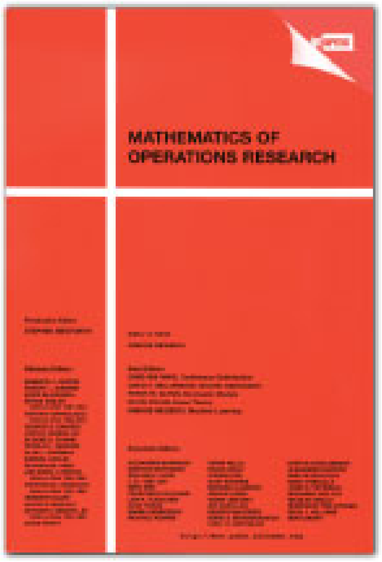

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Zvi Drezner, Saharon Shelah,

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ON THE COMPLEXITY OF THE ELZINGA-HEARN ALGORITHM FOR THE 1-CENTER PROBLEM*

ZVI DREZNER[†] AND SAHARON SHELAH[‡]

We show that the complexity of the Elzinga-Hearn algorithm for the 1-center (single facility minimax) problem is $\Omega(n^2)$. We also show that the algorithm reaches a solution within any given accuracy in $O(n)$ time.

Introduction. The 1-center problem, sometimes called the single facility minimax problem, is to find the best location on the plane for a new facility with the objective of minimizing the maximal distance between the new facility and a set of n given demand points. For a survey of solution methods see Hearn and Vijay [4], and a comprehensive list of references can be found in Domschke and Drexl [1]. An Optimal $O(n)$ Algorithm is given in [6].

The Elzinga and Hearn algorithm [3] is very easy to program. It was found by testing randomly generated problems [2, 4] that this algorithm is very efficient. Each iteration is of complexity $O(n)$ and the number of iterations seems not to go up with n. Therefore, it is the most widely used method. The algorithm is defined in the next section.

Many researchers are interested in the complexity of this algorithm [5], but no satisfactory result is known yet.

In this paper we show that the number of iterations can be as high as $O(n)$ and therefore, the complexity of the algorithm is $\Omega(n^2)$. We also prove that the algorithm reaches an ϵ-approximate solution in $O(n)$ time. We show that this time is bounded by $O(n \log^2 \epsilon)$.

The algorithm. The algorithm is based on the theorem [2, 3] that the optimal location of the 1-center problem is based on a subset of two or three points from which the optimal solution point is equidistant. Three points are randomly selected, and the center minimizing the maximal distance to these three points is calculated. Then the demand point farthest from this center is added to the group, and one point drops from it, so another group of three points is defined. The algorithm stops with the solution when the farthest demand point is not farther than the three demand points of the group. Let $N = \{1, \dots, n\}$ be the index set for all points.

The Elzinga-Hearn Algorithm.

Step (1) Set k (the iteration counter) to zero. Select a group I_0 of 3 points.

Step (2) Find O_k, the center of the smallest circle covering I_k, with a radius of r_k.

Step (3) Find $j \in N$, the farthest point from O_k, with the distance D_k. If $D_k = r_k$, stop with the solution point O_k.

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[†]California State University, Fullerton.
[‡]Hebrew University and University of Michigan.

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Step (4) Find the largest of the 3 radii of circles covering 3 points (2 points out of I_k and the third one is point j). Set I_{k+1} to the group with the largest radius. This group also defines O_{k+1}, r_{k+1}.

Step (5) Set $k = k + 1$, and go to Step 3.

The following lemma is trivial

LEMMA 1. *One iteration of the algorithm is executed in $O(n)$ time.*

The $O(n^2)$ example. The example problem consists of two demand points located at $(\delta, -1)$ and $(-\delta, -1)$ for a positive δ much smaller than the ϵ defined below. In the following we prove inequalities using in the calculations one demand point located at $(0, -1)$. There always exist a slack in these inequalities. δ must be so small such that the changes in the calculation are smaller than the slack in the inequalities and thus the inequalities still hold. In addition to these two demand points the problem contains n more demand points (for a total of $n + 2$ demand points) located at (x_i, y_i) for $i = 1, \ldots, n$ where:

$$x_i = (-1)^i 2^i \epsilon, \quad y_i = 1 - 0.6 \cdot 2^{2i} \epsilon^2, \quad 0 < \epsilon < 0.001 \cdot 2^{-2n}. \tag{1}$$

We first show that if in any iteration the three extreme points are (x_i, y_i), $(\delta, -1)$, and $(-\delta, -1)$, then the three extreme points at the next iteration are (x_{i+1}, y_{i+1}), $(\delta, -1)$, and $(-\delta, -1)$. The following lemma proves it by showing that (x_{i+1}, y_{i+1}) is the farthest from the center defined by (x_i, y_i), $(\delta, -1)$, and $(-\delta, -1)$. Note that this center is at $(x_i/2, (y_i - 1)/2)$ if we ignore δ. Also note that (x_i, y_i) is on the same side as (x_{i+1}, y_{i+1}) and thus dropped from the set of extreme points.

LEMMA 2. *For $j \neq i + 1$:*

$$(x_{i+1} - x_i/2)^2 + (y_{i+1} - (y_i - 1)/2)^2 > (x_j - x_i/2)^2 + (y_j - (y_i - 1)/2)^2. \tag{2}$$

PROOF. By rearranging (2): moving the x's to the left and the y's to the right we get

$$(x_{i+1} - x_j)(x_{i+1} + x_j - x_i) > (y_j - y_{i+1})(y_j + y_{i+1} - y_i + 1).$$

Substituting (1) leads to:

$$\left[-(-1)^i 2^{i+1}\epsilon - (-1)^j 2^j \epsilon\right]\left[-(-1)^i 2^{i+1}\epsilon + (-1)^j 2^j \epsilon - (-1)^i 2^i \epsilon\right]$$
$$> 0.6 \cdot \epsilon^2 [2^{2i+2} - 2^{2j}][2 - 0.6\epsilon^2 (2^{2j} + 2^{2i+2} - 2^{2i})].$$

Collecting terms and dividing by ϵ^2:

$$\left[(-1)^i 2^{i+1} + (-1)^j 2^j\right]\left[1.5(-1)^i 2^{i+1} - (-1)^j 2^j\right]$$
$$> 0.6[2^{2i+2} - 2^{2j}][2 - \epsilon^2(0.6 \cdot 2^{2j} + 0.45 \cdot 2^{2i-2})]$$

which leads to:

$$1.5 \cdot 2^{2i+2} - 2^{2j} + 0.5(-1)^{i+j} 2^{i+j+1}$$
$$> (2^{2i+2} - 2^{2j})(1.2 - \epsilon^2(0.36 \cdot 2^{2j} + 0.27 \cdot 2^{2i+2})),$$

$$0.3 \cdot 2^{2i+2} + 0.2 \cdot 2^{2j} + 0.5(-1)^{i+j} 2^{i+j+1}$$
$$> -(2^{2i+2} - 2^{2j})\epsilon^2(0.36 \cdot 2^{2j} + 0.27 \cdot 2^{2i+2}). \tag{3}$$

Note that the absolute value of the right-hand side of (3) is less than 10^{-6}. We consider the left-hand side of (3) and distinguish between two cases.

Case a. $j > i + 1$. If $j = i + 2$, then all the terms on the left-hand side are positive. If $j \geqslant i + 3$, then $2^{i+j+1} \leqslant 2^{2j-2} = 0.25 \cdot 2^{2j}$. Therefore, the sum of the last two terms is at least $0.075 \cdot 2^{2j}$ and thus positive. The sum on the left-hand side of (3) is greater than 0.3.

Case b. $j < i + 1$. $2^{i+j+1} \leqslant 2^{2i+1} = 0.5 \cdot 2^{2i+2}$. Therefore, the sum of the first and last terms is at least $0.05 \cdot 2^{2i+2}$ and thus positive. The sum of the terms on the left-hand side of (3) is greater than 0.2.

In conclusion, inequality (3) holds, and the lemma is proved. ∎

It is interesting to note that the choice of "0.6" in (1) is essential for the proof of Lemma 2.

THEOREM 1. *If the Elzinga-Hearn algorithm starts with the choice of* (x_1, y_1), $(\delta, -1)$, *and* $(-\delta, -1)$, *then the algorithm requires* $n - 1$ *iterations, and therefore it requires* $O(n^2)$ *time.*

PROOF. Trivial by the Lemmas 1 and 2.

Note that if the initial set of demand points is picked at random, then the expected number of iterations is still $O(n)$.

ϵ-approximate solutions. With reference to the Elzinga-Hearn algorithm given above define: $R_k = \min_{1 \leqslant i \leqslant k}\{D_i\} = \min\{R_{k-1}, D_k\}$. F^*, the optimal value of the objective function, fulfills: $r_k \leqslant F^* \leqslant R_k$. The error at iteration k, ϵ_k, is defined as: $\epsilon_k = R_k/r_k - 1$. An ϵ-approximate solution is found when $\epsilon_k \leqslant \epsilon$. Note that the r_k are strictly increasing, and the R_k are decreasing (not necessarily strictly decreasing). In a finite number of iterations, K ($K \leqslant n(n^2 + 5)/6$ [2]), $\epsilon_K = 0$ and the algorithm terminates. This is because there are at most $n(n^2 + 5)/6$ different radii that enclose 2 or 3 points out of n. Since the radius r_k increases every iteration, the number of iterations is bounded by $n(n^2 + 5)/6$. Note that a small change in the algorithm can reduce the bound for the number of iterations to $O(n^2)$. In Step 3 choose j as the point that results in the largest radius r_{k+1} (rather than the largest distance). This way only $O(n^2)$ different radii can be selected at the end of Step 3. This is because only radii which are maximal for a pair of points and all possible third ones are possible as r_{k+1}. Therefore, the number of iterations is bounded by $O(n^2)$.

THEOREM 2. $\epsilon_2 \leqslant 1$.

PROOF. Since O_0 is in the convex hull of the 3 points of I_0, the distance between point j and at least one of the points of I_0 is at least D_0. Thus, the circle covering $I_0 \cup \{j\}$ has a radius of at least $D_0/2$. Therefore at the second iteration: $1 + \epsilon_2 \leqslant D_0/(D_0/2) = 2$, and therefore $\epsilon_2 \leqslant 1$. ∎

THEOREM 3.

$$\epsilon_{k+1} \leqslant \frac{1 + \epsilon_k}{\left[1 + \left\{\dfrac{\epsilon_k(2 + \epsilon_k)}{2 + 2\epsilon_k}\right\}^2\right]^{0.5}} - 1.$$

PROOF. Without loss of generality we can assume that $r_k = 1$ and that the two remaining points of I_k are located at $(\cos\theta_k, -\sin\theta_k)$, $(-\cos\theta_k, -\sin\theta_k)$, and point j is at: $[(1 + \epsilon_k)\cos\phi_k, (1 + \epsilon_k)\sin\phi_k]$ for $0 \leqslant \theta_k \leqslant \pi/2$, $0 \leqslant \phi_k \leqslant \pi/2$. The center of the smallest circle covering these points is at $(0, Y_k)$. We get:

$$\cos^2\theta_k + (y_k + \sin\theta_k)^2 = (1 + \epsilon_k)^2\cos^2\phi_k + [(1 + \epsilon_k)\sin\phi_k - y_k]^2$$

which yields:

$$y_k = \frac{\epsilon_k(1+\epsilon_k/2)}{\sin\theta_k + (1+\epsilon_k)\sin\phi_k}. \tag{4}$$

The radius, r_{k+1}, of this circle is:

$$r_{k+1}^2 = \cos^2\theta_k + (y_k + \sin\theta_k)^2 = 1 + 2y_k\sin\theta_k + y_k^2, \tag{5}$$

$$r_{k+1}^2 = 1 + \frac{\epsilon_k(2+\epsilon_k)\left[\sin^2\theta_k + (1+\epsilon_k)\sin\phi_k\sin\theta_k + \epsilon_k(2+\epsilon_k)/4\right]}{\left[\sin\theta_k + (1+\epsilon_k)\sin\phi_k\right]^2}. \tag{6}$$

y_k and consequently r_{k+1}^2 decrease when $\sin\phi_k$ increases. Therefore, the minimal r_{k+1}^2 is achieved when $\sin\phi_k = 1$. It can be easily verified that for $\sin\phi_k = 1$, $dr_{k+1}^2/d\theta_k > 0$. Therefore, the minimal r_{k+1} occurs at $\theta_k = 0$. i.e.:

$$r_{k+1}^2 \geqslant 1 + \frac{\epsilon_k^2(2+\epsilon_k)^2}{4(1+\epsilon_k)^2}.$$

Since

$$\epsilon_{k+1} \leqslant \frac{1+\epsilon_k}{r_{k+1}} - 1$$

the theorem follows. ∎

LEMMA 3. $\epsilon_{k+1} \leqslant \epsilon_k - \epsilon_k^2/4$.

PROOF. Follows by straightforward calculations on the formula of Theorem 3, and assuming $\epsilon_k \leqslant 1$ by Theorem 2. ∎

LEMMA 4. $\epsilon_k \leqslant 4/(k+2)$ *for* $k \geqslant 2$.

PROOF. The proof follows by mathematical induction. The lemma is true for $k = 2$ by Theorem 2. The function $y = x - x^2/4$ is increasing for $x \leqslant 1$, therefore:

$$\epsilon_{k+1} \leqslant \epsilon_k - \epsilon_k^2/4 \leqslant 4/(k+2) - [4/(k+2)]^2/4 = 4(k+1)/(k+2)^2.$$

Now:

$$4(k+1)/(k+2)^2 < 4(k+1)/\left[(k+2)^2 - 1\right]$$
$$= 4(k+1)/[(k+1)(k+3)] = 4/(k+3),$$

and the lemma follows by induction. ∎

The following lemma and theorem are evident by Lemma 4.

LEMMA 5. *After* $4/\epsilon - 2$ *iterations, the accuracy of the solution is less than* ϵ.

THEOREM 4. *For a given* ϵ, *the Elzinga-Hearn algorithm reaches an* ϵ-*approximate solution in* $O(n)$ *time.*

THEOREM 5. *For a given* ϵ, *the worst problem needs at least* $O(\log\epsilon)$ *iterations to reach an* ϵ-*approximate solution, and therefore* $O(n\log\epsilon)$ *time is necessary for reaching it.*

PROOF. In the $O(n^2)$ example problem: after $n-2$ iterations the error is $O(1/2^n)$. For $\epsilon = O(1/2^n)$ (ϵ less than the error after $n-2$ iterations), $n-1 = O(\log\epsilon)$ iterations are required to get an ϵ-approximate solution. ∎

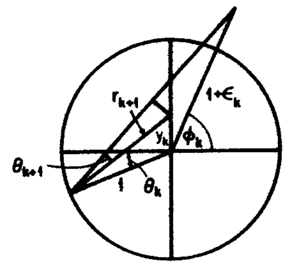

FIGURE 1. Calculating θ_{k+1}.

A better bound. In the previous section we showed that finding an ϵ-approximate solution is bounded by $O(n/\epsilon)$. In the following we show a better bound, namely $O(n(\log \epsilon)^2)$. First we find a relationship between θ_k and θ_{k+1}. By direct calculations (see Figure 1):

$$\sin \theta_{k+1} = \frac{(1+\epsilon_k)\sin(\phi_k - \theta_k) - y_k\left[(1+\epsilon_k)\cos\phi_k + \cos\theta_k\right]}{r_k\sqrt{(1+\epsilon_k)^2 + 1 + 2(1+\epsilon_k)\cos(\phi_k - \theta_k)}}. \tag{7}$$

Define: $\delta_k = (r_{k+1}^2 - 1)/(2\epsilon_k)$. By (5): $r_{k+1}^2 > 1 + 2y_k \sin\theta_k$. Therefore, by (4):

$$\delta_k > \frac{y_k \sin\theta_k}{\epsilon_k} = \frac{(1+\epsilon_k/2)\sin\theta_k}{\sin\theta_k + (1+\epsilon_k)\sin\phi_k}.$$

For a small ϵ_k, δ_k:

$$\delta_k > \theta_k. \tag{8}$$

In the following we assume small ϵ_k, δ_k and therefore also small θ_k. All equalities that follow are approximations assuming these small values. By (6):

$$\delta_k = \frac{\theta_k^2 + \theta_k \sin\phi_k + \epsilon_k/2}{(\theta_k + \sin\phi_k)^2}. \tag{9}$$

Or:

$$\sin\phi_k = \frac{\theta_k + \sqrt{\theta_k^2 + 2\delta_k\epsilon_k}}{2\delta_k}. \tag{10}$$

Since δ_k is small, θ_k is negligible when compared with $\sin\phi_k$ by equation (10). Substituting (4), (10) into (7):

$$\theta_{k+1} = \frac{\theta_k + \sqrt{\theta_k^2 + 2\delta_k\epsilon_k} - 4(1+\cos\phi_k)\delta_k^2\epsilon_k/\left(\theta_k + \sqrt{\theta_k^2 + 2\delta_k\epsilon_k}\right)}{2\delta_k\sqrt{2 + 2\cos\phi_k}}. \tag{11}$$

Since θ_{k+1} increases when θ_k increases, substituting $\theta_k = 0$ yields:

$$\theta_{k+1} \geqslant \frac{\sqrt{2\delta_k\epsilon_k} - 4(1+\cos\phi_k)\delta_k^2\epsilon_k/\sqrt{2\delta_k\epsilon_k}}{2\delta_k\sqrt{2+2\cos\phi_k}}, \quad \theta_{k+1} \geqslant \sqrt{\epsilon_k/(8\delta_k)}. \tag{12}$$

Also, by another approximation of (11):

$$\theta_{k+1} \geqslant \frac{2\theta_k}{2\delta_k\sqrt{2 + 2\cos\phi_k}} \geqslant \frac{\theta_k}{2\delta_k}. \tag{13}$$

Let $\delta = \max_{0 \leqslant s \leqslant t}\{\delta_{k+s}\}$. By (12), (13):

$$\theta_{k+t} \geqslant \frac{\theta_{k+1}}{(2\delta)^{t-1}} > \frac{\sqrt{\epsilon_k}/2}{(2\delta)^{t-0.5}}.$$

By (8) and by the definition of δ:

$$\delta \geqslant \delta_{k+t} > \theta_{k+t} > \frac{\sqrt{\epsilon_k}/2}{(2\delta)^{t-0.5}}.$$

which yields:

$$2\delta > \epsilon_k^{1/(2t+1)}. \tag{14}$$

There exists a $0 \leqslant v \leqslant t$ for which $\delta_{k+v} = \delta$, and thus:

$$2\delta_{k+v} > \epsilon_k^{1/(2t+1)},$$

$$\epsilon_{k+v+1} \leqslant \frac{1 + \epsilon_{k+v}}{r_{k+v+1}} - 1 = \frac{1 + \epsilon_{k+v}}{1 + \delta_{k+v}\epsilon_{k+v}} - 1$$

$$= \epsilon_{k+v} - \delta_{k+v}\epsilon_{k+v} < \epsilon_{k+v}\left(1 - 0.5\epsilon_k^{1/(2t+1)}\right).$$

Since (by Lemma 3) $\epsilon_{k+t+1} \leqslant \epsilon_{k+v+1}$, $\epsilon_{k+v} \leqslant \epsilon_k$:

$$\epsilon_{k+t+1} < \epsilon_k\left(1 - 0.5\epsilon_k^{1/(2t+1)}\right). \tag{15}$$

After t iterations where $2t + 1 = -\log_2\epsilon_k$:

$$\epsilon_k^{1/(2t+1)} = 2^{-\log_2\epsilon_k/\log_2\epsilon_k} = 0.5.$$

Therefore, after t iterations (t as above):

$$\epsilon_{k+t+1} < 0.75\epsilon_k. \tag{16}$$

Theorem 6. *In $O(\log^2\epsilon)$ iterations the Elzinga-Hearn algorithm reaches an ϵ-approximate solution. This means that the Elzinga-Hearn algorithm reaches an ϵ-approximate solution in $O(n\log^2\epsilon)$ time.*

Proof. By (16): in $O(\log\epsilon_k) \leqslant O(\log\epsilon)$ iterations ϵ_k is multiplied by at most 0.75. Therefore, in $O(\log\epsilon)$ times $O(\log\epsilon)$ iterations, an ϵ-approximate solution is reached. ∎

Note that when a finite length of numbers is used to calculate the values in the algorithm, the optimum for that accuracy is achieved in $O(n)$ time. The number of iterations does not exceed the order of the square of the number of digits.

Also note that if the coordinates of the points of the problem are integers in the range $[0, N]$, i.e. the number of digits for each coordinate is bounded by $\log_2 N$, then the centers of circles covering two or three points are rational numbers with a denominator bounded by $O(N^2)$. Therefore, the solution point and a point at the end of any iteration are rational numbers with a denominator bounded by $O(N^2)$. Therefore, r_k^2 and R_k^2 are rational numbers with denominator bounded by $O(N^4)$. Therefore, $(1 + \epsilon_k)^2 - 1 = (R_k^2 - r_k^2)/r_k^2$ is bounded by $1/O(N^8)$. Therefore, when $\epsilon_k < 1/O(N^8)$, the optimal solution must have been found. This yields a time of $O(n \log^2 N)$ for finding the optimal solution.

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DREZNER: SCHOOL OF BUSINESS ADMINISTRATION & ECONOMICS, CALIFORNIA STATE UNIVERSITY, FULLERTON, CALIFORNIA 92634

SHELAH: INSTITUTE OF MATHEMATICS & COMPUTER SCIENCE, HEBREW UNIVERSITY, JERUSALEM, ISRAEL AND ELECTRICAL ENGINEERING & COMPUTER SCIENCE & MATHEMATICS DEPARTMENTS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109