text{supp}(z) \mid z \in Z\} \in \mathcal{I}^+_*$$

and define Z to be $\mathcal{I}$-small otherwise.

This definition should be understood to apply only to subsets Z consisting of vectors in the unit sphere with finite support since if supp(z) is not finite then it is not even in the domain of $\mathcal{I}_*$. So, for example, $\mathcal{I}$-large sets will never be large subsets of $S(\mathbb{H})$ in the sense of category or measure.

Lemma 2.3. *If $W \subseteq S(\mathbb{H})$ is $\mathcal{I}$-large then for all $\epsilon > 0$ and all bounded operators Φ there is some k such that $\{w \in W \mid \|P_{[k,m)}\Phi(w)\| < \epsilon\}$ is $\mathcal{I}$-large for all $m \geq k$.*

Theorem 2.4. *If there is a strongly separable almost disjoint family then there is a masa in the Calkin algebra generated by its projections which does not lift to a masa of $\mathfrak{B}(\mathbb{H})$.*

Proof. If $\mathcal{A}$ is strongly separable then it is possible to choose for each $A \in \mathcal{A}$ a sequence $\{w_n^A\}_{n=0}^\infty \subseteq S(\mathbb{H})$ such that

(i) supp(w_n^A) is a finite subset of A for each n,

(ii) $\max(\text{supp}(w_n^A)) < \min(\text{supp}(w_{n+1}^A))$ for each n,

(iii) for each $W \subseteq S(\mathbb{H})$ which is $\mathcal{I}(\mathcal{A})$-large there is some $A \in \mathcal{A}$ such that $w_n^A \in W$ for each n.

This is accomplished by a transfinite induction of length $\mathfrak{c}$. Let $\{W_\xi\}_{\xi\in\mathfrak{c}}$ list all the sets $W \subseteq S(\mathbb{H})$ which are $\mathcal{I}(\mathcal{A})$-large. For each ξ there are $2^{\aleph_0}$ sets $A \in \mathcal{A}$ such that $\{w \in W_\xi \mid \text{supp}(w) \subseteq A\}$ is infinite. Hence, at each stage ξ of the transfinite induction it is possible to find $A \in \mathcal{A}$ for which the w_n^A have not yet been defined and such that $\{w \in W_\xi \mid \text{supp}(w) \subseteq A\}$ is infinite. Let $\{w_n^A\}_{n\in\omega}$ enumerate an appropriate infinite subset of this set.

Now, for each $A \in \mathcal{A}$ extend $\{w_n^A\}_{n\in\omega}$ to an orthonormal basis $\mathcal{B}_{A,n}$ on the space $\ell^2(\text{supp}(w_n^A))$ and let $\mathcal{B}_A = \bigcup_{n=0}^{\infty} \mathcal{B}_n^A$. Let $\mathfrak{A}_0$ be the subalgebra of $B(\mathbb{H})$ generated by

$$\bigcup_{A\in\mathcal{A}} \mathfrak{D}(\mathcal{B}_A).$$

Let $\mathfrak{A}_1$ be the quotient $\mathfrak{A}_0/\mathfrak{C}(\mathbb{H})$ and let $\mathfrak{A}$ be the quotient norm closure of $\mathfrak{A}_1$. The almost disjointness of $\mathcal{A}$ guarantees that $\mathfrak{A}$ is abelian as well as self adjoint.

To see that $\mathfrak{A}$ is maximal abelian let $\Phi \in \mathfrak{B}(\mathbb{H})$ and suppose that Φ commutes modulo a compact operator with every member of $\mathfrak{A}_0$ but $\pi(\Phi)$ is not in $\mathfrak{A}$. Let $\Phi_m = P_m^\perp \Phi P_m^\perp$ for any $m \in \mathbb{N}$. For any $m \in \mathbb{N}$ and $\epsilon > 0$ let

$$B(\epsilon, m) = \left\{x \in S(\mathbb{H}) \mid \|P_{\text{supp}(x)}^\perp \Phi_m(x)\| > \epsilon\right\}$$

and let

$$Z(\epsilon, m) = \left\{x \in S(\mathbb{H}) \mid \|\Phi_m(x) - \langle \Phi_m(x), x\rangle x\| > \epsilon\right\}$$

noting that $\langle \Phi_m(x), x\rangle x$ is simply the projection of $\Phi_m(x)$ onto the subspace generated by x. The next two lemmas establish that for sufficiently large m both $B(\epsilon, m)$ and $Z(\epsilon, m)$ are $\mathcal{I}$-small.

Claim 2. *For every $\epsilon > 0$ there is some $k = k(\epsilon)$ such that $B(\epsilon, k)$ is $\mathcal{I}(\mathcal{A})$-small.*

Proof. For any $w \in S(\mathbb{H})$ define

$$m(w) = \min(\text{supp}(w))$$

and suppose that the conclusion fails for ϵ. In other words, $B(\epsilon, k)$ is $\mathcal{I}(\mathcal{A})$-large for every k. Then define

$$B = \left\{w \in S(\mathbb{H}) \mid \left\| P_{[m(w),\infty)}\Phi(w) - \langle P_{[m(w),\infty)}\Phi(w), w\rangle w\right\| > \frac{\epsilon}{2\sqrt{2}}\right\}$$

and observe that B is $\mathcal{I}(\mathcal{A})$-large. To see this, let $A \in \mathcal{A}$ and let k be the least element of $\mathbb{N}$ not belonging to A and choose $j > k$ so large that $\|P_{[j,\infty)}\Phi(e_k)\| < \epsilon/2$. Since $B(\epsilon, j)$ is $\mathcal{J}(\mathcal{A})$-large it is possible to find $w \in B(\epsilon, j)$ such that

$$\text{supp}(w) \cap (A \cup j) = \emptyset$$

and then it follows that $\bar{w} = (w + e_k)/\sqrt{2} \in B$. To see this, note that $m(\bar{w}) = k$ and

$$\begin{aligned}
&\left\| P_{[k,\infty)}\Phi(\bar{w}) - \langle P_{[k,\infty)}\Phi(\bar{w}), \bar{w}\rangle \bar{w} \right\| \\
&\qquad \geq \left\| P^{\perp}_{\mathrm{supp}(\bar{w})} P_{[k,\infty)}\Phi(w + e_k)\right\| / \sqrt{2} \\
&\qquad \geq \left\| P_{[k+1,\infty)\setminus \mathrm{supp}(w)}\Phi(w + e_k)\right\| / \sqrt{2} \\
&\qquad \geq \left\| P_{[j,\infty)\setminus \mathrm{supp}(w)}(\Phi(w) + \Phi(e_k))\right\| / \sqrt{2} \\
&\qquad \geq \left(\left\| P_{[j,\infty)\setminus \mathrm{supp}(w)}\Phi(w)\right\| - \left\| P_{[j,\infty)}\Phi(e_k)\right\|\right) / \sqrt{2} \\
&\qquad = \left(\left\| P^{\perp}_{\mathrm{supp}(w)}\Phi_j(w)\right\| - \epsilon/2\right) / \sqrt{2} \\
&\qquad > \frac{\epsilon}{2\sqrt{2}}.
\end{aligned}$$

Then there is $A \in \mathcal{A}$ such that $w^A_n \in B$ for each n. Choose an increasing sequence $\{y_n\}_{n=0}^{\infty}$ from $\mathbb{N}$ such that if $m_n = m(w^A_{y_n})$ and $w_n = w^A_{y_n}$ then

$$\left\| P_{[m_n,m_{n+1})}\Phi(w_n) - \langle P_{[m_n,m_{n+1})}\Phi(w_n), w_n\rangle w_n \right\| > \epsilon/2\sqrt{2}$$

and $\mathrm{supp}(w_n) \subseteq [m_n, m_{n+1})$ for each n. Then let Q be the projection onto the space spanned by $\{w_n\}_{n=0}^{\infty}$. It then follows that

$$\begin{aligned}
&\left\| P_{[m_n,m_{n+1})}(\Phi Q - Q\Phi)(w_n)\right\| \\
&\qquad = \left\| P_{[m_n,m_{n+1})}\Phi(w_n) - P_{[m_n,m_{n+1})}\left(\textstyle\sum_{j=0}^{\infty}\langle \Phi(w_n), w_j\rangle w_j\right)\right\| \\
&\qquad = \left\| P_{[m_n,m_{n+1})}\Phi(w_n) - \langle \Phi(w_n), w_n\rangle w_n\right\| \\
&\qquad = \left\| P_{[m_n,m_{n+1})}\Phi(w_n) - \langle \Phi(w_n), P_{[m_n,m_{n+1})}w_n\rangle w_n\right\| \\
&\qquad = \left\| P_{[m_n,m_{n+1})}\Phi(w_n) - \langle P_{[m_n,m_{n+1})}\Phi(w_n), w_n\rangle w_n\right\| \\
&\qquad > \frac{\epsilon}{2}
\end{aligned}$$

contradicting the compactness of $Q\Phi - \Phi Q$. □

Corollary 2.5. *For every* $\epsilon > 0$ *and* $k \geq k(\epsilon)$ *the set* $B(\epsilon, k)$ *is* $\mathcal{I}(\mathcal{A})$*-small.*

Proof. If $B(\epsilon, k)$ is $\mathcal{I}(\mathcal{A})$-large then note that

$$B = B(\epsilon, k(\epsilon)) \cup \{w \in S(\mathbb{H}) \mid P_k(w) \neq 0\}$$

is $\mathcal{I}(\mathcal{A})$-small. Hence it is possible to choose $w \in B(\epsilon, k) \setminus B$ and for such a w the equality

$$P^{\perp}_{\mathrm{supp}(w)}\Phi_k(w) = P^{\perp}_{\mathrm{supp}(w)}\Phi(w) = P^{\perp}_{\mathrm{supp}(w)}\Phi_{k(\epsilon)}(w)$$

holds, contradicting that $w \in B(\epsilon, k) \setminus B(\epsilon, k(\epsilon))$. □

Claim 3. *For every $\epsilon > 0$ and $k \geq k(\epsilon/2)$ the set $Z(\epsilon, k)$ is $\mathcal{I}(\mathcal{A})$-small.*

Proof. If the conclusion fails let $\epsilon > 0$ witness this. From Corollary 2.5 it follows that $W = Z(\epsilon, k) \setminus B(\epsilon/2, k)$ is also $\mathcal{I}(\mathcal{A})$-large. Choose $A \in \mathcal{A}$ such that $w_n^A \in W$ for each n and let Q be the projection onto the space spanned by $\{w_n^A\}_{n=0}^{\infty}$. Note that it follows that if $n > k$ then

$$\begin{aligned}
&\left\| P_{\mathrm{supp}(w_n^A)}(Q\Phi - \Phi Q)(w_n^A)\right\| \\
&\quad = \left\| \langle \Phi(w_n^A), w_n^A\rangle w_n^A - P_{\mathrm{supp}(w_n^A)}\Phi(w_n^A)\right\| \\
&\quad = \left\| \langle \Phi_k(w_n^A), w_n^A\rangle w_n^A - \Phi_k(w_n^A) + \Phi_k(w_n^A) - P_{\mathrm{supp}(w_n^A)}\Phi_k(w_n^A)\right\| \\
&\quad \geq \left\| \langle \Phi_k(w_n^A), w_n^A\rangle w_n^A - \Phi_k(w_n^A)\right\| - \left\| P^{\perp}_{\mathrm{supp}(w_n^A)}\Phi_k(w_n^A)\right\| \\
&\quad > \frac{\epsilon}{2}
\end{aligned}$$

and this contradicts the compactness of $Q\Phi - \Phi Q$. □

Let $B_\delta(z)$ denote the open disk of radius δ and centre z in the complex plane. Let $\rho > \|\Phi\|$ and for $\lambda \in B_\rho(0)$ and $\delta > 0$ and $m \in \mathbb{N}$ let

$$X(\epsilon, \delta, m, \lambda) = \left\{ w \in S(\mathbb{H}) \setminus Z(\epsilon, m) \mid |\langle w, \Phi_m(w)\rangle - \lambda| < \delta \right\}$$

and let

$$C(\epsilon, m) = \left\{ \lambda \in \overline{B_\rho(0)} \mid (\forall \delta > 0) X(\epsilon, \delta, m, \lambda) \text{ is } \mathcal{I}(\mathcal{A})\text{-large} \right\}$$

and note that $C(\epsilon, m)$ is closed. It will be shown that $C(\epsilon, m)$ does not depend in a significant way on ϵ or m. After this has been established it is worth first remarking that $C(\epsilon, m)$ can be considered an essential spectrum of Φ_m with respect to $\mathcal{A}$.

To begin, note that if $k \geq k(\epsilon/2)$ then $X(\epsilon, \delta, k, \lambda)$ is $\mathcal{I}(\mathcal{A})$-large if and only if

$$X(\delta, \lambda) = \left\{ w \in S(\mathbb{H}) \mid |\langle w, \Phi(w)\rangle - \lambda| < \delta \right\}$$

is $\mathcal{I}(\mathcal{A})$-large. To see this, suppose that $X(\delta, \lambda)$ is $\mathcal{I}(\mathcal{A})$-large and let $A \in \mathcal{I}(\mathcal{A})$. Of course, $A \cup k$ is also in $\mathcal{I}(\mathcal{A})$ and so, by Claim 3, it is possible to find $w \in$

$S(\mathbb{H}) \setminus Z(\epsilon, k)$ with finite support such that $|\langle w, \Phi(w)\rangle - \lambda| < \delta$ and $\mathrm{supp}(w) \cap (A \cup k) = \emptyset$. It follows from the last equality that $\langle w, \Phi_k(w)\rangle = \langle w, \Phi(w)\rangle$ and hence $|\langle w, \Phi_k(w)\rangle - \lambda| < \delta$. Therefore $X(\epsilon, \delta, k, \lambda)$ is $\mathcal{I}(\mathcal{A})$-large. The other implication follows from a similar argument.

Hence, if $k \geq k(\epsilon/2)$ then

$$C(\epsilon, m) = \left\{\lambda \in \overline{B_\rho(0)} \mid (\forall \delta > 0) X(\delta, \lambda) \text{ is } \mathcal{I}(\mathcal{A})\text{-large}\right\}$$

and note that $C(\epsilon, m)$ depends on neither ϵ nor m. Now defining

$$C = \left\{\lambda \in \overline{B_\rho(0)} \mid (\forall \delta > 0) X(\delta, \lambda) \text{ is } \mathcal{I}(\mathcal{A})\text{-large}\right\}$$

it follows that $C(\epsilon, m) = C$ for any $k \geq k(\epsilon/2)$.

It will be shown that C is not empty. If not, it is possible to choose for each $\lambda \in \overline{B_\rho(0)}$ a $\delta_\lambda > 0$ such that $X(\delta_\lambda, \lambda)$ is $\mathcal{I}(\mathcal{A})$-small. Compactness then yields a finite $F \subseteq \overline{B_\rho(0)}$ such that $\bigcup_{\lambda \in F} B_{\delta_\lambda}(\lambda) \supseteq \overline{B_\rho(0)}$. It is possible to choose $w \in S(\mathbb{H}) \setminus \bigcup_{\lambda \in F} X(\epsilon, \delta_\lambda, k, \lambda)$ with finite support. Note that $\langle w, \Phi(w)\rangle$ must have some value in $\overline{B_\rho(0)}$ and, hence there is some $\lambda \in F$ such that $w \in B_{\delta_\lambda}(\lambda)$. This contradicts that $w \notin X(\delta_\lambda, \lambda)$.

Now, with the aim of obtaining a contradiction, suppose that λ and λ' are distinct elements of C. Let $\theta > 0$ be such that $9\theta < |\lambda - \lambda'|/\sqrt{2}$ and let M be so large that $M \geq k(\theta/2)$ and $M \geq k(\theta)$. Define W to be the set of all $w \in S(\mathbb{H})$ such that

- $w = x_0 + x_1$ such that $\mathrm{supp}(x_0) \cap \mathrm{supp}(x_1) = \emptyset$,
- $\|x_0\| = \|x_1\| = 1/\sqrt{2}$,
- $\|\langle x_0, \Phi_M(x_0)\rangle x_0 - \Phi_M(x_0)\| \leq \theta$,
- $\|\langle x_1, \Phi_M(x_1)\rangle x_1 - \Phi_M(x_1)\| \leq \theta$,
- $\|\langle x_0, \Phi_M(x_0)\rangle - \lambda\| < \theta$,
- $\|\langle x_1, \Phi_M(x_1)\rangle - \lambda'\| < \theta$,
- $\|P^\perp_{\mathrm{supp}(w)} \Phi_M(w)\| \leq \theta$.

It will first be shown that W is $\mathcal{I}(\mathcal{A})$-large. To see this, let $A \in \mathcal{I}(\mathcal{A})$. Then choose $x_0 \in X(\theta, \theta, M, \lambda) \setminus B(\theta, M)$ such that $\mathrm{supp}(x_0) \cap A = \emptyset$ and then choose $x_1 \in X(\theta, \theta, M, \lambda') \setminus B(\theta, M)$ such that $\mathrm{supp}(x_1) \cap (A \cup \mathrm{supp}(x_0)) = \emptyset$. Letting $w = (x_0 + x_1)/\sqrt{2}$ it is immediate that $w \in W$.

Now let $A \in \mathcal{A}$ be such that each w_n^A belongs to W and let Q be the projection onto the space spanned by $\{w_n^A\}_{n>M}$ and let $w_n^A = x_n^0 + x_n^1$ be the decomposition witnessing that w_n^A belongs to W. Before proceeding it is worth noting that for any n the following inequality holds:

$$\left\| Q(\lambda x_n^0 + \lambda' x_n^1) - (\lambda x_n^0 + \lambda' x_n^1) \right\| \geq \frac{\lambda - \lambda'}{\sqrt{2}} \tag{2}$$

as can be seen by noting that $Q(\lambda x_n^0 + \lambda' x_n^1) = \lambda x_n^0 + \lambda x_n^1 - Q((\lambda' - \lambda)x_n^1)$ and $\|Q(x_n^1)\| = 1/\sqrt{2}$.

Then for any n greater than M the projections $P_{\mathrm{supp}(w_n^A)}$ and Q commute and $\Phi_M(w_n^A) = P_M^{\perp}\Phi(w_n^A)$. Hence from the inequalities defining W and using inequality (2) at the end it follows that

$$\begin{aligned}
&\left\| P_{\mathrm{supp}(w_n^A)}(Q\Phi - \Phi Q)(w_n^A) \right\| \\
&\quad = \left\| Q(P_{\mathrm{supp}(w_n^A)}(\Phi_M(w_n^A)) - P_{\mathrm{supp}(w_n^A)}(\Phi_M(x_n^0) + \Phi_M(x_n^1)) \right\| \\
&\quad = \left\| Q(P_{\mathrm{supp}(w_n^A)}(\Phi_M(x_n^0) + \Phi_M(x_n^1))) \right. \\
&\qquad\qquad \left. - P_{\mathrm{supp}(w_n^A)}(\Phi_M(x_n^0) + \Phi_M(x_n^1)) \right\| \\
&\quad \geq \left\| Q(P_{\mathrm{supp}(w_n^A)}(\langle \Phi_M(x_n^0), x_n^0\rangle x_n^0 + \langle \Phi_M(x_n^1), x_n^1\rangle x_n^1)) \right. \\
&\qquad\qquad \left. - P_{\mathrm{supp}(w_n^A)}(\langle \Phi_M(x_n^0), x_n^0\rangle x_n^0 + \langle \Phi_M(x_n^1), x_n^1\rangle x_n^1) \right\| - 4\theta \\
&\quad \geq \left\| Q(\langle \Phi_M(x_n^0), x_n^0\rangle x_n^0 + \langle \Phi_M(x_n^1), x_n^1\rangle x_n^1) \right. \\
&\qquad\qquad \left. - \langle \Phi_M(x_n^0), x_n^0\rangle x_n^0 - \langle \Phi_M(x_n^1), x_n^1\rangle x_n^1 \right\| - 4\theta \\
&\quad \geq \left\| Q(\lambda x_n^0 + \lambda' x_n^1) - (\lambda x_n^0 + \lambda' x_n^1) \right\| - 8\theta \\
&\quad \geq \frac{|\lambda - \lambda'|}{\sqrt{2}} - 8\theta \geq \theta
\end{aligned}$$

and this contradicts the compactness of $Q\Phi - \Phi Q$.

It will be shown that for any $\epsilon > 0$ there is $k \in \mathbb{N}$ and some $T \in \mathfrak{A}_0$ such that $\|\Phi_k - T + \lambda I\| < \epsilon$. To see this, let $k = k(\epsilon/4)$. For each $\sigma \in \overline{B_{\|\Phi\|+1}(0)} \setminus \overline{B_{\epsilon/3}(\lambda)}$ choose $\delta_\sigma > 0$ such that $X(\epsilon/2, \delta_\sigma, k, \sigma)$ is $\mathcal{I}(\mathcal{A})$-small. Let $E \subseteq \overline{B_{\|\Phi\|+1}(0)} \setminus B_{\epsilon/3}$ be a finite set such that $\bigcup_{\sigma \in E} B_{\delta_\sigma}(\sigma) \supset \overline{B_{\|\Phi\|+1}(0)} \setminus B_{\epsilon/3}$. For each $\sigma \in E$ choose $A_\sigma \in \mathcal{I}(\mathcal{A})$ witnessing that $X(\epsilon/2, \delta_\sigma, k, \sigma)$ is $\mathcal{I}(\mathcal{A})$-small. Let $A \in \mathcal{I}(\mathcal{A})$ witness that $Z(\epsilon/2, k) \in \mathcal{I}(\mathcal{A})$ and let $C = A \cup \bigcup_{\sigma \in E} A_\sigma$.

Then for any $x \in \mathbb{H}$ such that $\mathrm{supp}(x) \cap C = \emptyset$ it must be that $\|\langle x, \Phi_k\rangle x - \Phi_k(x)\| \leq \epsilon/2$ since $x \cap A = \emptyset$. Moreover, $|\langle x, \Phi_k(x)\rangle - \sigma| \geq \delta_\sigma$ for each $\sigma \in E$ and, since $|\langle x, \Phi_k(x)\rangle| \leq \|\Phi\|$, it follows that $|\langle x, \Phi_k(x)\rangle - \lambda| < \epsilon/2$. In other words,

$$\left\| \Phi_k P_C^{\perp}(x) - \lambda P_C^{\perp} x \right\| < \epsilon \tag{3}$$

for all x. Since $P_C \in \mathfrak{D}(\mathcal{B}_C) \subseteq \mathfrak{A}_0$ it follows that $P_C \Phi_k P_C$ commutes with all members of $\mathfrak{D}(\mathcal{B}_C)$ modulo a compact set. By Theorem 1.1 it follows that $\pi(P_C \Phi_k P_C) \in \mathfrak{A}_0$. Furthermore, $P_C \Phi_k P_C^\perp$ and $P_C^\perp \Phi_k P_C$ are both compact because P_C commutes with Φ_k modulo a compact operator. Therefore

$$\begin{aligned}(\Phi_k - \lambda I)(x) &= (P_C + P_C^\perp)(\Phi_k - \lambda I)(P_C + P_C^\perp)(x)\\ &= P_C^\perp \Phi_k P_C^\perp(x) + P_C \Phi_k P_C(x) + \lambda P_C(x) + \lambda P_C^\perp(x)\end{aligned}$$

and combining this with inequality (3) yields that

$$\begin{aligned}&\|(\Phi_k - \lambda I)(x) - (P_C \Phi_k P_C(x) + \lambda P_C(x))\|\\ &\qquad \le \|P_C^\perp \Phi_k P_C^\perp(x) + \lambda P_C^\perp(x)\| < \epsilon.\end{aligned}$$

Since $P_C \Phi_k P_C + \lambda P_C \in \mathfrak{A}_0$ this yields the desired conclusion.

The final thing to show is that $\mathfrak{A}$ is not the lifting of a masa on Hilbert space. To begin notice that if $\mathfrak{A}$ is the lifting of a masa on Hilbert space then Theorem 1.2 yields that masa is of the form $L^\infty(\mu)$ acting on $L^2(\mu)$ where μ is a probability measure on a locally compact subset of $\mathbb{R}$. The first thing to notice is that μ must be atomic because otherwise there is some set X such that $\mu(X) > 0$ and the restriction of μ to X is atomless. However, it is immediate from the definition of $\mathfrak{A}$ that for every projection $P \in \mathfrak{A}$ there is a projection $Q \in \mathfrak{A}$ such that $QP = Q$ and $Q\mathfrak{A}$ is isomorphic to $\pi(\ell^\infty) = \mathcal{C}(\beta\mathbb{N} \setminus \mathbb{N})$. However, if $L^\infty(\mu)$ is atomless then $\pi(L^\infty(\mu)) = \mathcal{C}(X)$ where X is the Stone space of the measure algebra and it is known that $\mathcal{C}(X)$ and $\mathcal{C}(\beta\mathbb{N} \setminus \mathbb{N})$ are not isomorphic because X and $\beta\mathbb{N} \setminus \mathbb{N}$ are not homeomorphic.

Now suppose that $\mathcal{X} = \{x_n\}_{n\in\omega}$ is an orthonormal basis for $\mathbb{H}$ such that $\mathcal{A} = \pi(\mathfrak{D}(\mathcal{X}))$. Let R_s denote the projection onto the subspace spanned by $\{x_n \mid n \in s\}$. Now let $V(\epsilon) = \{w \in S(\mathbb{H}) \mid (\exists j)\|w - x_j\| < \epsilon\}$. It will be shown that $V(\epsilon)$ is not $\mathcal{I}(\mathcal{A})$-large for any $\epsilon > 0$. If this fails for some $\epsilon > 0$ then the complement of $V(\epsilon)$ is $\mathcal{I}(\mathcal{A})$-large. It follows that there is some $X \in \mathcal{A}$ such that $X \cap A$ is finite and $\|w_n^X - x_j\| > \epsilon$ for each n and j.

It is then possible to choose for all but finitely many n a finite set $s(n)$ such that

$$\epsilon/2 < \|R_{s(n)} w_n^X\| < 1 - \epsilon/2. \tag{4}$$

In order to see this, let $w_n^X = \sum_{i=0}^\infty \gamma_i x_i$. Then first observe that $|\gamma_j| < 1 - \epsilon/2$ for each j because otherwise

$$1 \ge \left(\sum_{i \ne j} |\gamma_i|^2\right) + |\gamma_j|^2 \ge \left(\sum_{i \ne j} |\gamma_i|^2\right) + (1 - \epsilon/2)^2$$

and hence $\sum_{i \neq j} |\gamma_i|^2 < \epsilon - (\epsilon/2)^2$ and therefore

$$\|w_n^X - x_j\|^2 = \sum_{i \neq j} |\gamma_i|^2 + (1 - \gamma_j)^2 < \epsilon$$

contradicting that w_n^X does not belong to $V(\epsilon)$. If there is some j such that $|\gamma_j| > \epsilon/2$ then let $s(n) = \{j\}$. Otherwise let $s(n)$ be a minimal set such that $\sum_{i \in s(n)} |\gamma_i|^2 > \epsilon/2$. Then, letting $j \in s(n)$ be arbitrary

$$\sum_{i \in s(n)} |\gamma_i|^2 = \sum_{i \in s(n) \setminus \{j\}} |\gamma_i|^2 + |\gamma_j|^2 < \epsilon/2 + (\epsilon/2)^2 < 1 - \epsilon/2$$

yielding inequality (4).

There is then an infinite $Y \subseteq \mathbb{N}$ such that

- $\sum_{m \neq n} \sum_{i \in s(m)} |\langle x_i, w_n^X \rangle|^2 < \epsilon^3$,
- $\sum_{m \neq n} |\langle w_n^X, w_m^X \rangle|^2 < \epsilon^3$

for each $n \in Y$. Let Ψ be the projection onto the space spanned by $\{w_n^X\}_{n \in Y}$ and let $C = \bigcup_{n \in Y} s(n)$. It will be shown that R_C and Ψ do not commute modulo compact.

To see this, note that

$$R_C \Psi(w_n^X) = R_C(w_n^X) + w_1 = R_{s(n)}(w_n^X) + w_2 + w_1$$

where $\|w_1\| < \epsilon^3$ and $\|w_2\| < \epsilon^3$. Hence $\|R_C \Psi(w_n^X)\| \geq \|R_{s(n)}(w_n^X)\| - 2\epsilon^3$. On the other hand,

$$\Psi R_C(w_n^X) = \Psi(R_{s(n)}(w_n^X) + w_2) = \langle w_n^X, R_{s(n)}(w_n^X) \rangle w_n^X + w_3 + \Psi(w_2)$$

where $\|w_3\| < \epsilon^3$. So

$$\|\Psi R_C(w_n^X)\| < |\langle w_n^X, R_{s(n)}(w_n^X) \rangle| + 2\epsilon^3 = \|R_{s(n)}(w_n^X)\|^2 + 2\epsilon^3$$

and hence

$$\begin{aligned} \|R_C \Psi(w_n^X) - \Psi R_C(w_n^X)\| &\geq \|R_C \Psi(w_n^X)\| - \|\Psi R_C(w_n^X)\| \\ &\geq \|R_{s(n)}(w_n^X)\| - 2\epsilon^3 - (\|R_{s(n)}(w_n^X)\|^2 + 2\epsilon^3). \end{aligned}$$

Since $\epsilon/2 < \|R_{s(n)}(w_n^X)\| < 1 - \epsilon/2$, it follows that

$$\|R_C \Psi(w_n^X) - \Psi R_C(w_n^X)\| > \epsilon/2 - \epsilon^2/4 - 4\epsilon^3 > 0$$

provided $0 < \epsilon < 1/4$. This contradicts the compactness of the commutator.

Therefore

$$\left\{\frac{z+z'}{\sqrt{2}} \mid \operatorname{supp}(z) \cap \operatorname{supp}(z') = \emptyset \text{ and } z \in V(\epsilon) \text{ and } z' \in V(\epsilon)\right\}$$

is also $\mathcal{I}(\mathcal{A})$-large. Let $A \in \mathcal{A}$ be such that for each n there are z_n and z'_n in $V(\epsilon)$ such that

- $w_n^A = (z_n + z'_n)/\sqrt{2}$,
- $\operatorname{supp}(z_n) \cap \operatorname{supp}(z'_n) = \emptyset$.

Let $m(n)$ and $m'(n)$ be the integers satisfying that $\|z_n - x_{m(n)}\| < \epsilon$ and $\|z' - e_{m'(n)}\| < \epsilon$. Let Ψ be the projection onto the space spanned by all the w_n^A and let $M = \{m(n)\}_{n\in\mathbb{N}}$. It is routine to check that R_M and Ψ do not commute modulo a compact provided that ϵ has been chosen sufficiently small. Since $\Psi \in \mathfrak{A}$ this shows that $\pi(\mathfrak{D}(\mathcal{X}) \neq \mathfrak{A}$. □

It is not known whether it is possible to construct an almost disjoint family which is strongly separable without assuming some extra set theoretic axioms. However, there are many models of set theory known in which there is such a family. Certainly assuming that $\mathfrak{a} = \mathfrak{c}$ suffices. This provides an easy way to see that some axiom like the Continuum Hypothesis is necessary for Anderson's construction in [4] if it is to rely on almost central families.

Corollary 2.6. *It is consistent that there are no almost central families yet there is a masa in $\mathcal{C}$ which is generated by its projections and not the quotient of any masa in $\mathfrak{B}(\mathbb{H})$.*

Proof. Assume that the union of $\aleph_1$ meagre sets never covers the reals and that there is an almost disjoint family which is strongly separable. (Martin's Axiom and $2^{\aleph_0} > \aleph_1$ will do.) There is then a masa in $\mathcal{C}$ which is not the quotient of one in $\mathfrak{B}(\mathbb{H})$. Moreover, the space S of all orthonormal sequences is a closed subspace of $\mathbb{H}^\omega$ with the product topology. Given any projection $P \in \mathfrak{B}(\mathbb{H})$ such that $\pi(P) \neq 0$ and $k \in \omega$, the set $D(k,P)$ of all $\sigma \in S$ such that there is some $n > k$ such that the distance from $\sigma(n)$ to the range of P is greater than $1/2$ is co-meagre. Hence, given any family $\mathcal{P}$ of projections of cardinality $\aleph_1$ the intersection

$$\bigcap_{k\in\omega} \bigcap_{P\in\mathcal{P}} D(k,p)$$

is not empty. Moreover any orthonormal sequence in this intersection yields a projection not commuting modulo a compact with any projection in $\mathcal{P}$. Hence there are no almost central families in this model. □

It must be remarked that in the absence of the Continuum Hypothesis it is natural to define a family of commuting projections $\mathcal{P}$ in a C^*-algebra $\mathfrak{A}$ to be κ-*almost central* if and only if $|\mathcal{P}| > \kappa$ and for every $a \in \mathfrak{A}$ the cardinality of $\{p \in \mathcal{P} \mid pa \neq ap\}$ is less than κ. So, with this terminology, an almost central family is an $\aleph_1$-almost central family. It can be shown that Anderson's argument from [4] extends to show that if $\mathfrak{A} \subseteq \mathfrak{B}(\mathbb{H})$ is a masa then $\pi(\mathfrak{A})$ does not contain a κ-almost central family for any uncountable κ; in particular, it does not contain a $2^{\aleph_0}$-almost central family. Moreover, the argument of Corollary 2.6 does not rule out the existence of $2^{\aleph_0}$-almost central families. However, Anderson's construction of a masa with an almost central family will only yield a masa containing an $\aleph_1$-almost central family because it relies on Voiculescu's theorem which does not generalize beyond the countable. This points to relevance of Corollary 2.6 and begs the following question.

Question 4. Can one prove, without assuming the continuum hypothesis or any similar axiom, that there is a masa in the Calkin algebra containing a κ-almost central family for some uncountable κ?

Question 5. Is there an almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ such that for every $\mathcal{H} \in \mathcal{I}(\mathcal{A})^+_*$ there is at least one $X \in \mathcal{A}$ such that for any $n \in \mathbb{N}$ there is $h \in \mathcal{H}$ such that $h \subseteq X \setminus n$?

Question 6. If there is an almost disjoint family such as in Question 5 does it follow that there is a strongly separable one?

Petr Simon has constructed [5] an almost disjoint family $\mathcal{A}$ such that if $X \in \mathcal{I}(\mathcal{A} \cup \mathcal{I}(\mathcal{A})^\perp)^+$ then there are $2^{\aleph_0}$ sets $A \in \mathcal{A}$ such that $A \cap X$ is infinite. Recently, Saharon Shelah has constructed a maximal almost disjoint such family assuming only that $2^{\aleph_0} < \aleph_\omega$.

Question 7. Does there exist an almost disjoint family $\mathcal{A}$ such that if $X \in \mathcal{I}(\mathcal{A} \cup \mathcal{I}(\mathcal{A})^\perp)^+_*$ then there are $2^{\aleph_0}$ sets $A \in \mathcal{A}$ such that $A \cap X$ is infinite? Does the existence of such a family allow the construction of a masa in $\mathcal{C}$ which is not the quotient of one in $\mathfrak{B}(\mathbb{H})$?

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Author information

Saharon Shelah, Department of Mathematics, Rutgers University, Hill Center,
Piscataway, New Jersey, 08854-8019, USA.
E-mail: shelah@math.rutgers.edu

Juris Steprāns, Department of Mathematics, York University, 4700 Keele Street,
Toronto, Ontario, M3J 1P3, Canada.
E-mail: steprans@yorku.ca